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Quantile-based information generating functions and their properties and uses

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Abstract
Information generating functions (IGFs) have been of great interest to researchers due to their ability to generate various information measures. The IGF of an absolutely continuous random variable (see Golomb, S. (1966). The information generating function of a probability distribution. *IEEE Transactions in Information Theory*, 12(1), 75–77) depends on its density function. But, there are several models with intractable cumulative distribution functions, but do have explicit quantile functions. For this reason, in this work, we propose quantile version of the IGF, and then explore some of its properties. Effect of increasing transformations on it is then studied. Bounds are also obtained. The proposed generating function is studied especially for escort and generalized escort distributions. Some connections between the quantile-based IGF (Q-IGF) order and well-known stochastic orders are established. Finally, the proposed Q-IGF is extended for residual and past lifetimes as well. Several examples are presented through out to illustrate the theoretical results established here. An inferential application of the proposed methodology is also discussed

1. Introduction

The notion of entropy, especially the Shannon entropy due to [24], has seen a great importance in many fields. Shannon entropy quantifies the amount of information needed to accurately send and receive messages in a communication channel. We refer to [24] and [21] for several important properties and applications of Shannon entropy, and its generalizations. For a random variable (RV) X with mass function $\{p_i \geq 0, i = 1, \ldots, n\}$, such that $\sum_{i=1}^{n} p_i = 1$, the Shannon entropy is given by $H(X) = -\sum_{i=1}^{n} p_i \log p_i$. For a nonnegative absolutely continuous RV X with probability density function (PDF) f_X, the Shannon entropy (or differential entropy) is analogously given by

$$S(X) = -\int_0^\infty f_X(x) \log f_X(x) dx, \tag{1.1}$$

which may take values in $(-\infty, \infty)$. For uniform RV in $(0, \theta)$, $S(X)$ is negative when $\theta < 1$. In (1.1), we have considered nonnegative RVs, but for an RV with support $(-\infty, \infty)$, Shannon entropy can be defined by changing the limits of integration.

Motivated by the concepts of moment and probability generating functions, Golomb [4] proposed the information generating function (IGF) of an RV X. For a nonnegative continuous RV X, the IGF is

$$I_\beta(X) = \int_0^\infty f_X^\beta(x) dx, \quad \beta \geq 1. \tag{1.2}$$

The derivative of IGF in (1.2), evaluated at 1, yields the negative Shannon entropy or negentropy. Further, $I_{\beta=1}(X) = 1$, and when $\beta = 2$, we get informational energy, which has been widely used in Physics. Golomb [4] explained the reason for considering $\beta \geq 1$ in (1.2) instead of $\beta \geq 0$. Along the same lines, throughout this work, we also consider $\beta \geq 1$. Recently, many authors have studied IGFs due to their importance in information theory; one may refer to [10], [12], [11], [13], [29], [14], and [15] for some IGFs and their diverse properties and applications.

We observe that the IGF in (1.2) depends on the PDF of the distribution. However, there are cases for which the quantile function (QF) is available in an explicit form, but not the density function as such. For example, lambda distributions [23], power-Pareto distributions [5], and Govindarajulu distribution [19] do not have exact distribution functions. For these cases, it is impossible to generate information measures using (1.2). To overcome this difficulty, the IGF in (1.2) needs to be re-defined. In this regard, we consider here IGF based on QF and then explore its properties. We also study the proposed quantile-based IGF (Q-IGF) for residual and past lifetimes, which are important characteristics in reliability theory. For some early developments regarding quantile-based information measures, one may refer to [25], [26], [1], [6], [7], [8], [17], [9], [30], and the references therein. The QF of an RV X, with cumulative distribution function (CDF) F_X, is given by

$$Q_X(p) = F_X^{-1}(p) = \inf\{x|F_X(x) \geq p\}, \ \ 0 \leq p \leq 1. \tag{1.3}$$

Note that the QF does not share all of its properties with the distribution function. For example, the sum of two QFs is a QF and the product of two positive QFs is also a QF. Interested readers may refer to [20] for some more features and properties of QF.

The key contributions of this work are as follows:

(a) There are distributions for which the QF is available in an explicit form, but not the distribution or density function. In order to study the IGF for such distributions, in this work, we propose Q-IGF of a nonnegative absolutely continuous RV. We then show that the Q-IGF can be represented in terms of hazard and reversed hazard QFs;

(b) The effect of monotone increasing transformations on Q-IGF is examined. An order based on Q-IGF is then introduced, and its connection to some existing stochastic orders is established. We have also studied the proposed Q-IGF for quantile-based escort and generalized escort distributions;

(c) Residual lifetime is an important mathematical concept, usually applied in the study of predicting future performance of a working system. The past lifetime is also a useful concept while dealing with the past performance of a failed system. In this work, we finally study Q-IGF for both residual and past lifetimes, and then establish various properties of them. Several examples are presented throughout this work for illustrating the theoretical results established here.

The rest of this paper is as follows. In Section 2, we introduce the IGF based on QF. We show that the Q-IGF can be represented in terms of quantile-based fractional Shannon entropy. Further, the effect of monotone increasing transformations on Q-IGF is studied. Quantile-based escort and generalized escort distributions are also considered. Based on the newly proposed IGF, an ordering is introduced and its connection to hazard and reversed hazard quantile orders is shown. The Q-IGF for residual lifetime is discussed in Section 3, while Section 4 deals with past lifetime. Some of their properties are also established in these sections. Further, an inferential application of the proposed methodology has been illustrated in Section 5. Finally, Section 6 presents some concluding remarks.

We assume throughout the paper that the nonnegative RVs are absolutely continuous. All the involved integration and differentiation are assumed to exist. Moreover, the words increasing and decreasing are used in a wider sense.

2. Quantile-based IGF

This section discusses briefly the IGF due to [4] based on the QF. Denote the PDF of X by f_X and the CDF by F_X. We recall that the quantile density function (QDF) of X, denoted by q_X, can be obtained from QF Q_X as $\frac{d}{dp}Q_X(p) = q_X(p)$. It is then easy to see that $F_X(Q_X(p)) = p$, implying $f_X(Q_X(p))q_X(p) = 1$. We then present the following definition.

Definition 2.1. *Suppose X is an RV with QF Q_X and QDF q_X. Then, for $\beta \geq 1$, the Q-IGF of X is given by*

$$I_\beta^Q(X) = \int_0^1 f_X^\beta(Q_X(p))dQ_X(p) = \int_0^1 f_X^{\beta-1}(Q_X(p))dp = \int_0^1 q_X^{1-\beta}(p)dp. \qquad (2.1)$$

Note that $I_\beta^Q(X)$ in (2.1) provides a quantile version of the IGF, which measures information contained in a distribution through QDF. From (2.1), the following facts are evident:

(i) $I_\beta^Q(X)|_{\beta=1} = 1$; $\quad I_\beta^Q(X)|_{\beta=2} = \int_0^1 (q_X(p))^{-1}dp = -2J^Q(X)$,

(ii) $\frac{\partial}{\partial \beta}I_\beta^Q(X)|_{\beta=1} = -\int_0^1 \log q_X(p)dp = -S^Q(X)$,

where $J^Q(X)$ and $S^Q(X)$ are, respectively, the quantile-based extropy (see (6) of [16]) and quantile-based Shannon entropy (see (7) of [25]). Using the hazard QF given by $H_X(p) = ((1-p)q_X(p))^{-1}$, and the reversed hazard QF given by $\tilde{H}_X(p) = (pq_X(p))^{-1}$, the Q-IGF in (2.1) can be rewritten as

$$I_\beta^Q(X) = \int_0^1 \left\{(1-p)H_X(p)\right\}^{\beta-1}dp \text{ and } I_\beta^Q(X) = \int_0^1 \left\{p\tilde{H}_X(p)\right\}^{\beta-1}dp, \ \beta \geq 1. \qquad (2.2)$$

The following example gives closed-form expressions for the Q-IGF for different distributions. Some plots of Q-IGF exponential density functions, for example, are presented in Figure 1.

Example 2.2.

- For an RV X following Exponential(θ) distribution, $Q_X(p) = -\frac{\log(1-p)}{\theta}$, and so $q_X(p) = \frac{1}{\theta(1-p)}$. Thus, from (2.1), we obtain $I_\beta^Q(X) = \frac{\theta^{\beta-1}}{\beta}$. The plots of the Q-IGF of exponential distribution are presented in Figure 1 for different choices of θ.
- For an RV X following Uniform(a,b) distribution, $Q_X(p) = a + (b-a)p$, and so $q_X(p) = b - a$. From (2.1), we get $I_\beta^Q(X) = (b-a)^{1-\beta}$;
- For an RV X having Pareto-I(a,b) distribution, $Q_X(p) = \frac{b}{(1-p)^{\frac{1}{a}}}$, and so $q_X(p) = \frac{b}{a(1-p)^{\frac{1}{a}+1}}$. From (2.1), we obtain $I_\beta^Q(X) = \frac{(a/b)^{\beta-1}}{(\frac{1}{a}+1)(\beta-1)+1}$;
- Let X follow inverted exponential distribution with $Q_X(p) = -\frac{\lambda}{\log x}$, and so $q_X(p) = \frac{\lambda}{p(\log p)^2}$. From (2.1), we find the Q-IGF as $I_\beta^Q(X) = \frac{\Gamma(2\beta-1)}{\lambda^{\beta-1}\beta^{2\beta-1}}$.

Next, we consider some distributions which do not have closed-form distribution functions, and then discuss their Q-IGFs.

Example 2.3.

- Consider Davies distribution with $Q_X(p) = cp^{\lambda_1}(1-p)^{-\lambda_2}$, $c > 0$, $\lambda_1, \lambda_2 > 0$. Here,

$$q_X(p) = c\lambda_1 \frac{p^{\lambda_1-1}}{(1-p)^{\lambda_2}} + c\lambda_2 \frac{p^{\lambda_1}}{(1-p)^{\lambda_2+1}}. \qquad (2.3)$$
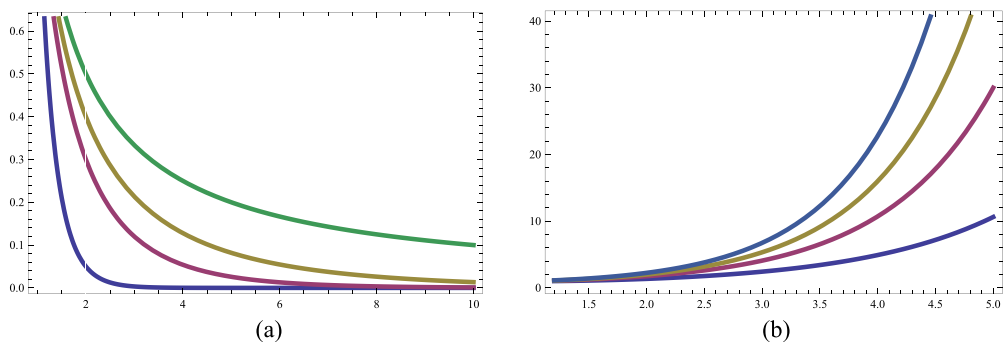
Figure 1. *Plots of Q-IGF for exponential distribution considered in Example 2.2, for (a) $\theta = 0.1, 0.6, 0.8, 1$ (presented from below) and (b) $\theta = 2.7, 3.5, 4, 4.5$ (presented from below). Along the x-axis, we have taken the values of β.*

Thus, the Q-IGF is given by

$$I_\beta^Q(X) = c^{1-\beta} \int_0^1 \left[\lambda_1 \frac{p^{\lambda_1 - 1}}{(1-p)^{\lambda_2}} + \lambda_2 \frac{p^{\lambda_1}}{(1-p)^{\lambda_2 + 1}} \right]^{1-\beta} dp, \tag{2.4}$$

which is difficult to obtain in closed-form. Suppose $\lambda_1 = 1$ and $\lambda_2 = 1$. Then, from (2.4), we obtain

$$I_\beta^Q(X) = c^{1-\beta} \int_0^1 (1-p)^{2(\beta-1)} dp = \frac{c^{1-\beta}}{2\beta - 1}, \ \beta > 1. \tag{2.5}$$

- Next, consider Govindarajulu's distribution with QF $Q_X(p) = a\{(b+1)p^b - bp^{b+1}\}$, $0 \leq p \leq 1$, $a, b > 0$. The QDF is given by

$$q_X(p) = ab(b+1)(1-p)p^{b-1}. \tag{2.6}$$

Thus, the Q-IGF is obtained as

$$
\begin{aligned}
I_\beta^Q(X) &= \int_0^1 \left[ab(b+1)(1-p)p^{b-1} \right]^{1-\beta} dp \\
&= \{ab(b+1)\}^{1-\beta} \int_0^1 (1-p)^{1-\beta} p^{(b-1)(1-\beta)} dp \\
&= \{ab(b+1)\}^{1-\beta} B(2 - \beta, (b-1)(1-\beta) + 1), \tag{2.7}
\end{aligned}
$$

provided $2 - \beta > 0$ and $(b-1)(1-\beta) + 1 > 0$, where $B(\cdot, \cdot)$ is the complete beta function.
- Consider the QF (see [18]) $Q_X(p) = -(c + \mu)\log(1 - p) - 2cp$, where $\mu > 0$ and $-\mu \leq c < \mu$, corresponding to the linear mean residual QF. In this case,

$$q_X(p) = \frac{c + \mu}{1 - p} - 2c. \tag{2.8}$$

Thus, the Q-IGF is obtained as

$$I_\beta^Q(X) = \int_0^1 \left[\frac{c + \mu}{1 - p} - 2c \right]^{1-\beta} dp, \ \beta > 1. \tag{2.9}$$
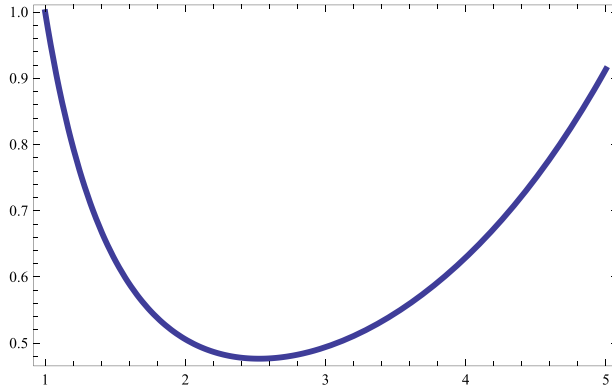
Figure 2. *Plot of Q-IGF for the QDF given by (2.8) considered in Example 2.3, for* c = 1 *and* μ = 1.5. *Along the* x-*axis, we have taken the values of* β.

Note that it is difficult to evaluate the integral in (2.9) in an explicit form. Thus, to have a rough idea about the behavior of the Q-IGF in (2.9), we have plotted the function in Figure 2 with respect to β for $c = 1$ and $\mu = 1.5$.

The fractional order Shannon entropy (FSE) was introduced by [27], which was subsequently extended by [28] and [3] to residual lifetime and past lifetime. We now provide a new representation for the Q-IGF.

Proposition 2.4. *Suppose the QDF of X is denoted by q_X. Then,*

$$I_\beta^Q(X) = \sum_{k=0}^{\infty} \frac{(1-\beta)^k}{k!} S_k^Q(X),$$ (2.10)

where $S_k^Q(X) = \int_0^1 \{\log q_X(p)\}^k \, dp$ is the quantile-based FSE of order k.

Proof. Using Maclaurin's theorem, we obtain from (2.1) that

$$
\begin{aligned}
I_\beta^Q(X) = \int_0^1 e^{(1-\beta)\log q_X(p)} dp &= \int_0^1 \sum_{k=0}^{\infty} \frac{(1-\beta)^k}{k!} \{\log q_X(p)\}^k dp \\
&= \sum_{k=0}^{\infty} \frac{(1-\beta)^k}{k!} \int_0^1 \{\log q_X(p)\}^k dp = \sum_{k=0}^{\infty} \frac{(1-\beta)^k}{k!} S_k^Q(X),
\end{aligned}
$$

as required. ☐

In the following, we present lower and upper bounds for Q-IGF in terms of the quantile-based Shannon entropy and the hazard QF.

Proposition 2.5. *For an RV X with QDF q_X, we have*

$$L(\beta) \le I_\beta^Q(X) \le U(\beta),$$ (2.11)

where $L(\beta) = \max\{0, (1-\beta)S^Q(X)\}$ and $U(\beta) = \int_0^1 \{H_X(p)\}^{\beta-1} dp$.

Proof. Making use of the inequality $x^{1-\beta} \geq (1 - \beta)\log x + 1$, the lower bound is easily obtained. The upper bound can be obtained from (2.2) by using the fact that $1 - p \leq 1$.   □

Next, we consider monotone transformations of RVs to see their effect on Q-IGF. Suppose ψ is an increasing function and $Y = \psi(X)$, where X is an RV with PDF f_X and QF Q_X. Then, it is known that the PDF of Y is $f_Y(y) = \frac{f_X(\psi^{-1}(y))}{\psi'(\psi^{-1}(y))}$. Moreover, $F_Y(y) = F_X(\psi^{-1}(y)) \Rightarrow F_Y(Q_Y(p)) = F_X(\psi^{-1}(Q_Y(p))) \Rightarrow \psi^{-1}(Q_Y(p)) = F_X^{-1}(p) = Q_X(p)$. Upon using this, the PDF of Y can be expressed as

$$f_Y(Q_Y(p)) = \frac{f_X(\psi^{-1}(Q_Y(p)))}{\psi'(\psi^{-1}(Q_Y(p)))} = \frac{f_X(Q_X(p))}{\psi'(Q_X(p))} = \frac{1}{q_X(p)\psi'(Q_X(p))} = \frac{1}{q_Y(p)}. \tag{2.12}$$

Theorem 2.6. *Suppose X is an RV with QF Q_X and QDF q_X. Further, suppose ψ is a positive-valued increasing function. Then,*

$$I_\beta^Q(\psi(X)) = \int_0^1 \frac{q_X^{1-\beta}(p)}{\{\psi'(Q_X(p))\}^{\beta-1}} dp. \tag{2.13}$$

Proof. The proof follows readily from (2.1) upon using (2.12).   □

The following example provides an illustration for the result in Theorem 2.6.

Example 2.7. Consider exponentially distributed RV X, as in Example 2.2. Further, consider an increasing transformation $\psi(X) = X^\alpha$, $\alpha > 0$. Then, it is known that $Y = \psi(X)$ follows a Weibull distribution with QF $Q_{\psi(X)}(p) = \theta^{-\alpha}\{-\log(1-p)\}^\alpha$. Now, by using (2.13), we obtain the Q-IGF for the Weibull distribution as

$$I_\beta^Q(\psi(X)) = \int_0^1 \frac{(\theta(1-p))^{\beta-1}}{(-\alpha\log(1-p)/\theta)^{\beta-1}} dp = \frac{\theta^{2\beta-2}}{\alpha^{\beta-1}} \int_0^1 \frac{(1-p)^{\beta-1}}{(-\log(1-p))^{\beta-1}} dp = \frac{\theta^{2\beta-2}\Gamma(2-\beta)}{\beta^{2-\beta}\alpha^{\beta-1}}, \tag{2.14}$$

provided $1 \leq \beta < 2$.

Weighted distributions are useful in many areas, such as renewal theory, reliability theory, and ecology. For an RV X with PDF f_X, the PDF of the weighted RV X_ω is $f_\omega(x) = \frac{\omega(x)f_X(x)}{E(\omega(X))}$, where $\omega(x)$ is a positive-valued weight function having finite expectation. We now consider a particular case of the weighted RV, known as an escort RV. Associated with X, the PDF of the escort distribution is given by

$$f_{X_{e,c}}(x) = \frac{f_X^c(x)}{\int_0^\infty f_X^c(x)dx}, \quad c > 0, \tag{2.15}$$

provided the involve integral exists. Observe that the escort distribution can be obtained as a weighted distribution with a suitable weight function. We now study the Q-IGF for the escort distribution in (2.15). Using the QF Q_X in (2.15), we obtain the density QF of $X_{e,c}$ as

$$f_{X_{e,c}}(Q_X(p)) = \frac{f_X^c(Q_X(p))}{\int_0^1 f_X^c(Q_X(p))dQ_X(p)} = \frac{f_X^c(Q_X(p))}{\int_0^1 f_X^{c-1}(Q_X(p))dp} = \frac{1}{q_X^c(p)\int_0^1 q_X^{1-c}(p)dp} = \frac{1}{q_{X_{e,c}}(p)}, \tag{2.16}$$

where $q_{X_{e,c}}(p)$ is the QDF of $X_{e,c}$.

Proposition 2.8. *Suppose $X_{e,c}$ is an escort RV with QDF $q_{e,c}$ corresponding to an RV X with QDF q_X and PDF f_X. Then,*

$$I_{c\beta}^Q(X) = I_\beta^Q(X_{e,c})(I_c^Q(X))^\beta. \tag{2.17}$$

Proof. From (2.1), the Q-IGF of $X_{e,c}$ is given by

$$I_\beta^Q(X_{e,c}) = \int_0^1 f_{X_{e,c}}^\beta(Q_X(p))dQ_X(p). \tag{2.18}$$

Now, using $f_{X_{e,c}}(Q_X(p)) = \dfrac{f_X^c(Q_X(p))}{\int_0^1 f_X^c(Q_X(p))dQ_X(p)}$ from (2.16) into (2.18) we obtain

$$I_\beta^Q(X_{e,c}) = \frac{\int_0^1 f_X^{c\beta}(Q_X(p))dQ_X(p)}{\{\int_0^1 f_X^c(Q_X(p))dQ_X(p)\}^\beta} = \frac{I_{c\beta}^Q(X)}{\{I_c^Q(X)\}^\beta},$$

as required. □

For two RVs *X* and *Y*, with respective PDFs f_X and f_Y, the PDF of a generalized escort distribution is given by

$$f_{Z_{ge,c}}(x) = \frac{f_X^c(x)f_Y^{1-c}(x)}{\int_0^\infty f_X^c(x)f_Y^{1-c}(x)dx}, \quad c > 0, \tag{2.19}$$

provided the involve integral exists. Like escort distributions, the generalized escort distributions can also be derived as a weighted distribution with a suitable weight function. The quantile version of the generalized escort distribution is given by

$$f_{Z_{ge,c}}(Q_X(p)) = \frac{f_X^c(Q_X(p))f_Y^{1-c}(Q_X(p))}{\int_0^1 f_X^c(Q_X(p))f_Y^{1-c}(Q_X(p))dQ_X(p)} = \frac{f_X^c(Q_X(p))f_Y^{1-c}(Q_X(p))}{\int_0^1 f_X^{c-1}(Q_X(p))f_Y^{1-c}(Q_X(p))dp}. \tag{2.20}$$

Theorem 2.9. *Let $X_{e,\beta}$ and $Y_{e,\beta}$ be two escort RVs associated with X and Y, respectively. Further, let $Z_{ge,c}$ be the generalized escort RV. Then,*

$$I_\beta^Q(Z_{ge,c}) = \frac{(I_\beta^Q(X))^c(I_\beta^Q(Y))^{1-c}}{\{R_c^Q(X,Y)\}^\beta}R_c^Q(X_{e,\beta},Y_{e,\beta}).$$

Proof. From (2.1), the Q-IGF of $Z_{ge,c}$ is given by

$$I_\beta^Q(Z_{ge,c}) = \int_0^1 f_{Z_{ge,c}}^\beta(Q_X(p))dQ_X(p). \tag{2.21}$$

Further, using (2.20) in (2.21), we obtain

$$
\begin{aligned}
I_\beta^Q(Z_{ge,c}) &= \frac{\int_0^1 \{f_X^c(Q_X(p))f_Y^{1-c}(Q_X(p))\}^\beta dQ_X(p)}{\{\int_0^1 f_X^{c-1}(Q_X(p))f_Y^{1-c}(Q_X(p))dp\}^\beta} \\
&= \frac{1}{\{R_c^Q(X,Y)\}^\beta} \int_0^1 \left(\frac{f_X^\beta(Q_X(p))}{\int_0^1 f_X^\beta(Q_X(p))dQ_X(p)}\right)^c \left(\frac{f_Y^\beta(Q_X(p))}{\int_0^1 f_Y^\beta(Q_X(p))dQ_X(p)}\right)^{1-c} dQ_X(p) \\
&\quad \times \left(\int_0^1 f_X^\beta(Q_X(p))dQ_X(p)\right)^c \left(\int_0^1 f_Y^\beta(Q_X(p))dQ_X(p)\right)^{1-c} \\
&= \frac{(I_\beta^Q(X))^c(I_\beta^Q(Y))^{1-c}}{\{R_c^Q(X,Y)\}^\beta} \int_0^1 f_{X_{e,\beta}}^c(Q_X(p))f_{Y_{e,\beta}}^{1-c}(Q_X(p))dQ_X(p) \\
&= \frac{(I_\beta^Q(X))^c(I_\beta^Q(Y))^{1-c}}{\{R_c^Q(X,Y)\}^\beta} R_c^Q(X_{e,\beta}, Y_{e,\beta}),
\end{aligned}
$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now introduce Q-IGF order between two RVs X and Y.

Definition 2.10. *A RV X is said to be smaller than Y in the sense of Q-IGF order, denoted by $X \leq_{qgf} Y$, if $I_\beta^Q(X) \leq I_\beta^Q(Y)$, for all $\beta \geq 1$.*

Example 2.11. Let X and Y have QFs $Q_X(p) = -\frac{\log(1-p)}{\theta_1}$ and $Q_Y(p) = -\frac{\log(1-p)}{\theta_2}$, with $\theta_1 \leq \theta_2$. Now, for $\beta \geq 1$, it can be easily shown that $I_\beta^Q(X) = \frac{\theta_1^{\beta-1}}{\beta} \leq \frac{\theta_2^{\beta-1}}{\beta} = I_\beta^Q(Y)$. Hence, $X \leq_{qgf} Y$, thus providing an example for Q-IGF order.

We now establish a relation between the hazard QF order, denoted by $\leq_{hq}$, and the Q-IGF order. For hazard QF order, one may refer to Definition 2.1(v) of [16]. The following example is presented to illustrate the hazard QF order.

Example 2.12. Consider two inverted exponential RVs X and Y with QDFs $q_X(p) = \frac{\lambda_1}{p(\log p)^2}$ and $q_Y(p) = \frac{\lambda_2}{p(\log p)^2}$, respectively, with $\lambda_1 \leq \lambda_2$. Then,

$$
H_X(p) = \frac{1}{(1-p)q_X(p)} = \frac{p(\log p)^2}{(1-p)q_X(p)} \geq \frac{p(\log p)^2}{(1-p)q_Y(p)} = H_Y(p), \tag{2.22}
$$

implying that $X \leq_{hq} Y$.

Theorem 2.13. *We have $X \leq_{hq} Y \Rightarrow X \geq_{qgf} Y$.*

Proof. From Definition 2.1(v) of [16], we obtain

$$
X \leq_{hq} Y \Rightarrow \frac{1}{(1-p)q_X(p)} \geq \frac{1}{(1-p)q_Y(p)} \Rightarrow \left(\frac{1}{q_X(p)}\right)^{\beta-1} \geq \left(\frac{1}{q_Y(p)}\right)^{\beta-1}
$$

$$
\Rightarrow \int_0^1 \left(\frac{1}{q_X(p)}\right)^{\beta-1} dp \geq \int_0^1 \left(\frac{1}{q_Y(p)}\right)^{\beta-1} dp \Rightarrow I_\beta^Q(X) \geq I_\beta^Q(Y) \Rightarrow X \geq_{qgf} Y,
$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It can be easily proved that the hazard QF order and reversed hazard QF order are equivalent, that is $X \leq_{hq} Y \Leftrightarrow X \geq_{rhq} Y$. One may refer to Definition 2.1(vi) of [16] for the definition of reversed hazard QF order. Hence, the condition "$X \leq_{hq} Y$" in Theorem 2.13 can be replaced by "$X \geq_{rhq} Y$" in order to get $X \geq_{qgf} Y$.

Definition 2.14. *A RV X is said to be smaller than Y in the sense of dispersive ordering, denoted by $X \leq_{disp} Y$, if $Q_Y(p) - Q_X(p)$ is increasing in $p \in (0, 1)$.*

Theorem 2.15. *Let X and Y be two RVs with QFs Q_X and Q_Y, respectively.*

(i) *Let $X \leq_{rhq} Y$ and ψ be increasing and concave. Then, $X \leq_{disp} Y \Rightarrow \psi(X) \leq_{qgf} \psi(Y)$;*
(ii) *Let $X \leq_{hq} Y$ and ψ be increasing and convex. Then, $X \leq_{disp} Y \Rightarrow \psi(X) \geq_{qgf} \psi(Y)$.*

Proof. We prove the first part of the theorem, while the second part can be proved in an analogous manner. First, we have $X \leq_{disp} Y$ implying $Q_X(p) \leq Q_Y(p)$. Then, as ψ is increasing and concave, we have

$$\psi'(Q_X(p)) \geq \psi'(Q_Y(p)) \Rightarrow 0 \leq \frac{1}{\psi'(Q_X(p))} \leq \frac{1}{\psi'(Q_Y(p))}. \tag{2.23}$$

Further,

$$X \leq_{rhq} Y \Rightarrow \frac{1}{q_X(p)} \leq \frac{1}{q_Y(p)}. \tag{2.24}$$

Upon combining (2.23) and (2.24), we obtain

$$\frac{1}{q_X(p)\psi'(Q_X(p))} \leq \frac{1}{q_Y(p)\psi'(Q_Y(p))} \Rightarrow \int_0^1 \frac{dp}{\{q_X(p)\psi'(Q_X(p))\}^{\beta-1}} \leq \int_0^1 \frac{dp}{\{q_X(p)\psi'(Q_X(p))\}^{\beta-1}}. \tag{2.25}$$

Hence, the required result follows from (2.25) and (2.13). □

3. Quantile-based residual IGF

The residual lifetime of a system with lifetime X, given that the system is working at time $t > 0$, is defined as $X_t = [X - t | X > t]$. The IGF has been studied for residual lifetimes by [12]. We now consider here the residual IGF based on QF. We first give the definition of quantile-based residual IGF (Q-RIGF).

Definition 3.1. *Let Q_X and q_X be the QF and QDF of an RV X. Then, the Q-RIGF of X is defined as*

$$I_\beta^Q(X; Q_X(u)) = \frac{1}{(1-u)^\beta} \int_u^1 f_X^\beta(Q_X(p)) dQ_X(p) = \frac{1}{(1-u)^\beta} \int_u^1 q_X^{1-\beta}(p) dp, \ \beta \geq 1. \tag{3.1}$$

The Q-RIGF in (3.1) can be expressed in terms of hazard and reversed hazard QFs as

$$I_\beta^Q(X; Q_X(u)) = \frac{1}{(1-u)^\beta} \int_u^1 \{(1-p)H_X(p)\}^{\beta-1} dp = \frac{1}{(1-u)^\beta} \int_u^1 (p\tilde{H}_X(p))^{\beta-1} dp. \tag{3.2}$$

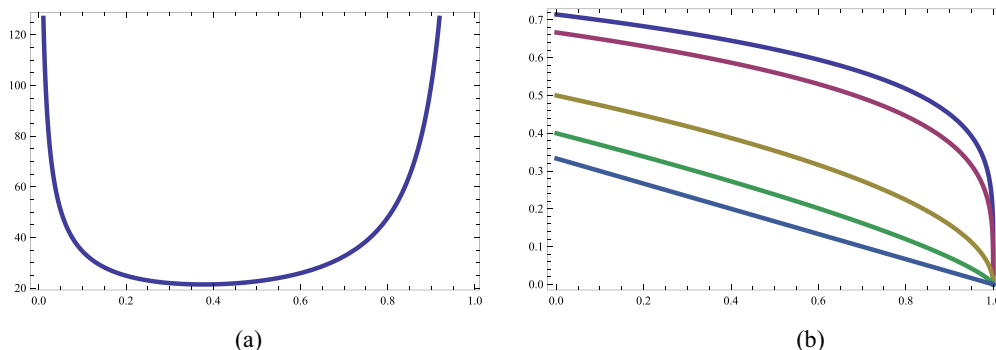Upon taking the derivative of (3.1) with respect to β, we get

Figure 3. (a) *Plot of Q-RIGF for power distribution considered in Example 3.2, for $\alpha = 0.1$, $\beta = 2.2$, and $\delta = 2.3$; (b) Plot of Q-RIGF for Davies distribution considered in Example 3.2 for $\beta = 1.2, 1.25, 1.5, 1.75, 2$. Here, along the x-axis, we take the values of $u \in (0, 1)$.*

$$\frac{\partial}{\partial \beta} I_\beta^Q(X; Q_X(u)) = -\left[\frac{1}{(1-u)^\beta} \int_u^1 q_X^{1-\beta}(p) \log q_X(p) dp + \log(1-u) I_\beta^Q(X; Q_X(u)) \right], \quad (3.3)$$

from which the following observations can be readily made:

(i) $I_\beta^Q(X; Q_X(u))|_{\beta=1} = 1$; $I_\beta^Q(X; Q_X(u))|_{\beta=2} = -2J^Q(X; Q_X(u))$,

(ii) $I_\beta^Q(X; Q_X(u))|_{u=0} = I_\beta^Q(X)$; $\frac{\partial}{\partial \beta} I_\beta^Q(X; Q_X(u))|_{\beta=1} = -S^Q(X; Q_X(u))$,

where $J^Q(X; Q_X(u)) = -\frac{1}{2(1-u)^2} \int_u^1 \frac{dp}{q_X(p)}$ is the quantile-based residual extropy (see Eq. (4.1) of [16]) and $S^Q(X; Q_X(u)) = \log(1-u) + \frac{1}{1-u} \int_u^1 \log q_X(p) dp$ is the quantile-based residual Shannon entropy (see Eq. (8) of [25]). The following example presents closed-form expressions for the Q-RIGF for some distributions.

Example 3.2.

- Consider exponential distribution with hazard rate θ. Then, from (3.1), we obtain $I_\beta^Q(X; Q_X(u)) = \frac{\theta^{\beta-1}}{\beta}$;

- For the power distribution with QF $Q_X(p) = \alpha p^\delta$, $\alpha, \delta > 0$, we obtain from (3.1) that $I_\beta^Q(X; Q_X(u)) = \frac{(\alpha\delta)^{1-\beta}}{\{(\delta-1)(1-\beta)+1\}(1-u)^\beta}[1 - u^{(\delta-1)(1-\beta)+1}]$;

- For Davies distribution with QF $Q_X(p) = \frac{cp}{1-p}$, $c > 0$, from (3.1), we obtain $I_\beta^Q(X; Q_X(u)) = \frac{c^{1-\beta}(1-u)^{\beta-1}}{2(\beta-1)+1}$;

- For the re-scaled beta distribution with $Q_X(p) = r[1 - (1-p)^{\frac{1}{c}}]$, $c, r > 0$, from (3.1), we obtain $I_\beta^Q(X; Q_X(u)) = (\frac{r}{c})^{1-\beta} \frac{(1-u)^{\frac{1-\beta}{c}}}{(\frac{1}{c}-1)(1-\beta)+1}$, provided $(\frac{1}{c} - 1)(1 - \beta) + 1 > 0$.

In Figure 3, we have plotted the Q-RIGF of power distribution and Davies distribution to show that it is not monotone in general with respect to u.

In the following example, we consider linear mean residual QF family of distributions, having no tractable distribution function (see [18]), and it includes exponential and uniform distributions as special cases.
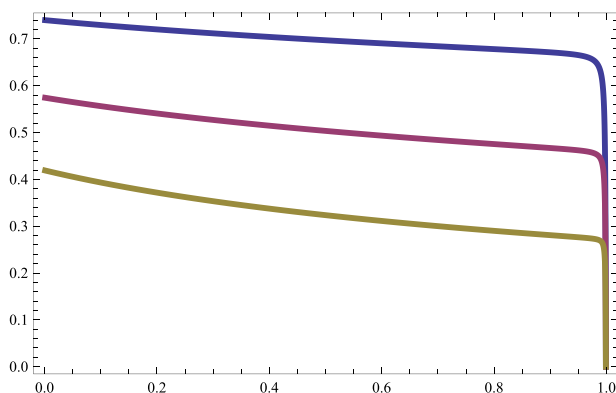
Figure 4. *Plot of Q-RIGF for the distribution with QF in Example 3.3 with c = 1 and μ = 2. Here, along the x-axis, we take the values of $u \in (0, 1)$. Three values of β have been considered, viz., β = 1.2, 1.4, and 1.7 (presented from above).*

Example 3.3. Let $Q_X(p) = -(c + \mu) \log(1 - p) - 2cp$, $\mu > 0$, $-\mu \leq c < \mu$, $0 < p < 1$. Then, the Q-RIGF is given by

$$I_\beta^Q(X; Q_X(u)) = \frac{1}{(1-u)^\beta} \int_u^1 \left\{ \frac{c+\mu}{1-p} - 2c \right\}^{1-\beta} dp. \tag{3.4}$$

Note that it is not possible to obtain a closed-form expression for the integral in (3.4). However, in order to get an idea regarding its behavior, the graph of $I_\beta^Q(X; Q_X(u))$ in (3.4) has been plotted in Figure 4 for some specific values of β, c, and μ.

Further, by differentiating (3.1) with respect to u, we get

$$q_X(u) = \left[\beta(1-u)^{\beta-1} I_\beta^Q(X; Q_X(u)) - (1-u)^\beta \frac{\partial}{\partial u} I_\beta^Q(X; Q_X(u)) \right]^{\frac{1}{1-\beta}}. \tag{3.5}$$

The expression in (3.5) can be utilized in two directions. First, it shows that the distribution of X is indeed characterized based on $I_\beta^Q(X; Q_X(u))$. Second, it shows that a new QF can be generated based on the assumed functional form of $I_\beta^Q(X; Q_X(u))$. From the expression in the first equality in (3.2), we obtain

$$H_X(u) = \left[(u - 1) \frac{\partial}{\partial u} I_\beta^Q(X; Q_X(u)) + \beta I_\beta^Q(X; Q_X(u)) \right]^{\frac{1}{\beta-1}}, \tag{3.6}$$

which is useful in determining $H_X(u)$ based on $I_\beta^Q(X; Q(u))$. Further, from the expression in the second equality in (3.2), we obtain

$$\tilde{H}_X(u) = \frac{1-u}{u} \left[(u - 1) \frac{\partial}{\partial u} I_\beta^Q(X; Q_X(u)) + \beta I_\beta^Q(X; Q_X(u)) \right]^{\frac{1}{\beta-1}}. \tag{3.7}$$

Analogous to (3.6), the expression in (3.7) can be utilized for determining the reversed hazard QF $\tilde{H}_X(u)$ of X. We now introduce two nonparametric classes of life distributions based on Q-RIGF.

Definition 3.4. *A RV X is said to have increasing (decreasing) Q-RIGF, that is, IQ-RIGF (DQ-RIGF) if $I_\beta^Q(X; Q_X(u))$ is increasing (decreasing) with respect to u.*

Based on the proposed nonparametric classes, the following bounds can be provided in terms of the hazard and reversed hazard QFs:

$$I_\beta^Q(X; Q_X(p)) \begin{cases} \geq (\leq) \dfrac{(\frac{u}{1-u})^{\beta-1}}{\beta} \tilde{H}_X^{\beta-1}(u) & \text{if } X \text{ is IQ-RIGF (DQ-RIGF)}, \\ \leq (\geq) \dfrac{1}{\beta} H_X^{\beta-1}(u) & \text{if } X \text{ is IQ-RIGF (DQ-RIGF)}. \end{cases} \quad (3.8)$$

Further, from (3.5), we obtain bounds for Q-IGF as

$$I_\beta^Q(X; Q_X(p)) \begin{cases} \geq \dfrac{((1-u)q_X(u))^{1-\beta}}{\beta} & \text{if } X \text{ is IQ-RIGF}, \\ \leq \dfrac{((1-u)q_X(u))^{1-\beta}}{\beta} & \text{if } X \text{ is DQ-RIGF}. \end{cases} \quad (3.9)$$

For exponential distribution, Q-RIGF is independent of *u* (see Example 3.2), implying that this distribution is the boundary of IQ-RIGF and DQ-RIGF classes. Similar to Theorem 2.6, the following result can be established, which provides the effect of increasing transformations on Q-RIGF.

Theorem 3.5. *Suppose X is an RV with QF Q_X and QDF q_X. Further, suppose ψ is a positive-valued increasing function. Then,*

$$I_\beta^Q(\psi(X); Q_X(u)) = \frac{1}{(1-u)^\beta} \int_u^1 \frac{q_X^{1-\beta}(p)}{\{\psi'(Q_X(p))\}^{\beta-1}} dp. \quad (3.10)$$

Proof. The proof is similar to that of Theorem 2.6, and is therefore omitted. □

Similar to Definition 2.10, we now present the definition of Q-RIGF order for two RVs *X* and *Y*.

Definition 3.6. *A RV X is said to be smaller than Y in the sense of Q-RIGF order, denoted by $X \leq_{q-ri} Y$, if $I_\beta^Q(X; Q_X(u)) \leq I_\beta^Q(Y; Q_Y(u))$, for all $\beta \geq 1$.*

Next, we obtain a relation between dispersive and Q-RIGF orders.

Theorem 3.7. *We have $X \leq_{disp} Y \Rightarrow X \geq_{q-ri} Y$.*

Proof. Note that $X \leq_{disp} Y$ implies $Q_Y(p) - Q_X(p)$ is increasing with respect to $p \in (0,1)$. So, $\frac{d}{dp}(Q_Y(p) - Q_X(p)) \geq 0$, implying $q_Y(p) \geq q_X(p)$. Hence, we have

$$\frac{1}{(1-u)^p} \int_u^1 q_Y^{1-\beta}(p)dp \leq \frac{1}{(1-u)^p} \int_u^1 q_X^{1-\beta}(p)dp \Rightarrow X \geq_{q-ri} Y, \quad (3.11)$$

as required. □

Next, we discuss stochastic orders connecting two random lifetimes *X* and *Y* with Q-RIGFs $I_\beta^Q(X; Q_X(u))$ and $I_\beta^Q(Y; Q_Y(u))$, respectively.

Theorem 3.8. *For two RVs X and Y with QFs $Q_X(.)$ and $Q_Y(.)$ and QDFs $q_X(.)$ and $q_Y(.)$, respectively, we have $X \leq_{hq} Y \Rightarrow X \geq_{q-ri} Y$.*

Proof. Under the assumptions made, we have

$$
\begin{aligned}
X \leq_{hq} Y \quad &\Rightarrow \quad \frac{1}{(1-p)q_X(p)} \geq \frac{1}{(1-p)q_Y(p)} \\
&\Rightarrow \quad \frac{1}{(1-u)^\beta q_X(p)^{\beta-1}} \geq \frac{1}{(1-u)^\beta q_Y(p)^{\beta-1}} \\
&\Rightarrow \quad \frac{1}{(1-u)^\beta} \int_u^1 (q_X(p))^{1-\beta} dp \geq \frac{1}{(1-u)^\beta} \int_u^1 (q_Y(p))^{1-\beta} dp \\
&\Rightarrow \quad I_\beta^Q(X; Q_X(u)) \geq I_\beta^Q(Y; Q_Y(u)),
\end{aligned}
$$

as required.   □

Moreover, since the hazard QF order and reversed hazard QF order are equivalent, in Theorem 3.8, we can also consider $\leq_{rhq}$ instead of $\leq_{hq}$. Note that the reverse implication in Theorem 3.8 may not hold.

Theorem 3.9. *Let $\frac{I_\beta^Q(X;Q_X(u))}{I_\beta^Q(Y;Q_Y(u))}$ be increasing with respect to $u \in (0,1)$. Then, $X \leq_{q-ri} Y \Rightarrow X \leq_{hq} Y$.*

Proof. Under the assumptions made, we have

$$
\frac{q_X^{1-\beta}(u)}{q_Y^{1-\beta}(u)} \leq \frac{\int_u^1 q_X^{1-\beta}(p)dp}{\int_u^1 q_Y^{1-\beta}(p)dp} \leq 1. \tag{3.12}
$$

Thus,

$$
\frac{1}{q_X(u)} \leq \frac{1}{q_Y(u)} \Rightarrow \frac{1}{(1-u)q_X(u)} \leq \frac{1}{(1-u)q_Y(u)} \Rightarrow X \leq_{hq} Y, \tag{3.13}
$$

as required.   □

We end this section with a characterization result of the exponential distribution in connection with the Q-RIGF.

Theorem 3.10. *The Q-RIGF of a nonnegative RV is constant (independent of time) if and only if it is exponentially distributed.*

Proof. The "if" part is clear from Example 3.2. To establish the "only if" part, we consider the Q-RIGF to be constant, that is, $I_\beta^Q(X; Q_X(u)) = k$, where k is a constant (here, independent of u). Further, differentiating (3.1) with respect to u, and then substituting $I_\beta^Q(X; Q_X(u)) = k$, we obtain after some simplification

$$
q_X(u) = \frac{(k\beta)^{1-\beta}}{1-u} = \frac{k^*}{1-u}, \tag{3.14}
$$

which is indeed the QF of the exponential distribution. This completes the proof of the theorem.   □

4. Quantile-based past IGF

Just as the concept of residual lifetimes in reliability, the past lifetime of a system also plays an important role in studying the past history of a failed system. The past lifetime of a system with lifetime X is given by $\widetilde{X}_t = [t - X | X < t]$, where t is the prefixed inspection time. We now define the quantile-based past IGF (Q-PIGF).

Definition 4.1. *Let Q_X and q_X be the QF and QDF of X. Then, the Q-PIGF of X is defined as*

$$\widetilde{I}_\beta^Q(X; Q_X(u)) = \frac{1}{u^\beta} \int_0^u f_X^\beta(Q_X(p)) dQ_X(p) = \frac{1}{u^\beta} \int_0^u q_X^{1-\beta}(p) dp, \ \beta \geq 1. \tag{4.1}$$

Similar to (3.2), the Q-PIGF can be expressed in terms of hazard and reversed hazard QFs as follows:

$$\widetilde{I}_\beta^Q(X; Q_X(u)) = \frac{1}{u^\beta} \int_0^u \{(1-p)H_X(p)\}^{\beta-1} dp = \frac{1}{u^\beta} \int_0^u \left\{ p\widetilde{H}_X(p) \right\}^{\beta-1} dp. \tag{4.2}$$

By differentiating (4.1) with respect to β, we obtain

$$\frac{\partial}{\partial \beta} \widetilde{I}_\beta^Q(X; Q_X(u)) = - \left[\frac{1}{u^\beta} \int_0^u q_X^{1-\beta}(p) \log q_X(p) dp + \widetilde{I}_\beta^Q(X; Q_X(u)) \log u \right], \tag{4.3}$$

from which the following observations can be easily made:

(i) $\widetilde{I}_\beta^Q(X; Q_X(u))|_{\beta=1} = 1$; $\widetilde{I}_\beta^Q(X; Q_X(u))|_{\beta=2} = -2\widetilde{J}^Q(X; Q_X(u))$,

(ii) $\widetilde{I}_\beta^Q(X; Q_X(u))|_{u=1} = I_\beta^Q(X)$; $\frac{\partial}{\partial \beta} \widetilde{I}_\beta^Q(X; Q_X(u))|_{\beta=1} = -\widetilde{S}^Q(X; Q_X(u))$,

where $\widetilde{J}^Q(X; Q_X(u)) = -\frac{1}{2u^2} \int_0^u \frac{dp}{q_X(p)}$ is the quantile-based past extropy (see Eq. (5.5) of [16]) and $\widetilde{S}^Q(X; Q_X(u)) = \log u + \frac{1}{u} \int_0^u \log q_X(p) dp$ is the quantile-based residual Shannon entropy (see Eq. (8) of [26]). In the following example, we present closed-form expressions for the Q-PIGF for some distributions.

Example 4.2.

- Consider power distribution with QF $Q_X(p) = \alpha p^\delta$, $\alpha, \delta > 0$. Then, the Q-PIGF can be obtained as $\widetilde{I}_\beta^Q(X; Q_X(u)) = \frac{(\alpha\delta)^{1-\beta}}{u^\beta} \frac{u^{(\delta-1)(1-\beta)+1}}{(\delta-1)(1-\beta)+1}$, provided $(\delta - 1)(1 - \beta) + 1 > 0$;
- For exponential distribution with hazard rate θ, the Q-PIGF is obtained from (4.1) as $\widetilde{I}_\beta^Q(X; Q_X(u)) = \frac{\theta^{\beta-1}}{\beta u^\beta} \left\{ 1 - (1-u)^\beta \right\}$.
- For the uniform distribution with QF $Q_X(p) = a + (b - a)p$, $0 < a < b$, the Q-PIGF can be obtained as $\widetilde{I}_\beta^Q(X; Q_X(u)) = \{u(b - a)\}^{1-\beta}$;
- For the half-logistic distribution with QF $Q_X(p) = \sigma \log(\frac{1+p}{1-p})$, $\sigma > 0$, the Q-IGF can be obtained as $\widetilde{I}_\beta^Q(X; Q_X(u)) = \frac{1}{u^\beta} \int_0^u (\frac{2\sigma}{1-p^2})^{1-\beta} dp$.

Further, differentiating (4.1) with respect to u, we obtain

$$q_X(u) = \left[\beta u^{\beta-1} \widetilde{I}_\beta^Q(X; Q_X(u)) + u^\beta \frac{\partial}{\partial} \widetilde{I}_\beta^Q(X; Q_X(u)) \right]^{\frac{1}{1-\beta}}, \tag{4.4}$$

which is useful in obtaining a characterization of a distribution based on the Q-PIGF. Similar to Q-RIGF, (4.4) can be used to produce a new QF. The following two relations that have been derived from (4.2) are useful in determining hazard QF and reversed hazard QF, respectively:

$$H_X(u) \quad = \quad \frac{1}{1-u} \left[\beta u^{\beta-1} \widetilde{I}_\beta^Q(X; Q_X(u)) + u^\beta \frac{\partial}{\partial u} \widetilde{I}_\beta^Q(X; Q_X(u)) \right]^{\frac{1}{\beta-1}}, \tag{4.5}$$

$$\widetilde{H}_X(u) \quad = \quad \frac{1}{u} \left[\beta u^{\beta-1} \widetilde{I}_\beta^Q(X; Q_X(u)) + u^\beta \frac{\partial}{\partial u} \widetilde{I}_\beta^Q(X; Q_X(u)) \right]^{\frac{1}{\beta-1}}. \tag{4.6}$$

Now, two nonparametric classes of distributions based on the Q-PIGF can be constructed, analogous to Definition 4.1.

Definition 4.3. *A RV X is said to have increasing (decreasing) Q-PIGF, that is, IQ-PIGF (DQ-PIGF) if $\widetilde{I}_\beta^Q(X; Q_X(u))$ is increasing (decreasing) with respect to u.*

Similar to (3.8) and (3.9), we can provide bounds for Q-PIGF as follows:

$$\widetilde{I}_\beta^Q(X; Q_X(u)) \begin{cases} \leq (\geq) \dfrac{(\frac{1-u}{u})^{\beta-1}}{\beta} H_X^{\beta-1}(u) & \text{if } X \text{ is IQ-PIGF (DQ-PIGF)}, \\[4mm] \leq (\geq) \dfrac{1}{\beta} \tilde{H}_X^{\beta-1}(u) & \text{if } X \text{is IQ-PIGF (DQ-PIGF)}, \end{cases} \tag{4.7}$$

and

$$\widetilde{I}_\beta^Q(X; Q_X(u)) \begin{cases} \geq \dfrac{(uq_X(u))^{1-\beta}}{\beta} & \text{if } X \text{ is IQ-PIGF}, \\[4mm] \leq \dfrac{(uq_X(u))^{1-\beta}}{\beta} & \text{if } X \text{ is DQ-PIGF}. \end{cases} \tag{4.8}$$

Definition 4.4. *A RV X is said to be smaller than Y in the sense of Q-PIGF order, denoted by $X \leq_{q-pi} Y$, if $\widetilde{I}_\beta^Q(X; Q_X(u)) \leq \widetilde{I}_\beta^Q(Y; Q_Y(u))$, for all $\beta \geq 1$.*

Theorem 4.5. *We have $X \leq_{disp} Y \Rightarrow X \geq_{q-pi} Y$.*

Proof. The proof is analogous to that of Theorem 3.7, and is therefore omitted. ☐

Theorem 4.6. *For two RVs X and Y with QFs $Q_X(.)$ and $Q_Y(.)$ and QDFs $q_X(.)$ and $q_Y(.)$, respectively, we have $X \leq_{rq} Y \Rightarrow X \geq_{q-ri} Y$.*

Proof. The proof is similar to that of Theorem 3.8, and is therefore omitted. ☐

Theorem 4.7. *Let $\dfrac{\widetilde{I}_\beta^Q(X; Q_X(u))}{\widetilde{I}_\beta^Q(Y; Q_Y(u))}$ be increasing with respect to $u \in (0, 1)$. Then, $X \leq_{q-pi} Y \Rightarrow X \leq_{rq} Y$.*

Proof. The proof follows along the lines of Theorem 3.9, and is therefore omitted. ☐

We finish this subsection with a result which shows that Q-RIGF and Q-PIGF can be related to each other. The advantage of the following theorem is that one of these concepts is sufficient to study the other.

Theorem 4.8. *We have*

$$I_\beta^Q(X; Q_X(u)) \quad = \quad (1-u)^{-\beta} \left[\widetilde{I}_\beta^Q(X; Q_X(1)) - u^\beta \widetilde{I}_\beta^Q(X; Q_X(u)) \right], \tag{4.9}$$

$$\widetilde{I}_\beta^Q(X; Q_X(u)) \quad = \quad u^{-\beta} \left[I_\beta^Q(X; Q_X(0)) - (1-u)^\beta I_\beta^Q(X; Q_X(u)) \right], \tag{4.10}$$

where $\widetilde{I}_\beta^Q(X; Q_X(1)) = I_\beta^Q(X; Q_X(0)) = I_\beta^Q(X).$

Proof. From (3.1) and (4.1), we obtain

$$q_X^{1-\beta}(u) = -\frac{d}{du}[(1-u)^\beta I_\beta^Q(X; Q_X(u))] \text{ and } q_X^{1-\beta}(u) = \frac{d}{du}[u^\beta \widetilde{I}_\beta^Q(X; Q_X(u))].$$

Now, equating these two and then integrating, we obtain

$$(1-u)^\beta I_\beta^Q(X; Q_X(u)) = -u^\beta \widetilde{I}_\beta^Q(X; Q_X(u)) + l, \tag{4.11}$$

where l is a constant. Further, when u tends to 0, we have $l = I_\beta^Q(X; Q_X(0))$, and when u tends to 1, $l = \widetilde{I}_\beta^Q(X; Q_X(1))$. Upon using these facts, the desired identities follow.   □

5. Application of the Q-IGF

This section focuses on the construction of an empirical estimator of Q-IGF and examines its usefulness using a real-life data set. In this regard, consider a random sample of size n as $X_1, \ldots, X_n$. Further, let $X_{(1)} \leq \cdots \leq X_{(n)}$ be the order statistics of this random sample. Then, the empirical QF is given by (see [22])

$$\hat{Q}_X(v) = n\left(\frac{j}{n} - v\right) X_{(j-1)} + n\left(v - \frac{j-1}{n}\right) X_{(j)}, \tag{5.1}$$

where $\frac{j-1}{n} \leq v \leq \frac{j}{n}$, for $j = 1, \ldots, n$. Thus, the corresponding empirical estimator of the QDF is

$$\hat{q}_X(v) = n(X_{(j)} - X_{(j-1)}), \tag{5.2}$$

for $\frac{j-1}{n} \leq v \leq \frac{j}{n}$, for $j = 1, \ldots, n$. Using (5.2), the empirical Q-IGF estimator is obtained as

$$\hat{I}_\beta^Q(X) = \int_0^1 \hat{q}_X^{1-\beta}(p) dp, \tag{5.3}$$

where $\hat{q}_X(p) = n(X_{(j)} - X_{(j-1)})$. Thus, we have the empirical Q-IGF estimator as

$$\hat{I}_\beta^Q(X) = \frac{1}{n} \sum_{j=1}^n \left[n\left(X_{(j)} - X_{(j-1)}\right) \right]^{1-\beta}, \quad \beta \geq 1. \tag{5.4}$$
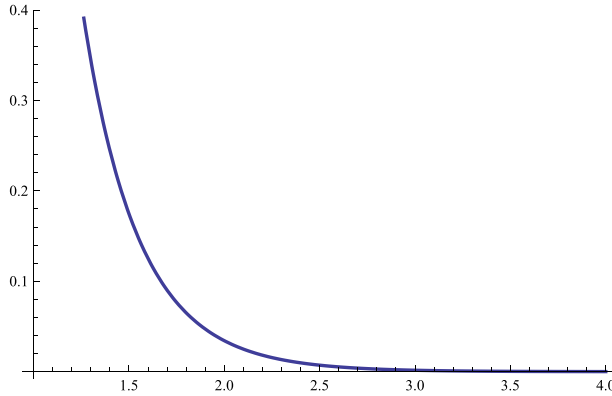
Figure 5. *Plot of the parametric estimate of the Q-IGF (given by (5.5)) for Davies distribution with respect to β. Here, we have considered β from 1 to 4.*

Table 1. *The estimated values of Q-IGF for different values of β.*

β	1.2	1.5	1.8	2	2.5	3	5
$\widehat{I}_{\beta}^{Q}(X)$	0.4879	0.1748	0.0651	0.0343	0.0072	0.0017	0.0001

Further, in order to see the usefulness of the proposed estimator in (5.4), we compute its value based on a real data set (see [31]), which represents the time (in months) to first failure of twenty electric carts:

0.9, 1.5, 2.3, 3.2, 3.9, 5.0, 6.2, 7.5, 8.3, 10.4, 11.1, 12.6, 15.0, 16.3, 19.3, 22.6, 24.8, 31.8, 38.1, 53.0.

Using chi-square goodness of fit test and Q-Q plot, Krishnan [16] showed that the data set is well fitted by Davies distribution with QF $Q_X(p) = \frac{cp^a}{(1-p)^b}$, $a, b, c > 0$. We recall that Davies distribution does not have tractable CDF, but has a closed-form QF. Further, equating sample L-moments with population L-moments, Krishnan [16] obtained the estimated values of the parameters of Davies distribution to be

$$\hat{a} = 1.1255, \ \hat{b} = 0.2911, \ \hat{c} = 18.6139.$$

We note that for Davies distribution, the parametric estimate of Q-IGF is obtained as

$$\widehat{I}_{\beta}^{Q}(X) = c^{1-\beta} \int_0^1 \left[\frac{\hat{a}p^{\hat{a}-1}}{(1-p)^{\hat{b}}} + \frac{\hat{b}p^{\hat{a}}}{(1-p)^{\hat{b}+1}} \right]^{1-\beta} dp, \ \ \beta \geq 1. \tag{5.5}$$

Using Mathematica software, the parametric estimates of the Q-IGF for the Davies family of distributions, with $\hat{a} = 1.1255$, $\hat{b} = 0.2911$, and $\hat{c} = 18.6139$, for different values of $\beta \geq 1$ are plotted in Figure 5. In addition to it, we have also computed the empirical estimate of the Q-IGF given in (5.5) for some values of β, which are presented in Table 1.

6. Concluding remarks with discussion on a future problem

The concept of IGF gained much attention recently even though it was introduced by [4] more than five decades ago. The importance of this function is that it helps to produce various information measures for models having closed-form probability density functions. However, there are many distributions which do not have closed-form density functions. In this paper, we have proposed Q-IGF and studied various properties of it. Some bounds, its connection to reliability theory, and effects under monotone

transformations have also been discussed. The proposed IGFs have been studied in particular for escort and generalized escort distributions. Orders based on the newly proposed measure have also been introduced. Finally, we have extended the proposed concept to residual and past lifetimes, and discussed them under different contexts.

Very recently, Capaldo [2] introduced cumulative IGF of an RV X with CDF F_X and survival function $\bar{F}_X$ as

$$G_{\alpha,\beta}(X) = \int_l^r \{F_X(x)\}^\alpha \{1 - F_X(x)\}^\beta dx, \ \alpha, \beta \in \mathcal{R}, \tag{6.1}$$

where $l = \inf\{x \in \mathcal{R} | F_X(x) > 0\}$ and $r = \sup\{x \in \mathcal{R} | \bar{F}_X(x) > 0\}$. The quantile-based cumulative IGF of X is obtained as

$$G_{\alpha,\beta}^Q(X) = \int_0^1 p^\alpha (1 - p)^\beta q_X(p) dp, \tag{6.2}$$

where α and β are real numbers. We propose to explore the properties of this quantile-based measure in (6.2) in our future work.

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