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MEREOLOGICAL BIMODAL LOGICS

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Abstract. In this paper, using a propositional modal language extended with the window modality, we capture the first-order properties of various mereological theories. In this setting, $\Box\varphi$ reads *all the parts* (*of the current object*) *are* φ, interpreted on the models with a *whole-part* binary relation under various constraints. We show that all the usual mereological theories can be captured by modal formulas in our language via frame correspondence. We also correct a mistake in the existing completeness proof for a basic system of mereology by providing a new construction of the canonical model.

§1. Introduction. *Mereology*, as an umbrella term for the general study of parts and wholes, dates back to the early days of philosophy and constitutes an important active sub-field of metaphysics (cf., e.g., the survey by Varzi [47]). In a narrower sense, mereology refers to the formal theory of the *part-whole* relation initiated by Stanisław Leśniewski (cf., [31, 47]), aiming at an alternative to set theory as a foundation for mathematics. Despite the fact that it did not quite achieve its initial goal, mereology has become a well-established field on its own. It can also provide tools for formal ontology focusing on the general structure of what there is, regardless of the actual ontological stance, which can be traced back to Husserl's writings [12, 30].

Formulations of mereological theories change over time. The original theory proposed by Leśniewski was formulated in a special logical language based on his system *Ontology*, which was then recast as a second-order theory by Tarski [34] and Leonard & Goodman [19]. However, what most logicians nowadays are familiar with might be the first-order approximations of the classical theory, which were started by Goodman [7] and developed further by Eberle [5] and Casati & Varzi [2], among others. Moreover, there is a series of fruitful research aiming to characterize precisely the classical theory of mereology with finitely many first-order principles, such as [26, chap. 7], [24, 25] and the recent work of Tsai [40]. Also, there are other important traditions of formulations of theories for the parthood relation, including those involving plural quantification that began with [20]. For an excellent introduction to the basics of mereology, we refer to [47].

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In this article, we restrict ourselves to the first-order theories of mereology using a language with equality symbol[1] $\equiv$ and a special binary predicate symbol P meant to represent 'part of' or 'a part of' relation.[2] Various theories have been proposed and studied in the literature featuring different first-order properties of P. For example, it is widely held that the part-whole relation is at least reflexive, transitive and anti-symmetric (see [47] for exceptions), i.e., it is a *partial order*. The theory with the corresponding axioms for P is called the *ground mereology* (GM) in the literature of mereology. On top of this, various axioms can be added to capture the composition and decomposition of the parts, resulting in *minimal mereology* (MM), *extensional mereology* (EM), *extensional closure mereology* (CEM), *general extensional mereology* (GEM), and so on (see, e.g., [38, sec. 1], [47]).

Since mereological theories are about properties of the binary relation of part-whole, it is very natural, at least for a modal logician, to ask whether it is possible to use a modal language, instead of the first-order language, to capture those first-order mereological properties of the part-whole relation and obtain the corresponding modal theories. This is the basic motivation behind the *mereological modal theories* proposed by this paper. In fact, if we reverse the part-whole relation to *whole-part* relation and use the standard Kripke semantics for the modality with a set of objects as the domain, then $\Box\varphi$ has a very natural reading: *all parts of the current object are* φ. Note that all the modal formulas are evaluated with respect to an object in the domain, and this object exactly is what 'current' refers to. Similarly, a modal formula φ captures a property of an object.[3] $\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$ not only capture the corresponding first-order properties of Reflexivity and Transitivity, but also provide some intuitive readings in the context of mereology, e.g., $\Box\varphi \to \Box\Box\varphi$ says that if all parts are φ, then all the parts of the parts are also φ.

However, one may complain that the usual modal language seems to be inadequate to capture even the ground mereology, since Anti-symmetry, as a frame property, is not modally definable [1, sec. 3.3]. This motivates us to find a more expressive language for mereological modal theories.

A natural candidate to be added into the modal language is the so-called *window modality* $\boxminus$ studied by van Benthem [42], [45, p. 412], Humberstone [17, 18], and Gargov et al. [6].[4] In our setting, $\boxminus\varphi$ also has a natural reading: *all* φ*-things are parts of the current object*. With the help of $\boxminus$, many modally undefinable properties become definable in the extended language, such as Irreflexivity, Trichotomy, and Anti-symmetry [9, sec. 5]. Actually, as also shown by Goranko [9], every *universal* first-order formula with the equality $\equiv$ and a binary predicate R can be defined by a modal language with both $\Box$ and $\boxminus$ (see also [8, 10], [44, p. 155] for further results).

1 In contrast, we use $=$ for the identity relation in the *meta* language.

2 As suggested by Sharvy [28], 'is a part of' and 'is part of' are very different, but in this article we treat them as the same relation.

3 Mathematically, Kripke models are simply relational models with a domain and some relations between the elements in the domain. Although the domain is often taken as a set of *possible worlds* in various philosophical settings, such a philosophical interpretation of those elements can vary in different applications. Technically, the so-called *standard translation* connects every modal formula with an equivalent first-order formula with a single free variable, where every propositional letter is translated into a unary predicate symbol in the first-order language (cf., e.g., [1, chap. 3]).

4 See the notes of [1, chap. 7] for the history.

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Note that the existing results about $\boxminus$ do not make the modal rendering of the mereological theories easier. First of all, all the non-trivial mereological properties that we will discuss are not expressible in universal formulas due to the non-trivial quantifier alternations. Technically speaking, as we will see later, most of the first-order properties of mereology are not closed under taking substructures, thus by the Łoś–Tarski Theorem (see, e.g., [14, p. 143]), the general correspondence method proposed by Goranko [9] for the universal fragment of first-order logic is not applicable. In contrast, it is highly non-trivial to find the corresponding modal formulas for the first-order mereological properties (or the combinations of several such properties), even if they are indeed definable by the language with $\Box$ and $\boxminus$. Moreover, having the corresponding modal formulas for the desired first-order properties does not mean the resulting logical system is complete. Actually, showing the completeness for systems using both $\Box$ and $\boxminus$ is also highly non-trivial and error-prone (see discussions in Section 4.2).

Despite the difficulties of using an (extended) modal language to specify mereological theories, there are also some potential technical advantages of using the modal language. First of all, as it is well-known, some modal formulas can express properties that are not first-order definable, such as the McKinsey formula $\Box\Diamond p \to \Diamond\Box p$. Over transitive frames, the McKinsey formula boils down to the first-order definable property of *Atomicity* in mereology. However, over arbitrary frames, the modal formula does not have a first-order correspondence. Another interesting example is that the Grzegorczyk formula $\Box(\Box(p \to \Box p) \to p) \to p$ defines the class of frames that are reflexive, transitive, and anti-symmetric without any infinite ascending chain of distinct elements, which is also not definable but relevant to mereology.[5] As another example, the mereological property *Fusion* is essentially second-order, and we will define it by our modal language based on extensional mereology. Another potential advantage is that by using a modest modal language, we may obtain the decidability of the resulting modal logic over the relevant mereology-inspired frames, which we leave for further studies.[6]

In this paper, we mainly focus on the modal definability of existing first-order mereological theories. There can be various taxonomies of mereological properties, e.g., Hovda [16] discussed different but equivalent formulations of the classical mereology using various versions of *fusion*. In this paper, we follow the classification by [38, sec. 1] and [37, sec. 1]. Note that, we take the whole-part relation R, the converse of the part-whole relation P often used in the mereological theories, as the primitive relation, to have a more natural reading of the $\Box$ modality. Table 1 is a summary of the first-order theories of mereology (based on R instead of P) that will be studied in this article.[7]

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5 Holliday [15] used it to capture set-theoretic inclusion in the setting of Medvedev frames.

6 For the details of the (un)decidability results on the first-order mereological theories, we refer the reader to [37–39].

7 It is important to recognize that the way to present these systems need not be unique: we may also obtain the equivalent theories with principles different from those in Table 1. Usually, different principles themselves convey different standpoints of philosophy. In this article, we will not address those debates between different choices. For readers interested in this background, we refer to the survey [47]. More generally, beyond the theories listed here, there are also many other theories of mereology. For instance, a number of mereological theories do not assume Anti-symmetry for important philosophical reasons. For more on this direction, the reader may consult [3, 4].

Table 1. *First-order theories of mereology*

Auxiliary notions:		
Proper Part $PPxy := Ryx \wedge \neg Rxy$		
Overlap $Oxy := \exists z(Rxz \wedge Ryz)$		
Underlap $Uxy := \exists z(Rzx \wedge Rzy)$		
Atom (or Simple) $Ax := \neg\exists y PPyx$		
Axioms:		
$M1.$	$\forall x Rxx$	*Reflexivity*
$M2.$	$\forall x \forall y \forall z(Rxy \wedge Ryz \to Rxz)$	*Transitivity*
$M3.$	$\forall x \forall y(Rxy \wedge Ryx \to x \equiv y)$	*Anti-symmetry*
$M4.$	$\forall x \forall y(PPxy \to \exists z(Ryz \wedge \neg Ozx))$	*Supplementation*
$M5.$	$\forall x \forall y(\neg Ryx \to \exists z(Rxz \wedge \neg Ozy))$	*Strong Supplementation*
$M6.$	$\forall x \forall y(Oxy \to \exists z \forall w(Rzw \leftrightarrow (Rxw \wedge Ryw)))$	*Finite Product*
$M7.$	$\forall x \forall y(Uxy \to \exists z \forall w(Owz \leftrightarrow (Owx \vee Owy)))$	*Finite Sum*
$M8.$	$\exists x \alpha \to \exists z \forall y(Oyz \leftrightarrow \exists x(\alpha \wedge Oyx))$	*Fusion*
$M9.$	$\forall x \exists y(Ay \wedge Rxy)$	*Atomicity*
$M10.$	$\forall x \exists y PPyx$	*Atomlessness*
where α is an arbitrary formula, and variables y, z do not occur free in α.		
Theories of mereology:		
Ground Mereology GM:= $M1 + M2 + M3$		
Minimal Mereology MM:=GM+ $M4$		
Extensional Mereology EM:=GM+ $M5$		
Extensional Closure Mereology CEM:=EM+ $M6 + M7$		
General Extensional Mereology GEM:=EM+ $M8$		
For any X $\in$ {GM,MM,EM,CEM,GEM}, AX:= X+ $M9$, and $\widetilde{\text{A}}$X:=X+ $M10$.		

The main contributions of the paper are:

- Modal correspondences of the mereological properties *Anti-symmetry*, *Supplementation*, *Strong Supplementation*, *Finite Product*, *and Atomlessness*.
- Relative modal correspondences[8] of the mereological properties *Atomicity*, *Finite Sum*, and (the second-order version of) *Fusion*.
- Based on the above correspondences, we modally define all the usual mereological theories according to the classification by Tsai [37, 38].
- We also show that all the aforementioned mereological properties are *not* definable in the modal logic with $\Box$ only. In particular, *Atomicity* cannot even be (absolutely) defined by the language with both $\Box$ and $\boxminus$.
- Incompleteness of the corresponding modal system *MGM* over frames of the ground mereology and the completeness proof of its extension MGM^+, where we correct an error in the literature.

[8] More precisely, we find the modal formulas defining the first-order mereological properties w.r.t. other frame properties that are assumed in various mereological theories.

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Related work. Modality and mereology can interact in various ways. As a sub-field of metaphysics, in the philosophical study of mereology, the so-called (*weak*) *modal mereology* refers to the view that a possible whole exists in virtue of its parts and the internal relations among its parts, but an actual whole does not [48]. One can also ask where mereological fusions have their parts necessarily [41]. The formal discussions in this line often involve a language extending the first-order language of mereology with modalities for necessity or tense (e.g., [29, 41]). In contrast with such work, the point of our work is to capture the standard first-order mereological theories by using a modal language.

Instead of reasoning about mereological objects, there is a line of research on the modal reasoning about space or spatial regions reflecting some mereological features. Originating from [33, 32], modal logics have been very successful in spatial reasoning (cf., e.g., [46] and the references therein). Compared with the necessity-/tense-readings of modalities in the above philosophical discussions, modal operators in this line of research are often interpreted as spatial concepts, such as the *topological interior*.

Combining the ideas of topology, geometry and mereology, the research fields of *mereotopology* and *mereogeometry* study extensions of the basic ground mereology with various relations of geometrical and topological nature, where modalities also play a role. For instance, to reason about spatial regions, both Lutz & Wolter [21] and Nenov & Vakarelov [23] developed modal logics with the corresponding modalities to the eight well-known RCC8 (or Egenhofer-Franzosa) relations about how regions are connected, and study in-depth their meta-properties involving expressiveness and computational complexity. Also, Torrini et al. [36] proposed a powerful framework extending intuitionistic propositional logics with propositional quantification and a strong modality denoting global truth, which can express a number of mereotopological relations.

In contrast to the above work, where many new modalities capturing various mereotopological relations are used, we stick to the theories of mereology themselves that have the part-whole relation as the only primitive relation, and resort to a simple modal language to describe its properties.

Technically, the closest work to ours in the literature is [15], where the class of *Medvedev frames*[9] is defined by a formula of the modal language extended with a converse modality[10] and any one of the following: *nominals* in hybrid logic, a *difference modality* $[\neq]$, or a *complement modality* $\boxminus$. The interpretation of $\Box\varphi$ in [15] is similar to ours: *all the non-empty subsets of the current set are* φ. The complement modality $\boxminus$ is inter-definable with the window modality, i.e., $\boxminus\varphi \leftrightarrow \boxdot\neg\varphi$ is valid. According to [15], a Medvedev frame can be characterized as a finite poset with some properties about *separativity*, *reversed convergence*, and *union*. In Section 3.8, we will show that our language can also define all these properties except the finiteness. In [15], the finiteness is captured by the Grzegorczyk axiom for $\Box$ and its converse variant, in presence of other axioms.

[9] Medvedev frames are the Kripke frames isomorphic to $\langle \mathcal{P}(W) \setminus \emptyset, \supseteq \rangle$ for some non-empty finite W, i.e., non-empty subsets of a finite set with the superset relation [22].

[10] The language with $\Box$ and the converse modality can be viewed as a language of tense logic.

As for the (in)completeness results, we follow the general strategy elaborated in the in-depth study by Goranko [8] on the modal logics with both $\Box$ and $\boxminus$ over various frame classes.

Structure of the paper. In Section 2, we introduce the modal language and its semantics. The main correspondence results are presented in Section 3 and the (in)completeness results are proved in Section 4. Finally we conclude with future directions in Section 5.

§2. Language and semantics.

Our language $\mathcal{L}_{\Box\boxminus}$ extends the standard modal language $\mathcal{L}_{\Box}$ with the window modality $\boxminus$.

DEFINITION 2.1 (Language). *Let* **P** *be a countable set of propositional atoms. Formulas of* $\mathcal{L}_{\Box\boxminus}$ *are defined as follows*:

$$\mathcal{L}_{\Box\boxminus} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \boxminus\varphi,$$

where $p \in \mathbf{P}$. *The abbreviations* $\top, \bot, \vee, \rightarrow$ *and* $\leftrightarrow$ *are defined as usual. Also, we define* $\blacksquare\varphi := \Box\varphi \wedge \boxminus\varphi$ *and* $\Box_u\varphi := \Box\varphi \wedge \boxminus\neg\varphi$. *In particular, modalities* $\Diamond$ *and* $\Diamond_u$ *are the dual operators of* $\Box$ *and* $\Box_u$, *respectively.*

Intuitively, $\Box\varphi$ says that *all the parts of the current object are* φ;[11] $\boxminus\varphi$ means that *all the objects that are* φ *are parts of the current object*; $\Box_u\varphi$ reads *all objects are* φ; $\blacksquare\varphi$ states that *the parts of the current object are exactly the objects that are* φ, i.e., *an object is part of the current object iff it is* φ; $\Diamond\varphi$ says that *at least one part of the current object* is φ; and $\Diamond_u\varphi$ means *at least one object is* φ.

Formulas of $\mathcal{L}_{\Box\boxminus}$ are evaluated in standard relational models $\mathcal{M} = \langle W, R, V\rangle$, where W is a non-empty set of objects, $R \subseteq W \times W$ is a binary relation, and $V : \mathbf{P} \rightarrow 2^W$ is a valuation function. Intuitively, the relation R is intended to be the whole-part relation, i.e., Rxy means that y is part of x. As for the valuation function, $w \in V(p)$ means that the object w has the property p. For any $w \in W$, $\langle\mathcal{M}, w\rangle$ is called a *pointed model*. For brevity, we usually write $\mathcal{M}, w$ instead of $\langle\mathcal{M}, w\rangle$. A *frame* $\mathcal{F} = \langle W, R\rangle$ is a model without the valuation function. Thus a model can be written as $\langle\mathcal{F}, V\rangle$ where $\mathcal{F}$ is a frame.

REMARK 2.2. *Note that to have an intuitive reading for* $\Box\varphi$, *we take* R *as the converse of the usual part-whole relation* P *in the standard mereological theories.*

DEFINITION 2.3 (Semantics.) *Given a pointed model* $\langle\mathcal{M}, w\rangle$ *and a formula* φ *of* $\mathcal{L}_{\Box\boxminus}$, *we say that* φ *is true in* $\mathcal{M}$ *at* w, *written as* $\mathcal{M}, w \vDash \varphi$, *when*

$\mathcal{M}, w \vDash p$	$\Leftrightarrow$	$w \in V(p)$,
$\mathcal{M}, w \vDash \neg\varphi$	$\Leftrightarrow$	$\mathcal{M}, w \nvDash \varphi$,
$\mathcal{M}, w \vDash (\varphi \wedge \psi)$	$\Leftrightarrow$	$\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$,
$\mathcal{M}, w \vDash \Box\varphi$	$\Leftrightarrow$	for each $v \in W$, if Rwv, then $\mathcal{M}, v \vDash \varphi$,
$\mathcal{M}, w \vDash \boxminus\varphi$	$\Leftrightarrow$	for each $v \in W$, if $\mathcal{M}, v \vDash \varphi$, then Rwv.

[11] Or simply read it as "all *my* parts are φ" where "my" refers to the current object.

Based on the above semantics we have the following derived semantics for $\Box_u$ (the *universal modality*) and $\blacksquare$ in accordance with their intuitive readings:

$\mathcal{M}, w \vDash \Box_u \varphi$	$\Leftrightarrow$	for each $v \in W$, $\mathcal{M}, v \vDash \varphi$,
$\mathcal{M}, w \vDash \blacksquare \varphi$	$\Leftrightarrow$	for each $v \in W$, Rwv iff $\mathcal{M}, v \vDash \varphi$.

Notions of *satisfiability*, *validity* and *logical consequence* are defined as usual. In particular, for any formula φ of $\mathcal{L}_{\Box\boxminus}$, we write $\mathcal{F} \vDash \varphi$, i.e., φ is valid on frame $\mathcal{F}$, if $\langle \mathcal{F}, V\rangle, w \vDash \varphi$ for all $w \in W$ and all the valuations V on W.

We now introduce some useful notions. Let $\mathcal{F} = \langle W, R\rangle$ be a frame and $w \in W$, and $R(w) := \{v \in W \mid Rwv\}$ is *the set of successors* of w. Moreover, for all $\Phi \subseteq \mathcal{L}_{\Box\boxminus}$, $\mathcal{F} \vDash \Phi$ iff $\mathcal{F} \vDash \varphi$ for all formulas $\varphi \in \Phi$. In addition, we say that $\mathcal{F}$ is a Φ-*frame* if $\mathcal{F} \vDash \Phi$. For any two classes $\mathcal{A}, \mathcal{B}$ of frames and formula $\varphi \in \mathcal{L}_{\Box\boxminus}$, φ (*absolutely*) *defines* $\mathcal{A}$ iff for each frame $\mathcal{F}$, $\mathcal{F} \in \mathcal{A}$ iff φ is valid on $\mathcal{F}$; φ *defines* $\mathcal{A}$ *relative to* $\mathcal{B}$ iff for any frame $\mathcal{F} \in \mathcal{B}$, $\mathcal{F} \in \mathcal{A}$ iff φ is valid on $\mathcal{F}$.

§3. Correspondences. In this section, we define the standard theories of mereology listed in Table 1 with $\mathcal{L}_{\Box\boxminus}$. Recall that Rxy means Pyx in the standard mereology, i.e., *y is part of x*.

3.1. Correspondences of ground mereology. *Ground mereology* (GM) is the ground of all standard first-order theories of mereology. It consists of the following three first-order principles:

$M1$.	$\forall x Rxx$	*Reflexivity*
$M2$.	$\forall x \forall y \forall z (Rxy \wedge Ryz \to Rxz)$	*Transitivity*
$M3$.	$\forall x \forall y (Rxy \wedge Ryx \to x \equiv y)$	*Anti-symmetry*

Namely, the relation R is a partial order: it is reflexive, transitive and anti-symmetric. The modal correspondences of these properties are familiar to us: Reflexivity and Transitivity correspond to $\Box p \to p$ (**T**) and $\Box p \to \Box\Box p$ (**4**), respectively, but it is well-known that Anti-symmetry cannot be defined by $\Box$(cf., e.g., [1, sec. 3.3]). With $\boxminus$, we can also define Anti-symmetry by $\Diamond(\boxminus\neg p \wedge p) \to p$ as noted by Goranko [9, p. 97]. However, to ease the completeness proof in Section 4.2, we will use another apparently stronger formula $\Diamond(\boxminus p \wedge q) \to (p \to q)$ to define this property.[12]

In what follows, we denote by $\Vdash$ the satisfaction relation for first-order logic, i.e., taking a frame $\langle W, R\rangle$ as a first-order structure where W is the domain and R is the interpretation for the corresponding binary predicate (cf., e.g., [1, sec. 3]).

THEOREM 3.4. *For each frame $\mathcal{F}$, $\mathcal{F} \Vdash$ Anti-symmetry if and only if $\mathcal{F} \vDash \Diamond(\boxminus p \wedge q) \to (p \to q)$.*

[12] We can obtain $\Diamond(\boxminus\neg p \wedge p) \to p$ by replacing p with $\neg p$ and replacing q with p in $\Diamond(\boxminus p \wedge q) \to (p \to q)$. The canonicity of $\Diamond(\boxminus p \wedge q) \to (p \to q)$ can be proved easily (cf. Proposition 4.43), but it is not so clear in the case of $\Diamond(\boxminus\neg p \wedge p) \to p$.

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Proof. Let $\mathcal{F} = \langle W, R\rangle$ be a frame. For the direction from left to right, suppose that $\mathcal{F} \not\models \Diamond(\boxbar p \wedge q) \to (p \to q)$. Then there are $w, w' \in W$ and a valuation V such that $\langle \mathcal{F}, V\rangle, w \models p \wedge \neg q$, $\langle \mathcal{F}, V\rangle, w' \models \boxbar p \wedge q$ and Rww'. By the semantics, it holds that $Rw'w$. Now we have Rww' and $Rw'w$. However, by $w \notin V(q)$ and $w' \in V(q)$, it follows that $w \neq w'$. Therefore, $\mathcal{F} \not\Vdash$ *Anti-symmetry*.

For the converse direction, assume that $\mathcal{F} \not\Vdash$ *Anti-symmetry*. Then there are $w, w' \in W$ such that $w \neq w'$, Rww' and $Rw'w$. Define a valuation V such that $V(p) = \{w\}$ and $V(q) = \{w'\}$. Consequently, we obtain $\langle \mathcal{F}, V\rangle, w \models p \wedge \neg q$ and $\langle \mathcal{F}, V\rangle, w' \models \boxbar p \wedge q$. Thus, $\langle \mathcal{F}, V\rangle, w \models \Diamond(\boxbar p \wedge q) \wedge p \wedge \neg q$. The proof is completed. □

On top of GM, there are further plausible mereological properties, which are the subjects of other subsections. As mentioned earlier, the original goal of mereology was to replace set theory, and many of the properties in mereology have similar functions to those in set theory: say, in what follows you may find that mereological theories include operations on objects which look similar to the set-theoretic '*complementation*', '*union*', '*intersection*', and so on. However, this by no means indicates mereological concepts can be viewed precisely as set-theoretic ones. Technically, the mereological properties are closer to the properties studied in the theory of partial orders.[13]

3.2. Correspondences of minimal mereology. *Minimal mereology* (MM) is the minimal standard theory of mereology. It extends GM with the following axiom:

$M4.$	$\forall x \forall y(PPxy \to \exists z(Ryz \wedge \neg Ozx))$	*Supplementation*

where $PPxy := Ryx \wedge \neg Rxy$ means *x is a proper part of y*, and $Oxy := \exists z(Rxz \wedge Ryz)$ states *x overlaps y*, i.e., *they have common parts*.

Supplementation states that if x is a proper part of y, then y has a part that is disjoint from x. Supplementation is essentially the formula $\forall x \forall y(Ryx \wedge \neg Rxy \to \exists z(Ryz \wedge \neg \exists w(Rxw \wedge Rzw)))$. The following result shows that it corresponds to the $\mathcal{L}_{\Box\boxbar}$-formula $\neg p \wedge \Diamond \blacksquare p \to \Diamond \Box \neg p$, which says if the current object s does not have property p and there is a part w of s which has exactly the p-objects as the parts, then there is some part t of s such that all its parts do not have p thus t and w are disjoint.

THEOREM 3.5. *For each frame $\mathcal{F}$, $\mathcal{F} \Vdash$ Supplementation if and only if $\mathcal{F} \models \neg p \wedge \Diamond \blacksquare p \to \Diamond \Box \neg p$.*

Proof. Let $\mathcal{F} = \langle W, R\rangle$ be a frame. For the direction from left to right, suppose that $\mathcal{F} \not\models \neg p \wedge \Diamond \blacksquare p \to \Diamond \Box \neg p$. Then there exist $w \in W$ and a valuation V such that $\langle \mathcal{F}, V\rangle, w \models \neg p \wedge \Diamond \blacksquare p \wedge \Box \Diamond p$. Consequently, there exists $w_1 \in R(w)$ such that $\langle \mathcal{F}, V\rangle, w_1 \models \blacksquare p$. Thus it holds that $\neg Rw_1 w$. Furthermore, from $\langle \mathcal{F}, V\rangle, w \models \Box \Diamond p$, we know that each $v \in R(w)$ overlaps w_1. Thus, we do not have $\mathcal{F} \Vdash$ *Supplementation*.

For the direction from right to left, assume that $\mathcal{F} \models \neg p \wedge \Diamond \blacksquare p \to \Diamond \Box \neg p$ and $\mathcal{F} \not\Vdash$ *Supplementation*. Then there are $w, w_1 \in W$ such that Rww_1, $\neg Rw_1 w$, and for all $o \in R(w)$, there exists $o' \in W$ with $Rw_1 o' \wedge Roo'$. Define a valuation V such that $V(p) = R(w_1)$. Then we obtain $\langle \mathcal{F}, V\rangle, w \models \neg p \wedge \Diamond \blacksquare p$. By $\mathcal{F} \models \neg p \wedge \Diamond \blacksquare p \to$

13 For the reader interested in this direction, we refer to, e.g., [11, 26] and pages 7, 30, 41, 51–54, 241, and 242 in [27] for specific correspondences of properties in mereology and in order (and lattice) theory.

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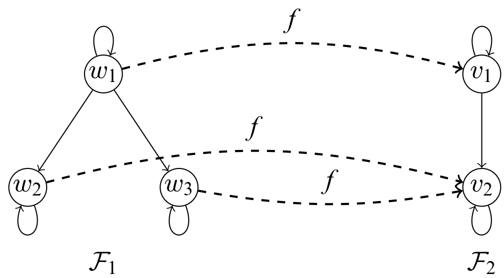

Fig. 1. A bounded morphism (for Supplementation).

$\Diamond\Box\neg p$, it holds that $\langle\mathcal{F}, V\rangle, w \vDash \Diamond\Box\neg p$. Namely, there exists $v \in R(w)$ such that $\langle\mathcal{F}, V\rangle, v \vDash \Box\neg p$, so v is disjoint from w_1. This completes the proof. □

REMARK 3.6. *If one prefers to define the proper parthood $PPxy$ as $Ryx \wedge x \not\equiv y$ (see, e.g., [47]), Supplementation then becomes $\forall x\forall y(Ryx \wedge x \not\equiv y \to \exists z(Ryz \wedge \neg Ozx))$. These two versions are not equivalent if we do not assume Anti-symmetry (the reader may find it interesting to see which one is stronger). For the alternative formulation, the reader can check that it corresponds to the modal principle $\neg p \wedge \Diamond(p \wedge \blacksquare q) \to \Diamond\Box\neg q$.*[14]

In the result above, the modality ◫ is used to define Supplementation. But, is this first-order property also definable with the standard modal language? The following result is a negative answer.

PROPOSITION 3.7. *Supplementation cannot be defined by $\mathcal{L}_\Box$.*

Proof. Consider two frames $\mathcal{F}_1 = \langle W_1, R_1\rangle$ and $\mathcal{F}_2 = \langle W_2, R_2\rangle$ depicted in Figure 1. Define a function $f : W_1 \to W_2$ such that $f(w_1) = v_1$ and $f(w_2) = f(w_3) = v_2$. Then, for any $s, t \in W_1$ and $v \in W_2$, if R_1st, then $R_2f(s)f(t)$; if $R_2f(s)v$, then there exists $u \in W_1$ such that $f(u) = v$ and R_1sv. Hence, f is a *bounded morphism* from $\mathcal{F}_1$ to $\mathcal{F}_2$ (cf. [1, sec. 3.3]). So, for any $\varphi \in \mathcal{L}_\Box$, if $\mathcal{F}_1 \vDash \varphi$ then $\mathcal{F}_2 \vDash \varphi$. If Supplementation can be defined by some formula φ' of $\mathcal{L}_\Box$, then for each frame $\mathcal{F}$, $\mathcal{F} \Vdash$ *Supplementation* iff $\mathcal{F} \vDash \varphi'$. However, since $\mathcal{F}_1 \Vdash$ *Supplementation* and $\mathcal{F}_2 \nVdash$ *Supplementation*, we know $\mathcal{F}_1 \vDash \varphi'$ and $\mathcal{F}_2 \nvDash \varphi'$. By the *Goldblatt-Thomason Theorem*, Supplementation cannot be defined by $\mathcal{L}_\Box$. □

3.3. Correspondences of extensional mereology. *Extensional mereology (EM)* is stronger than MM, which results from replacing Supplementation in MM with Strong Supplementation.

Strong Supplementation states that if object x is not a part of y, then x has a part disjoint from y. This principle is stronger than Supplementation, and it can be

[14] Here is the proof. ⇒. Suppose $\mathcal{F} \vDash \neg p \wedge \Diamond(p \wedge \blacksquare q) \to \Diamond\Box\neg q$ and there are s, t such that $s \neq t$, Rst and for all $o \in W$, if Rso then Oot. As $s \neq t$, it is reasonable to define a valuation function V such that $t \in V(p)$ and $s \notin V(p)$. Also, let $V(q) = \{o \in W \mid Rto\}$. Consequently, from $\neg p \wedge \Diamond(p \wedge \blacksquare q) \to \Diamond\Box\neg q$ we know that s has a part that is disjoint from t. A contradiction. ⇐. Assume $\mathcal{F} \nvDash \neg p \wedge \Diamond(p \wedge \blacksquare q) \to \Diamond\Box\neg q$. Then we have an object s and a valuation function V such that $\langle\mathcal{F}, V\rangle, s \vDash \neg p \wedge \Diamond(p \wedge \blacksquare q) \wedge \Box\Diamond q$. Thus, there is $t \in R(s)$ satisfying $p \wedge \blacksquare q$. From $t \in V(p)$ we know $s \neq t$. Moreover, as t is $\blacksquare q$, $\Box\Diamond q$ indicates that every part of s overlaps t. Now we have already known that the first-order property fails. The proof is completed.

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$M5.$	$\forall x \forall y(\neg Ryx \to \exists z(Rxz \wedge \neg Ozy))$	*Strong Supplementation*

easily checked that Strong Supplementation entails Supplementation. Furthermore, formula $\forall x \forall y(\exists z PPzx \vee \exists z PPzy) \to (x \equiv y \leftrightarrow \forall z(PPzx \leftrightarrow PPzy)))$ is provable in EM, which intuitively states that no composite objects (i.e., objects including proper parts) with the same proper parts can be distinguished (see [47]). By the definition of O, it is not hard to see that Strong Supplementation is essentially the formula $\forall x \forall y(\neg Ryx \to \exists z(Rxz \wedge \neg \exists v(Rzv \wedge Ryv))$.

THEOREM 3.8. *For each frame $\mathcal{F}$, $\mathcal{F} \Vdash$ Strong Supplementation if and only if $\mathcal{F} \vDash \blacksquare p \to \Box_u(\neg p \to \Diamond\Box\neg p)$.*

Proof. Let $\mathcal{F} = \langle W, R\rangle$ be a frame. From left to right, suppose that $\mathcal{F} \nvDash \blacksquare p \to \Box_u(\neg p \to \Diamond\Box\neg p)$. Then there exist $w \in W$ and a valuation V such that $\langle \mathcal{F}, V\rangle, w \vDash \blacksquare p$ and $\langle \mathcal{F}, V\rangle, w \nvDash \Box_u(\neg p \to \Diamond\Box\neg p)$. Thus, there exists $w_1 \in W$ with $\langle \mathcal{F}, V\rangle, w_1 \vDash \neg p \wedge \Box\Diamond p$. Since $\langle \mathcal{F}, V\rangle, w \vDash \blacksquare p$, we have $\neg Rww_1$. Moreover, for any $v_1 \in R(w_1)$, there exists $v_2 \in R(v_1)$ such that Rwv_2.

Now, we have $\neg Rww_1$, and for all $v_1 \in W$, if Rw_1v_1, then there exists $v_2 \in W$ such that Rv_1v_2 and Rwv_2. Consequently, $\mathcal{F} \nVdash$ *Strong Supplementation*.

For the converse direction, assume that $\mathcal{F} \nVdash$ *Strong Supplementation*. Therefore, there are $w, w_1 \in W$ with $\neg Rww_1 \wedge \forall z(Rw_1z \to \exists v(Rzv \wedge Rwv)))$.

Define a valuation V such that $V(p) = R(w)$; hence $\langle \mathcal{F}, V\rangle, w \vDash \blacksquare p$. By $\neg Rww_1$, it holds that $\langle \mathcal{F}, V\rangle, w_1 \vDash \neg p$. Furthermore, for all $v_1 \in R(w_1)$, there exists $v_2 \in R(v_1)$ such that Rwv_2. By $V(p)$, it follows that $\langle \mathcal{F}, V\rangle, w_1 \vDash \neg p \wedge \Box\Diamond p$. Consequently, $\langle \mathcal{F}, V\rangle, w \nvDash \Box_u(\neg p \to \Diamond\Box\neg p)$. Thus, $\langle \mathcal{F}, V\rangle, w \nvDash \blacksquare p \to \Box_u(\neg p \to \Diamond\Box\neg p)$. ☐

In addition, by the same reasoning as in the proof of Proposition 3.7, we know that Strong Supplementation is not definable in $\mathcal{L}_\Box$, either.

PROPOSITION 3.9. *Strong Supplementation cannot be defined by $\mathcal{L}_\Box$.*

3.4. Correspondences of extensional closure mereology. *Extensional closure mereology (CEM)* is an extension of EM, with the following two axioms which allow us to compose objects:

$M6.$	$\forall x \forall y(Oxy \to \exists z \forall w(Rzw \leftrightarrow (Rxw \wedge Ryw)))$	*Finite Product*
$M7.$	$\forall x \forall y(Uxy \to \exists z \forall w(Owz \leftrightarrow (Owx \vee Owy)))$	*Finite Sum*

where $Uxy := \exists z(Rzx \wedge Rzy)$ means x *underlaps* y, i.e., *there exists an object including both x and y as parts*.

Before introducing the correspondence results, let us briefly comment on this theory. By Finite Product, if x overlaps y, then there exists an object whose parts are exactly the common parts of x and y. On the other hand, Finite Sum shows that if x underlaps y, then there exists an object such that the objects overlapping it are exactly those overlapping x or y. Since CEM is an extension of EM, it is extensional, too. Thus the sum and product of two objects are unique.[15]

[15] For instance, suppose for reductio that there are two different products u, v of x, y. Since $\neg u \equiv v$, by Anti-symmetry it follows that $\neg Ruv \vee \neg Rvu$. W.l.o.g., we assume that $\neg Ruv$.

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Unravelling the abbreviations, Finite Product is $\forall x \forall y(\exists z(Rxz \wedge Ryz) \to \exists z \forall w(Rzw \leftrightarrow (Rxw \wedge Ryw)))$. We can define it by the intuitive formula $\blacksquare p \wedge \Diamond_u \blacksquare q \wedge \Diamond(p \wedge q) \to \Diamond_u \blacksquare(p \wedge q)$, which says *if* the current object has exactly the p-objects as parts and there is one object with exactly the q-objects as parts and they indeed overlap *then* there is one object with exactly those objects in both as parts.

THEOREM 3.10. *For each frame $\mathcal{F}$, $\mathcal{F} \Vdash$ Finite Product iff $\mathcal{F} \vDash \blacksquare p \wedge \Diamond_u \blacksquare q \wedge \Diamond(p \wedge q) \to \Diamond_u \blacksquare(p \wedge q)$.*

Proof. Let $\mathcal{F} = \langle W, R\rangle$ be a frame. From left to right, suppose that $\mathcal{F} \Vdash$ *Finite Product* and $\mathcal{F} \nvDash \blacksquare p \wedge \Diamond_u \blacksquare q \wedge \Diamond(p \wedge q) \to \Diamond_u \blacksquare(p \wedge q)$. Then there exist $w \in W$ and a valuation V such that $\langle \mathcal{F}, V\rangle, w \vDash \blacksquare p \wedge \Diamond_u \blacksquare q \wedge \Diamond(p \wedge q) \wedge \Box_u \neg \blacksquare(p \wedge q)$. Therefore, there exists $v \in W$ with $\langle \mathcal{F}, V\rangle, v \vDash \blacksquare q$. Additionally, for any $o \in W$, it holds that $\langle \mathcal{F}, V\rangle, o \nvDash \blacksquare(p \wedge q)$, i.e., $\langle \mathcal{F}, V\rangle, o \vDash \neg\Box(p \wedge q) \vee \neg \boxminus (p \wedge q)$.

From $\langle \mathcal{F}, V\rangle, w \vDash \blacksquare p \wedge \Diamond(p \wedge q)$, we know that w overlaps v. By $\mathcal{F} \Vdash$ *Finite Product*, there exists $s \in W$ such that for any $t \in W$, $Rst \leftrightarrow (Rwt \wedge Rvt)$. Now let us consider the two cases of s. First, when $\langle \mathcal{F}, V\rangle, s \vDash \neg\Box(p \wedge q)$, there exists $s' \in R(s)$ such that $\langle \mathcal{F}, V\rangle, s' \vDash \neg p \vee \neg q$. We now conclude that $\neg Rws' \vee \neg Rvs'$, which entails a contradiction. Next, if $\langle \mathcal{F}, V\rangle, s \vDash \neg \boxminus (p \wedge q)$, then there exists $s' \in W$ such that $\langle \mathcal{F}, V\rangle, s' \vDash p \wedge q$ and $\neg Rss'$. However, by $\langle \mathcal{F}, V\rangle, s' \vDash p \wedge q$, we obtain $Rws' \wedge Rvs'$ which also contradicts $\mathcal{F} \Vdash$ *Finite Product*.

For the converse direction, assume for reductio that $\mathcal{F} \vDash \blacksquare p \wedge \Diamond_u \blacksquare q \wedge \Diamond(p \wedge q) \to \Diamond_u \blacksquare(p \wedge q)$ and $\mathcal{F} \nVdash$ *Finite Product*. Then there are $w_1, w_2, w_3 \in W$ with Rw_1w_3 and Rw_2w_3, and for each $u \in W$ there exists $t \in W$ such that $\neg(Rut \leftrightarrow (Rw_1t \wedge Rw_2t))$.

Define a valuation V such that $V(p) = R(w_1)$ and $V(q) = R(w_2)$, thus $\langle \mathcal{F}, V\rangle, w_1 \vDash \blacksquare p \wedge \Diamond_u \blacksquare q$. By Rw_1w_3 and Rw_2w_3, it holds that $\langle \mathcal{F}, V\rangle, w_1 \vDash \Diamond(p \wedge q)$. By assumption, there exists $w \in W$ such that $\langle \mathcal{F}, V\rangle, w \vDash \blacksquare(p \wedge q)$. Moreover, there exists $w' \in W$ such that $\neg(Rww' \leftrightarrow (Rw_1w' \wedge Rw_2w'))$.

When $\neg Rww' \wedge Rw_1w' \wedge Rw_2w'$, it holds that $\langle \mathcal{F}, V\rangle, w \nvDash \boxminus(p \wedge q)$. Also, we can obtain $\langle \mathcal{F}, V\rangle, w \nvDash \Box(p \wedge q)$ if $Rww' \wedge \neg(Rw_1w' \wedge Rw_2w')$. Each of them implies a contradiction. Now the proof is completed. □

Furthermore, this property cannot be defined by the standard modal language:

PROPOSITION 3.11. *Finite Product cannot be defined by $\mathcal{L}_\Box$.*

Proof. Consider the two frames $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{F}' = \langle W', R'\rangle$ depicted in Figure 2. Observe that $\mathcal{F}' = \langle W', R'\rangle$ is a subframe of $\mathcal{F}$. Moreover, for any $s \in W'$, if Rst, then $t \in W'$. Therefore, $\mathcal{F}'$ is a *generated subframe* of $\mathcal{F}$. By the well-known Goldblatt–Thomason Theorem, for any $\varphi \in \mathcal{L}_\Box$, $\mathcal{F}' \vDash \varphi$ follows from $\mathcal{F} \vDash \varphi$ (see, e.g., [1, sec. 3.3]). However, we have $\mathcal{F} \Vdash$ *Finite Product* and $\mathcal{F}' \nVdash$ *Finite Product*. Specifically, in frame $\mathcal{F}'$, w_2 and w_3 overlap, but they have no product. Hence the property cannot be defined by $\mathcal{L}_\Box$. □

Compared with Finite Product, Finite Sum is much more complicated. If Finite Product can be viewed as an analog of intersection in set theory then Finite Sum is an analog of union. Based on this intuition, it seems natural to define it as: if x

Then by Strong Supplementation, v has a part o that is disjoint from u. By Finite Product, o is a common part of x and y. However, by Finite Product again, o should also be a part of u. Now we have arrived at a contradiction.

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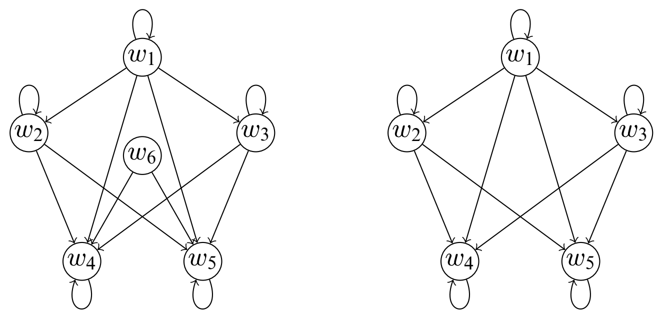

$\mathcal{F}$ $\mathcal{F}'$

Fig. 2. A generated subframe (for Finite Product).

underlaps y, then there exists an object z whose parts are exactly the parts of x or y, i.e., $\forall x \forall y(Uxy \to \exists z \forall w(Rzw \leftrightarrow (Rxw \vee Ryw)))$. This property can be intuitively captured by $\Diamond \blacksquare p \wedge \Diamond \blacksquare q \to \Diamond_u \blacksquare (p \vee q)$.

PROPOSITION 3.12. *For each frame* $\mathcal{F}$, $\mathcal{F} \vDash \Diamond \blacksquare p \wedge \Diamond \blacksquare q \to \Diamond_u \blacksquare (p \vee q)$ *iff* $\mathcal{F} \Vdash \forall x \forall y(\exists z(Rzx \wedge Rzy) \to \exists z \forall w(Rzw \leftrightarrow (Rxw \vee Ryw)))$.

Proof. Let $\mathcal{F} = \langle W, R \rangle$ be a frame. From left to right, suppose that $\mathcal{F} \nVdash \forall x \forall y(\exists z(Rzx \wedge Rzy) \to \exists z \forall w(Rzw \leftrightarrow (Rxw \vee Ryw)))$. Then there are $w_1, w_2, w_3 \in W$ such that Rw_3w_1, Rw_3w_2, and for each $o \in W$, there is a $w \in W$ with $\neg(Row \leftrightarrow (Rw_1w \vee Rw_2w))$.

Define a valuation V such that $V(p) = R(w_1)$ and $V(q) = R(w_2)$. Now it holds that $\langle \mathcal{F}, V \rangle, w_1 \vDash \blacksquare p$, $\langle \mathcal{F}, V \rangle, w_2 \vDash \blacksquare q$ and $\langle \mathcal{F}, V \rangle, w_3 \vDash \Diamond \blacksquare p \wedge \Diamond \blacksquare q$. Furthermore, for all $o \in W$, if there exists $w \in W$ such that Row and $\neg Rw_1w \wedge \neg Rw_2w$, then $\langle \mathcal{F}, V \rangle, o \vDash \neg \Box (p \vee q)$, and if there exists $w \in W$ with $\neg Row$ and $Rw_1w \vee Rw_2w$, then $\langle \mathcal{F}, V \rangle, o \vDash \neg \boxbar (p \vee q)$. Each of them shows that for all $o \in W$, $\langle \mathcal{F}, V \rangle, o \vDash \neg \blacksquare (p \vee q)$. Thus we conclude that $\langle \mathcal{F}, V \rangle, w_3 \vDash \neg \Diamond_u \blacksquare (p \vee q)$.

For the converse direction, assume that $\mathcal{F} \nvDash \Diamond \blacksquare p \wedge \Diamond \blacksquare q \to \Diamond_u \blacksquare (p \vee q)$. Then there exist $w, w_1, w_2 \in W$ and a valuation V such that Rww_1, Rww_2, $V(p) = R(w_1)$ and $V(q) = R(w_2)$. For all $o \in W$, it holds that $\langle \mathcal{F}, V \rangle, o \vDash \neg \Box (p \vee q) \vee \neg \boxbar (p \vee q)$. If $\langle \mathcal{F}, V \rangle, o \vDash \neg \Box (p \vee q)$, then there exists $o' \in W$ such that Roo', $\neg Rw_1o'$ and $\neg Rw_2o'$. Thus, $\neg(Roo' \to (Rw_1o' \vee Rw_2o'))$. If $\langle \mathcal{F}, V \rangle, o \vDash \neg \boxbar (p \vee q)$, then there exists $o' \in W$ with $\neg Roo'$ and $(Rw_1o' \vee Rw_2o')$; consequently, $\neg((Rw_1o' \vee Rw_2o') \to Roo')$. So we have $\forall o \exists o' \neg (Roo' \leftrightarrow (Rw_1o' \vee Rw_2o'))$. This completes the proof. $\Box$

Unfortunately, property $\forall x \forall y(Uxy \to \exists z \forall w(Rzw \leftrightarrow (Rxw \vee Ryw)))$ is not an ideal alternative to Finite Sum. By this formula, when x underlaps y, there exists z such that $\forall w(Rzw \leftrightarrow (Rxw \vee Ryw))$. With Reflexivity, we have Rzz; consequently, $(Rxz \vee Ryz)$. However, this is not our initial intention: when z is the sum of two different objects x and y, z should not be a part of x or y. Instead, what we expect is that if z is the sum of x and y, then every part of z overlaps x or y, i.e.,

$\forall x \forall y(Uxy \to \exists z \forall w(Owz \leftrightarrow (Owx \vee Owy)))$.[16] Here we obtain a relative definability result based on EM-frames.

THEOREM 3.13. *For each EM-frame* $\mathcal{F}$, $\mathcal{F} \Vdash$ *Finite Sum iff* $\mathcal{F} \vDash \Diamond\blacksquare p \wedge \Diamond\blacksquare q \to \Diamond_u(\boxplus(p \vee q) \wedge \Box(\Diamond p \vee \Diamond q))$.

Proof. Let $\mathcal{F} = \langle W, R\rangle$ be an EM-frame. For the direction from left to right, suppose that $\mathcal{F} \nvDash \Diamond\blacksquare p \wedge \Diamond\blacksquare q \to \Diamond_u(\boxplus(p \vee q) \wedge \Box(\Diamond p \vee \Diamond q))$. Then there exist $w \in W$ and a valuation V such that $\langle \mathcal{F}, V\rangle, w \vDash \Diamond\blacksquare p \wedge \Diamond\blacksquare q \wedge \Box_u(\neg \boxplus (p \vee q) \vee \Diamond(\Box\neg p \wedge \Box\neg q))$.

By $\langle \mathcal{F}, V\rangle, w \vDash \Diamond\blacksquare p \wedge \Diamond\blacksquare q$, there are $w_1, w_2 \in R(w)$ such that $\langle \mathcal{F}, V\rangle, w_1 \vDash \blacksquare p$ and $\langle \mathcal{F}, V\rangle, w_2 \vDash \blacksquare q$. By $\langle \mathcal{F}, V\rangle, w \vDash \Box_u(\neg \boxplus (p \vee q) \vee \Diamond(\Box\neg p \wedge \Box\neg q))$, for all $o \in W$, either $\langle \mathcal{F}, V\rangle, o \nvDash \boxplus(p \vee q)$ or $\langle \mathcal{F}, V\rangle, o \vDash \Diamond(\Box\neg p \wedge \Box\neg q)$. Now given an arbitrary $w' \in W$, we consider these two cases.

If $\langle \mathcal{F}, V\rangle, w' \vDash \Diamond(\Box\neg p \wedge \Box\neg q)$, then there exists $w'' \in R(w')$ such that $\langle \mathcal{F}, V\rangle, w'' \vDash \Box\neg p \wedge \Box\neg q$. Thus w'' is disjoint from both w_1 and w_2. Furthermore, by Reflexivity, $Rw'w''$ entails that objects w' and w'' overlap.

If $\langle \mathcal{F}, V\rangle, w' \nvDash \boxplus(p \vee q)$, then there exists $w'' \in V(p) \cup V(q)$ such that $\neg Rw'w''$. W.l.o.g., suppose $w'' \in V(p)$. It follows that Rw_1w'' since $\langle \mathcal{F}, V\rangle, w_1 \vDash \blacksquare p$. By Strong Supplementation, w'' has a successor w_1'' that is disjoint from w'. By Transitivity, Rw_1w_1''. Thus, w_1'' overlaps w_1 but w_1'' is disjoint from w'. The case for supposing $w'' \in V(q)$ is similar.

By these two cases, we can conclude that for all $o \in W$, there exists $o' \in W$ such that either o overlaps o' that is disjoint from both w_1 and w_2, or that o' is disjoint from o and overlaps w_1 or w_2. Each of them indicates that $\mathcal{F} \nVdash$ *Finite Sum*.

For the other direction, assume that $\mathcal{F} \vDash \Diamond\blacksquare p \wedge \Diamond\blacksquare q \to \Diamond_u(\boxplus(p \vee q) \wedge \Box(\Diamond p \vee \Diamond q))$ and $\mathcal{F} \nVdash$ *Finite Sum*. Then there are $w, w_1, w_2 \in W$ with Rww_1 and Rww_2. Moreover, for all $o \in W$, there exists $o' \in W$ such that o overlaps o', and o' is disjoint from both w_1 and w_2; or that o' overlaps w_1 or w_2, and o is disjoint from o' $(\star)$.

Define a valuation V such that $V(p) = R(w_1)$ and $V(q) = R(w_2)$. Then, $\langle \mathcal{F}, V\rangle, w \vDash \Diamond\blacksquare p \wedge \Diamond\blacksquare q$. By the assumption, there exists $o_1 \in W$ such that $\langle \mathcal{F}, V\rangle, o_1 \vDash \boxplus(p \vee q) \wedge \Box(\Diamond p \vee \Diamond q)$. By $(\star)$, there is an o' for this o_1. Now we consider the two cases stated in $(\star)$.

First, o_1 overlaps o', and o' is disjoint from both w_1 and w_2. It holds directly that $\langle \mathcal{F}, V\rangle, o' \vDash \Box\neg p \wedge \Box\neg q$. Additionally, since o_1 overlaps o', by Transitivity, we have $\langle \mathcal{F}, V\rangle, o_1 \vDash \Diamond(\Box\neg p \wedge \Box\neg q)$.

Next, o' overlaps w_1 or w_2, and o' is disjoint from o_1. So, $\langle \mathcal{F}, V\rangle, o' \vDash \Diamond p \vee \Diamond q$. Furthermore, since o' is disjoint from o_1, we can obtain $\langle \mathcal{F}, V\rangle, o_1 \nvDash \boxplus(p \vee q)$.

Each of these two cases entails a contradiction. Thus we conclude that if $\mathcal{F} \vDash \Diamond\blacksquare p \wedge \Diamond\blacksquare q \to \Diamond_u(\boxplus(p \vee q) \wedge \Box(\Diamond p \vee \Diamond q))$, then $\mathcal{F} \Vdash$ *Finite Sum*. □

PROPOSITION 3.14. *Finite Sum cannot be defined by* $\mathcal{L}_\Box$.

Proof. Consider the two frames $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{F}' = \langle W', R'\rangle$ depicted in Figure 3. As an observation, notice $\mathcal{F}' = \langle W', R'\rangle$ is a generated subframe of $\mathcal{F}$. However, we have $\mathcal{F} \Vdash$ *Finite Sum* and $\mathcal{F}' \nVdash$ *Finite Sum*. For instance, in the frame $\mathcal{F}'$, s_3 underlaps s_4, and both of them are parts of s_6. However, s_6 also overlaps s_5 which is disjoint from both s_3 and s_4. Hence it cannot be defined by $\mathcal{L}_\Box$. □

16 For discussions on possible alternative definitions of Finite Sum, we refer the reader to [31, 47].

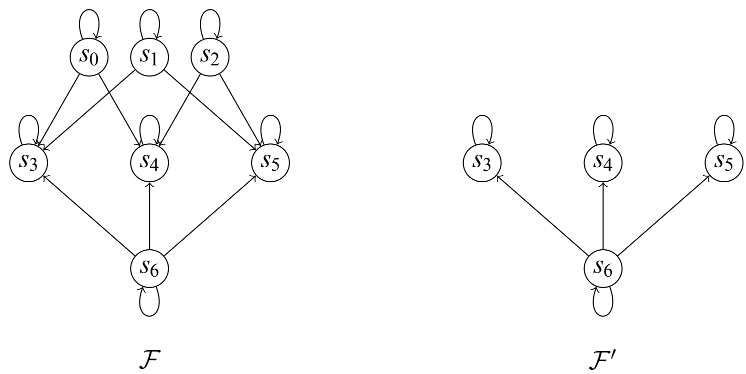

Fig. 3. A generated subframe (for Finite Sum).

Theorem 3.13 shows that we can define Finite Sum relative to the class of EM-frames. Also, Proposition 3.14 illustrates that the principle cannot be defined by $\mathcal{L}_{\Box}$. But, a natural question is:

Open Problem. Can we define Finite Sum absolutely with $\mathcal{L}_{\Box\boxminus}$?

3.5. Correspondences of general extensional mereology. *General extensional mereology* (GEM) is a generalization of CEM: when the latter theory allows us to operate finite objects, GEM enables us to operate infinite ones. It is the strongest standard theory of mereology. In particular, GEM is also called *classical mereology*, since it corresponds to the initial theory of mereology proposed by Leśniewski. Specifically, it extends theory EM with the following principle:

$M8$.	$\exists x\alpha \to \exists z\forall y(Oyz \leftrightarrow \exists x(\alpha \wedge Oyx))$	*Fusion*

where α is an arbitrary first-order formula, and variables y, z do not occur free in α.

Fusion is also called *Unrestricted Sum*, since there is almost no restriction on the property α. Intuitively, the axiom states that for any non-empty (first-order) property α, there exists an object such that the objects overlapping it are exactly those overlapping the α-objects. This axiom might be the most debatable principle among those we have introduced so far. There are many (equivalent) alternatives to the formulation of GEM employed in the literature, which usually resort to different principles for the notion of fusion, e.g., [16, sec. 5] and [40, sec. 2]. All those formulations for fusion are interesting, but as a first attempt, in what follows we just work with the above formula Fusion and leave all others for future inquiry.

Replacing the abbreviations, Fusion is $\exists x\alpha \to \exists z\forall y(\exists v(Ryv \wedge Rzv) \leftrightarrow \exists x(\alpha \wedge \exists v(Ryv \wedge Rxv)))$. Unlike the previous axioms, it is not a single first-order formula but an axiom schema w.r.t. an arbitrary first-order formula α.[17] This makes it impossible to have a modal correspondence precisely, since we cannot really capture all the first-order

[17] As mentioned at the very beginning of the article, many efforts have been made to develop first-order theories that are as powerful as GEM. Perhaps surprisingly, with some other first-order principles, finitely many instances of the schema are enough capture the full strength

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formulas using the modal language. However, we can reformulate Fusion a little bit, while keeping its essential idea, with a monadic second-order formula

$$\forall S(\exists x Sx \to \exists z \forall y(\exists v(Ryv \wedge Rzv) \leftrightarrow \exists x(Sx \wedge \exists v(Ryv \wedge Rxv)))),$$

where S denotes a second-order set variable. We call it *SO-Fusion*. Compared with Fusion, SO-Fusion is a stronger principle since the set S does not need to be first-order definable, which actually brings us closer to the original idea of Leśniewski that *every* non-empty subset should have a fusion (cf., e.g., [40, p. 811]). On the technical side, the validity of a modal formula is essentially second-order as discussed in the *Correspondence Theory* (cf., e.g., [43]); thus it is still possible to have a corresponding modal formula.

In the following we will abuse the satisfaction relation $\Vdash$ to accommodate monadic second-order formulas in the most natural manner. We have the following result:

THEOREM 3.15. *For each EM-frame $\mathcal{F}$, $\mathcal{F} \Vdash$ SO-Fusion if and only if $\mathcal{F} \vDash p \to \Diamond_u(\boxminus p \wedge (\Diamond\Box q \to \Diamond_u(p \wedge \Diamond q)))$.*

Proof. Let $\mathcal{F} = \langle W, R\rangle$ be an EM-frame. From left to right, assume towards a contradiction that $\mathcal{F} \Vdash$ *SO-Fusion* and $\mathcal{F} \nvDash p \to \Diamond_u(\boxminus p \wedge (\Diamond\Box q \to \Diamond_u(p \wedge \Diamond q)))$. Then there exist $w \in W$ and a valuation V such that $\langle \mathcal{F}, V\rangle, w \vDash p$ and $\langle \mathcal{F}, V\rangle, w \nvDash \Diamond_u(\boxminus p \wedge (\Diamond\Box q \to \Diamond_u(p \wedge \Diamond q)))$. Consequently, for every $o \in W$, either (1) or (2) holds: (1) $\langle \mathcal{F}, V\rangle, o \nvDash \boxminus p$; and (2) o has a successor $o' \in W$ such that $\langle \mathcal{F}, V\rangle, o' \vDash \Box q$ and every $v \in V(p)$ has no q-successors. Now, consider $V(p)$ as the extension for the predicate S in $\mathcal{F}$, clearly the antecedent $\exists x Sx$ of SO-Fusion holds; thus we have $\exists z \forall y(Oyz \leftrightarrow \exists x(Sx \wedge Oyx))(\star)$, too. Given an arbitrary $o \in W$, let us consider the above two cases to see whether o can be the witness for z in $(\star)$.

For (1), if $\langle \mathcal{F}, V\rangle, o \nvDash \boxminus p$, then there exists $o_1 \in V(p)$ such that $\neg Roo_1$. By Strong Supplementation, o_1 has a successor o_1' that is disjoint from o. Therefore, we have $\neg Oo_1'o$, but $(So_1 \wedge Oo_1'o_1)$. Thus, o cannot be the witness for z in $(\star)$.

For (2), consider the case that Roo', $\langle \mathcal{F}, V\rangle, o' \vDash \Box q$, and for each $v \in W$, $\langle \mathcal{F}, V\rangle, v \vDash p \to \Box\neg q$. Since $o' \in R(o)$, the two objects overlap trivially. Moreover, for any $v \in V(p)$, it is not hard to see that o' is disjoint from v. Therefore we have $Oo'o$ but $\neg\exists x(Sx \wedge Oo'x)$. Thus o cannot be the witness for z in $(\star)$ either.

In sum, no $o \in W$ can be the witness of z in $(\star)$, contradiction.

From right to left, suppose that $\mathcal{F} \vDash p \to \Diamond_u(\boxminus p \wedge (\Diamond\Box q \to \Diamond_u(p \wedge \Diamond q)))$ and $\mathcal{F} \nVdash$ *SO-Fusion*. From the latter there is a non-empty set $X \subseteq W$ such that $\forall z \exists y \neg(Oyz \leftrightarrow \exists x(Sx \wedge Oyx))(\dagger)$ holds when assigning S to X. Now, pick one $w \in X$, let $V(p) = X$, $V(q) = \{o \in W \mid \neg\exists x(Xx \wedge Oxo)\}$, and we have $\langle \mathcal{F}, V\rangle, w \vDash \Diamond_u(\boxminus p \wedge (\Diamond\Box q \to \Diamond_u(p \wedge \Diamond q)))$. Thus there is a $v \in W$, such that $\langle \mathcal{F}, V\rangle, v \vDash \boxminus p \wedge (\Diamond\Box q \to \Diamond_u(p \wedge \Diamond q))(\circ)$. Now by $(\dagger)$, there is v' such that either (1) $(\neg Ovv' \wedge \exists x(Sx \wedge Ov'x))$ or (2) $Ovv' \wedge \forall x(Sx \to \neg Ov'x)$.

Now consider case (1). Formula $\exists x(Sx \wedge Ov'x)$ states that there exists $o \in X$ such that Oov', i.e., there is some o' such that $Rv'o'$ and Roo'. Furthermore, since $\langle \mathcal{F}, V\rangle, v \vDash \boxminus p$, we have Rvo. Now by Transitivity of EM, Rvo'. However then we obtain Ovv', contradicting the first conjunct of (1).

of the high-order theory (see, e.g., [24, 25], [26, chap. 7], [40]). It is important to study the modal correspondence of all those theories, but Fusion is of interest on its own.

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For case (2), by the second conjunct, we have $v' \in V(q)$. Moreover, from the definition of the valuation V, it follows that $\langle \mathcal{F}, V\rangle, v' \vDash \Box q$. Now due to the first conjunct of (2), there is an o such that Rvo and $Rv'o$. Then, by Transitivity, we have $\langle \mathcal{F}, V\rangle, o \vDash \Box q$. Thus, $\langle \mathcal{F}, V\rangle, v \vDash \Diamond\Box q$. Now by ($\circ$), $\langle \mathcal{F}, V\rangle, v \vDash \Diamond_u(p \wedge \Diamond q)$, which indicates that there exists an $o' \in X$ that has a successor disjoint from any X-point. However, this is impossible since due to Reflexivity, o' overlaps with any successor. Now we have arrived at a contradiction. This completes the proof. □

Here a question arises:

Open Problem. Is there a formula of $\mathcal{L}_{\Box\boxminus}$ that corresponds to SO-Fusion?

Instead of answering this question, we now show that the standard modal language is not expressive enough to define Fusion.

PROPOSITION 3.16. *Fusion is not definable in* $\mathcal{L}_\Box$.

Proof. To prove this result, we consider a special case where α is Rxx. Define three frames as follows: $\mathcal{F}_1 = \langle\{w_1\}, \{\langle w_1, w_1\rangle\}\rangle$, $\mathcal{F}_2 = \langle\{w_2\}, \{\langle w_2, w_2\rangle\}\rangle$ and $\mathcal{F}_3 = \langle\{w_1, w_2\}, \{\langle w_1, w_1\rangle, \langle w_2, w_2\rangle\}\rangle$. Each of $\mathcal{F}_1$ and $\mathcal{F}_2$ consists of a reflexive point, and frame $\mathcal{F}_3$ is the *disjoint union* of $\mathcal{F}_1$ and $\mathcal{F}_2$. Now it is not hard to see that when $i \in \{1, 2\}$, we have $\mathcal{F}_i \Vdash$ *Fusion*. However, in frame $\mathcal{F}_3$ both w_1 and w_2 satisfy the property Rxx but they are disjoint from each other. So, $\mathcal{F}_3 \nVdash$ *Fusion*. Again, by the Goldblatt–Thomason Theorem, the property cannot be defined by $\mathcal{L}_\Box$. □

3.6. Correspondences of atomicity and atomlessness. For each standard theory mentioned above, we can also extend it with other properties. For instance, we can stipulate that every object has a part that has no proper part, or that every object has some proper part. The former property is called *Atomicity*, and the latter one is named as *Atomlessness*. When an object x has no proper part, we call it an *atom* (notation, Ax), which is formally defined as $Ax := \neg\exists y PPyx$. The definitions of the above two principles are as follows:

$M9$.	$\forall x\exists y(Ay \wedge Rxy)$	*Atomicity*
$M10$.	$\forall x\exists y PPyx$	*Atomlessness*

An atom is also called *simple*. By its definition, any two different atoms have no common parts. It is important to recognize that Atomicity does not state that everything is ultimately composed of atoms, and it just says that every object contains some atomic part. Furthermore, Atomicity is not compatible with Atomlessness, since an atom has no proper parts. For any X ∈ {GM, MM, EM, CEM, GEM}, we denote by AX and $\widetilde{\text{A}}$X the theories enriching X with Atomicity and Atomlessness respectively.

With the definition of Proper Part, we know that Ax is equivalent to $\forall y(Rxy \rightarrow Ryx)$ (*Symmetry*), which corresponds to formula $p \rightarrow \Box\Diamond p$ (**B**). In addition, Atomicity is equivalent to $\forall x\exists y(Rxy \wedge \forall z(Ryz \rightarrow Rzy))$, and Atomlessness is equivalent to $\forall x\exists y(Rxy \wedge \neg Ryx)$. As noted by Blackburn et al. [1, example 3.57], we have the following result:

PROPOSITION 3.17. *For any transitive frame* $\mathcal{F}$, $\mathcal{F} \Vdash \forall x\exists y(Rxy \wedge \forall z(Ryz \rightarrow z \equiv y))$ *if and only if* $\mathcal{F} \vDash \Box\Diamond p \rightarrow \Diamond\Box p$.

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Recall that a GM-frame is transitive and anti-symmetric; thus we have:

THEOREM 3.18. *For each GM-frame* $\mathcal{F}$, $\mathcal{F} \Vdash$ *Atomicity iff* $\mathcal{F} \vDash \Box\Diamond p \to \Diamond\Box p$.

Namely, relative to GM-frames, we can define Atomicity with the McKinsey formula, which is an $\mathcal{L}_\Box$-formula. But, is this principle absolutely definable with the standard modal language? Unfortunately, although this property is closed under taking bounded morphic images, disjoint unions, and generated subframes, the following result still provides us with a negative answer to this question:

PROPOSITION 3.19. *Atomicity cannot be defined by* $\mathcal{L}_\Box$.

Proof. To prove this, by the Goldblatt–Thomason Theorem, it suffices to give an example showing that the class of Atomicity-frames does not *reflect*[18] *ultrafilter extensions*. Consider the frame $\mathcal{F} = \langle \mathbb{N}, < \rangle$ (the natural numbers with the usual $<$ relation) and its ultrafilter extension $ue(\mathcal{F})$ (cf. [1, sec. 3.3]). It is a matter of direct checking that $ue(\mathcal{F}) \Vdash$ *Atomicity* and $\mathcal{F} \nVdash$ *Atomicity*. Now the proof is completed. □

Furthermore, as the following result shows, Atomicity essentially cannot even be defined by our language $\mathcal{L}_{\Box\boxplus}$.

PROPOSITION 3.20. *Atomicity is undefinable in* $\mathcal{L}_{\Box\boxplus}$.

The result can be proved with the same reasoning as that for Example 4.15 in [9], which indicated that the first-order property $\exists x Rxx$ is undefinable in $\mathcal{L}_{\Box\boxplus}$. However, the details of the proof involve some results on the expressivity of the language, and it is beyond the scope of this article to explain them. For the reader interested in the details of the proof and the expressivity of $\mathcal{L}_{\Box\boxplus}$, we refer to [9].

As for Atomlessness, the correspondence result is as follows:

THEOREM 3.21. *For each frame* $\mathcal{F}$, $\mathcal{F} \Vdash$ *Atomlessness iff* $\mathcal{F} \vDash p \to \Diamond\neg\boxplus p$.

Proof. Let $\mathcal{F} = \langle W, R \rangle$ be a frame. For the direction from left to right, suppose for reductio that $\mathcal{F} \Vdash$ *Atomlessness* and $\mathcal{F} \nvDash p \to \Diamond\neg\boxplus p$. Then there exist $w \in W$ and a valuation V such that $\langle \mathcal{F}, V \rangle, w \vDash p \wedge \Box\boxplus p$. Since $\mathcal{F} \Vdash$ *Atomlessness*, there exists $v \in W$ with $Rwv \wedge \neg Rvw$. By Rwv, it holds that $\langle \mathcal{F}, V \rangle, v \vDash \boxplus p$. However, from $\neg Rvw$ and $w \in V(p)$, we know $\langle \mathcal{F}, V \rangle, v \nvDash \boxplus p$, which entails a contradiction.

For the other direction, suppose that $\mathcal{F} \vDash p \to \Diamond\neg\boxplus p$ and $\mathcal{F} \nVdash$ *Atomlessness*. Then there exists $w \in W$ such that for all $w' \in W$, Rww' entails $Rw'w$. Now define a valuation V such that $V(p) = \{w\}$. Since formula $p \to \Diamond\neg\boxplus p$ is valid, there exists $v \in R(w)$ such that v is not $\boxplus p$. Then we conclude that $\neg Rvw$, which entails a contradiction, too. This completes the proof. □

Additionally, the following result shows that this property cannot be defined by the standard modal language:

PROPOSITION 3.22. *Atomlessness is not definable in* $\mathcal{L}_\Box$.

Proof. Consider frames $\mathcal{F}_1 = \langle W_1, R_1 \rangle$ and $\mathcal{F}_2 = \langle W_2, R_2 \rangle$ depicted in Figure 4. Define a function $f : W_1 \to W_2$ such that $f(w) = v$ for all $w \in W_1$. Then f is a bounded morphism from $\mathcal{F}_1$ to $\mathcal{F}_2$. However, we have $\mathcal{F}_1 \Vdash$ *Atomlessness* and $\mathcal{F}_2 \nVdash$ *Atomlessness*; therefore by the Goldblatt–Thomason Theorem, Atomlessness is not definable in $\mathcal{L}_\Box$. □

[18] We say a class $\mathcal{A}$ of frames *reflects* ultrafilter extensions if, for any frame $\mathcal{F}$, if the ultrafilter extension of $\mathcal{F}$ belongs to $\mathcal{A}$, then $\mathcal{F}$ is also an element of $\mathcal{A}$.

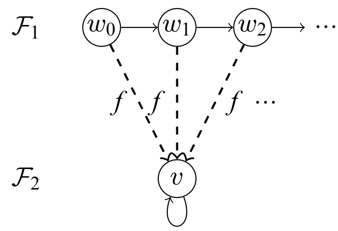

Fig. 4. A bounded morphism (for Atomlessness).

3.7. Summary. So far, we have shown that all the following properties cannot be defined by the standard modal language: Supplementation (Proposition 3.7), Strong Supplementation (Proposition 3.9), Finite Product (Proposition 3.11), Finite Sum (Proposition 3.14), Fusion (Proposition 3.16), Atomicity (Proposition 3.19) and Atomlessness (Proposition 3.22). Also, we have shown that the principle Atomicity cannot be defined by $\mathcal{L}_{\Box\boxbar}$ (Proposition 3.20). But, all these properties are definable (absolutely or relative to some particular class of frames) in $\mathcal{L}_{\Box\boxbar}$. Now we summarize the correspondence results on these properties.

For each frame $\mathcal{F}$,

$\mathcal{F} \Vdash$ *Reflexivity*	$\Leftrightarrow$	$\mathcal{F} \vDash \Box p \to p$,
$\mathcal{F} \Vdash$ *Transitivity*	$\Leftrightarrow$	$\mathcal{F} \vDash \Box p \to \Box\Box p$,
$\mathcal{F} \Vdash$ *Anti-symmetry*	$\Leftrightarrow$	$\mathcal{F} \vDash \Diamond(\boxbar p \wedge q) \to (p \to q)$,
$\mathcal{F} \Vdash$ *Supplementation*	$\Leftrightarrow$	$\mathcal{F} \vDash \neg p \wedge \Diamond\blacksquare p \to \Diamond\Box\neg p$,
$\mathcal{F} \Vdash$ *Strong Supplementation*	$\Leftrightarrow$	$\mathcal{F} \vDash \blacksquare p \to \Box_u(\neg p \to \Diamond\Box\neg p)$,
$\mathcal{F} \Vdash$ *Finite Product*	$\Leftrightarrow$	$\mathcal{F} \vDash \blacksquare p \wedge \Diamond_u\blacksquare q \wedge \Diamond(p \wedge q) \to \Diamond_u\blacksquare(p \wedge q)$,
$\mathcal{F} \Vdash$ *Atomlessness*	$\Leftrightarrow$	$\mathcal{F} \vDash p \to \Diamond\neg \boxbar p$.

For each GM-frame $\mathcal{F}$,

$\mathcal{F} \Vdash$ *Atomicity*	$\Leftrightarrow$	$\mathcal{F} \vDash \Box\Diamond p \to \Diamond\Box p$.

For each EM-frame $\mathcal{F}$,

$\mathcal{F} \Vdash$ *Finite Sum*	$\Leftrightarrow$	$\mathcal{F} \vDash \Diamond\blacksquare p \wedge \Diamond\blacksquare q \to \Diamond_u(\boxbar(p \vee q) \wedge \Box(\Diamond p \vee \Diamond q))$,
$\mathcal{F} \Vdash$ *SO-Fusion*	$\Leftrightarrow$	$\mathcal{F} \vDash p \to \Diamond_u(\boxbar p \wedge (\Diamond\Box q \to \Diamond_u(p \wedge \Diamond q)))$.

Finally, it is worth noting that our results can also be generalized to define some other theories of mereology, say, minimal extensional mereology, which extends theory MM with Finite Product [38, p. 990].

3.8. A digression: Medvedev frames. To conclude this section, we show that we can define Medvedev frames by $\mathcal{L}_{\Box\boxbar}$ relative to the class of finite frames. According to [15], a frame $\langle W, R\rangle$ is a Medvedev frame if it is a finite poset satisfying the following properties:

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- $\forall x\forall y\exists z(Rzx \wedge Rzy)$ (*Reversed Convergence*);
- $\forall x\forall y(\forall y'(Ryy' \to \exists y''(Ry'y'' \wedge Rxy'')) \to Rxy)$ (*Separativity*); and
- $\forall x\forall y_1\forall y_2((Rxy_1 \wedge Rxy_2) \to \exists u(Rxu \wedge Ruy_1 \wedge Ruy_2 \wedge \forall v(Ruv \to \exists w(Rvw \wedge (Ry_1w \vee Ry_2w)))))$ (*Union*).

We only need to define the first-order property for Reversed Convergence, and the other formulas are already defined by $\Box$ and $\boxminus$ by Holliday [15], which can be turned into formulas in our language since $\boxminus$ and $\boxdot$ are inter-definable.[19]

PROPOSITION 3.23. *For each frame $\mathcal{F}$, $\mathcal{F} \Vdash \forall x\forall y\exists z(Rzx \wedge Rzy)$ iff $\mathcal{F} \vDash p \wedge \Diamond_u q \to \Diamond_u(\Diamond p \wedge \Diamond q)$.*

Proof. Let $\mathcal{F} = \langle W, R\rangle$ be a frame. From left to right, suppose that $\mathcal{F} \nvDash p \wedge \Diamond_u q \to \Diamond_u(\Diamond p \wedge \Diamond q)$. Then there exist $w, v \in W$ and a valuation V such that $\langle \mathcal{F}, V\rangle, w \vDash p$, $\langle \mathcal{F}, V\rangle, v \vDash q$, and for all $o \in W$, $\langle \mathcal{F}, V\rangle, o \nvDash \Diamond p \wedge \Diamond q$. We conclude that w and v have no common predecessor, i.e., $\mathcal{F} \nVdash \forall x\forall y\exists z(Rzx \wedge Rzy)$.

From right to left, assume that $\mathcal{F} \vDash p \wedge \Diamond_u q \to \Diamond_u(\Diamond p \wedge \Diamond q)$ and $\mathcal{F} \nVdash \forall x\forall y\exists z(Rzx \wedge Rzy)$. Then there are $w, v \in W$ that have not common predecessor. Define the valuation V such that $V(p) = \{w\}$ and $V(q) = \{v\}$. Consequently, $\langle \mathcal{F}, V\rangle, w \vDash p \wedge \Diamond_u q$. Since $\mathcal{F} \vDash p \wedge \Diamond_u q \to \Diamond_u(\Diamond p \wedge \Diamond q)$, there exists $o \in W$ with $\langle \mathcal{F}, V\rangle, o \vDash \Diamond p \wedge \Diamond q$, i.e., o is a common predecessor of w and v, which entails a contradiction. □

Given that our language is able to define those properties of mereology, it is not a surprise that we can also define Medvedev frames relative to finite frames. Essentially, when taking 'part of' P as the primitive relation, theory GEM is isomorphic to the inclusion relation restricted to the set of all non-empty subsets of a given set, which is to say a complete Boolean algebra with the zero element removed [13, 35]. Although we have this general result, the findings in [15] are not enough to achieve our goals. At the end of this part, we briefly comment on the modal correspondences of Separativity and Union in [15].

The principle Separativity is equivalent to Strong Supplementation. Its modal correspondence in [15] is $\neg \boxdot\, p \wedge \boxdot\neg q \to \neg \boxdot\, (p \wedge \Diamond\Box q)$ (reformulated with window modality), which does correspond to Strong Supplementation. However, formula $\blacksquare p \to \Box_u(\neg p \to \Diamond\Box\neg p)$, corresponding to Strong Supplementation, has a more natural reading in the context of mereology. In addition, our modal formula contains only one propositional atom, which looks much simpler.

The modal correspondence of Union in [15] is $\Diamond(p_1 \wedge \Box q) \wedge \Diamond(p_2 \wedge \Box q) \to \Diamond(\Diamond p_1 \wedge \Diamond p_2 \wedge \Box\Diamond q)$. Although the function of mereological 'sum' is similar to that of set-theoretic 'union' in some sense, this standard modal formula does not correspond to Finite Sum, which follows from Proposition 3.14 directly.

§4. Mereological modal logics.

In this section, we introduce the mereological modal logics, and show that they are sound. Corresponding to the first-order theories, the modal systems are called *modal ground mereology* (*MGM*), *modal minimal mereology* (*MMM*), *modal extensional mereology* (*MEM*), *modal extensional closure mereology* (*MCEM*), and *modal general extensional mereology* (*MGEM*), respectively. Also, we

[19] $\boxminus\varphi$ holds at w iff for all v, if not wRv, then φ holds at v.

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prove the incompleteness of MGM over frames and the completeness of an extension of this modal system.

4.1. Logical systems. To get the mereological modal systems, we first introduce two logical systems K^* and $K^\sim$: the former one is the minimal system of $\mathcal{L}_{\boxminus}$, which has only the operator $\boxminus$ (besides Boolean connectives $\neg$ and $\wedge$), and the latter one is the minimal normal system of $\mathcal{L}_{\Box\boxminus}$. These two systems are well introduced in [6, secs. 1 and 2], and we reformulate them with two equivalent systems. Let us begin with K^*, which is defined formally as follows:

System K^*	
Axioms:	
$A^*1.$	All the instances of tautologies
$A^*2.$	$\boxminus\varphi \wedge \boxminus(\neg\varphi \wedge \psi) \to \boxminus\psi$
Inference rules:	
$R^*1.$	From φ and $\varphi \to \psi$, infer ψ.
$R^*2.$	If $\vdash_{K^*} \varphi$, then $\vdash_{K^*} \boxminus\neg\varphi$.

Let K be the basic normal modal system, we have the following results:

THEOREM 4.24 [6]. *For any $\mathcal{M} = \langle W, R, V\rangle$, $w \in W$ and $\varphi \in \mathcal{L}_{\Box}$, let φ^* be the formula obtained by substituting every $\Box$ occurring in φ with $\boxminus\neg$, and $\mathcal{M}^* := \langle W, W^2\backslash R, V\rangle$. Then, $\vdash_K \varphi$ iff $\vdash_{K^*} \varphi^*$, $\mathcal{M}, w \vDash \varphi$ iff $\mathcal{M}^*, w \vDash \varphi^*$, and $\vDash \varphi$ iff $\vDash \varphi^*$.*

THEOREM 4.25 [6]. *Logic K^* is sound, complete, decidable and compact.*

Compared with K^*, system $K^\sim$ is much more complicated: its axioms include not only principles involving $\Box$ and $\boxminus$, but also those involving the universal modality $\Box_u$ that can be defined by $\Box$ and $\boxminus$. Here is the definition:

System $K^\sim$	
Axioms:	
$A1.$	All the instances of tautologies
$A2.$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$A3.$	$\Box_u\varphi \to \varphi$
$A4.$	$\Box_u\varphi \to \Box_u\Box_u\varphi$
$A5.$	$\varphi \to \Box_u\Diamond_u\varphi$
$A6.$	$\boxminus\varphi \wedge \boxminus\psi \to \boxminus(\varphi \vee \psi)$
$A7.$	$\boxminus\bot$
Inference rules:	
$R1.$	From φ and $\varphi \to \psi$, infer ψ.
$R2.$	If $\vdash_{K^\sim} \varphi$, then $\vdash_{K^\sim} \Box\varphi$.
$R3.$	If $\vdash_{K^\sim} \varphi \to \psi$, then $\vdash_{K^\sim} \boxminus\psi \to \boxminus\varphi$.

Observe that $\Box_u$ is an S5-modality. Moreover, the rules are below derivable in $K^\sim$:

- If $\vdash_{K^\sim} \varphi \to \psi$ and $\vdash_{K^\sim} \psi \to \chi$, then $\vdash_{K^\sim} \varphi \to \chi$. (RS)

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- If $\vdash_{K^{\sim}} \varphi \to \psi$ and $\vdash_{K^{\sim}} \varphi \to \chi$, then $\vdash_{K^{\sim}} \varphi \to \psi \wedge \chi$. (RCC)
- If $\vdash_{K^{\sim}} \psi \leftrightarrow \chi$ and $\vdash_{K^{\sim}} \varphi$, then $\vdash_{K^{\sim}} \varphi(\psi/\chi)$, where $\varphi(\psi/\chi)$ is the formula obtained by substituting each ψ occurring in φ with χ. (RES)

We omit the proof here, which is by a simple adaption of standard arguments (cf. [1]). Furthermore, as to the relation between $K^{\sim}$ and K^*, we have the following result:

THEOREM 4.26. *$K^{\sim}$ is a proper extension of K^*.*

Proof. First, we prove that all inference rules of K^* are derivable in $K^{\sim}$. To do so, we only need to show that R^*2 is derivable in $K^{\sim}$. Suppose that $\neg\varphi$, i.e., $\varphi \to \bot$. From $R3$, it follows that $\boxbar\bot \to \boxbar\varphi$. As $\boxbar\bot$ is $A7$, we conclude $\boxbar\varphi$ by $R1$.

Next, we show that $\vdash_{K^{\sim}} \boxbar\varphi \wedge \boxbar(\neg\varphi \wedge \psi) \to \boxbar\psi$:

(1)	$\boxbar\varphi \wedge \boxbar(\neg\varphi \wedge \psi) \to \boxbar(\varphi \vee (\neg\varphi \wedge \psi))$	$A6$
(2)	$\varphi \vee (\neg\varphi \wedge \psi) \leftrightarrow \varphi \vee \psi$	$A1$
(3)	$\boxbar\varphi \wedge \boxbar(\neg\varphi \wedge \psi) \to \boxbar(\varphi \vee \psi)$	(1)(2)RES
(4)	$\psi \to \varphi \vee \psi$	$A1$
(5)	$\boxbar(\varphi \vee \psi) \to \boxbar\psi$	(4) $R3$
(6)	$\boxbar\varphi \wedge \boxbar(\neg\varphi \wedge \psi) \to \boxbar\psi$	(3)(5)RS

Moreover, K^* has no formula containing modality $\Box$, but $K^{\sim}$ has $R2$ as an inference rule. Thus the proof is completed. □

By Theorems 4.24 and 4.26, it holds immediately that:

THEOREM 4.27. *For any $\mathcal{L}_{\Box}$-formula φ, if $\vdash_K \varphi$, then $\vdash_{K^{\sim}} \varphi^*$, where φ^* is the formula obtained by substituting every $\Box$ occurring in φ with $\boxbar\neg$.*

Consequently, the following result holds:

PROPOSITION 4.28. $\vdash_{K^{\sim}} \boxbar\neg(\varphi \to \psi) \to (\boxbar\neg\varphi \to \boxbar\neg\psi)$.

Also, we can prove that:

PROPOSITION 4.29. $\vdash_{K^{\sim}} \Box\varphi \wedge \boxbar\psi \to \Box_u(\psi \to \varphi)$.

Proof.

(1)	$\varphi \to (\psi \to \varphi)$	$A1$
(2)	$\Box(\varphi \to (\psi \to \varphi))$	(1) $R2$
(3)	$\Box(\varphi \to (\psi \to \varphi)) \to (\Box\varphi \to \Box(\psi \to \varphi))$	$A2$
(4)	$\Box\varphi \to \Box(\psi \to \varphi)$	(2)(3) $R1$
(5)	$\neg\psi \to (\psi \to \varphi)$	$A1$
(6)	$\Box(\neg\psi \to (\psi \to \varphi))$	(5) $R2$
(7)	$\Box(\neg\psi \to (\psi \to \varphi)) \to (\Box\neg\psi \to \Box(\psi \to \varphi))$	$A2$
(8)	$\Box\neg\psi \to \Box(\psi \to \varphi)$	(6)(7) $R1$
(9)	$\boxbar\psi \to \boxbar\neg(\psi \to \varphi)$	(8)Theorem 4.27
(10)	$\Box\varphi \wedge \boxbar\psi \to \Box\varphi$	$A1$
(11)	$\Box\varphi \wedge \boxbar\psi \to \Box(\psi \to \varphi)$	(10)(4)RS
(12)	$\Box\varphi \wedge \boxbar\psi \to \boxbar\psi$	$A1$
(13)	$\Box\varphi \wedge \boxbar\psi \to \boxbar\neg(\psi \to \varphi)$	(12)(9)RS
(14)	$\Box\varphi \wedge \boxbar\psi \to \Box_u(\psi \to \varphi)$	(11)(13)RCC

□

Note that formula (8) in the proof above is provable in K, so Theorem 4.27 is applicable. Furthermore, as noted by Gargov et al. [6, sec. 2], we have the following result:

THEOREM 4.30 [6]. *$K^{\sim}$ is sound, complete and decidable.* □

Now it is time to introduce the definition of our modal systems. Each of the modal systems is an extension of $K^{\sim}$.

DEFINITION 4.31 (Mereological modal systems). *For any $X \in \{GM, MM, EM, CEM, GEM\}$, we denote by MX the logical system extending $K^{\sim}$ with the schemata of the modal correspondences of X. Formally, they are defined as follows*:

$$\begin{aligned}
MGM &:= K^{\sim} + \mathbf{T} + \mathbf{4} + \Diamond(\boxminus\varphi \wedge \psi) \to (\varphi \to \psi),\\
MMM &:= MGM + \neg\varphi \wedge \Diamond\blacksquare\varphi \to \Diamond\Box\neg\varphi,\\
MEM &:= MGM + \blacksquare\varphi \to \Box_u(\neg\varphi \to \Diamond\Box\neg\varphi),\\
MCEM &:= MEM + \blacksquare\varphi \wedge \Diamond_u\blacksquare\psi \wedge \Diamond(\varphi \wedge \psi) \to \Diamond_u\blacksquare(\varphi \wedge \psi)\\
&\quad + \Diamond\blacksquare\varphi \wedge \Diamond\blacksquare\psi \to \Diamond_u(\boxminus(\varphi \vee \psi) \wedge \Box(\Diamond\varphi \vee \Diamond\psi)),\\
MGEM &:= MEM + \varphi \to \Diamond_u(\boxminus\varphi \wedge (\Diamond\Box\psi \to \Diamond_u(\varphi \wedge \Diamond\psi))).
\end{aligned}$$

Also, define $AMX := MX + \Box\Diamond\varphi \to \Diamond\Box\varphi$ and $\tilde{A}MX := MX + \varphi \to \Diamond\neg\boxminus\varphi$.

Recall that the formula used in MGM to capture Anti-symmetry is slightly different from the one used by Goranko [9].

With our findings in Section 3, it is easy to see that all the resulting modal systems are sound:

PROPOSITION 4.32 (Soundness). *For any theory $X \in \{GM, MM, EM, CEM, GEM\}$, systems MX, AMX and $\tilde{A}MX$ are sound with respect to X-frames, AX-frames and $\tilde{A}X$-frames, respectively.*

4.2. Incompleteness of MGM. Now we proceed to study the completeness of MGM. Actually, it is not frame-complete. To prove this result, we need to find a formula φ that is valid on GM-frames but not provable in MGM.

Consider the formula $\Diamond\boxminus\varphi \to \boxminus\varphi$ that is proposed by Goranko [8, p. 321], which in fact also corresponds to Transitivity.[20] In the context of mereology, it intuitively states that if an object has a part having all φ-objects as its parts, then all those objects are also the parts of this object. First, we show that it is valid on GM-frames.

PROPOSITION 4.33. *Formula $\Diamond\boxminus\varphi \to \boxminus\varphi$ is valid on GM-frames.*

Proof. Let $\mathcal{F} = \langle W, R\rangle$ be a GM-frame, V a valuation and $w \in W$. Suppose that $\langle\mathcal{F}, V\rangle, w \vDash \Diamond\boxminus\varphi$. Then there exists $v \in R(w)$ such that for any $u \in W$, if u is φ, then Rvu. By Transitivity, all φ-points are successors of w. Therefore, $\langle\mathcal{F}, V\rangle, w \vDash \boxminus\varphi$. This completes the proof. □

Next, we show that $\Diamond\boxminus\varphi \to \boxminus\varphi$ is not provable in MGM. To do so, we introduce an auxiliary notion of '*generalized model*' proposed by Gargov et al. [6]:

[20] The formula was used to show that $K^{\sim} + \mathbf{4} + \mathbf{T}$ is not complete over reflexive and transitive frames, but adding it to the system completes the logic [8, theorem 8].

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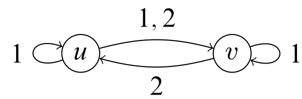

Fig. 5. A generalized frame (R_1 is labelled with ‘1’, and R_2 with ‘2’).

Definition 4.34 (Generalized model and generalized frame). *A generalized model $\mathcal{M}_g = \langle W, R_1, R_2, V\rangle$ is a tuple, where W is a non-empty set of objects, R_1 and R_2 are two binary relations such that $R_1 \cup R_2 = W \times W$, and V is a valuation function defined as usual. Moreover, we call $\mathcal{F}_g = \langle W, R_1, R_2\rangle$ a generalized frame.*

It is important to notice that generalized models and frames have two distinguishing features. On the one hand, it always holds that $R_1 \cup R_2 = W \times W$. On the other hand, generally $R_1 \cap R_2 \neq \emptyset$. From now on, we will use the notation ‘$\Vdash$’ to denote the satisfaction relation w.r.t. generalized models. The truth conditions for Boolean cases are as usual. The semantics for $\Box$ and $\boxbar$ are as follows:

$\mathcal{M}_g, w \Vdash \Box\varphi$	$\Leftrightarrow$	for each $v \in W$, if R_1wv, then $\mathcal{M}_g, v \Vdash \varphi$
$\mathcal{M}_g, w \Vdash \boxbar\,\varphi$	$\Leftrightarrow$	for each $v \in W$, if R_2wv, then $\mathcal{M}_g, v \Vdash \neg\varphi$

Now we are able to show the following result:

Proposition 4.35. *$\Diamond \boxbar\, \varphi \to \boxbar\varphi$ is not provable in MGM.*

Proof. Consider the generalized frame $\mathcal{F}_g$ depicted in Figure 5. It holds that $R_1 \cup R_2 = W \times W$. Let $\varphi := p$. We now show that $\mathcal{F}_g \nVdash \Diamond \boxbar\, p \to \boxbar p$. Define a valuation V such that $V(p) = \{v\}$. Then it follows that $\langle \mathcal{F}_g, V\rangle, v \Vdash \boxbar\, p$ and $\langle \mathcal{F}_g, V\rangle, u \Vdash \neg \boxbar\, p$. Also, we have $\langle \mathcal{F}_g, V\rangle, u \Vdash \Diamond \boxbar\, p$. Consequently, $\langle \mathcal{F}_g, V\rangle, u \nVdash \Diamond \boxbar\, p \to \boxbar p$.

Next, we show that MGM is valid on $\mathcal{F}_g$. Here we only prove that formulas $A3$, $A6$ and $\Diamond(\boxbar\varphi \wedge \psi) \to (\varphi \to \psi)$ are valid on $\mathcal{F}_g$, and that the validity of $\mathcal{L}_{\Box\boxbar}$-formulas is invariant under $R3$ in $\mathcal{F}_g$.

(1) For each $x \in \{u, v\}$, suppose that x is $\Box_u\varphi$. Thus, x is $\Box\varphi$. Since R_1 in the frame $\mathcal{F}_g$ is reflexive, x is φ. Consequently, x is $\Box_u\varphi \to \varphi$. Thus, $A3$ is valid on $\mathcal{F}_g$.

(2) If u is $\boxbar\varphi \wedge \boxbar\psi$, then v is $\neg\varphi \wedge \neg\psi$, i.e., $\neg(\varphi \vee \psi)$. So u is $\boxbar(\varphi \vee \psi)$. Similarly, when v is $\boxbar\varphi \wedge \boxbar\psi$, we can also prove that v is $\boxbar(\varphi \vee \psi)$. Thus, $A6$ is valid on $\mathcal{F}_g$.

(3) Suppose that u is $\Diamond(\boxbar\varphi \wedge \psi)$. Then, at least one of u and v is $\boxbar\varphi \wedge \psi$. If u is $\boxbar\varphi \wedge \psi$, u is ψ. Thus, $\Diamond(\boxbar\varphi \wedge \psi) \to (\varphi \to \psi)$ is true at u. If v is $\boxbar\varphi \wedge \psi$, then u is $\neg\varphi$. So formula $\Diamond(\boxbar\varphi \wedge \psi) \to (\varphi \to \psi)$ is always true at u. Similarly, we can prove that $\Diamond(\boxbar\varphi \wedge \psi) \to (\varphi \to \psi)$ is also true at v.

(4). Assume that $\varphi \to \psi$ is valid on $\mathcal{F}_g$. If u is $\boxbar\psi \wedge \neg \boxbar\, \varphi$, then v is $\neg\psi \wedge \varphi$, i.e., $\neg(\varphi \to \psi)$, which contradicts our assumption. Similarly, we can prove that v is also $\boxbar\psi \to \boxbar\varphi$. Therefore the validity of $\mathcal{L}_{\Box\boxbar}$-formulas is invariant under $R3$ in $\mathcal{F}_g$.

Hence we conclude that all formulas provable in MGM are valid on $\mathcal{F}_g$, but $\Diamond \boxbar\, \varphi \to \boxbar\varphi$ is not. Now the proof is completed. $\Box$

By Propositions 4.33 and 4.35, we can obtain the following result:

Theorem 4.36 (Incompleteness of MGM). *MGM is not complete over GM-frames.*

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At the end of this part, with respect to generalized frames, we show the frame-conditions characterized by formulas $\Diamond \boxbar p \to \boxbar p$ and $\Diamond(\boxbar p \wedge q) \to (p \to q)$.

Proposition 4.37 [8]. *For any generalized frame* $\mathcal{F}_g = \langle W, R_1, R_2 \rangle$, $\mathcal{F}_g \Vvdash \Diamond \boxbar p \to \boxbar p$ *if and only if* $\mathcal{F}_g \Vdash \forall x \forall y \forall z (R_1 xy \wedge R_2 xz \to R_2 yz)$.

Proof. From left to right, suppose for reductio that $\mathcal{F}_g \Vvdash \Diamond \boxbar p \to \boxbar p$ and $\mathcal{F}_g \nVdash \forall x \forall y \forall z (R_1 xy \wedge R_2 xz \to R_2 yz)$. Then there are $w_1, w_2, w_3 \in W$ such that $R_1 w_1 w_2$, $R_2 w_1 w_3$ and $\neg R_2 w_2 w_3$. Define a valuation V such that $V(p) = \{w \in W \mid \neg R_2 w_2 w\}$. Consequently, w_2 is $\boxbar p$. Since $R_1 w_1 w_2$, w_1 is $\Diamond \boxbar p$. Consequently, w_1 is $\boxbar p$. Then, from $R_2 w_1 w_3$ we know that w_3 is $\neg p$. Therefore, it holds that $R_2 w_2 w_3$, contradiction.

From right to left, assume that $\mathcal{F}_g \not\Vvdash \Diamond \boxbar p \to \boxbar p$. There exist a valuation V and $w_1 \in W$ such that $\langle \mathcal{F}_g, V \rangle, w_1 \Vvdash \Diamond \boxbar p \wedge \neg \boxbar p$. So there exists $w_2 \in W$ such that $R_1 w_1 w_2$ and w_2 is $\boxbar p$. Moreover, as w_1 is $\neg \boxbar p$, there exists $w_3 \in W$ with $R_2 w_1 w_3$ and $w_3 \in V(p)$. However, since w_2 and w_3 are, respectively, $\boxbar p$ and p, it holds that $\neg R_2 w_2 w_3$. Thus, $\mathcal{F}_g \nVdash \forall x \forall y \forall z (R_1 xy \wedge R_2 xz \to R_2 yz)$. Now the proof is completed. $\square$

Proposition 4.38. *For each generalized frame* $\mathcal{F}_g$, $\mathcal{F}_g \Vvdash \Diamond(\boxbar p \wedge q) \to (p \to q)$ *if and only if* $\mathcal{F}_g \Vdash \forall x \forall y (R_1 xy \wedge R_1 yx \to R_2 yx \vee x \equiv y)$.

Proof. Let $\mathcal{F}_g = \langle W, R_1, R_2 \rangle$ be a generalized frame. From left to right, suppose that $\mathcal{F}_g \Vvdash \Diamond(\boxbar p \wedge q) \to (p \to q)$ and $\mathcal{F}_g \nVdash \forall x \forall y (R_1 xy \wedge R_1 yx \to R_2 yx \vee x \equiv y)$. Then there are $w_1, w_2 \in W$ such that $R_1 w_1 w_2$, $R_1 w_2 w_1$, $w_1 \neq w_2$ and $\neg R_2 w_2 w_1$. Define a valuation V such that $V(p) = \{w_1\}$ and $V(q) = \{w_2\}$. Then, w_1 is $p \wedge \neg q$, and w_2 is $\boxbar p \wedge q$. Therefore, $\langle \mathcal{F}_g, V \rangle, w_1 \Vvdash \Diamond(\boxbar p \wedge q)$. Consequently, $\langle \mathcal{F}_g, V \rangle, w_1 \Vvdash p \to q$, which entails a contradiction.

From right to left, suppose that $\mathcal{F}_g \not\Vvdash \Diamond(\boxbar p \wedge q) \to (p \to q)$. Then there exist a valuation V and $w_1 \in W$ such that $\langle \mathcal{F}_g, V \rangle, w_1 \Vvdash \Diamond(\boxbar p \wedge q) \wedge p \wedge \neg q$. Therefore, there exists $w_2 \in W$ such that $R_1 w_1 w_2$ and w_2 is $\boxbar p \wedge q$. Also, we have $w_1 \neq w_2$. Furthermore, since w_2 is $\boxbar p$, we know $\neg R_2 w_2 w_1$ from $w_1 \in V(p)$. With Definition 4.34, $R_1 \cup R_2 = W \times W$. Thus we obtain $R_1 w_2 w_1$. Immediately, we conclude that $\mathcal{F}_g \nVdash \forall x \forall y (R_1 xy \wedge R_1 yx \to R_2 yx \vee x \equiv y)$. The proof is completed. $\square$

4.3. Completeness of MGM$^+$.

We will show that MGM^+, the extension of MGM with $\Diamond \boxbar \varphi \to \boxbar \varphi$, is strongly complete with respect to GM-frames. A version of this result was first given by Goranko [8, theorem 8], but, as we will show later, the proof was flawed.[21]

In this section, we fix the proof by giving a different construction of the canonical model. First, we introduce some preliminary notions.

Definition 4.39 (Generalized canonical model). *The generalized canonical model is a tuple* $\mathcal{M}^c_g = \langle W^c, R^c_1, R^c_2, V^c \rangle$:

- W^c *is the class of all maximal consistent sets of* $\mathcal{L}_{\Box\boxbar}$*-formulas,*
- $R^c_1 wv$ *iff* $\Box w \subseteq v$,

[21] As mentioned before, we use a slightly different (but stronger) axiom to capture Anti-symmetry than the original axiom $\Diamond(\boxbar \neg p \wedge p) \to p$ used by Goranko [8]. The counterexample to show the failure of Transitivity of the constructed canonical model still works for the original setting, given $\Diamond(\boxbar \neg p \wedge p) \to p$ is canonical.

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- $R_2^c wv$ *iff* $\boxplus\neg w \subseteq v$,
- $w \in V^c(p)$ *iff* $p \in w$,

where $\Box w := \{\varphi \mid \Box\varphi \in w\}$ *and* $\boxplus\neg w := \{\varphi \mid \boxplus\neg\varphi \in w\}$. *In addition, since* $\Box_u\varphi = \Box\varphi \wedge \boxplus\neg\varphi$, *we define that* $T^c wv$ *iff* $\Box_u w \subseteq v$, *where* $\Box_u w := \{\varphi \mid \Box_u\varphi \in w\}$.

For any $w, v \in W^c$, if $R_1^c wv$, then we call v an R_1-*subordinate set* of w; if $R_2^c wv$, then v is an R_2-*subordinate set* of w; and if $T^c wv$, then v is a T-*subordinate set* of w. Also, $\mathcal{F}_g^c = \langle W^c, R_1^c, R_2^c\rangle$ is the *generalized canonical frame*. As a routine step, we now prove the following '*existence lemma*' for this new device:

LEMMA 4.40 (Existence lemma for generalized canonical frame). *For the generalized canonical frame* $\mathcal{F}_g^c = \langle W^c, R_1^c, R_2^c\rangle$ *and* $w \in W^c$, *we have*:

- *If* $\Diamond\varphi \in w$, *then there exists an* R_1^c-*subordinate set v of w such that* $\varphi \in v$ *and*
- *If* $\neg\boxplus\neg\varphi \in w$, *then there exists an* R_2^c-*subordinate set v of w such that* $\neg\varphi \in v$.

Proof. 1. Assume that $\Diamond\varphi \in w$, i.e., $\neg\Box\neg\varphi \in w$. If $\Box w \cup \{\varphi\}$ is inconsistent, then there exists a finite subset $\{\varphi_1, \cdots, \varphi_n\}(n \in \mathbb{N})$ of $\Box w$ such that $\vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \varphi)$. If $n = 0$, then by $R2$, it holds that $\vdash \Box\neg\varphi$, which contradicts $\neg\Box\neg\varphi \in w$. If $n > 0$, then it holds that $\vdash \Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n \to \Box\neg\varphi$. However, from $\Box\varphi_1, \dots, \Box\varphi_n \in w$, we can obtain $\Box\neg\varphi \in w$, which contradicts $\neg\Box\neg\varphi \in w$. Thus, $\Box w \cup \{\varphi\}$ is consistent. Consequently, there exists a maximal consistent set v with $\Box w \cup \{\varphi\} \subseteq v$. Therefore, we conclude that w has an R_1^c-subordinate set v with $\varphi \in v$.

2. Suppose that $\neg\boxplus\neg\varphi \in w$. If $\boxplus\neg w \cup \{\neg\varphi\}$ is inconsistent, then there exists a finite subset $\{\varphi_1, \dots, \varphi_n\}(n \in \mathbb{N})$ of $\boxplus\neg w$ such that $\vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \neg\varphi)$. If $n = 0$, then $\vdash \varphi$, i.e., $\vdash \neg\varphi \to \bot$. By $R3$, it holds that $\vdash \boxplus\bot \to \boxplus\neg\varphi$. From $A7$ and $R1$, we know $\vdash \boxplus\neg\varphi$, which contradicts $\neg\boxplus\neg\varphi \in w$. If $n > 0$, then by Theorem 4.26 and the rule R^*2, it holds that $\vdash \boxplus\neg(\varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi)$. By Proposition 4.28, it follows that $\vdash \boxplus\neg(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \boxplus\neg\varphi$. Moreover, given Theorem 4.27, it is not hard to see that $\vdash \boxplus\neg\varphi_1 \wedge \cdots \wedge \boxplus\neg\varphi_n \to \boxplus\neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. Therefore, we have $\vdash \boxplus\neg\varphi_1 \wedge \cdots \wedge \boxplus\neg\varphi_n \to \boxplus\neg\varphi$. However, since $\boxplus\neg\varphi_1, \dots, \boxplus\neg\varphi_n \in w$, it holds that $\boxplus\neg\varphi \in w$, which contradicts $\neg\boxplus\neg\varphi \in w$. So $\boxplus\neg w \cup \{\neg\varphi\}$ is consistent, and there exists a maximal consistent set v with $\boxplus\neg w \cup \{\neg\varphi\} \subseteq v$. Thus, w has an R_2^c-subordinate set v such that $\neg\varphi \in v$. □

With the result above, by a simple induction on the structure of φ, we can show the following '*truth lemma*' for the generalized canonical model:

LEMMA 4.41 (Truth lemma for generalized canonical model). *For the generalized canonical model* $\mathcal{M}_g^c = \langle W^c, R_1^c, R_2^c, V^c\rangle$, $w \in W^c$ *and* $\varphi \in \mathcal{L}_{\Box\boxplus}$, *it holds that* $\mathcal{M}_g^c, w \Vdash \varphi$ *iff* $\varphi \in w$.

Now we are going to show that all axioms of MGM^+ are canonical w.r.t. its generalized canonical frame, which will be useful below. In particular, the canonicity of axioms **4** and **T** is obvious, and we only show the canonicity of the principles $\Diamond\boxplus\varphi \to \boxplus\varphi$ and $\Diamond(\boxplus\varphi \wedge \psi) \to (\varphi \to \psi)$. With Proposition 4.37 and Proposition 4.38, we only need to show Propositions 4.42 and 4.43 below.

PROPOSITION 4.42. *Let* $\mathcal{F}_g^c = \langle W^c, R_1^c, R_2^c\rangle$ *be the generalized canonical frame such that for all* $w \in W^c$, $\Diamond\boxplus\varphi \to \boxplus\varphi \in w$. *Then for any* $w_1, w_2, w_3 \in W^c$, *it holds that* $R_1^c w_1 w_2 \wedge R_2^c w_1 w_3 \to R_2^c w_2 w_3$.

Proof. Assume towards a contradiction that there are $w_1, w_2, w_3 \in W^c$ such that $R_1^c w_1 w_2 \wedge R_2^c w_1 w_3 \wedge \neg R_2^c w_2 w_3$. Consequently, there exist formulas $\boxplus\varphi$ and φ such

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that $\boxminus\varphi \in w_2$ and $\varphi \in w_3$. Also, from $R_1^c w_1 w_2$ we know $\Diamond \boxminus \varphi \in w_1$. Since $\Diamond \boxminus \varphi \to \boxminus\varphi \in w_1$, it holds that $\boxminus\varphi \in w_1$. In addition, by $R_2^c w_1 w_3$, it follows that $\neg\varphi \in w_3$. Now we have arrived at a contradiction. □

PROPOSITION 4.43. *Let $\mathcal{F}_g^c = \langle W^c, R_1^c, R_2^c\rangle$ be the generalized canonical frame such that for all $w \in W^c$, $\Diamond(\boxminus\varphi \wedge \psi) \to (\varphi \to \psi) \in w$. Then for any $w, v \in W^c$, it holds that $R_1^c wv \wedge R_1^c vw \to R_2^c vw \vee w \equiv v$.*

Proof. If not, then there are $w, v \in W^c$ such that $R_1^c wv \wedge R_1^c vw \wedge \neg R_2^c vw \wedge \neg w \equiv v$. Since $\neg w \equiv v$, there exists a formula ψ such that $\neg\psi \in w$ and $\psi \in v$. By $\neg R_2^c vw$, there exist formulas $\boxminus\varphi$ and φ such that $\boxminus\varphi \in v$ and $\varphi \in w$. So, $\Diamond(\boxminus\varphi \wedge \psi) \wedge \varphi \wedge \neg\psi \in w$, which contradicts the assumption. □

Therefore, we now can conclude that:

PROPOSITION 4.44. *All axioms of MGM$^+$ are canonical w.r.t. its generalized canonical frame.*

So far so good. However, when considering the relation T^c introduced in Definition 4.39, we will face another crucial challenge, which indicates that we need further ingredients to enrich our framework. Now, to really understand the features of that relation, we introduce the following result:

PROPOSITION 4.45. *Let $\mathcal{F}_g^c = \langle W^c, R_1^c, R_2^c\rangle$ be the generalized canonical frame. Then*:

- $T^c = R_1^c \cup R_2^c$ *and*
- T^c *is an equivalence relation.*

Proof. 1. We now prove that $T^c = R_1^c \cup R_2^c$. From left to right, suppose that there exist $w, v \in W^c$ such that $T^c wv$, $\neg R_1^c wv$ and $\neg R_2^c wv$. Then it holds that $\Box_u w \subseteq v$, $\Box w \not\subseteq v$ and $\boxminus\neg w \not\subseteq v$. So there exist $\Box\varphi, \boxminus\neg\psi \in w$ such that $\varphi, \psi \notin v$. Since both w and v are maximal consistent sets, we have $\Box\varphi \wedge \boxminus\neg\psi \in w$ and $\neg\varphi \wedge \neg\psi \in v$. Furthermore, by Proposition 4.29, it holds that $\Box_u(\neg\psi \to \varphi) \in w$. However, from $\Box_u w \subseteq v$, we know $\neg\psi \to \varphi \in v$, which contradicts $\neg\varphi \wedge \neg\psi \in v$.

From right to left, we first consider the case that there exist $w, v \in W^c$ with $R_1^c wv$ and $\neg T^c wv$. Then we have $\Box w \subseteq v$ and $\Box_u w \not\subseteq v$. So there exist $\Box_u\varphi \in w$ and $\neg\varphi \in v$. However, from $\Box_u\varphi = \Box\varphi \wedge \boxminus\neg\varphi$, $\Box\varphi \in w$ and $\neg\varphi \in v$, we obtain $\neg R_1^c wv$, which entails a contradiction. Similarly, when $R_2^c wv$, we can also obtain $T^c wv$.

2. Next we show that T^c is an equivalence relation. For each $w \in W^c$, if $\Box_u\varphi \in w$, then by $A3$, we have $\varphi \in w$. Thus T^c is reflexive. For any $w_1, w_2, w_3 \in W^c$, if $T^c w_1 w_2$ and $T^c w_2 w_3$, then $\Box_u w_1 \subseteq w_2$ and $\Box_u w_2 \subseteq w_3$. For each $\Box_u\varphi \in w_1$, by $A4$ it follows $\Box_u\Box_u\varphi \in w_1$. Consequently, $\Box_u\varphi \in w_2$. Then we have $\varphi \in w_3$, i.e., $T^c w_1 w_3$. So, T^c is transitive. For any $w, v \in W^c$, if $T^c wv$ and $\neg T^c vw$, then $\Box_u w \subseteq v$ and $\Box_u v \not\subseteq w$. By $\Box_u v \not\subseteq w$, there exists φ such that $\Box_u\varphi \in v$ and $\varphi \notin w$. Since both w and v are maximal consistent sets, we have $\Diamond_u\neg\varphi \notin v$ and $\neg\varphi \in w$. Moreover, from $\neg\varphi \in w$ and $A5$, it follows that $\Box_u\Diamond_u\neg\varphi \in w$. Furthermore, $\Diamond_u\neg\varphi \in v$ follows from $T^c wv$, which entails contradiction. Therefore, the relation T^c is symmetric. Now the proof is completed. □

Intuitively, the results in Proposition 4.45 are in line with the definition of T^c and our axioms for the universal modality $\Box_u$. However, although T^c forms a partition of the domain W^c, it does not necessarily 'cover' the whole model, in the sense that the

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relation T^c may be not a total relation, i.e., $T^c \neq W^c \times W^c$. In other words, generally the occurrence of a formula $\Box_u\varphi$ in one maximal consistent set cannot guarantee that φ occurs in all maximal consistent sets. This definitely has a different spirit with the universal modality that is intended to quantify all objects in a given model. So, the generalized canonical model looks too large. But, how to make it 'smaller'? To deal with this, following [6], we introduce a notion of '*generated generalized canonical model*' as follows:[22]

Definition 4.46 (Generated generalized canonical model). *For the generalized canonical model $\mathcal{M}_g^c = \langle W^c, R_1^c, R_2^c, V^c\rangle$ and $w \in W^c$, we say $\mathcal{M}_{gw}^c = \langle W_w^c, R_{1w}^c, R_{2w}^c, V_w^c\rangle$ is its generated generalized canonical model, when*

- $W_w^c = \{v \in W^c \mid T^c wv\}$,
- $R_{1w}^c = R_1^c \cap (W_w^c \times W_w^c)$,
- $R_{2w}^c = R_2^c \cap (W_w^c \times W_w^c)$, *and*
- $V_w^c(p) = V^c(p) \cap W_w^c$.

We call $\mathcal{F}_{gw}^c = \langle W_w^c, R_{1w}^c, R_{2w}^c\rangle$ a *generated generalized canonical frame*. Also, define $T_w^c := \{v \in W^c \mid T^c wv\}$. Then, in the generated generalized canonical frame, it holds that $R_{1w}^c \cup R_{2w}^c = T_w^c = W_w^c \times W_w^c$ (recall Proposition 4.45). Furthermore, it is not hard to see the following desirable observation: for any $v \in W_w^c$ and $\Box_u\varphi$, when $\Box_u\varphi \in v$, we have $\varphi \in o$ for any $o \in W_w^c$. Here it is important to recognize that both the existence lemma (Lemma 4.40) and the truth lemma (Lemma 4.41) for the generalized canonical model also apply to the generated generalized canonical model.

Lemma 4.47 (Existence lemma for generated generalized canonical frame). *For the generated generalized canonical frame $\mathcal{F}_{gw}^c = \langle W_w^c, R_{1w}^c, R_{2w}^c\rangle$ and $s \in W_w^c$, we have*:

- *If $\Diamond\varphi \in s$, then there exists some $t \in W_w^c$ such that $R_{1w}^c st$ and $\varphi \in t$ and*
- *If $\neg \boxminus \neg\varphi \in s$, then there exists some $t \in W_w^c$ such that $R_{2w}^c st$ and $\neg\varphi \in v$.*

Proof. Instead of a precise proof for the result, we merely give a few hints why we still have the results for the generated generalized canonical frame. Let $\mathcal{F}_g^c = \langle W^c, R_1^c, R_2^c\rangle$ be the generalized canonical frame, $w \in W^c$, and $\mathcal{F}_{gw}^c = \langle W_w^c, R_{1w}^c, R_{2w}^c\rangle$ the generated generalized canonical frame. Let $s \in W_w^c$ and $\Diamond\varphi, \neg \boxminus \neg\psi \in \mathcal{L}_{\Box\boxminus}$.

Let $\Diamond\varphi \in s$. With Lemma 4.40, there exists some $t \in W^c$ such that $R_1^c st$ and $\varphi \in t$. Therefore, by Proposition 4.45, it follows that $T^c ws$ and $T^c wt$. So, $t \in W_w^c$. Now, it is easy to see $R_{1w}^c st$. Similarly, we can show the case for $\neg \boxminus \neg\psi \in s$. □

Lemma 4.48 (Truth lemma for generated generalized canonical model). *For the generated generalized canonical model $\mathcal{M}_{gw}^c = \langle W_w^c, R_{1w}^c, R_{2w}^c, V_w^c\rangle$, $s \in W_w^c$ and $\varphi \in \mathcal{L}_{\Box\boxminus}$, it holds that $\mathcal{M}_{gw}^c, s \Vdash \varphi$ iff $\varphi \in s$.*

Similar to that for the generalized canonical model, this can be proved with the help of Lemma 4.47, and we omit the details here. Moreover, for any formula φ and $u \in W_w^c$, it holds that $\mathcal{M}_g^c, u \Vdash \varphi$ if and only if $\mathcal{M}_{gw}^c, u \Vdash \varphi$. We leave its proof to the reader.

Note that we have already shown the canonicity of the axioms in MGM^+ (Proposition 4.44). As noted by Goranko [8], if the axioms of a logical system X in $\mathcal{L}_{\Box\boxminus}$

[22] For more details on the universal modality, we refer to [10].

are canonical, we can prove its completeness by making an extension $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$[23] of its generated generalized canonical frame $\mathcal{F}^c_{gw} = \langle W^c_w, R^c_{1w}, R^c_{2w}\rangle$, where $\mathcal{F}$ satisfies the frame-conditions of $\mathcal{F}^c_{gw}$, and

- $\mathcal{W} = \bigcup_{v\in W^c_w} \{v\} \times I_v$, where I_v is an index set for each $v \in W^c_w$, and
- $\mathcal{R}\langle s,i\rangle\langle t,j\rangle$ iff for any $s,t \in W^c_w$, $R^c_{1w}st \wedge (R^c_{2w}st \Rightarrow S\langle s,i\rangle\langle t,j\rangle)$, where S is a binary relation s.t. $R^c_{1w}st \wedge R^c_{2w}st \Rightarrow \forall i \in I_s(\exists j \in I_t\ S\langle s,i\rangle\langle t,j\rangle \wedge \exists j \in I_t\ \neg S\langle s,i\rangle\langle t,j\rangle)$.

The index set I_v in the definition of $\mathcal{W}$ makes copies of objects in W^c_w, in order to separate pairs of objects which are connected by both R^c_{1w} and R^c_{2w}. Essentially, the second condition above is to ensure that, if $R^c_{1w}st$ and $R^c_{2w}st$, then for all $i \in I_s$, there exist $j, j' \in I_t$ with $\mathcal{R}\langle s,i\rangle\langle t,j\rangle$ and $\neg\mathcal{R}\langle s,i\rangle\langle t,j'\rangle$. With the definition of $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$, when $\mathcal{R}\langle s,i\rangle\langle t,j\rangle$, we have $R^c_{1w}st$. On the other hand, since $R^c_{1w} \cup R^c_{2w}$ is a total relation, i.e., $R^c_{1w} \cup R^c_{2w} = W^c_w \times W^c_w$, it is easy to see that $\neg\mathcal{R}\langle s,i\rangle\langle t,j\rangle$ is equivalent to $R^c_{2w}st \wedge (R^c_{1w}st \Rightarrow \neg S\langle s,i\rangle\langle t,j\rangle)$.

Let $\mathcal{F}_g = \langle W, R_1, R_2\rangle$ be a generalized frame and $w \in W$. Following [8, p. 317], we say that w has an *entry defect* $(d_i(w))$, if there exists $v \in W$ such that R_1vw and R_2vw. Furthermore, define $D_i(W) := \{w \in W \mid d_i(w)\}$.

If $\mathcal{F}^c_g = \langle W^c, R^c_1, R^c_2\rangle$ is the generalized canonical frame such that for each $w \in W^c$, $\Diamond \boxminus \varphi \to \boxminus\varphi \in w$, then by Proposition 4.42, for all $v \in D_i(W^c_w)$, it holds that $R^c_{2w}vv$.

As mentioned above, [8, theorem 13(ii)] showed the completeness of a version of MGM^+, but the proof is flawed as shown below. Let $\mathcal{F}^c_{gw} = \langle W^c_w, R^c_{1w}, R^c_{2w}\rangle$ be a generated generalized canonical frame of MGM^+. By [8], $\mathcal{F}^c_{gw}$ is extended to $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where:

- $\mathcal{W} = W^c_w \times \{0\} \cup D_i(W^c_w) \times \mathbb{Z}$ and
- $S\langle s,i\rangle\langle t,j\rangle$ iff $(Est \wedge (i < j \vee (i = j \wedge s \leq' t))) \vee (\neg Est \wedge j \geq 0)$, where $\leq'$ is some linear order in W^c_w, and $Est := R^c_{1w}st \wedge R^c_{1w}ts$.

However, the following example shows that the frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ defined above is not a partial order. More specifically, it is not transitive.

Example 4.49. *Let* $w_1, w_2, w_3 \in D_i(W^c_w)$. *Then for any* $j \in \{1,2,3\}$, *we have* $R^c_{2w}w_jw_j$. *Also, since* $\mathcal{F}^c_{gw}$ *is the generated generalized canonical frame of* MGM^+, *the relation* R^c_{1w} *is reflexive. Moreover, consider the following situation* s:

- $R^c_{1w}w_1w_2$, $\neg R^c_{2w}w_1w_2$, $R^c_{1w}w_1w_3$, $R^c_{2w}w_1w_3$;
- $R^c_{1w}w_2w_3$, $R^c_{2w}w_2w_3$, $\neg R^c_{1w}w_2w_1$, $R^c_{2w}w_2w_1$; *and*
- $R^c_{1w}w_3w_2$, $R^c_{2w}w_3w_2$, $\neg R^c_{1w}w_3w_1$, $R^c_{2w}w_3w_1$.

The relations R^c_{1w} *and* R^c_{2w} *are depicted in Figure 6. For any* $i, j \in \{1,2,3\}$, *we have* $R^c_{1w}w_iw_j$ *or* $R^c_{2w}w_iw_j$. *However, with the definition of* $\mathcal{R}$ *defined in* [8], *it holds that* $\mathcal{R}\langle w_1,1\rangle\langle w_2,-2\rangle$, $\mathcal{R}\langle w_2,-2\rangle\langle w_3,-1\rangle$ *and* $\neg\mathcal{R}\langle w_1,1\rangle\langle w_3,-1\rangle$. *Thus the frame is not transitive.*

To fix the problem, we extend the generated generalized canonical frame of MGM^+ by giving a different S. Although this S still cannot establish the transitivity of the relation in the canonical frame directly, it is close enough to ease our later construction by taking the transitive closure.

[23] This kind of extension is called *standard extension* by Goranko [8, p. 315].

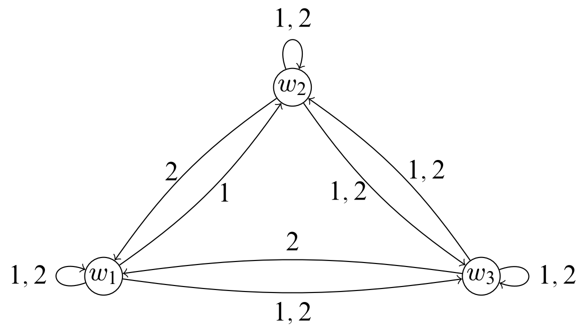


Fig. 6. A counterexample (R^c_{1w} and R^c_{2w} are labelled with '1' and '2', respectively).

DEFINITION 4.50. *Given the generated generalized canonical frame of MGM* $^+\mathcal{F}^c_{gw} = \langle W^c_w, R^c_{1w}, R^c_{2w}\rangle$, *we define the frame* $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ *where*:

- $\mathcal{W} = W^c_w \times \{0\} \cup D_i(W^c_w) \times \mathbb{Z}$ *and*
- $\mathcal{R}\langle s, i\rangle\langle t, j\rangle$ *iff for any* $s, t \in W^c_w$, $R^c_{1w}st \wedge (R^c_{2w}st \Rightarrow S\langle s, i\rangle\langle t, j\rangle)$ *where*:

$$S\langle s, i\rangle\langle t, j\rangle \text{ iff } (i < j \wedge j/2 \in \mathbb{Z}) \vee (s = t \wedge i = j).$$

S tells us how we should connect $\langle s, i\rangle$ and $\langle t, j\rangle$ when $R^c_{1w}st$ and $R^c_{2w}st$. S defined above is reflexive, and it connects $\langle s, i\rangle$ and $\langle t, j\rangle$ if j is even and i is smaller than j.

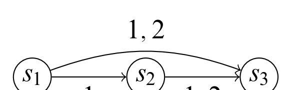
Let $\mathcal{F}^+ = \langle \mathcal{W}, \mathcal{R}^+\rangle$ be the transitive closure of the frame $\mathcal{F}$ defined in Definition 4.50. We will use $\mathcal{F}^+$ instead of $\mathcal{F}$ to show the completeness. Before the completeness proof we need the following crucial lemma about when we really need to add edges according to the transitive closure.

LEMMA 4.51. *If there is an* $\mathcal{R}$*-path* $\langle s_1, i_1\rangle\langle s_2, i_2\rangle \dots \langle s_k, i_k\rangle$ *in* $\mathcal{F}$ *(i.e., for all* $1 \leq j < k$, $\mathcal{R}\langle s_j, i_j\rangle\langle s_{j+1}, i_{j+1}\rangle$*) where* $3 \leq k$ *such that it is not the case that* $\mathcal{R}\langle s_1, i_1\rangle\langle s_k, i_k\rangle$, *then there exists* $\langle s_m, i_m\rangle$ *for some* $m < k$ *such that*:

- $\mathcal{R}\langle s_1, i_1\rangle\langle s_m, i_m\rangle$ *and* $\mathcal{R}\langle s_m, i_m\rangle\langle s_k, i_k\rangle$ *in* $\mathcal{F}$, *and*
- $R^c_{1w}s_1s_m$, $R^c_{1w}s_ms_k$, $R^c_{2w}s_1s_k$, $R^c_{2w}s_ms_k$ *but not* $R^c_{2w}s_1s_m$ *in* $\mathcal{F}^c_{gw}$.

Proof. We first prove the result given $k = 3$ and then show the general case can be reduced to this basic case.

Suppose $\mathcal{R}\langle s_1, i_1\rangle\langle s_2, i_2\rangle$, $\mathcal{R}\langle s_2, i_2\rangle\langle s_3, i_3\rangle$, but not $\mathcal{R}\langle s_1, i_1\rangle\langle s_3, i_3\rangle$. Then by the definition of $\mathcal{R}$, $R^c_{1w}s_1s_2$ and $R^c_{1w}s_2s_3$. By the canonicity of Axiom **4** w.r.t. the generalized canonical $\mathcal{F}^c_{gw}$, we have $R^c_{1w}s_1s_3$. In the following, we show that $R^c_{2w}s_1s_3$, $R^c_{2w}s_2s_3$ but not $R^c_{2w}s_1s_2$, namely, the edges between s_1, s_2, s_3 are illustrated as below:



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where we label R^c_{1w} and R^c_{2w} with '1' and '2', respectively, in the graph, and omit the reflexive links.

Since it is not the case that $\mathcal{R}\langle s_1, i_1\rangle\langle s_3, i_3\rangle$ but $R^c_{1w}s_1s_3$, by the definition of $\mathcal{R}$ we have $R^c_{2w}s_1s_3$. Now since $R^c_{1w}s_1s_2$, by Proposition 4.37 we have $R^c_{2w}s_2s_3$.

Finally, towards contradiction let us assume that $R^c_{2w}s_1s_2$. Since $\mathcal{R}\langle s_1, i_1\rangle\langle s_2, i_2\rangle$, we have either $\langle s_1, i_1\rangle = \langle s_2, i_2\rangle$ or $i_1 < i_2$ and i_2 is even, according to the definition of $\mathcal{R}$. The first case is not possible since $\mathcal{R}\langle s_2, i_2\rangle\langle s_3, i_3\rangle$ but it is not the case that $\mathcal{R}\langle s_1, i_1\rangle\langle s_3, i_3\rangle$. Therefore $i_1 < i_2$ and i_2 is even. Similarly, since $\mathcal{R}\langle s_2, i_2\rangle\langle s_3, i_3\rangle$, $R^c_{1w}s_2s_3$ and $R^c_{2w}s_2s_3$ we can show that $i_2 < i_3$ and i_3 is even. Therefore, $i_1 < i_3$ and i_3 is even. However, this then means that $\mathcal{R}\langle s_1, i_1\rangle\langle s_3, i_3\rangle$ according to the definition of $\mathcal{R}$, which contradicts the assumption. In sum, it cannot be the case that $R^c_{2w}s_1s_2$.

For the general case, suppose there is an $\mathcal{R}$-path from $\langle s, i\rangle$ to $\langle s', i'\rangle$ in $\mathcal{F}$ such that it is not the case that $\mathcal{R}\langle s, i\rangle\langle s', i'\rangle$. Clearly we can shorten the path while keeping the starting and ending points the same, if two non-adjacent points in the path are also connected by $\mathcal{R}$. Now let us take one of the shortest subpaths[24] $\langle s_1, i_1\rangle \ldots \langle s_k, i_k\rangle$ for some k, such that $\langle s_1, i_1\rangle = \langle s, i\rangle$, $\langle s_k, i_k\rangle = \langle s', i'\rangle$, and for all $j < k$, $\mathcal{R}\langle s_j, i_j\rangle\langle s_{j+1}, i_{j+1}\rangle$. Note that since it is the shortest in length, it is not the case that $\mathcal{R}\langle s_j, i_j\rangle\langle s_{j'}, i_{j'}\rangle$ for any $j < j' \leq k$.

Note that $k > 2$, since it is not the case that $\mathcal{R}\langle s, i\rangle\langle s', i'\rangle$. In the following, we show that k must be 3, which will reduce the general case to the previous basic case. Suppose $k > 3$ and the shortest subpath starts with $\langle s_1, i_1\rangle\langle s_2, i_2\rangle\langle s_3, i_3\rangle\langle s_4, i_4\rangle \ldots$[25]. Since it is not the case that $\mathcal{R}\langle s_1, i_1\rangle\langle s_3, i_3\rangle$ but $\mathcal{R}\langle s_1, i_1\rangle\langle s_2, i_2\rangle$, and $\mathcal{R}\langle s_2, i_2\rangle\langle s_3, i_3\rangle$, we can follow the discussion for the basic 3-point case and show $R^c_{2w}s_2s_3$. However, since it is also not the case that $\mathcal{R}\langle s_2, i_2\rangle\langle s_4, i_4\rangle$ but $\mathcal{R}\langle s_2, i_2\rangle\langle s_3, i_3\rangle$, and $\mathcal{R}\langle s_3, i_3\rangle\langle s_4, i_4\rangle$, by the conclusion of the basic 3-point case we have $\neg R^c_{2w}s_2s_3$. Now we have a contradiction, thus k can only be 3 if the shortest subpaths are taken. □

The above lemma shows that if there is a path such that the starting point and the ending point are not connected by $\mathcal{R}$ then it is a 3-point path. This means when we build the transitive closure, we just need to add edges in such simple cases, and this is the key to our next lemma.

LEMMA 4.52. *$\mathcal{F}^+$ is a GM-frame.*

Proof. Since $\mathcal{R}^+$ is the transitive closure of $\mathcal{R}$, we just need to show that $\mathcal{R}^+$ is reflexive and anti-symmetric.

We first show that $\mathcal{R}^+$ is reflexive. For any $v \in W^c_w$ and $i \in I_v$, from $v = v$ and $i = i$, we know $S\langle v, i\rangle\langle v, i\rangle$. By the canonicity of Axiom **T**, we have $R^c_{1w}vv$. Consequently, $\mathcal{R}\langle v, i\rangle\langle v, i\rangle$. Therefore, $\mathcal{R}$ is reflexive, so is $\mathcal{R}^+$.

We then prove that $\mathcal{F}^+$ is anti-symmetric. If not, then there are $\langle w_1, i\rangle, \langle w_2, j\rangle \in \mathcal{W}$ such that $\langle w_1, i\rangle \neq \langle w_2, j\rangle$, $\mathcal{R}^+\langle w_1, i\rangle\langle w_2, j\rangle$ and $\mathcal{R}^+\langle w_2, j\rangle\langle w_1, i\rangle$. By the definition of $\mathcal{R}$ and the discussion above, we know that $R^c_{1w}w_1w_2$ and $R^c_{1w}w_2w_1$. Consider the following situations.

First, consider the case that $\mathcal{R}\langle w_1, i\rangle\langle w_2, j\rangle$ and $\mathcal{R}\langle w_2, j\rangle\langle w_1, i\rangle$, i.e., both the edges $\mathcal{R}^+\langle w_1, i\rangle\langle w_2, j\rangle$ and $\mathcal{R}^+\langle w_2, j\rangle\langle w_1, i\rangle$ are not new edges added to the frame when taking the transitive closure. Since $R^c_{1w}w_1w_2$ and $R^c_{1w}w_2w_1$, from Proposition 4.43 it follows that $R^c_{2w}w_2w_1$ or $w_1 \equiv w_2$. If $w_1 \equiv w_2$, then from $\langle w_1, i\rangle \neq \langle w_2, j\rangle$ it follows

[24] In general, there can be multiple subpaths of the same minimal length.
[25] Note that $\langle s_4, i_4\rangle$ might well be the ending point if the length is exactly 4.

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that $i \neq j$, which suggests that $w_1 \in D_i(W^c_w)$ for otherwise $i = j = 0$. It then means that there exists $v \in W^c_w$ such that $R^c_{1w}vw_1$ and $R^c_{2w}vw_1$. By Proposition 4.42, $R^c_{2w}w_1w_1$. W.l.o.g., assume $i < j$. From $\mathcal{R}\langle w_2, j\rangle\langle w_1, i\rangle$ and $R^c_{2w}w_1w_1$, we can also obtain $j < i$, which entails a contradiction. So it can only be the case that $R^c_{2w}w_2w_1$. Similarly, we have $R^c_{2w}w_1w_2$. However from the definition of $\mathcal{R}$ and $\langle w_1, i\rangle \neq \langle w_2, j\rangle$, it follows that $(i < j \wedge j/2 \in \mathbb{Z})$ and $(j < i \wedge i/2 \in \mathbb{Z})$, which is a contradiction, too.

Next, consider the case that only one of the two edges is a new edge added when taking transitive closure. W.l.o.g., assume that $\neg\mathcal{R}\langle w_1, i\rangle\langle w_2, j\rangle$ and $\mathcal{R}\langle w_2, j\rangle\langle w_1, i\rangle$. Then the edge from $\langle w_1, i\rangle$ to $\langle w_2, j\rangle$ is added in the transitive closure, i.e., there is an $\mathcal{R}$-path (of a length greater than 2) from $\langle w_1, i\rangle$ to $\langle w_2, j\rangle$ in $\mathcal{F}$. Now by Proposition 4.51, there must be some $\langle w_3, k\rangle$ such that $\mathcal{R}\langle w_1, i\rangle\langle w_3, k\rangle$, $\mathcal{R}\langle w_3, k\rangle\langle w_2, j\rangle$, and moreover $R^c_{1w}w_1w_3$, $\neg R^c_{2w}w_1w_3$, $R^c_{1w}w_3w_2$, $R^c_{2w}w_3w_2$ and $R^c_{2w}w_1w_2$. Since $\mathcal{R}\langle w_2, j\rangle\langle w_1, i\rangle$, we have $R^c_{1w}w_2w_1$. Now by $R^c_{1w}w_3w_2$ and $R^c_{1w}w_2w_1$, we have $R^c_{1w}w_3w_1$. From Proposition 4.43 and $R^c_{1w}w_1w_3$, it holds that $R^c_{2w}w_1w_3 \vee w_1 \equiv w_3$. Note that we already have $\neg R^c_{2w}w_1w_3$, so $w_1 \equiv w_3$. Thus $\mathcal{R}\langle w_1, i\rangle\langle w_1, k\rangle$ and $\mathcal{R}\langle w_1, k\rangle\langle w_2, j\rangle$. From $\neg\mathcal{R}\langle w_1, i\rangle\langle w_2, j\rangle$ and $\mathcal{R}\langle w_1, k\rangle\langle w_2, j\rangle$, we know $i \neq k$. Consequently, we have $w_1 \in D_i(W^c_w)$, i.e., there exists $v \in W^c_w$ such that $R^c_{1w}vw_1$ and $R^c_{2w}vw_1$. However, as $R^c_{1w}vw_1$ and $R^c_{2w}vw_1$, by Proposition 4.42 it holds that $R^c_{2w}w_1w_1$, which contradicts $\neg R^c_{2w}w_1w_3$ (i.e., $\neg R^c_{2w}w_1w_1$).

Finally, let us consider the case that $\neg\mathcal{R}\langle w_1, i\rangle\langle w_2, j\rangle$ and $\neg\mathcal{R}\langle w_2, j\rangle\langle w_1, i\rangle$. Since the edges are added in the transitive closure, by Proposition 4.51 there are $w_3, w_4 \in W^c_w$ in the initial frame $\mathcal{F}^c_{gw}$, and $k, h \in \mathbb{Z}$, such that

(1) $\mathcal{R}\langle w_1, i\rangle\langle w_3, k\rangle$, $\mathcal{R}\langle w_3, k\rangle\langle w_2, j\rangle$, $R^c_{1w}w_1w_3$, $\neg R^c_{2w}w_1w_3$, $R^c_{1w}w_3w_2$, $R^c_{2w}w_3w_2$, $R^c_{2w}w_1w_2$; and
(2) $\mathcal{R}\langle w_2, i\rangle\langle w_4, h\rangle$, $\mathcal{R}\langle w_4, h\rangle\langle w_1, j\rangle$, $R^c_{1w}w_2w_4$, $\neg R^c_{2w}w_2w_4$, $R^c_{1w}w_4w_1$, $R^c_{2w}w_4w_1$, $R^c_{2w}w_2w_1$.

From $R^c_{1w}w_3w_2$ in (1) and $R^c_{1w}w_2w_1$ in (2), we have $R^c_{1w}w_3w_1$. By Proposition 4.43 and $R^c_{1w}w_1w_3$, it follows that $R^c_{2w}w_1w_3 \vee w_1 \equiv w_3$. From $\neg R^c_{2w}w_1w_3$ in (1), we obtain $w_1 \equiv w_3$. Again from (1) we have $\mathcal{R}\langle w_1, i\rangle\langle w_1, k\rangle$ and $\mathcal{R}\langle w_1, k\rangle\langle w_2, j\rangle$. From $\neg\mathcal{R}\langle w_1, i\rangle\langle w_2, j\rangle$ and $\mathcal{R}\langle w_1, k\rangle\langle w_2, j\rangle$, it holds that $i \neq k$. Consequently, we know $w_1 \in D_i(W^c_w)$, i.e., there exists $v \in W^c_w$ such that $R^c_{1w}vw_1$ and $R^c_{2w}vw_1$. However, as $R^c_{1w}vw_1$ and $R^c_{2w}vw_1$, we know $R^c_{2w}w_1w_1$ from Proposition 4.42, which contradicts $\neg R^c_{2w}w_1w_3$, i.e., $\neg R^c_{2w}w_1w_1$ in (1). Similarly, (2) entails a contradiction, too.

Therefore, for any $\langle w_1, i\rangle, \langle w_2, j\rangle \in \mathcal{W}$, if $\mathcal{R}^+\langle w_1, i\rangle\langle w_2, j\rangle$ and $\mathcal{R}^+\langle w_2, j\rangle\langle w_1, i\rangle$, then $\langle w_1, i\rangle = \langle w_2, j\rangle$. Thus, $\mathcal{F}^+$ is anti-symmetric.

All in all, $\mathcal{F}^+$ is reflexive and anti-symmetric. Additionally, it is a transitive closure, so it is a GM-frame. □

Now we move to the next step. With the construction of $\mathcal{F}$ introduced in Definition 4.50, it is an easy exercise to verify that for any $s, t \in W^c_w$, if $R^c_{1w}st$ and $R^c_{2w}st$, then for each $i \in I_s$, there exist $j, j' \in I_t$ such that $\mathcal{R}\langle s, i\rangle\langle t, j\rangle$ and $\neg\mathcal{R}\langle s, i\rangle\langle t, j'\rangle$. Furthermore, as the relation $\mathcal{R}^+$ is an extension of $\mathcal{R}$, we know $\mathcal{R}^+\langle s, i\rangle\langle t, j\rangle$ from $\mathcal{R}\langle s, i\rangle\langle t, j\rangle$. However, when $\mathcal{R}^+\langle s, i\rangle\langle t, j'\rangle$, is there always another $j'' \in I_t$ such that $\neg\mathcal{R}^+\langle s, i\rangle\langle t, j''\rangle$? The following result gives us a positive answer to this:

LEMMA 4.53. *For all $s, t \in W^c_w$, if $R^c_{1w}st$ and $R^c_{2w}st$, then for each $i \in I_s$, there exist $j, j' \in I_t$ such that $\mathcal{R}^+\langle s, i\rangle\langle t, j\rangle$ and $\neg\mathcal{R}^+\langle s, i\rangle\langle t, j'\rangle$.*

Proof. As stated above, we only need to show that when $\neg\mathcal{R}\langle s,i\rangle\langle t,j\rangle$ and $\mathcal{R}^+\langle s,i\rangle\langle t,j\rangle$, there exists $j' \in I_t$ such that $\neg\mathcal{R}^+\langle s,i\rangle\langle t,j'\rangle$. Let us begin.

Since the edge from $\langle s,i\rangle$ to $\langle t,j\rangle$ is one of those added when building the transitive closure, there must be some $v \in W^c_w$ such that $R^c_{1w}sv$, $\neg R^c_{2w}sv$, $R^c_{1w}vt$, $R^c_{2w}vt$ and $R^c_{2w}st$. If $s = t$, then by Proposition 4.43, we know $R^c_{2w}sv \vee s \equiv v$ from $R^c_{1w}sv$ and $R^c_{1w}vt$. We already have $\neg R^c_{2w}sv$, so $s = v$. However, now $\neg R^c_{2w}sv$ contradicts $R^c_{2w}st$. Moreover, as $R^c_{1w}st$ and $R^c_{2w}st$, it holds that $I_t = \mathbb{Z}$. Consider the case that $j' = 1$. Obviously, $j'/2 \notin \mathbb{Z}$. Also, since $s \neq t$, we have $\neg\mathcal{R}\langle s,i\rangle\langle t,j'\rangle$. If $\mathcal{R}^+\langle s,i\rangle\langle t,j'\rangle$, then there exists $k \in I_v$ with $\mathcal{R}\langle s,i\rangle\langle v,k\rangle$ and $\mathcal{R}\langle v,k\rangle\langle t,1\rangle$. In addition, since $R^c_{2w}st$ and $\neg R^c_{2w}sv$, $t \neq v$. Therefore, from $\mathcal{R}\langle v,k\rangle\langle t,1\rangle$, we know $k < 1 \wedge 1/2 \in \mathbb{Z}$ which entails a contradiction. Hence, $\neg\mathcal{R}^+\langle s,i\rangle\langle t,j'\rangle$. Now the proof is completed. □

As mentioned, with the valuable findings in [8], Lemma 4.53 itself has already illustrated the completeness of MGM$^+$. However, the reader may still have no feeling for how this result can help us to establish the completeness. Therefore, to understand how the lemma above works, we go one step further and show the result more directly in what follows.

THEOREM 4.54. *MGM$^+$ is strongly complete with respect to GM-frames.*

Proof. Let $\mathcal{M} = \langle \mathcal{F}^+, V\rangle$ where $V(\langle s,i\rangle) = V^c_w(s)$. By Lemma 4.52, $\mathcal{F}^+$ is indeed a GM-frame. In the following we show that

$$\mathcal{M}, \langle s,i\rangle \vDash \varphi \text{ iff } \mathcal{M}^c_{gw}, s \Vdash \varphi. (\star).$$

Note that the strong completeness follows if the above claim is true: for each set of MGM$^+$-consistent formulas, we can extend it to a maximal MGM$^+$-consistent set s by a Lindenbaum-like argument. Then by the truth lemma for the generalized generated canonical model (Lemma 4.48), we have $\mathcal{M}^c_{gw}, s \Vdash \varphi$ iff $\varphi \in s$. Finally we have $\mathcal{M}, \langle s,0\rangle \vDash \varphi$ for all $\varphi \in s$ by $(\star)$.

Now we prove the claim $(\star)$. The proof goes by induction on $\varphi \in \mathcal{L}_{\Box\boxplus}$. The Boolean cases are routine, and we only show the cases for $\Box$ and $\boxplus$.

Let us begin with that for $\Box\varphi$. From left to right, assume towards a contradiction that $\mathcal{M}, \langle s,i\rangle \vDash \Box\varphi$ and $\mathcal{M}^c_w, s \nVdash \Box\varphi$. By the latter assumption, there exists $t \in W^c_{gw}$ such that $R^c_{1w}st$ and $\mathcal{M}^c_{gw}, t \Vdash \neg\varphi$. If $\neg R^c_{2w}st$, then by the construction of $\mathcal{R}^+$, for all $j \in I_t$, $\mathcal{R}^+\langle s,i\rangle\langle t,j\rangle$ follows directly from $R^c_{1w}st$. By the inductive hypothesis, for any $j \in I_t$, we have $\mathcal{M}, \langle t,j\rangle \vDash \varphi$ iff $\mathcal{M}^c_{gw}, t \Vdash \varphi$. Therefore, we have $\mathcal{M}, \langle s,i\rangle \nvDash \Box\varphi$ contradicting to our assumption. On the other hand, when $R^c_{2w}st$, it holds that $t \in D_i(W^c_w)$, so $I_t = \mathbb{Z}$. By Lemma 4.53, there exists $j' \in I_t$ such that $\mathcal{R}^+\langle s,i\rangle\langle t,j'\rangle$. Consequently, by the inductive hypothesis, we know $\mathcal{M}, \langle s,i\rangle \nvDash \Box\varphi$ that again entails a contradiction.

From right to left, suppose that $\mathcal{M}^c_{gw}, s \Vdash \Box\varphi$ and $\mathcal{M}, \langle s,i\rangle \nvDash \Box\varphi$. Hence there exists $\langle t,j\rangle \in \mathcal{W}$ with $\mathcal{R}^+\langle s,i\rangle\langle t,j\rangle$ and $\mathcal{M}, \langle t,j\rangle \vDash \neg\varphi$. Note that $R^c_{1w}st$ follows from $\mathcal{R}^+\langle s,i\rangle\langle t,j\rangle$. Therefore, by the inductive hypothesis, it is not hard to see that $\mathcal{M}^c_{gw}, s \nVdash \Box\varphi$, which contradicts the assumption.

Now we move to the case for $\boxplus\varphi$. For the direction from left to right, suppose for reductio that $\mathcal{M}, \langle s,i\rangle \vDash \boxplus\varphi$ and $\mathcal{M}^c_{gw}, s \nVdash \boxplus\varphi$. Then, there exists $t \in W^c_w$ such that $R^c_{2w}st$ and $\mathcal{M}^c_{gw}, t \Vdash \varphi$. When $\neg R^c_{1w}st$, we have $\neg\mathcal{R}^+\langle s,i\rangle\langle t,j\rangle$ for any $j \in I_t$. By the inductive hypothesis, for each $j \in I_t$, it holds that $\mathcal{M}, \langle t,j\rangle \vDash \varphi$ iff $\mathcal{M}^c_{gw}, t \Vdash \varphi$. Thus, we have $\mathcal{M}, \langle s,i\rangle \nvDash \boxplus\varphi$ that contradicts the assumption. On the other hand, if $R^c_{1w}st$, then by Lemma 4.53, that there exists $j' \in I_t$ such that $\neg\mathcal{R}^+\langle s,i\rangle\langle t,j'\rangle$. Again by the inductive hypothesis, we have $\mathcal{M}, \langle s,i\rangle \nvDash \boxplus\varphi$.

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For the converse direction, assume that $\mathcal{M}^c_{gw}, s \Vdash \boxminus \varphi$ and $\mathcal{M}, \langle s, i\rangle \nvDash \boxminus\varphi$. Then there exists $\langle t, j\rangle \in \mathcal{W}$ such that $\neg\mathcal{R}^+\langle s, i\rangle\langle t, j\rangle$ and $\mathcal{M}, \langle t, j\rangle \vDash \varphi$. By the inductive hypothesis, we have $\mathcal{M}^c_{gw}, t \Vdash \varphi$. However, $R^c_{2w}st$ follows from $\neg\mathcal{R}^+\langle s, i\rangle\langle t, j\rangle$, which contradicts $\mathcal{M}^c_{gw}, s \Vdash \boxminus \varphi$. Now the proof is completed. □

§5. Future work. In this paper, we propose a modal approach to mereological theories. By a modal language extended with the window modality, we modally capture the first-order properties of various mereological theories via frame correspondence. We also correct a mistake in the existing completeness proof for a basic system of mereology by providing a new construction of the canonical model.

We have left several questions open, e.g., the absolute modal correspondences (if any) of Atomicity, Finite Sum and Fusion. Our discussion of the modal axiomatization over the GM-frames (i.e., Ground Mereology) demonstrates that complete mereological modal systems are very hard to obtain. In particular, the incompleteness of the intuitive MGM shows that the extra expressive power brought in by the window modality may help the modal language to define the same frame property in different ways, which may lead to the incompleteness. This suggests that the other modal systems proposed in Definition 4.31 may well be incomplete, even when the extra axiom of MGM^+ is added. Another significant difficulty behind the completeness proof for a system using both $\Box$ and $\boxminus$ is that we need to make the two relations in the generalized canonical frame complement each other, while keeping the important frame properties intact. This is already very hard in the case of GM frames, let alone EM and other much more complicated frames. We probably need new general techniques. Besides completeness, the decidability of our modal systems is also one important issue which we left open for further investigations. We conjecture that using a modal language rather than the first-order language may lead to more decidable (modal) logics.

Moreover, if we go back to the basics of our modal approach, there are also some options to be explored. For example, in order to have a more intuitive reading of the $\Box$ modality, we use the whole-part relation as the primitive relation in our models, in contrast to the part-whole relation used in the literature. It seems to be a non-essential design choice, but it could well affect the modal definability of various mereological properties. Similarly, if we take the proper whole-part relation as primitive, the correspondences of various properties may also change. Finally, instead of modally characterizing the existing first-order theories of mereology, we may also propose new modal theories for its own sake.

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