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The functional-differential equation $\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{y}(qt) + C\mathbf{y}'(qt) + \mathbf{f}(t)$

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An initial value problem for the functional differential equation

$$\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{y}(qt) + C\mathbf{y}'(qt) + \mathbf{f}(t), \quad t \geqslant t_0 > 0,$$

where A, B, C are complex matrices, $q \in (0, 1)$, and $\mathbf{f}$ is a vector of continuous functions, is considered in this paper. Its solution is represented in terms of the fundamental solution via the variation-of-constants formula. For some special cases, the fundamental solutions are formulated as piecewise Dirichlet series. The variation-of-constants formula is used to analysis the asymptotic behaviour of the solutions of some scalar equations, including one with variable coefficients related to coherent states of the q-oscillator algebra in quantum mechanics.

1 Introduction

This paper is concerned with the initial value problem of the nonhomogeneous functional differential equation

$$\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{y}(qt) + C\mathbf{y}'(qt) + \mathbf{f}(t), \quad t \geqslant t_0, \tag{1.1}$$

$$\mathbf{y}t(t) = \phi(t), \quad t \in [qt_0, t_0], \tag{1.2}$$

where A, B, C are $d \times d$ complex matrices, $q \in (0, 1)$, $t_0 > 0$, $\mathbf{f} \in C[t_0, \infty)$, and $\phi \in C^1[qt_0, t_0]$. A (vector-valued) function $\mathbf{y}$ is defined as a solution of (1.1, 1.2) if $\mathbf{y} \in C[t_0, \infty)$ such that it coincides with ϕ in the interval $[qt_0, t_0]$ and $\mathbf{y}(t) - q^{-1} C\mathbf{y}(qt) \in C^1(t_0, \infty)$ obeys the equation

$$[\mathbf{y}(t) - q^{-1} C\mathbf{y}(qt)]' = A\mathbf{y}(t) + B\mathbf{y}(qt) + \mathbf{f}(t), \quad t \geqslant t_0. \tag{1.3}$$

This definition is in line with the one for functional differential equations with constant delays given in Hale and Verduyn Lunel [32]. It can be shown by the method-of-steps that the solution of (1.1, 1.2) exists and is unique. In general, the solution is not differentiable at $t = q^{-n} t_0$ for $n \in \mathbb{Z}^+$, unless ϕ satisfies the consistency condition

$$\phi'(t_0) = A\phi(t_0) + B\phi(qt_0) + C\phi'(qt_0) + \mathbf{f}(t_0).$$

Functional differential equations with proportional delays are usually referred to as pantograph equations or generalized pantograph equations. The name *pantograph* originated from the work of Ockendon & Tayler [1] on the collection of current by the pantograph head of an electric locomotive, where the mathematical model was reduced to a system (of four equations) of the form

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$$\mathbf{y}'(x) = A\mathbf{y}(\lambda x) + B\mathbf{y}(x), \quad x \geqslant 0 \tag{1.4}$$

with A and B being constant matrices and $\lambda \in (0,1)$ is a parameter. Equations of a similar form appear in many applications such as astrophysics [2], nonlinear dynamical systems [3], probability theory on algebraic structures [4], spectral problem of the Schrödinger equations [5] and quantum mechanics [6]. In particular, the nonhomogeneous scalar equation

$$y'(x) = ay(\lambda x) + by(x) + f(x), \quad x \geqslant x_0 > 0, \tag{1.5}$$

where a and b are constants, $\lambda \in (0,1)$, $f \in C[x_0, \infty)$, was derived by Fox *et al.* [7] as the mathematical model for a problem in electric locomotion.

The homogeneous case of (1.5) has been studied in great detail by De Bruijn [8], Kato [9] and Kato & McLeod [10]. Various expansions or representations of its solution have been obtained in Fox *et al.* [7], Frederickson [11], Hahn [12] and Vogl [13]. Some special cases of the nonhomogeneous equation (1.5) have been investigated by Lim [14], whereas Carr & Dyson [16, 17] have treated some special cases of the pantograph equation (1.4).

The generalized pantograph equations

$$\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{y}(qt) + C\mathbf{y}'(qt), \quad t \geqslant 0, \tag{1.6}$$

where A, B and C are $d \times d$ complex constant matrices and $q \in (0,1)$ has been treated in Iserles [17] and Liu [18]. Linear equations with more than one proportional delay are discussed in Derfel [19, 20], Derfel & Iserles [21], Derfel & Vogl [22], Frederickson [11], Hahn [12], Iserles & Liu [23–25], Kuang & Feldstein [26] and Liu [18]. Equations with variable coefficients are treated in Derfel & Vogl [22], Morris *et al.* [27] and Feldstein & Liu [28]. Some nonlinear equations are studied by Iserles [29] and Liu [30], for instance, the functional-Riccati equation

$$y'(t) + q^2 y'(qt) + y^2(t) - q^2 y^2(qt) = \mu, \quad t \in \mathbb{R}$$

with q and μ being real parameters, which was introduced in Shabat [5] as a simple reduction of the so-called dressing chain of Schrödinger operators.

The subject matter of this paper is the initial value problem (1.1, 1.2). We are mainly interested in the asymptotic behaviour of the solution. Since (1.1) is different from (1.4) and (1.6) in the sense that its solution is in general not smooth, many established techniques for analysing equations such as (1.4) and (1.6) do not apply to (1.1). The basic idea of this paper is to formulate the variation-of-constants formula for the solution of (1.1, 1.2) using the fundamental solution. In some cases, the fundamental solution can be represented by a piecewise Dirichlet series, and subsequently optimal asymptotic bounds can be obtained. To demonstrate the usefulness of the variation-of-constants formula, we investigate in the last section the asymptotic behaviour of the solutions of two scalar equations; one of them has variable coefficients and is related to the coherent states of the q-oscillator algebra in quantum mechanics.

2. The fundamental solution

The fundamental solution of (1.1) is the unique solution of the initial value problem

$$\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{y}(qt) + C\mathbf{y}'(qt), \quad t \geqslant t_0, \tag{2.1}$$

$$\mathbf{y}'(t) = \begin{cases} 0, & qt_0 \leqslant t < t_0, \\ I, & t = t_0, \end{cases} \tag{2.2}$$

where I is the $d \times d$ identity matrix. The fundamental solution is denoted throughout this paper by $G(t, t_0)$. It should be noted that $G(t, t_0) = e^{A(t-t_0)}$, $t \in [t_0, q^{-1} t_0]$, is smooth even though $G(t, t_0) \notin C^1[qt_0, t_0]$.

To see the asymptotic behaviour of the fundamental solution $G(t, t_0)$ as t tends to infinity, it is desirable to have an explicit representation of the fundamental solution $G(t, t_0)$. Motivated by the Dirichlet series expansion in Iserles [17], we seek a piecewise-Dirichlet series expansion of the form

$$G(t, t_0) = \sum_{k=0}^{n} \sum_{\ell=0}^{k} D_{k,\ell} e^{q^{-\ell} A(q^k t - t_0)}, \quad t \in [q^{-n} t_0, q^{-n-1} t_0], \quad n = 0, 1, \ldots, \tag{2.3}$$

where $D_{k,\ell}$, $\ell = 0, 1, \ldots, k$, $k = 0, 1, \ldots,$, are $d \times d$ matrices to be determined. Since

$$G(t, t_0) = e^{A(t-t_0)}, \quad t \in [t_0, q^{-1} t_0],$$

it is true that

$$D_{0,0} = I.$$

Applying the method-of-steps to the initial value problem (2.1, 2.2) yields

$$AD_{k,k} - D_{k,k} A = 0, \tag{2.4}$$

$$AD_{k,\ell} - q^{k-\ell} D_{k,\ell} A = -BD_{k-1,\ell} - q^{k-\ell-1} CD_{k-1,\ell} A, \quad \ell = 0, 1, \ldots, k-1, \tag{2.5}$$

whereas the continuity of $G(t, t_0)$ at $t = q^{-k} t_0$ gives the recurrence relation

$$D_{k,k} = -\sum_{\ell=0}^{k-1} D_{k,\ell}, \tag{2.6}$$

where $k = 1, 2, \ldots$.

Theorem 1 *If A is nonsingular and it commutes with B and C then the fundamental solution has the representation*

$$G(t, t_0) = \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^{k-\ell} \left(\prod_{j=1}^{k-\ell} \frac{A^{-1} B + q^{k-l-j} C}{1 - q^j} \right) \left(\prod_{j=1}^{l} \frac{C + q^{l-j} A^{-1} B}{1 - q^j} \right) \times e^{q^{-\ell} A(q^k t - t_0)}, \quad t \in [q^{-n} t_0, q^{-n-1} t_0]. \quad n = 0, 1, \ldots. \tag{2.7}$$

Proof It can be shown that the coefficients of $e^{q^{-\ell} A(q^k t - t_0)}$ on the right-hand side of (2.7) satisfy (2.4) and (2.5). To verify (2.6), the continuity condition, we need only to prove that

$$\sum_{\ell=0}^{k} (-1)^{k-\ell} \left(\prod_{j=1}^{k-\ell} \frac{A^{-1} B + q^{k-l-j} C}{1 - q^j} \right) \left(\prod_{j=1}^{l} \frac{C + q^{l-j} A^{-1} B}{1 - q^j} \right) = 0, \quad k \in \mathbb{N}. \tag{2.8}$$

Let

$$P(x) = \sum_{\ell=0}^{k} \left(\prod_{j=1}^{k-\ell} \frac{1 - q^{k-l-j} x}{1 - q^j} \right) \left(\prod_{j=1}^{l} \frac{x - q^{l-j}}{1 - q^j} \right).$$

It suffices to show that

$$P(x) \equiv 0, \tag{2.9}$$

since the left-hand side of (2.8) is $(-A^{-1} B)^k P(-AB^{-1} C^{-1})$ if B and C are nonsingular (the

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case where B or C is singular can be dealt with by a continuation argument). To prove (2.9), we note that

$$\left[\prod_{j=1}^{k}(1-q^j)\right]^2 P(x) = \sum_{l=0}^{k}(-1)^{\ell}\left(\prod_{j=1}^{k-l} b_{\ell+j}\right)\left(\prod_{j=1}^{l} a_{\ell-j}\right), \tag{2.10}$$

where

$$a_m = (q^{k-m}-1)\,x + q^m - q^k, \quad b_m = 1 - q^m + (q^k - q^{k-m})\,x.$$

The right-hand side of (2.10) is equal to the $(k+1)\times(k+1)$ determinant

$$\begin{vmatrix} 1 & a_0 & 0 & \cdots & 0 & 0 \\ 1 & b_1 & a_1 & \cdots & 0 & 0 \\ 1 & 0 & b_2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_{k-2} & 0 \\ 1 & 0 & 0 & \cdots & b_{k-1} & a_{k-1} \\ 1 & 0 & 0 & \cdots & 0 & b_k \end{vmatrix}.$$

Adding the third, fourth and all the rest of the columns of the preceding determinant to the second one yields a determinant whose first and second columns are linearly dependent, since

$$a_0 = a_1 + b_1 = \cdots = a_{k-1} + b_{k-1} = b_k.$$

Hence, $P(x) \equiv 0$. □

An alternative proof of the identity (2.9) based on hypergeometric functions are suggested by a referee of this paper, and it goes as follows. Using the two identities (cf. [31])

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{(q^{-k};q)_l}{(q;q)_l}(-1)^l\,q^{kl-(l-1)l/2},$$

$$(z;q)_{k-1} = \frac{(z;q)_k}{(q^{1-k}/z;q)_l}\left(-\frac{q}{z}\right)^l q^{-kl+(l-1)l/2}$$

we get

$$\begin{aligned} P(x) &= \frac{1}{(q;q)_k}\sum_{l=0}^{k}\begin{bmatrix} k \\ l \end{bmatrix}(1/x;q)_l\,(x;q)_{k-l}\,x^l \\ &= \frac{(x;q)_k}{(q;q)_k}\sum_{l=0}^{\infty}\frac{(q^{-k};q)_l\,(1/x;q)\,l}{(q;q)_l\,(q^{1-k}/x;q)_l}q^l \\ &= \frac{(x;q)_k}{(q;q)_k}\,{}_2\Phi_1\left[\begin{matrix} q^{-k}, 1/x; \\ q^{1-k}/x; \end{matrix}\,q, q\right]. \end{aligned}$$

We can now use the q-Chu-Vandermonde identity [31, p. 236] to obtain

$$P(x) = \frac{(x;q)_k\,(q^{1-k};q)_k}{(q;q)_k\,(q^{1-k}/x;q)_k}x^{-k} \equiv 0,$$

since $(q^{-m};q)_k \equiv 0$ for $m \leqslant k-1$.

The condition that A commutes with B and C in Theorem 1 is restrictive. However, without it, the fundamental solution cannot be represented by a piecewise Dirichlet series of the form (2.3) when A is nonsingular. To prove this, we note that when $k = 1$ and $k = 2$, equations (2.4), (2.5) and (2.6) are

$$\begin{aligned}
&AD_{1,1} - D_{1,1}A = 0,\\
&AD_{1,0} - qD_{1,0}A = -(B + CA),\\
&D_{1,0} + D_{1,1} = 0,\\
&AD_{2,2} - D_{2,2}A = 0,\\
&AD_{2,1} - qD_{2,1}A = -(BD_{1,1} + CD_{1,1}A),\\
&AD_{2,0} - q^2 D_{2,0}A = -(BD_{1,0} + CD_{1,0}A),\\
&D_{2,0} + D_{2,1} + D_{2,2} = 0.
\end{aligned}$$

After some tedious calculation it can be concluded that the above system of equations of the unknown matrices $D_{1,0}$, $D_{1,1}$, $D_{2,0}$, $D_{2,1}$ and $D_{2,2}$ is solvable if and only if A commutes with B and C.

3 The variation-of-constants formula

Denoting the solution of the initial value problem (1.1, 1.2) by $\mathbf{y}(t;\, t_0, \phi, \mathbf{f})$, the linearity of the initial value problem implies that

$$\mathbf{y}(t;\, t_0, \phi, \mathbf{f}) = \mathbf{y}(t;\, t_0, \phi, 0) + \mathbf{y}(t;\, t_0, 0, \mathbf{f}).$$

The variation-of-constants formula reads

$$\mathbf{y}(t;\, t_0 \phi, \mathbf{f}) = \mathbf{y}(t; t_0, \phi, 0) + \int_{t_0}^{t} G(t,s)\,\mathbf{f}(s)\,\mathrm{d}s, \quad t \geqslant t_0, \tag{3.1}$$

where

$$\begin{aligned}
\mathbf{y}(t;\, t_0, \phi, 0) = G(t, t_0)\,(\phi(t_0) - q^{-1}C\phi(qt_0)) + q^{-1}\int_{qt_0}^{t_0} G(t, q^{-1}s)\,B\phi(s)\,\mathrm{d}s\\
- q^{-1}\int_{qt_0}^{t_0} [\mathrm{d}G(t, q^{-1}s)]\,C\phi(s), \quad t \geqslant t_0.
\end{aligned} \tag{3.2}$$

The last integral in the preceding formula is of Riemann–Stieltjes type with s as the integral variable.

The validity of (3.1) and (3.2) are not difficult to verify directly, though care should be taken that the solution satisfies (1.3) instead of (1.1). We note in passing that those two formulas are similar to the classical formulas (7.12) and (7.13) in Hale & Verduyn Lunel [32] for a system of functional differential equations of neutral type with a constant delay.

Since (3.2) can be written as

$$\begin{aligned}
\mathbf{y}(t; t_0, \phi, 0) = G(t, t_0)\,(\phi(t_0) - q^{-1}C\phi(qt_0)) + q^{-1}\int_{qt_0}^{t_0} G(t, q^{-1}s)\,B\phi(s)\,\mathrm{d}s\\
- q^2\int_{qt_0}^{t_0} G_2(t, q^{-1}s)\,C\phi(s)\,\mathrm{d}s, \quad t \geqslant q^{-1}t_0,
\end{aligned} \tag{3.3}$$

where G_2 is the derivative of $G(t,s)$ with respect to the second variable s, one can obtain information about the asymptotic behaviour of the solution of (1.1, 1.2) through (3.1) and (3.3) by estimating the fundamental solution.

In the sequel, we use $\|\cdot\|$ to denote the matrix normal induced by a vector norm, likewise denoted by $\|\cdot\|$, which is arbitrary unless otherwise stated. Note that this assumption is independent of the norm, since all norms in a finite dimensional space are equivalent. For the matrix A, $\alpha[A]$ denotes its maximal real part of the eigenvalues (the *spectral abscissa*) and $\kappa[A]$ the maximal geometric multiplicity of those eigenvalues with the maximal real part $\alpha[A]$.

Theorem 2 *Assume that A is nonsingular, A commutes with B and C, and $B, C \neq 0$. Then the fundamental solution $G(t,s)$, $t_0 \leqslant s \leqslant t$, has the following asymptotic estimates (as $t \to \infty$ and $\tau = t/s \to \infty$):*

1. *If $\alpha[A] > 0$ then $\|G(t,s)\| = \mathcal{O}(t^{\kappa(A)-1} e^{\alpha[A]t})$;*
2. *If $\alpha[A] = 0$ then*

(a) *when $\|C\| < q^{\kappa(A)-1}\|A^{-1}B\|$,*

$$\|G(t,s)\| = \begin{cases} \mathcal{O}(\tau^{-\ln\|A^{-1}B\|/\ln q+\kappa[A]-1}), & \|A^{-1}B\| > q^{1-\kappa(A)}, \\ \mathcal{O}(\tau^{\kappa[A]-1}\ln\tau), & \|A^{-1}B\| = q^{1-\kappa(A)}, \\ \mathcal{O}(\tau^{\kappa[A]-1}), & \|A^{-1}B\| < q^{1-\kappa(A)}, \end{cases}$$

(b) *when $\|C\| = q^{\kappa(A)-1}\|A^{-1}B\|$,*

$$\|G(t,s)\| = \begin{cases} \mathcal{O}(\tau^{-\ln\|C\|/\ln q+\kappa[A]-1}), & \|C\| > 1, \\ \mathcal{O}((\tau^{\kappa[A]-1}\ln\tau)^2), & \|C\| = 1, \\ \mathcal{O}(\tau^{\kappa[A]-1}), & \|C\| < 1, \end{cases}$$

(c) *when $\|C\| > q^{\kappa(A)-1}\|A^{-1}B\|$,*

$$\|G(t,s)\| = \begin{cases} \mathcal{O}(\tau^{-\ln\|C\|/\ln q+\kappa[A]-1}), & \|C\| > 1, \\ \mathcal{O}(\tau^{\kappa[A]-1}\ln\tau), & \|C\| = 1, \\ \mathcal{O}(\tau^{\kappa[A]-1}), & \|C\| < 1; \end{cases}$$

3. *If $\alpha[A] < 0$ then $\|G(t,s)\| = \mathcal{O}(\tau^{-\ln\|A^{-1}B\|/\ln q})$.*

The above theorem can be proved by using the fact that

$$\|e^{At}\| = \mathcal{O}(t^{\kappa[A]-1} e^{\alpha[A]t}) \quad \text{as } t \to \infty.$$

For instance, when $\alpha[A] < 0$, we have

$$\|G(t,s)\| \leqslant \|A^{-1}B\|^n \sum_{k=0}^{n}\sum_{\ell=0}^{k} \|A^{-1}B\|^{-k}\left(\prod_{j=1}^{n-k-\ell}\frac{1+q^{n-k-l-j}\|AB^{-1}C\|}{1-q^j}\right)$$
$$\times\left(\prod_{j=1}^{l}\frac{\|AB^{-1}C\|+q^{l-j}}{1-q^j}\right)e^{\alpha_0 q^{-\ell}(q^{-k}-1)t_0}, \quad t \in [q^{-n}t_0, q^{-n-1}t_0], \quad n = 0, 1, \ldots.$$

The double summation on the right-hand side of the preceding inequality can be bounded uniformly. Hence,

$$\|G(t,s)\| = \mathcal{O}(\tau^{-\ln\|A^{-1}B\|/\ln q})$$

as $\tau = t/s \to \infty$.

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Remark 1 Theorem 2 still holds when B or C vanishes, except that $\|A^{-1}B\|$ or $\|C\|$ should be replaced by an arbitrarily small positive constant.

Remark 2 Theorem 2 still holds when we replace $G(t,s)$ by $G_2(t,s)$, the derivative of $G(t,s)$ with respect to the second variable s.

Remark 3 The asymptotic bounds given in Theorem 2 are optimal in the sense that they are consistent with those obtained by Kato [9], Kato & McLeod [10] and Carr & Dyson [33] for the scalar equation

$$y'(x) = ay(\lambda x) + by(x), \quad x \geqslant x_0 \geqslant 0,$$

where a, b are complex constants, $\lambda \in (0,1)$.

If A does not commute with B or C it is still possible to give some (probably not optimal) asymptotic bounds for the fundamental solution. For instance, the following lemma of Liu [30] can be used to obtain an asymptotic bound when $\alpha[A] < 0$.

Lemma 3 *Consider the equation*

$$[\mathbf{y}(t) - C_0(t)\,\mathbf{y}(qt)]' = -A_0(t)\,\mathbf{y}(t) + B_0(t)\,\mathbf{y}(qt) + \mathbf{f}(t), \quad t \geqslant t_0 \geqslant 0, \tag{3.4}$$

where A_0, B_0, C_0 are matrices of continuous functions and $\mathbf{f}$ *is a vector of continuous functions, all defined on $[t_0, \infty)$. Suppose that the matrices A_0, B_0 and C_0 have the asymptotic expansions*

$$A_0(t) \to A, \quad B_0(t) \to B, \quad C_0(t) \to C \quad \text{as } t \to \infty,$$

and that there exists a constant α such that

$$\rho(C) < q^{-\alpha}, \quad \lambda_1 > q^{2\alpha-1}|\lambda_2|, \quad \|B + A^T C\| < q^{1/2-\alpha}(\lambda_1 + q^{2\alpha-1}\lambda_2),$$

and

$$(t+1)^{-\alpha}\,\mathbf{f}(t) \in \mathrm{L}_2(t_0, \infty),$$

where λ_1 and λ_2 are the smallest eigenvalues of $(A + A^T)/2$ and $(C^T B + B^T C)/2$, respectively. If $\mathbf{y}(t) \in \mathrm{C}([qt_0, \infty))$ such that $\mathbf{y}(t) - C(t)\,\mathbf{y}(qt) \in \mathrm{C}^1[qt_0, \infty)$ and (3.4) is satisfied, then $\mathbf{y}(t) = \mathcal{O}(t^\alpha)$ as $t \to \infty$.

4 The asymptotic behaviour

In this section, we use the variation-of-constants formula (3.1) to investigate the asymptotic behaviour of the solution $\mathbf{y}(t;\, t_0, \phi, \mathbf{f})$ of the initial value problem (1.1, 1.2). It follows from (3.1) and (3.3) that

$$\|\mathbf{y}(t;\, t_0, \phi, \mathbf{f})\| \leqslant M \max_{s \in [qt_0, t_0]} \|\phi(s)\| \max_{s \in [t_0, q^{-1}t_0]} (\|G(t,s)\| + \|G_2(t,s)\|) + \int_{t_0}^{t} \|G(t,s)\,\mathbf{f}(s)\|\,\mathrm{d}s, \quad t \geqslant q^{-1}t_0, \tag{4.1}$$

where M is a positive constant that depends on A, B, C and q only. When A is nonsingular and commutes with B and C, Theorem 2 provides us with a very good estimate of $G(t,s)$.

Subsequently, we can easily formulate the corresponding result for the solution of (1.1, 1.2) via the inequality (4.1).

For simplicity, we consider the nonhomogeneous scalar equation

$$y'(t) = ay(t) + by(qt) + f(t), \quad t \geqslant t_0 > 0, \tag{4.2}$$

where a and b are nonzero complex constants, $q \in (0, 1)$, $f \in C(t_0, \infty)$. In the sequel, we write

$$\gamma = -\ln|b/a|/\ln q$$

and use the two notations

$$\alpha_+ = \begin{cases} \alpha, & \alpha > 0, \\ 0, & \alpha \leqslant 0, \end{cases} \quad \operatorname{sign}(\alpha) = \begin{cases} 1, & \alpha > 0, \\ 0, & \alpha = 0, \\ -1, & \alpha < 0, \end{cases}$$

for a constant α. Equation (4.2) has been discussed by Lim [14]. To make a comparison we list the results of Lim in the following two theorems:

Theorem A *Let $a < 0$. Assume that f' exists $f(t) = \mathcal{O}(t^{\beta})$ and $f'(t) = \mathcal{O}(t^{\beta-1})$ as $t \to \infty$ for some real constant β. Then every solution of* (4.2) *has the asymptotic bound $\mathcal{O}(t^{\max\{\gamma,\beta\}} \ln t)^{1-\operatorname{sign}(|\gamma-\beta|)})$ as $t \to \infty$.*

Theorem B *Let $a > 0$. Assume that $f(t) = \mathcal{O}(t^{\beta})$ as $t \to \infty$ for some real constant β. Then every solution of* (4.2) *is $\mathcal{O}(e^{\operatorname{Re} at})$ as $t \to \infty$.*

Using the estimate of $G(t, s)$ and $G_2(t, s)$ (see Theorem 2, Remarks 1 and 2), we can obtain from (4.1) the following result:

Theorem 4 *Assume that $f(t) = \mathcal{O}(t^{\beta})$ as t tends to ∞ for some real constant β. Then every solution of* (4.2) *has the asymptotic bound*

$$\begin{cases} \mathcal{O}(t^{\max\{\gamma,\beta+1\}} (\ln t)^{1-\operatorname{sign}(|\gamma-\beta-1|)}), & \operatorname{Re} a < 0, \\ \mathcal{O}(t^{\max\{\gamma_+,\beta+1\}} (\ln t)^{1-\operatorname{sign}(|\gamma_+-\beta-1|)}), & \operatorname{Re} a = 0, \\ \mathcal{O}(e^{\operatorname{Re} at}), & \operatorname{Re} a > 0 \end{cases}$$

as $t \to \infty$.

In the case $\operatorname{Re} a \leqslant 0$, we can obtain a better result under a stronger assumption.

Theorem 5 *Assume that $f(t) = \mathcal{O}(t^{\beta})$ and $f'(t) = \mathcal{O}(t^{\beta-1})$ as t tends to ∞ for some real constant β. Then every solution of* (4.2) *has the asymptotic bound*

$$\begin{cases} \mathcal{O}(t^{\max\{\gamma,\beta\}} (\ln t)^{1-\operatorname{sign}(|\gamma-\beta|)}), & \operatorname{Re} a < 0, \\ \mathcal{O}(t^{\max\{\gamma_+,\beta\}} (\ln t)^{2-\operatorname{sign}(|(\gamma-1)_+-\beta|)-\operatorname{sign}(|\gamma-\max\{(\gamma-1)_+,\beta\}|)}), & \operatorname{Re} a = 0 \end{cases}$$

as $t \to \infty$.

The preceding theorem does not follow directly from the estimates of $G(t,s)$ and $G_2(t,s)$. Instead, we apply Theorem 4 to the equation

$$x'(t) = ax(t) + qbx(qt) + f'(t), \quad t \geqslant q^{-1} t_0 > 0$$

to derive the estimate

$$y'(t) = \begin{cases} \mathcal{O}(t^{\max\{\gamma-1,\beta\}} (\ln t)^{1-\text{sign}(|\gamma-\beta-1|)}), & \operatorname{Re} a < 0, \\ \mathcal{O}(t^{\max\{(\gamma-1)_+,\beta\}} (\ln t)^{1-\text{sign}(|\gamma-1)_+-\beta|)}), & \operatorname{Re} a = 0 \end{cases}$$

as $t \to \infty$. The desired asymptotic bounds can be obtained by applying Lemma 11 and Lemma 12 of Liu [19] to the q-difference equation

$$y(t) - \frac{b}{a} y(qt) = \frac{1}{a}(y'(t) - f(t)), \quad t \geqslant t_0,$$

where y' is treated as given.

It is evident that Theorems 4 and 5 either improved or generalized the result of Lim [14]. We remark that the technique used in [14] does not apply to the case $\operatorname{Re} a \leqslant 0$. Moreover, our result can be easily generalized to include the neutral equation

$$y'(t) = ay(t) + by(qt) + cy'(qt) + f(t), \quad t \geqslant t_0 > 0,$$

with a, b, c being complex constants, $a \neq 0$, and $q \in (0,1)$.

Next, we consider the scalar equation

$$y'(t) = -u(t)\, y(t) + by(qt), \quad t \in \mathbb{R}, \tag{4.3}$$

where b is a nonzero complex constant, $q \in (0,1)$, and u is a solution of the functional-Riccati equation

$$u'(t) + q^2 u'(qt) + u^2(t) - q^2 u^2(qt) = \mu, \quad t \in \mathbb{R}, \tag{4.4}$$

with μ being a positive constant. Equation (4.3) arises in the study of coherent states of the q-oscillator (q-Heisenberg of q-Weyl) algebra in quantum mechanics [6]. We assume in the sequel that u is a regular solution of (4.3) in the sense that

$$u(t) = \pm \left(\frac{\mu}{1-q^2}\right)^{1/2} + \mathcal{O}(t^{-2}), \quad u'(t) = \mathcal{O}(t^{-3}) \quad \text{as } t \to \pm\infty.$$

We refer to the paper of Liu [30] for a comprehensive discussion of the existence of regular solutions and many other issues on equation (4.4). The following result was proposed in Spiridonov [6] based on an informal perturbation argument.

Theorem 6 *Let $y(t) \in C^1(\mathbb{R})$ be a solution of* (4.3). *Then*

$$y(t) = \mathcal{O}(t^{\alpha}) \quad \text{as } t \to \pm\infty,$$

where $\alpha = -\frac{1}{2\ln q} \ln(1 - q^2|b|^2/\mu$.

Proof The application of Lemma 3 to (4.3) yields

$$y(t) = \mathcal{O}(t^{\alpha+3/4}) \quad \text{as } t \to \infty.$$

By writing (4.3) in the form of (4.1) with

$$a = -\left(\frac{\mu}{1-q^2}\right)^{1/2}, \quad f(t) = a - u(t),$$

we derive from Theorem 4 that

$$y(t) = \mathcal{O}(t^{\alpha}) \quad \text{as } t \to \infty.$$

The simple change of the unknown function $u(t) \to u(-t)$ gives us the same result for the case where $t \to -r$. □

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