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A new equation describing travelling water waves

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A new single equation for the surface elevation of a travelling water wave in an incompressible, inviscid, irrotational fluid is derived. This new equation is derived without approximation from Euler's equations, valid for both a one- and two-dimensional travelling-wave surface. We show that this new formulation can be used to efficiently derive higher-order Stokes-wave approximations, and pose that this new formulation provides a useful framework for further investigation of travelling water waves.

Key words: mathematical foundations, surface gravity waves, waves/free-surface flows

1. Introduction

The objective of this paper is to introduce a new formulation describing travelling-wave solutions to Euler's equations. Assuming an irrotational, inviscid and incompressible flow, the governing equations for the fluid surface $\eta(\boldsymbol{x}, t)$ and velocity potential $\phi(\boldsymbol{x}, z, t)$ are given by

$$\Delta\phi + \phi_{zz} = 0, \quad (\boldsymbol{x}, z) \in S \times [-h, \eta], \tag{1.1}$$

$$\phi_z = 0, \quad z = -h, \tag{1.2}$$

$$\eta_t + \nabla\eta \cdot \nabla\phi = \phi_z, \quad z = \eta(\boldsymbol{x}, t), \tag{1.3}$$

$$\phi_t + \frac{1}{2}(|\nabla\phi|^2 + \phi_z^2) + g\eta = \frac{\sigma}{\rho}\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right), \quad z = \eta(\boldsymbol{x}, t), \tag{1.4}$$

where S is a subset of $\mathbb{R}^2$, $\boldsymbol{x} = (x_1, x_2)$ are the horizontal directions, z is the vertical coordinate, $\nabla = (\partial_{x_1}, \partial_{x_2})$, and $\Delta = \nabla^2$. Furthermore, g is the acceleration due to gravity, h is the constant depth of the fluid when at a state of rest, σ represents the coefficient of surface tension, and ρ is the constant fluid density.

To investigate travelling-wave solutions, we make the change of variables $\boldsymbol{x} \to \tilde{\boldsymbol{x}} - \boldsymbol{c}t$ and equate all t-derivatives to zero. The resulting equations of motion are

$$\Delta\phi + \phi_{zz} = 0, \quad (\boldsymbol{x}, z) \in S \times [-h, \eta], \tag{1.5}$$

$$\phi_z = 0, \quad z = -h, \tag{1.6}$$

$$-\boldsymbol{c} \cdot \nabla\eta + \nabla\eta \cdot \nabla\phi = \phi_z, \quad z = \eta(\boldsymbol{x}), \tag{1.7}$$

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$$-\boldsymbol{c}\cdot\nabla\phi+\frac{1}{2}(|\nabla\phi|^2+\phi_z^2)+g\eta=\frac{\sigma}{\rho}\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right),\quad z=\eta(\boldsymbol{x}),\tag{1.8}$$

where $\boldsymbol{c}=(c_1,c_2)$, and we have dropped the tildes for simplicity.

As written above, the equations of motion are challenging to work with directly: they are a free-boundary problem with nonlinear boundary conditions. Specifically, one must solve Laplace's equation inside an unknown domain while simultaneously satisfying the highly nonlinear boundary conditions applied at the unknown free surface η.

Various reformulations of (1.5)–(1.8) have been presented in the literature in order to simplify the equations of motion. For example, for one-dimensional surfaces (no x_2-variable), conformal mappings have been used to eliminate these problems (for an overview, see Dyachenko *et al.* 1996; Okamoto & Shoji 2001). However, the conformal mapping approach does not generalize to two-dimensional surfaces.

For both one- and two-dimensional surfaces, other formulations (such as the Hamiltonian formulation given in Zakharov 1968 or the Zakharov–Craig–Sulem formulation, Craig & Sulem 1993) reduce the Euler equations to a system of two equations, in terms of surface variables only, by introducing a Dirichlet-to-Neumann operator (DNO). In a similar spirit, Ablowitz, Fokas & Musslimani (2006) introduce a new non-local formulation of Euler's equation (henceforth referred to as the AFM formulation) that results in a system of two equations for the same variables as presented in the DNO formulation.

Both the DNO and AFM formulations reduce the problem from the full fluid domain to a system of equations that depend on the surface elevation η and the velocity potential evaluated at the surface $q(\boldsymbol{x})$, where

$$q(\boldsymbol{x})=\phi(\boldsymbol{x},\eta(\boldsymbol{x})).\tag{1.9}$$

While this simplification significantly reduces the computational domain, one could argue that the equations require solving for an additional function q, which is typically of less interest and not easily measured in experiments. The primary interest in applications is determining the surface elevation η.

For the one-dimensional case, this issue has been addressed by various authors. For instance, several authors have found a single nonlinear integro-differential equation which can be solved for the surface elevation (see for example Longuet-Higgins 1978; Babenko 1987; Dyachenko *et al.* 1996; Toland 2002). In Deconinck & Oliveras (2011), the authors found a different single nonlinear integro-differential equation in physical variables which accomplishes the same. However these formulations only apply to the one-dimensional problem.

In this paper, we present a new scalar equation that determines travelling-wave solutions valid for both one- and two-dimensional surfaces. Although we begin in a similar manner to the Hamiltonian, DNO, and AFM formulation (Zakharov 1968; Craig & Sulem 1993; Ablowitz *et al.* 2006), we propose (*a*) to replace the velocity potential q by the tangential component of the velocity at the free surface ∇q; and (*b*) to replace the DNO by the normal-to-tangential derivative operator $H(\eta,D)$, which can be regarded as a generalization of the Hilbert transform. These two key ideas are physically motivated, and significantly reduce the number of calculations involved to compute asymptotic expansions for travelling-wave solutions.

Combining (1.7) and (1.8) we obtain

$$-\boldsymbol{c}\cdot\nabla q+\frac{1}{2}\left|\nabla q\right|^2+g\eta-\frac{1}{2}\frac{(\nabla\eta\cdot(\nabla q-\boldsymbol{c}))^2}{1+\left|\nabla\eta\right|^2}=0. \tag{1.10}$$

Equation (1.10) is an equation for the two unknowns (q,η). In order to find a single equation for the surface η, we introduce the operator $\mathscr{H}(\eta,D)$ for the Laplace equation which defines the following normal-to-tangential derivative map:

$$\mathscr{H}(\eta,D)\{-\boldsymbol{c}\cdot\nabla\eta\}=\nabla q, \tag{1.11}$$

where we assume that $(q,\eta,\boldsymbol{c})$ is a solution set to (1.5)–(1.8), and D is the typical differential operator defined as $D=-\mathrm{i}\nabla$. The quantity $-\boldsymbol{c}\cdot\nabla\eta$ is the normal derivative of the potential due to (1.7). Using this operator and (1.10) we can write a single equation for the water-wave surface in a travelling coordinate frame as

$$\begin{aligned}-\boldsymbol{c}\cdot\mathscr{H}(\eta,D)\{-\boldsymbol{c}\cdot\nabla\eta\}+\frac{1}{2}\left|\mathscr{H}(\eta,D)\{-\boldsymbol{c}\cdot\nabla\eta\}\right|^2&\\+\frac{1}{2}\frac{(\nabla\eta\cdot(\mathscr{H}(\eta,D)\{-\boldsymbol{c}\cdot\nabla\eta\}-\boldsymbol{c}))^2}{1+\left|\nabla\eta\right|^2}+g\eta=0.&\end{aligned} \tag{1.12}$$

The above equation only depends on the surface elevation $\eta(\boldsymbol{x})$ and is completely independent of the velocity potential q. Thus, (1.12) represents a scalar equation for the water-wave surface expressed in terms of η which is valid for both a one- and two-dimensional travelling wave.

The usefulness of (1.12) is dependent on finding a representation for the operator $\mathscr{H}(\eta,D)$. In the following sections, we describe a method for determining $\mathscr{H}(\eta,D)$ based on the work of Ablowitz & Haut (2008). Using this representation and (1.12), we determine formal expansions of asymmetric travelling gravity water waves for a two-dimensional surface in finite depth.

Remark 1.1. *The approach presented here should be contrasted with the approach taken in Deconinck & Oliveras (2011). In that paper, the authors first solved the Bernoulli equation for q_x in terms of η and replaced the corresponding terms in the non-local equation of the AFM formulation. In the current work, we pose that we have reversed this order. Here we effectively solve the non-local equation of the AFM formulation for q_x in terms of η and substitute the corresponding term in the Bernoulli equation to obtain a single equation valid for both one- and two-dimensional waves.*

2. Taylor series for the operator $\mathscr{H}(\eta,D)$

In this section we discuss the relationship between $\mathscr{H}(\eta,D)$ and the Dirichlet-to-Neumann operator (DNO) given by Craig & Sulem (1993). Let $\mathscr{G}(\eta)$ represent the DNO for the water-wave problem. As described in Craig & Sulem (1993), the equations of motion for η and q implied by (1.5)–(1.8) are

$$-\boldsymbol{c}\cdot\nabla\eta=\mathscr{G}(\eta)q, \tag{2.1}$$

$$-\boldsymbol{c}\cdot\nabla q+\frac{1}{2}\left|\nabla q\right|^2+g\eta-\frac{1}{2}\frac{(\nabla\eta\cdot(\nabla q-\boldsymbol{c}))^2}{1+\left|\nabla\eta\right|^2}=0. \tag{2.2}$$

One could reduce the above system of equations to a single equation for η if we could find the inverse of the operator $\mathscr{G}(\eta)$. This would allow us to write q in terms of the inverse operator $\mathscr{G}^{-1}(\eta)$, η and $\boldsymbol{c}$. However, it is readily seen that $\mathscr{G}(\eta)$ does not have a unique inverse since solutions to the Neumann problem for the Laplace equation are unique up to a constant.

In the light of the way in which the Dirichlet data appear in the non-local equation of the AFM formulation (equation (I) of Ablowitz *et al.* 2006), a map from the Neumann data to the gradient of the Dirichlet data at the surface of the fluid, i.e. a normal-to-tangential derivative operator, is reasonable, particularly when we recognize that q appears in (2.2) through its gradient alone. The operator $\mathscr{H}(\eta, D)$ has the following formal relationship with the DNO: $\mathscr{H}(\eta, D)\mathscr{G}(\eta, D) \equiv \boldsymbol{\nabla}$.

2.1. *Taylor series of $\mathscr{H}(\eta)$ for the two-dimensional surface*

In order to compute travelling-wave surfaces, one would like to compute the operator $\mathscr{H}(\eta, D)$. There are several ways to find a representation for this operator including, but not limited to, the Taylor series approach taken for the DNO by Craig & Sulem (1993), or the different approach taken for the DNO given by Ablowitz & Haut (2008). We choose to follow the latter as it leads to an easier numerical implementation.

To determine an expression for $\mathscr{H}(\eta, D)$, we assume that $\mathscr{H}(\eta, D)$ has a Taylor series representation in η of the form

$$\mathscr{H}(\eta, D)\{f\} = \sum_{j=0}^{\infty} \mathscr{H}_j(\eta, D)\{f\}, \tag{2.3}$$

where each $\mathscr{H}_j(\eta, D)$ is homogeneous of order j in η, i.e. $\mathscr{H}_j(\lambda\eta, D) = \lambda^j \mathscr{H}_j(\eta, D)$.

Consider the following boundary value problem:

$$\Delta\phi = 0, \quad (\boldsymbol{x}, z) \in S \times [-h, \eta], \tag{2.4}$$

$$\frac{\partial\phi}{\partial n}(\boldsymbol{x}, \eta) = f(\boldsymbol{x}), \tag{2.5}$$

$$\frac{\partial\phi}{\partial z}(\boldsymbol{x}, -h) = 0, \tag{2.6}$$

where $\partial/\partial n$ represents the normal derivative and S is a subset of $\mathbb{R}^2$. In the horizontal direction, we consider either periodic boundary conditions or decay at infinity. For the infinite-domain case we assume f is suitably smooth and has appropriate decay.

Following Ablowitz *et al.* (2006) and Ablowitz & Haut (2008), it can be shown that $f(\boldsymbol{x})$ satisfies the relationship

$$\int_S \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \left[\mathrm{i}\cosh(|\boldsymbol{k}|(\eta + h)) f(\boldsymbol{x}) - \frac{\sinh(|\boldsymbol{k}|(\eta + h))}{|\boldsymbol{k}|} (\boldsymbol{k} \cdot \mathscr{H}(\eta, D)\{f(\boldsymbol{x})\}) \right] \mathrm{d}\boldsymbol{x} = 0, \tag{2.7}$$

where $\boldsymbol{k}$ is determined by the boundary conditions in $\boldsymbol{x}$. For the infinite-line case where (2.4)–(2.6) decay as $|\boldsymbol{x}| \to \infty$, $\boldsymbol{k} \in \mathbb{R}^2$. However, if we consider (2.4)–(2.6) with periodic boundary conditions (where $f(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{L})$), then the vector $\boldsymbol{k}$ is restricted to the dual lattice Λ' of the problem's period lattice Λ:

$$\Lambda = \{\boldsymbol{L} = m_1\boldsymbol{L}_1 + m_2\boldsymbol{L}_2 | m_j \in \mathbb{Z}, \boldsymbol{L}_j \cdot \boldsymbol{k}_l = 2\pi\delta_{jl}\}, \tag{2.8}$$

so that

$$\Lambda' = \{\boldsymbol{k} = n_1\boldsymbol{k}_1 + n_2\boldsymbol{k}_2 | n_j \in \mathbb{Z}, n_1^2 + n_2^2 > 0\}, \tag{2.9}$$

where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are linearly independent vectors in $\mathbb{R}^2$.

A calculation similar to the one presented in Ablowitz & Haut (2008) allows us to determine the following recursive relationship for $\mathscr{H}_j(\eta, D)$ in terms of lower-order terms:

$$\int_S \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \mathscr{H}_j(\eta, D)\{f\}\,\mathrm{d}\boldsymbol{x} = \mathrm{i}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\int_S \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \frac{(|\boldsymbol{k}|\eta)^j}{j!}\begin{bmatrix}\coth(|\boldsymbol{k}|h), & j \text{ even}\\ 1, & j \text{ odd}\end{bmatrix} f\,\mathrm{d}\boldsymbol{x}$$
$$-\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\int_S \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{m=1}^{j}\left(\frac{\boldsymbol{k}\cdot\mathscr{H}_{j-m}(\eta, D)\{f\}}{|\boldsymbol{k}|}\frac{(|\boldsymbol{k}|\eta)^m}{m!}\begin{bmatrix}1, & m \text{ even}\\ \coth(|\boldsymbol{k}|h), & m \text{ odd}\end{bmatrix}\right)\mathrm{d}\boldsymbol{x}. \quad (2.10)$$

In the above, we have used the brackets [ ] as a conditional multiplier at the appropriate index of summation. The only difference between infinite-domain and periodic boundary conditions is the allowable values of the vector $\boldsymbol{k}$.

We proceed to find $\mathscr{H}_0(\eta, D)$ by equating $j = 0$ in (2.10) and evaluating the Fourier transform of that expression to show that

$$\mathscr{H}_0(\eta, D)\{f(\boldsymbol{x}')\} = \int_{\mathbb{R}^2}\mathrm{d}\boldsymbol{k}\int_S \mathrm{d}\boldsymbol{x}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{i}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\coth(|\boldsymbol{k}|h)f, \quad (2.11)$$

in the case of the whole line, and

$$\mathscr{H}_0(\eta, D)\{f(\boldsymbol{x}')\} = \sum_{\boldsymbol{k}\in\Lambda}\int_S \mathrm{d}\boldsymbol{x}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{i}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\coth(|\boldsymbol{k}|h)f, \quad (2.12)$$

for the case of periodic boundary conditions. Since the above equation must be true for arbitrary $f(\boldsymbol{x})$ which satisfies our boundary conditions, we see that the symbol of the pseudo-differential operator $\mathscr{H}_0(\eta, D)$ is given by

$$\mathscr{H}_0(\eta, \boldsymbol{k}) = \mathrm{i}\coth(|\boldsymbol{k}|h)\frac{\boldsymbol{k}}{|\boldsymbol{k}|}. \quad (2.13)$$

We may now continue to first-order to find $\mathscr{H}_1(\eta, D)$ in terms η, $\mathscr{H}_0(\eta)$ and D.

Remark 2.1. *The approach taken above is different from that presented in Craig & Sulem (1993), which however leads to an equivalent expression. The expansion given by (2.10) is more suited for numerical implementation as one does not need to store the operator $\mathscr{H}_j(\eta, D)$ at every order but only the previously determined action on f: $\mathscr{H}_j(\eta, D)\{f\}$.*

2.2. *Restriction to a one-dimensional surface*

On restricting the problem to a one-dimensional surface, we find the following pseudo-differential operators:

$$\mathscr{H}_0(\eta, D) = \mathrm{i}\coth(hD), \quad (2.14a)$$
$$\mathscr{H}_1(\eta, D) = \mathrm{i}\left[D\eta - \coth(hD)D\eta\coth(hD)\right], \quad (2.14b)$$
$$\mathscr{H}_2(\eta, D) = \tfrac{1}{2}\left[i\eta^2 D^2\coth(hD) - \mathscr{H}_0(\eta)D\eta^2 D - 2\mathscr{H}_1(\eta)D\eta\coth(hD)\right], \quad (2.14c)$$

where we have written the operators in a form suitable for comparison with the DNO of Craig & Sulem (1993). The symbol of each operator has a singularity at $k = 0$; this singularity is always of order one. Since the operator $\mathscr{H}$ acts on the normal derivative of a function the singularity is cancelled.

3. Stokes-wave expansion for two-dimensional surfaces

The new formulation provides a single, scalar equation which can be used to rigorously investigate both one- and two-dimensional travelling waves. In this section, we show that the expected Stokes-wave asymptotic expansions for periodic waves can be obtained with arguably less effort than using other methods, especially for two-dimensional surfaces.

Consider travelling waves for a two-dimensional surface where the surface elevation $\eta(\boldsymbol{x})$ is doubly periodic with a period lattice given by (2.8). Using the traditional formulation, the resulting equations of motion are quite complicated and require tedious calculations to generate Stokes-wave solutions even to second- and third-order. In this section, we extend the results of Iooss & Plotnikov (2011) to finite-depth, non-symmetric water waves using our single equation.

We consider the expansion for the surface $\eta(\boldsymbol{x})$ and wave speed $\boldsymbol{c}$ given by

$$\eta(\boldsymbol{x}) = \sum_{j=1}^{\infty} \epsilon^j \eta_j(\boldsymbol{x}), \quad \boldsymbol{c} = \sum_{j=0}^{\infty} \epsilon^j \boldsymbol{c}_j, \tag{3.1}$$

where each $\eta_j(\boldsymbol{x})$ is a doubly periodic function of both x_1 and x_2 with (for now) undetermined periods L_1 and L_2. The parameter ϵ is treated as an accounting parameter that allows us to establish the asymptotic nature of the η_j and c_j (for example, $\eta_j \gg \eta_{j+1}$).

Substituting the above expansions into (1.12) and collecting like powers of ϵ, we find at leading order

$$\left(g - \frac{\coth(h|\boldsymbol{k}|)\,(\boldsymbol{k}\cdot\boldsymbol{c}_0)^2}{|\boldsymbol{k}|} \right) \hat{\eta}_1(\boldsymbol{k}) = 0, \tag{3.2}$$

for all $\boldsymbol{k} = (k_1, k_2)^{\mathrm{T}} \in \Lambda'$. We wish to find non-trivial solutions to the above equation. Indeed non-trivial Fourier coefficients of $\eta(\boldsymbol{x})$ are associated with the zeros of the term in parentheses.

At this point, we have two options to determine $\eta_1(\boldsymbol{x})$: (i) we may pick a value of c_0 which would then determine the non-trivial wavenumbers of the solution by forcing k_1 and k_2 to be in an appropriate lattice of wavenumbers and thereby determining the period; or (ii) we can fix the periods (equivalently the wavenumbers) of the solution and determine the corresponding wave speed c_0.

Following the work of Craig & Nicholls (2002) and Iooss & Plotnikov (2009) and Iooss & Plotnikov (2011), we consider two different wave vectors $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ that are linearly independent which allows (3.2) to be solved for precisely the same value of c_0. With this choice of the vectors $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$, the lattice Λ and corresponding dual lattice Λ' for the full solutions are determined. The chosen values of $\boldsymbol{k}^{(j)}$ for $j = 1, 2$ may not be the only vectors which give rise to a specific c_0 but this is not important for what follows.

For convenience, we introduce the parameters θ_1, θ_2, and τ such that our vectors $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ can be represented as follows:

$$\boldsymbol{k}^{(1)} = \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{pmatrix}, \quad \boldsymbol{k}^{(2)} = \tau \begin{pmatrix} \cos(\theta_2) \\ -\sin(\theta_2) \end{pmatrix}. \tag{3.3}$$

Remark 3.1. *Our choice of expansion includes possible solutions referred to as 'diamond' or 'short-crested' waves. For example, if $\theta_1 = \theta_2$ we restrict ourselves to the study of short-crested waves that are symmetric to the direction of propagation. Furthermore, if we restrict $\tau = 1$, we recover the diamond waves as described Bridges, Dias & Menasce (2001).*

Since (3.2) is chose to be valid for our two linearly independent vectors $\boldsymbol{k}^{(j)}$ for $j=1,2$, then we have the following condition that allows us to solve for $\boldsymbol{c}_0$ (the leading order wave speed):

$$\boldsymbol{c}_0 = \frac{\pm 1}{\sin(\theta_1+\theta_2)}\sqrt{\frac{g}{\tau}}\begin{pmatrix}\sin(\theta_2) & \sin(\theta_1)\\ \cos(\theta_2) & -\cos(\theta_1)\end{pmatrix}\begin{pmatrix}\sqrt{\tau\tanh(h)}\\ \sqrt{\tanh(\tau h)}\end{pmatrix}. \tag{3.4}$$

Using the relationships $|\boldsymbol{k}^{(1)}|=1$, $|\boldsymbol{k}^{(2)}|=\tau$ and $\boldsymbol{c}_0$ as given by (3.3), we have the basic form for $\eta_1(\boldsymbol{x})$ given by

$$\eta_1(x) = A_1 \mathrm{e}^{\mathrm{i}\boldsymbol{k}^{(1)}\cdot\boldsymbol{x}} + A_1^* \mathrm{e}^{-\mathrm{i}\boldsymbol{k}^{(1)}\cdot\boldsymbol{x}} + A_2 \mathrm{e}^{\mathrm{i}\boldsymbol{k}^{(2)}\cdot\boldsymbol{x}} + A_2^* \mathrm{e}^{-\mathrm{i}\boldsymbol{k}^{(2)}\cdot\boldsymbol{x}}. \tag{3.5}$$

Thus we have the following leading-order approximations:

$$\eta(\boldsymbol{x}) = \epsilon(A_1 \mathrm{e}^{\mathrm{i}\boldsymbol{k}^{(1)}\cdot\boldsymbol{x}} + A_2 \mathrm{e}^{\mathrm{i}\boldsymbol{k}^{(2)}\cdot\boldsymbol{x}} + \text{c.c.}) + O(\epsilon^2), \tag{3.6}$$

$$\boldsymbol{c}_0 = \frac{\pm 1}{\sin(\theta_1+\theta_2)}\sqrt{\frac{g}{\tau}}\begin{pmatrix}\sin(\theta_2) & \sin(\theta_1)\\ \cos(\theta_2) & -\cos(\theta_1)\end{pmatrix}\begin{pmatrix}\sqrt{\tau\tanh(h)}\\ \sqrt{\tanh(\tau h)}\end{pmatrix}, \tag{3.7}$$

where c.c. represents the complex conjugate. Proceeding to the next-order correction, we find

$$\begin{aligned}&\sum_{\boldsymbol{k}\in\Lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left(g - \frac{\coth(h|\boldsymbol{k}|)\,(\boldsymbol{k}\cdot\boldsymbol{c}_0)^2}{|\boldsymbol{k}|}\right)\hat{\eta}_2(\boldsymbol{k})\\ &\quad = 2\boldsymbol{c}_1\cdot\mathscr{H}_0(0)\{-\boldsymbol{c}_0\cdot\nabla\eta_1\} + \boldsymbol{c}_0\cdot\mathscr{H}_1(\eta_1)\{-\boldsymbol{c}_0\cdot\nabla\eta_1\} + \tfrac{1}{2}\left(\nabla\eta_1\cdot c_0\right)^2\\ &\qquad + \tfrac{1}{2}\left|\mathscr{H}_0(0)\{-\boldsymbol{c}_0\cdot\nabla\eta_1\}\right|^2\end{aligned} \tag{3.8}$$

where we have introduced the notation $\mathscr{H}_j$ to represent the vector generated by the jth Taylor series coefficient for the expansion of the $\mathscr{H}(\eta, D)$ operator. This allows us to determine an expression for the $\boldsymbol{k}$th Fourier coefficient of $\eta_2(\boldsymbol{x})$. Since $\hat{\eta}_1(\boldsymbol{k})=0$ unless $\boldsymbol{k}=\pm\boldsymbol{k}^{(j)}$ for $j=1,2$, we can greatly simplify the expression for the unknown function $\eta_2(\boldsymbol{x})$ by noting that the $\boldsymbol{k}$th Fourier coefficient of the right-hand side of (3.8) is identically zero unless $\boldsymbol{k}=\pm\boldsymbol{k}^{(j)}$ for $j=1,2$, $\boldsymbol{k}=\pm 2\boldsymbol{k}^{(j)}$ for $j=1,2$ and $\boldsymbol{k}=\pm(\boldsymbol{k}^{(1)}\pm\boldsymbol{k}^{(2)})$.

Remark 3.2. *There is a singularity for (3.8) when $\boldsymbol{k}=\boldsymbol{0}$. We do not need to consider this case as $\boldsymbol{k}=\boldsymbol{0}$ is not a vector in the dual lattice Λ'.*

Thus, only twelve possibilities must be considered for the $\boldsymbol{k}$th Fourier coefficient of $\eta(\boldsymbol{x})$. All other Fourier coefficients are identically zero. Under the first four cases (where $\boldsymbol{k}=\pm\boldsymbol{k}^{\pm}$), we find

$$(0)\hat{\eta}_2(\pm\boldsymbol{k}^{\pm}) = 2\boldsymbol{c}_1\cdot\mathscr{H}_0(0)\{-\boldsymbol{c}_0\cdot\nabla\eta_1\}. \tag{3.9}$$

Since all factors other than c_1 on the right-hand side of the above expression are known to be non-zero, this immediately gives $c_1=0$.

Considering the remaining eight cases, $\boldsymbol{k}=\pm 2\boldsymbol{k}^{\pm}$, $\boldsymbol{k}=\pm(\boldsymbol{k}^{(1)}+\boldsymbol{k}^{(2)})$ or $\boldsymbol{k}=\pm(\boldsymbol{k}^{(1)}-\boldsymbol{k}^{(2)})$, we find the appropriate Fourier coefficients for $\eta_2(\boldsymbol{x})$. Since the right-hand side of (3.8) is known, we can calculate the Fourier coefficients to find that the $\boldsymbol{k}$th Fourier coefficient of $\eta_2(\boldsymbol{x})$ must satisfy

$$\left(g - \frac{(\boldsymbol{k}\cdot\boldsymbol{c}_0)^2}{|\boldsymbol{k}|}\coth(|\boldsymbol{k}|h)\right)\hat{\eta}_2(\boldsymbol{k}) = \iint_{\Lambda} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} F(\eta_1)\,\mathrm{d}\boldsymbol{x}, \tag{3.10}$$

where the form of $F(\eta_1)$ can easily be read from (3.8).

Thus, we find that $\eta_2(\boldsymbol{x})$ is as follows:

$$\eta_2(\boldsymbol{x}) = B_1 \mathrm{e}^{2\mathrm{i}\boldsymbol{k}^{(1)}\cdot\boldsymbol{x}} + B_2 \mathrm{e}^{2\mathrm{i}\boldsymbol{k}^{(2)}\cdot\boldsymbol{x}} + B_3 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}^{(1)}+\boldsymbol{k}^{(2)})\cdot\boldsymbol{x}} + B_4 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}^{(1)}-\boldsymbol{k}^{(2)})\cdot\boldsymbol{x}} + \text{c.c.}, \tag{3.11}$$

where as before, c.c. represents the complex conjugate, and the B_n are given by the following expressions:

$$B_1 = \frac{A_1^2}{2} \coth(h)(\coth(h)^2 - 1), \tag{3.12a}$$

$$B_2 = \frac{\tau A_2^2}{2} \coth(h\tau)(\coth(h\tau)^2 - 1), \tag{3.12b}$$

$$\begin{aligned} B_3 = {} & \frac{A_1 A_2}{g\mu_{1,1} - \coth(h\mu_{1,1})\lambda_{1,1}^2} \\ & \times \left(g\alpha\mu_{1,1}(\cos(\theta_1+\theta_2)\coth(h)\coth(h\tau) - 1) + \mu_{1,1}\lambda_{1,1}^2 - g\coth(h\mu_{1,1}) \right. \\ & \left. \times \left(\mu_{1,1}^2 + \alpha(\boldsymbol{k}^{(1)} + \boldsymbol{k}^{(2)}) \cdot \left(\frac{\boldsymbol{k}^{(1)}}{\tanh(h)} + \frac{\boldsymbol{k}^{(2)}}{\tau\tanh(h\tau)} \right) \right) \right), \end{aligned} \tag{3.12c}$$

$$\begin{aligned} B_4 = {} & \frac{A_1 A_2}{g\mu_{1,-1} - \coth(h\mu_{1,-1})\lambda_{1,-1}^2} \\ & \times \left(g\alpha\mu_{1,-1}(\cos(\theta_1+\theta_2)\coth(h)\coth(h\tau) + 1) + \mu_{1,-1}\lambda_{1,-1}^2 - g\coth(h\mu_{1,-1}) \right. \\ & \left. \times \left(\mu_{1,-1}^2 - \alpha(\boldsymbol{k}^{(1)} + \boldsymbol{k}^{(2)}) \cdot \left(\frac{\boldsymbol{k}^{(1)}}{\tanh(h)} - \frac{\boldsymbol{k}^{(2)}}{\tau\tanh(h\tau)} \right) \right) \right), \end{aligned} \tag{3.12d}$$

where we have introduced the notation

$$\mu_{m,n} = \|m\cdot\boldsymbol{k}^{(1)} + n\cdot\boldsymbol{k}^{(2)}\|, \quad \lambda_{m,n} = m\sqrt{g\tanh(h)} + n\sqrt{g\tau\tanh(h\tau)}, \tag{3.13}$$

$$\alpha = \sqrt{\tau\tanh(h)\tanh(h\tau)}. \tag{3.14}$$

Remark 3.3. *We could continue this same procedure to determine higher-order corrections. In addition, it is possible to determine a recursive relationship that uses the expansions for the operator $\mathscr{H}(\eta, D)$ found in the previous section.*

4. Conclusion

In this paper we have derived a new equation for the water-wave surface in a travelling coordinate system. The primary advantage of this new single equation is the ease with which Stokes-wave solutions for both a one- and two-dimensional surface may be computed. By exploiting the spectral nature of these equations and the operator $\mathscr{H}(\eta, D)$, it becomes easy to see which modes will arise at any given order. In addition, the operator $\mathscr{H}(\eta, D)$ can be computed easily from (2.10).

As shown in the previous section, if we rotate the coordinate system in a manner such that $\boldsymbol{c} = (c, 0)$, we have a further simplification which allows us to write the equations of motion in the form

$$c^2\,|\mathscr{H}(\eta, D)\{\eta_x\} + \boldsymbol{e}_1|^2 + c^2\,(\nabla\eta \otimes (\mathscr{H}(\eta, D)\{\eta_x\} + \boldsymbol{e}_1))^2 = (c^2 - 2g\eta)(1 + |\nabla\eta|^2), \tag{4.1}$$

where, $\otimes$ represents the cross-product One can also readily extend the above equation to include the effects of surface tension. We believe the reduction to a single scalar equation will enable further insights into the travelling-water-wave problem from a theoretical point of view. The beauty of the above representation is that one can easily rewrite the equation in the form $c^2 = F(\eta)$. This formulation of the equation may be useful for studying the bifurcation structure of travelling waves for the two-dimensional surface. These same ideas can be applied to the one-dimensional surface problem. These ideas will be explored in future publications.

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