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CLASSES OF BARREN EXTENSIONS

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Abstract. Henle, Mathias, and Woodin proved in [21] that, provided that $\omega \rightarrow (\omega)^\omega$ holds in a model M of ZF, then forcing with $([\omega]^\omega, \subseteq^*)$ over M adds no new sets of ordinals, thus earning the name a "barren" extension. Moreover, under an additional assumption, they proved that this generic extension preserves all strong partition cardinals. This forcing thus produces a model $M[\mathcal{U}]$, where $\mathcal{U}$ is a Ramsey ultrafilter, with many properties of the original model M. This begged the question of how important the Ramseyness of $\mathcal{U}$ is for these results. In this paper, we show that several classes of σ-closed forcings which generate non-Ramsey ultrafilters have the same properties. Such ultrafilters include Milliken–Taylor ultrafilters, a class of rapid p-points of Laflamme, k-arrow p-points of Baumgartner and Taylor, and extensions to a class of ultrafilters constructed by Dobrinen, Mijares, and Trujillo. Furthermore, the class of Boolean algebras $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$, $2 \le \alpha < \omega_1$, forcing non-p-points also produce barren extensions.

§1. Introduction.

In their paper, *A barren extension* [21], Henle, Mathias, and Woodin proved that forcing with $([\omega]^\omega, \subseteq^*)$ does not add new subsets of ordinals and preserves strong partition cardinals, assuming the ground model satisfies certain properties. The first of these is the infinite partition relation

$$\omega \rightarrow (\omega)^\omega, \tag{1}$$

which means that for each coloring $c : [\omega]^\omega \rightarrow 2$, there is an infinite subset $x \subseteq \omega$ such that the restriction of c to $[x]^\omega$ is constant. This partition relation fails outright in the presence of the Axiom of Choice. However, it is consistent with fragments of Choice, as seen in the following: $\mathrm{AD}_\mathbb{R}$ implies $\omega \rightarrow (\omega)^\omega$. This was first proved by Prikry with the additional assumption of DC in [34], and soon after, Mathias showed DC was unnecessary in [28]. Similarly $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ implies $\omega \rightarrow (\omega)^\omega$. This was proved in the Cabal and can be seen to follow from Σ^2_1 reflection to the Suslin coSuslin sets (see [37] and Theorem 25 in [41]) and from the fact that every Suslin set of reals is Ramsey (see Theorem 2.2 of [18]). Hence $\mathrm{AD} + V = L(\mathbb{R})$ implies $\omega \rightarrow (\omega)^\omega$ (because $\mathrm{AD} + V = L(\mathbb{R})$ implies $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$), and so $\omega \rightarrow (\omega)^\omega$ holds in $L(\mathbb{R})$ assuming there are ω Woodin cardinals with a measurable above (see [35]). Furthermore, the partition relation $\omega \rightarrow (\omega)^\omega$ holds in the $L(\mathbb{R})$ of a model V of ZFC if V is a model obtained by collapsing a strongly inaccessible cardinal κ to ω_1 via the Lévy collapse (see [28]).

Henle, Mathias, and Woodin proved that forcing with $([\omega]^\omega, \subseteq^*)$ over a model satisfying the infinitary partition relation $\omega \rightarrow (\omega)^\omega$ does not add any new subsets of ordinals over the ground model, aptly calling this extension *barren*.

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Theorem 1.1 (Henle et al. [21]). *Let M be a transitive model of $ZF + \omega \rightarrow (\omega)^\omega$ and N be its extension via $([\omega]^\omega, \subseteq^*)$. Then M and N have the same sets of ordinals; moreover every wellordered sequence in N of elements of M lies in M.*

The other key properties utilized in [21] are called LU and EP. LU is a uniformization axiom, guaranteeing a function uniformizing a relation on the Baire space, relative to some infinite set. EP says that the intersection of any well-ordered collection of completely Ramsey positive sets is again completely Ramsey positive. These assumptions are shown to preserve infinite dimensional partition relations of the following type. Given cardinals κ, λ, μ with $2 \leq \mu < \kappa$ and $\lambda \leq \kappa$,

$$\kappa \rightarrow (\kappa)^\lambda_\mu \tag{2}$$

denotes that for each coloring $c : [\kappa]^\lambda \rightarrow \mu$, there is a subset $K \in [\kappa]^\kappa$ such that the restriction of c to $[K]^\lambda$ is constant. An uncountable cardinal κ satisfying (2) for $\lambda = \kappa$ and for every $2 \leq \mu < \kappa$ is called a *strong partition cardinal*.

Theorem 1.2 (Henle et al. [21]). *(ZF + EP + LU) Suppose $0 < \lambda = \omega \cdot \lambda \leq \kappa$, $2 \leq \mu < \kappa$, $\kappa \rightarrow (\kappa)^\lambda_\mu$, and that there is a surjection from $[\omega]^\omega$ onto $[\kappa]^\kappa$. Then in the forcing extension via $([\omega]^\omega, \subseteq^*)$, $\kappa \rightarrow (\kappa)^\lambda_\mu$ holds.*

Forcing with $([\omega]^\omega, \subseteq^*)$ adds an ultrafilter $\mathcal{U}$ which is *Ramsey*, meaning that given any $l, n \geq 2$, $X \in \mathcal{U}$, and coloring $c : [X]^n \rightarrow l$, there is some $U \in \mathcal{U}$ with $U \subseteq X$ such that the restriction of c to $[U]^n$ is constant. This is written as

$$\mathcal{U} \rightarrow (\mathcal{U})^n_l. \tag{3}$$

It is shown in Proposition 4.1 of [21] that the hypotheses, EP + LU, of Theorem 1.2 hold in V if it satisfies AD + $V = L(\mathbb{R})$. In this case, $V[\mathcal{U}]$ preserves the strong partition cardinals in V mentioned in that theorem.

One cannot help but wonder, how important is the Ramseyness of the generic ultrafilter $\mathcal{U}$ for these results? Are there forcings which add non-Ramsey ultrafilters for which the consequences of Theorems 1.1 and 1.2 still hold? The main tools of the proofs, $\omega \rightarrow (\omega)^\omega$, EP and LU, involve properties of $[\omega]^\omega$ as a topological space. Thus, we surmised that other forcings with similar topological properties would likely add ultrafilters with barren extensions. This turned out to be the case. In this paper, we prove that several collections of σ-closed partial orders forcing non-Ramsey ultrafilters produce the same conclusions as Theorems 1.1 and 1.2.

The natural place to look for forcings satisfying analogues of $\omega \rightarrow (\omega)^\omega$ is topological Ramsey spaces, as such spaces, by definition, satisfy analogues of this infinitary partition relation for definable sets. These spaces are defined in Section 2, which provides basic background and their connection with forced ultrafilters. Topological Ramsey spaces have been shown to form dense subsets of many forcings which add ultrafilters satisfying weak partition relations. This includes constructions in [10, 11, 14, 15, 16], which were motivated by questions on exact Rudin–Keisler and Tukey structures below such ultrafilters. An exposition of this area is found in [12]. In this paper, we utilize topological Ramsey space techniques to extend results of Henle, Mathias, and Woodin. We first state the results for general forcing posets.

The following two theorems extend Theorems 1.1 and 1.2. For the notions of extended coarsened poset and the Left-Right Axiom assumed in the next theorem, see Definitions 3.3 and 3.4. We say that *all cubes of a poset* $\langle X, \leq \rangle$ *are Ramsey* if the following holds: Given $x \in X$, a positive integer k, and a coloring $c : \{y \in X : y \leq x\} \to k$, there is some $y \leq x$ such that $c \restriction \{z \in X : z \leq y\}$ is constant.

THEOREM 1.3. *Let M be a transitive model of ZF. In M, let $\langle X, \leq \rangle$ be a forcing poset and assume that $\leq^*$ is a σ-closed coarsening of $\leq$ such that $\langle X, \leq \rangle$ and $\langle X, \leq^* \rangle$ have isomorphic separative quotients. Suppose that this coarsening (or a forcing equivalent one) satisfies the Left-Right Axiom, and suppose all cubes of $\langle X, \leq \rangle$ are Ramsey. Let N be a generic extension of M by the forcing $\langle X, \leq \rangle$. Then M and N have the same sets of ordinals; moreover, every sequence in N of elements of M lies in M.*

In Lemma 3.9, we prove that if either $\mathrm{AD}_{\mathbb{R}}$ or $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ hold, then all subsets of a topological Ramsey space are Ramsey, which makes such spaces a source of natural examples producing barren extensions.

In Section 4, we prove the extension of Theorem 1.2. In the following, LU $(\mathbb{P})$ and EP $(\mathbb{P})$ are generalizations of the axioms LU and EP of Henle, Mathias, and Woodin (see Definitions 4.3 and 4.8). A subset S of a partial ordering $\langle X, \leq \rangle$ is *Ramsey* if for each $p \in X$, there exists a $q \leq p$ such that $\{r \in X : r \leq q\}$ is either contained in or disjoint from S.

THEOREM 1.4. *Suppose $\kappa \to (\kappa)^{\lambda}_{\mu}$, where κ, λ, μ are nonzero ordinals such that $\lambda = \omega\lambda \leq \kappa$ and $2 \leq \mu < \kappa$. Suppose also that there is a surjection from ${}^{\omega}2$ to $[\kappa]^{\kappa}$. Let $\mathbb{P} = \langle X, \leq, \leq^* \rangle$ be a coarsened poset such that EP$(\mathbb{P})$ and each $=^*$-equivalence class is countable. Assume every $S \subseteq X$ is Ramsey. If LU$(\mathbb{P})$ holds and $\langle X, \leq \rangle$ adds no new sets of ordinals, then $\langle X, \leq \rangle$ forces $\kappa \to (\kappa)^{\lambda}_{\mu}$.*

It follows from these theorems that topological Ramsey spaces with natural σ-closed coarsenings force barren extensions containing ultrafilters and preserving the strong partition cardinals in the ground model. By an *axiomatized* topological Ramsey space, we mean one that satisfies the four axioms of Todorcevic in [39] and is closed as a subspace of an appropriate product space (see Definition 2.4). All known examples of topological Ramsey spaces are axiomatized. We will say that a forcing $\mathbb{P}$ is *Ramsey-like* if it is forcing equivalent to some axiomatized topological Ramsey space with an (σ-closed) extended coarsening satisfying the Left-Right Axiom. The next theorem follows from the previous two.

THEOREM 1.5. *Assume M satisfies ZF + either 1) $AD_{\mathbb{R}}$ or 2) $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $\mathbb{P}$ be a Ramsey-like forcing, and let $\mathcal{U}$ be an ultrafilter forced by $\mathbb{P}$. Then M and $M[\mathcal{U}]$ have the same sets of ordinals; moreover, every sequence in $M[\mathcal{U}]$ of elements of M lies in M. If, further, the σ-closed coarsening $\leq^*$ has countable $=^*$-equivalences classes, then $\kappa \to (\kappa)^{\lambda}_{\mu}$ holds in $M[\mathcal{U}]$ whenever $\kappa \to (\kappa)^{\lambda}_{\mu}$ holds in M, where κ, λ, μ are nonzero ordinals such that $\lambda = \omega\lambda \leq \kappa$, $2 \leq \mu < \kappa$, and there is a surjection in M from ${}^{\omega}2$ to $[\kappa]^{\kappa}$.*

Instances of Theorem 1.5 are seen in Sections 5 and 6, where $\mathcal{U}$ ranges over a large collection of non-Ramsey ultrafilters. The following examples are indicative of the

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types of ultrafilters for which our results guarantee barren extensions. First, there are the Milliken–Taylor ultrafilters studied by Mildenberger in [30] which form a hierarchy extending the stable ordered union ultrafilters of Blass in [4]. It is shown in Section 5.1 that these forcings are Ramsey-like.

In Section 5.2 we present a property called *Independent Sequencing* which, when satisfied, guarantees that a forcing is Ramsey-like. All of the forcings in the remainder of Section 5 have this property, and so their generic ultrafilters satisfy Theorem 1.5.

Our second class of forcings, seen in Section 5.3, is a collection of forcings constructed by Laflamme in [25] which extend the forcing $([\omega]^\omega, \subseteq^*)$ by restricting the partial order to produce a hierarchy of ultrafilters $\mathcal{U}_\alpha$, $\alpha < \omega_1$, for which the partition relations get weaker and weaker. Laflamme proved that these ultrafilters form a decreasing chain under Rudin–Keisler reduction of order type $(\alpha+1)^*$, where the minimum ultrafilter is Ramsey, and the next one above it is weakly Ramsey. For each $1 \leq k < \omega$, $\mathcal{U}_k$ satisfies the following partition relation: Given a coloring c of $[\omega]^2$ into finitely many colors, there is a member of $\mathcal{U}_k$ on which c takes no more than $k+1$ colors. All of the ultrafilters $\mathcal{U}_\alpha$ have interesting combinatorial properties, but for α infinite, there are no finite bounds for colorings of pairs.

A third collection of Ramsey-like forcings includes forcings of Baumgartner and Taylor in [2] which generate k-arrow, not $(k+1)$-arrow ultrafilters, as well as a forcing of Blass in [3] producing a p-point with two Rudin–Keisler-incomparable predecessors. In Section 5.4, we present these ultrafilters as well as a general class of forcings constructed in [14] which encompass these as special cases. These forcings are shown to be Ramsey-like, and hence, their forced ultrafilters satisfy Theorem 1.5.

In Section 6, we investigate another line of forcings of stratified complexity over $([\omega]^\omega, \subseteq^*)$. Noting that $([\omega]^\omega, \subseteq^*)$ is forcing equivalent to $\mathcal{P}(\omega)/\mathrm{Fin}$, we work with the natural hierarchy of Boolean algebras $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$, where for $1 \leq \alpha < \beta < \omega_1$, the projection of $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$ to the first β coordinates recovers $\mathcal{P}(\omega^\beta)/\mathrm{Fin}^{\otimes\beta}$. These forcings generate non-p-points for $\alpha \geq 2$, which still satisfy weak partition relations. For example, the generic ultrafilter $\mathcal{G}_2$ forced by $\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}$ satisfies the following partition property: Given a coloring c of $[\omega^2]^2$ into finitely many colors, there is a member of $\mathcal{G}_2$ on which c takes at most four colors. Each of these forcings has been shown to contain dense subsets forming topological Ramsey spaces (see [10, 11]), so Theorem 1.3 holds for these forcings. However, we do not know if they preserve strong partition cardinals, since their $=^*$-equivalence classes have cardinality continuum.

§2. Background: Topological Ramsey spaces, infinite partition relations, and associated ultrafilters.

A key assumption in the results in [21] is the infinitary partition relation $\omega \to (\omega)^\omega$. That this holds in models of $\mathrm{AD}_\mathbb{R}$ or $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R})$ is connected with a topological characterization of the Ramsey property due to Ellentuck. Let τ be the topology generated by basic open sets of the form

$$[a,x] = \{y \in [x]^\omega : a \sqsubset y\},$$

where $a \in [\omega]^{<\omega}$ and $x \in [\omega]^\omega$, and call $[\omega]^\omega$ with this topology the *Ellentuck space*. Here $a \sqsubset y$ means there is an $n \in \omega$ such that $a = \{0,1,\dots,n-1\} \cap y$ (a is an initial

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segment of y). Notice that τ refines the metric topology on the Baire space $[\omega]^\omega$. Culminating a line of work beginning with Nash-Williams as to which subsets of the Baire space satisfy an infinite partition relation (see for instance, [20, 31, 36]) Ellentuck proved the following theorem.

THEOREM 2.1 (Ellentuck, [17]). *A subset $S \subseteq [\omega]^\omega$ has the property of Baire with respect to the Ellentuck topology if and only if the following holds: For each nonempty Ellentuck basic open set $[a,x]$, there is a $y \in [a,x]$ such that either $[a,y] \subseteq S$ or else $[a,y] \cap S = \emptyset$.*

In particular, for each subset $S \subseteq [\omega]^\omega$ with the property of Baire in the Ellentuck topology, for any $x \in [\omega]^\omega$ there is some $y \in [x]^\omega$ such that $[y]^\omega$ is either contained in S or disjoint from S. Thus, $\omega \to (\omega)^\omega$ holds models of ZF where all sets of reals are sufficiently definable.

Carlson and Simpson in [6] extracted properties responsible for infinitary partition relations on more general spaces, for instance, spaces of infinite sequences of finite words, and called such spaces topological Ramsey spaces. Building on their work, Todorcevic presented four axioms in [39] which are responsible for similar infinitary partition relations on a wider array of topological spaces. The next subsection provides the minimal background necessary for understanding this paper.

2.1. Topological Ramsey spaces. Most of the material in this subsection comes from Chapter 5 in [39], with a few new definitions which will help the exposition of this paper. Axioms **A.1**–**A.4** below are defined for triples $(\mathcal{R}, \leq, r)$ of objects with the following properties: $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$, and $r : \mathcal{R} \times \omega \to \mathcal{AR}$ is a map producing the sequence $(r_n(\cdot) = r(\cdot, n))$ of restriction maps, where $\mathcal{AR}$ is the collection of all finite approximations to members of $\mathcal{R}$. For $u \in \mathcal{AR}$ and $X \in \mathcal{R}$,

$$[u, X] = \{Y \in \mathcal{R} : Y \leq X \text{ and } (\exists n)\, r_n(Y) = u\}. \tag{4}$$

A.1 (1) $r_0(X) = \emptyset$ for all $X \in \mathcal{R}$.
(2) $X \neq Y$ implies $r_n(X) \neq r_n(Y)$ for some n.
(3) $r_m(X) = r_n(Y)$ implies $m = n$ and $r_k(X) = r_k(Y)$ for all $k < n$.

Let $\mathcal{AR}_n = \{r_n(X) : X \in \mathcal{R}\}$. It follows from **A.1** (1) that $\mathcal{AR}_0 = \{\emptyset\}$. By **A.1** (3), the sets $\mathcal{AR}_m$ and $\mathcal{AR}_n$ are disjoint whenever $m \neq n$. For each $u \in \mathcal{AR}_n$ and $m \leq n$, let $r_m(u)$ be defined to be $r_m(X)$ where X is any element of $\mathcal{R}$ such that $r_n(X) = u$. Given $u \in \mathcal{AR}$, let $|u|$ denote the length of u. That is, $|u|$ equals the integer n such that $u \in \mathcal{AR}_n$. Said another way, $|u|$ is the integer n such that $r_n(u) = u$. For $u, v \in \mathcal{AR}$, we write $u \sqsubset v$ exactly when $r_{|u|}(v) = u$. For $n > |u|$, let $r_n[u, X]$ denote the collection of all $v \in \mathcal{AR}_n$ such that $u \sqsubset v$ and $v \leq_{\text{fin}} X$.

A.2 There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that

(1) $\{v \in \mathcal{AR} : v \leq_{\text{fin}} u\}$ is finite for all $u \in \mathcal{AR}$,
(2) $Y \leq X$ iff $(\forall n)(\exists m)\, r_n(Y) \leq_{\text{fin}} r_m(X)$,
(3) $\forall u, v, y \in \mathcal{AR}[y \sqsubset v \wedge v \leq_{\text{fin}} u \to \exists x \sqsubset u\ (y \leq_{\text{fin}} x)]$.

Given an $X \in \mathcal{R}$, we define $\mathcal{AR} \restriction X$ to be the set of all finite approximations to members $Y \in \mathcal{R}$ such that $Y \leq X$. Note that **A.2**(1) and (2) imply that for any $X \in \mathcal{R}$,

$$\mathcal{AR} \upharpoonright X = \{u \in \mathcal{AR} : (\exists m)\, u \leq_{\text{fin}} r_m(X)\} \tag{5}$$

and hence, $\mathcal{AR} \upharpoonright X$ is countable. This fact will be important throughout the paper.

The number $\text{depth}_X(u)$ is the least n, if it exists, such that $u \leq_{\text{fin}} r_n(X)$. If such an n does not exist, then we write $\text{depth}_X(u) = \infty$. If $\text{depth}_X(u) = n < \infty$, then $[\text{depth}_X(u), X]$ denotes $[r_n(X), X]$.

A.3 (1) If $\text{depth}_X(u) < \infty$ then $[u, Y] \neq \emptyset$ for all $Y \in [\text{depth}_X(u), X]$.
(2) $Y \leq X$ and $[u, Y] \neq \emptyset$ imply that there is $Y' \in [\text{depth}_X(u), X]$ such that $\emptyset \neq [u, Y'] \subseteq [u, Y]$.

A.4 If $\text{depth}_X(u) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{|u|+1}$, then there is $Y \in [\text{depth}_X(u), X]$ such that $r_{|u|+1}[u, Y] \subseteq \mathcal{O}$ or $r_{|u|+1}[u, Y] \subseteq \mathcal{O}^c$.

Axiom **A.1** implies that the map $\imath : \mathcal{R} \to \prod_{n<\omega} \mathcal{AR}_n$ defined by

$$\imath(X) = \langle r_n(X) : n < \omega \rangle, \tag{6}$$

for $X \in \mathcal{R}$, is an injection. Note that $\prod_{n<\omega} \mathcal{AR}_n$ is a subspace of $\mathcal{AR}^{\mathbb{N}}$, where the set $\mathcal{AR}$ is given the discrete topology; this slightly streamlined notation us commonly used in topological Ramsey space theory. Recalling the remark after axiom **A.2**, we may assume without loss of generality that $\mathcal{AR}$ is countable, especially since in practice, we will always be working below some member of $\mathcal{R}$.

The Ellentuck space is a good reference point for digesting this notation. In the Ellentuck space, $\mathcal{AR}_n$ is the set of increasing sequences of length n where the entries are natural numbers. Then $\imath[\mathcal{R}]$ is the subset of $\prod_{n<\omega} \mathcal{AR}_n$ consisting of all infinite sequences of finite sequences which cohere: For $X = \{x_0, x_1, x_2, \dots\} \in [\omega]^\omega$ enumerated in increasing order,

$$\imath(X) = \langle r_n(X) : n < \omega \rangle = \langle \emptyset, \{x_0\}, \{x_0, x_1\}, \{x_0, x_1, x_2\}, \dots \rangle. \tag{7}$$

Observe that the sequence on the right recovers X in its limit.

The *metric topology* on $\mathcal{R}$ is the topology generated by basic open sets of the form $\{X \in \mathcal{R} : r_n(X) = u\}$, where $n < \omega$ and $u \in \mathcal{AR}_n$. This corresponds to the topology on $\imath[\mathcal{R}]$ inherited as a subspace of $\mathcal{AR}^{\mathbb{N}}$, where the countable set $\mathcal{AR}$ has the discrete topology and $\mathcal{AR}^{\mathbb{N}}$ has the product (i.e., Tychonoff) topology. When we speak about $\mathcal{R}$ being *closed* in $\mathcal{AR}^{\mathbb{N}}$, we mean that the $\imath$-image of $\mathcal{R}$ is a closed subspace of $\mathcal{AR}^{\mathbb{N}}$.

The *Ellentuck topology* on $\mathcal{R}$ is the topology generated by the basic open sets $[u, X]$; it refines the metric topology on $\mathcal{R}$. Given the Ellentuck topology on $\mathcal{R}$, the notions of nowhere dense, and hence of meager are defined in the natural way. We say that a subset $\mathcal{X}$ of $\mathcal{R}$ has the *property of Baire* iff there is an open Ellentuck open set $\mathcal{O}$ such that the symmetric difference of $\mathcal{X}$ and $\mathcal{O}$ is meager.

Definition 2.2 ([39]). A subset $\mathcal{X}$ of $\mathcal{R}$ is *Ramsey* if for every $\emptyset \neq [u, X]$, there is a $Y \in [u, X]$ such that $[u, Y] \subseteq \mathcal{X}$ or $[u, Y] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey null* if for every $\emptyset \neq [u, X]$, there is a $Y \in [u, X]$ such that $[u, Y] \cap \mathcal{X} = \emptyset$.

A triple $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* if every subset of $\mathcal{R}$ with the property of Baire is Ramsey and if every meager subset of $\mathcal{R}$ is Ramsey null.

The following result is Theorem 5.4 in [39].

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THEOREM 2.3 (Abstract Ellentuck Theorem). *If* $(\mathcal{R}, \leq, r)$ *is closed* (*as a subspace of* $\mathcal{AR}^{\mathbb{N}}$) *and satisfies axioms* **A.1**, **A.2**, **A.3**, *and* **A.4**, *then every subset of* $\mathcal{R}$ *with the property of Baire is Ramsey, and every meager subset is Ramsey null*; *in other words, the triple* $(\mathcal{R}, \leq, r)$ *forms a topological Ramsey space.*

Rather than repeating the hypotheses of the Abstract Ellentuck Theorem many times throughout this paper, we will simply make the following definition.

DEFINITION 2.4. We say that a topological Ramsey spaces is *axiomatized* if it is closed as a subspace of $\mathcal{AR}^{\mathbb{N}}$ and satisfies axioms **A.1**, **A.2**, **A.3**, and **A.4**.

EXAMPLE 2.5. The *Ellentuck space* is the triple $([\omega]^\omega, \subseteq, r)$, where for each $X \in [\omega]^\omega$ and $n < \omega$, $r_n(X)$ denotes the set of the n least members of X. Here, $\leq_{\text{fin}}$ is simply the subset relation.

The Ellentuck space is the prototypical topological Ramsey space. All known examples of topological Ramsey spaces contain a copy of this space. Notice that $\subseteq^*$ is a σ-closed quasi-order coarsening the partial order $\subseteq$ on the Ellentuck space. This forcing $([\omega]^\omega, \subseteq^*)$ produces a Ramsey ultrafilter on base set ω, which is in one-to-one correspondence with $[\omega]^1$, also known as the set of first approximations of members of the Ellentuck space.

2.2. Forcing with topological Ramsey spaces, and the ultrafilters they generate. Given a topological Ramsey space $(\mathcal{R}, \leq, r)$, there are three related forcings. The first is the Mathias-like forcing, where conditions are of the form $[s, A]$ where $s \in \mathcal{AR}$, $A \in \mathcal{R}$, and $s \sqsubset A$. The second is $\langle \mathcal{R}, \leq \rangle$. The third is $\langle \mathcal{R}, \leq^* \rangle$ where $\leq^*$ is some σ-closed partial order which coarsens $\leq$ such that the separative quotients of $\langle \mathcal{R}, \leq \rangle$ and $\langle \mathcal{R}, \leq^* \rangle$ are isomorphic. These forcings were shown to have many properties in common with Mathias forcing in [8, 29]. Similarly to $\langle [\omega]^\omega, \subseteq^* \rangle$, forcing with $\langle \mathcal{R}, \leq^* \rangle$ adds a new ultrafilter on a countable base set as follows: By Axiom **A.2**, relativizing below some member of $\mathcal{R}$ if necessary, the collection $\mathcal{AR}_1$ of all first approximations to members of $\mathcal{R}$ is a countable set. We shall let

$$\mathcal{U}_{\mathcal{R}} = \{ Y \subseteq \mathcal{AR}_1 : \exists X \in G \, (\mathcal{AR}_1 \upharpoonright X \subseteq Y) \}, \tag{8}$$

where G is some generic filter forced by $\langle \mathcal{R}, \leq^* \rangle$. By genericity and the Abstract Ellentuck Theorem 2.3, one sees that $\mathcal{U}_{\mathcal{R}}$ is an ultrafilter on base set $\mathcal{AR}_1$. For all known topological Ramsey spaces, the collection $\mathcal{AR}_1$ of first approximations is a countable set. If not, then by Axiom **A.2**, restricting below some member $Z \in \mathcal{R}$ provides a countable set $\mathcal{AR}_1 \upharpoonright Z$. If $\langle \mathcal{R}, \leq^* \rangle$ is isomorphic to a dense subset of some σ-closed forcing $\mathbb{P}$ which forces a new ultrafilter, then the ultrafilter forced by $\mathbb{P}$ is isomorphic to $\mathcal{U}_{\mathcal{R}}$. In this way, when we have a forcing $\mathbb{P}$ with a Ramsey space as a dense subset, then results for forcing with $\mathbb{P}$ reduce to results for ultrafilters forced by Ramsey spaces.

Ultrafilters $\mathcal{U}_{\mathcal{R}}$ forced by topological Ramsey spaces satisfy interesting partition relations.

DEFINITION 2.6. Given an ultrafilter $\mathcal{U}$ on a countable base set S, for each $n \geq 2$, we define $t(\mathcal{U}, n)$ to be the least number t, if it exists, such that for any $\ell \geq 2$ and any coloring $c : [S]^n \to \ell$, there is a member $U \in \mathcal{U}$ such that $c \upharpoonright [U]^n$ takes no more than

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t colors. When this $t(\mathcal{U},n)$ exists, the standard notation is to write

$$\mathcal{U}\to(\mathcal{U})^n_{\ell,t(\mathcal{U},n)},$$

and call $t(\mathcal{U},n)$ the *Ramsey degree* of $\mathcal{U}$ for n.

Recall that $\mathcal{U}$ is a Ramsey ultrafilter if and only if $t(\mathcal{U},2)=1$ if and only if $t(\mathcal{U},n)=1$ for all $n\geq 2$. The ultrafilters for which our results hold all have $t(\mathcal{U},2)\geq 2$.

The role of topological Ramsey spaces in this paper is several-fold. First, we will need forcings which satisfy analogues of $\omega\to(\omega)^\omega$, and topological Ramsey spaces are natural candidates because of the Abstract Ellentuck Theorem 2.3. Second, many topological Ramsey spaces behave similarly to the Baire space in contexts of $\mathrm{AD}_\mathbb{R}$, or AD^+, or the Solovay model. Third, in previous papers, many known forcings producing ultrafilters with interesting Ramsey degrees were shown to contain dense sets forming topological Ramsey spaces (see [9, 10, 11, 14, 15, 16]). The original motivation for these constructions was to find exact Rudin–Keisler and Tukey structures below these forced ultrafilters, as Ramsey space structure and Theorem 2.3 make such results possible. See [12] for an overview of those results. These topological Ramsey spaces provided a variety of ultrafilters which were likely to have barren extensions. The topological Ramsey space approach makes the results in this paper possible, while streamlining proofs and making the results applicable to a variety of known forcings.

§3. No new sets of ordinals. We begin this section by defining the notion of extended coarsened posets. These are used in the definition of the Left-Right Axiom, which abstracts the key property of the forcing $([\omega]^\omega,\subseteq^*)$ which Henle, Mathias, and Woodin used in the proof of Theorem 1.1. This axiom along with the assumption of infinite dimensional partition relations will allow us to prove the more general Theorem 1.3. After proving in Lemma 3.9 that, assuming some determinacy, all subsets of a topological Ramsey space are Ramsey, we conclude this section with Theorem 3.10. It follows that a large collection of ultrafilters produce barren extensions, as will be discussed in Sections 5 and 6.

In what follows, given a quasi-order $\leq^*$, we write $x=^*y$ iff $x\leq^*y$ and $y\leq^*x$.

Definition 3.1. A *coarsened poset* $\mathbb{P}$ is a triple $\mathbb{P}=\langle X,\leq,\leq^*\rangle$ where $\leq$ is a partial order and $\leq^*$ is a quasi-order on X with the following properties:

1) For all $x,y\in X$, $x\leq y$ implies $x\leq^*y$;
2) For all $x,y\in X$, if $y\leq^*x$ then there is some $z\leq x$ such that $z=^*y$.

Given $x\in X$, define the following notation:

$$[x]:=\{y\in X:y\leq x\}$$
$$[x]^*:=\{y\in X:y\leq^*x\}.$$

Observation 3.2. *If $\mathbb{P}=\langle X,\leq,\leq^*\rangle$ is a coarsened poset, then the separative quotients of $\langle X,\leq\rangle$ and $\langle X,\leq^*\rangle$ are isomorphic, so we say that $\langle X,\leq\rangle$ and $\langle X,\leq^*\rangle$ are forcing equivalent.*

Proof. For $x,y\in X$, write $x\|y$ if x and y are compatible in $\langle X,\leq\rangle$, and write $x\|^*y$ if x and y are compatible in $\langle X,\leq^*\rangle$. If $x\|y$, then there is some $z\in X$ such

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that $z \leq x$ and $z \leq y$. Then $z \leq^* x$ and $z \leq^* y$, so $x \|^* y$. On the other hand if $x \|^* y$, then there is some $z \in X$ such that $z \leq^* x$ and $z \leq^* y$. Using the fact that $z \leq^* x$ and letting z play the role of y in 2) in Definition 3.1, we see that there is some $z' \leq x$ such that $z' =^* z$. By 1) and transitivity of $\leq^*$, $z' \leq^* y$, so by 2) there is some $z'' \leq y$ such that $z'' =^* z'$. In particular, z'' witnesses that $x \| y$. ⊣

A typical example of a coarsened poset has the form $\langle \mathcal{H}, \subseteq, \subseteq^{\mathcal{I}} \rangle$ where $\mathcal{I}$ is an ideal on ω and $\mathcal{H}$ is the coideal $\{X \subseteq \omega : X \notin \mathcal{I}\}$. Another typical example is a topological Ramsey space $\langle \mathcal{R}, \leq, \leq^* \rangle$, where $\leq$ is the partial-order on $\mathcal{R}$ and $\leq^*$ is a σ-closed quasi-order coarsening $\leq$, where $\langle \mathcal{R}, \leq \rangle$ and $\langle \mathcal{R}, \leq^* \rangle$ have the same separative quotient.

Definition 3.3. Given a coarsened poset $\mathbb{P} = \langle X, \leq, \leq^* \rangle$ and a quasi-ordered set $\mathbb{P}^* = \langle X^*, \prec \rangle$, we say that $\mathbb{P}^*$ is an *equivalent extension* of $\langle X, \leq^* \rangle$ iff the following hold:

1) $X \subseteq X^*$ and $\prec \restriction (X \times X)$ equals $\leq^*$; and
2) $\langle X, \leq^* \rangle$ is a dense subset of $\langle X^*, \prec \rangle$.

In this case, we write $\mathbb{P}^* = \langle X, X^*, \leq, \leq^* \rangle$, and say that $\mathbb{P}^*$ is an *extended coarsened poset*, or *EC poset* (we write $\leq^*$ for $\prec$). Given an EC poset, for $x \in X^*$ define the notation:

$$X^*[x]^* := \{y \in X^* : y \leq^* x\}.$$

In some cases X^* will simply be X, but for many of our applications in Sections 5 and 6, we shall need the flexibility of EC posets. Notice that 2) in Definition 3.3 and Observation 3.2 imply that $\langle X, \leq \rangle$, $\langle X, \leq^* \rangle$, and $\langle X^*, \leq^* \rangle$ are forcing equivalent.

Definition 3.4. Let $\mathbb{P}^* = \langle X, X^*, \leq, \leq^* \rangle$ be an extended coarsened poset. We say that $\mathbb{P}^*$ *satisfies the Left-Right Axiom* (*LRA*) iff there are functions $\text{Left} : X \to X^*$ and $\text{Right} : X \to X^*$ such that the following are satisfied:

1) For each $x \in X$, we have $\text{Left}(x), \text{Right}(x) \leq^* x$.
2) For each $x \in X$, there are $y, z \in [x]$ such that
   2a) $\text{Left}(y) =^* \text{Right}(z)$;
   2b) $\text{Right}(y) =^* \text{Left}(z)$;
3) Given $p \in X$, for each $x, y \in [p]$, there is some $z \in [p]$ such that
   3a) $\text{Left}(z) \leq^* x$;
   3b) $\text{Left}(\text{Right}(z)) \leq^* x$;
   3c) $\text{Right}(\text{Right}(z)) \leq^* y$;

We say that *all cubes of a poset* $\langle X, \leq \rangle$ *are Ramsey* if the following holds: Given $x \in X$, a positive integer k, and a coloring $c : [x] \to k$, there is some $y \leq x$ such that $c \restriction [y]$ is constant.

Theorem 1.3. *Let M be a transitive model of ZF. In M, let $\mathbb{P} = \langle X, X^*, \leq, \leq^* \rangle$ be an extended coarsened poset satisfying the Left-Right Axiom, and assume that all cubes of $\langle X, \leq \rangle$ are Ramsey. Let N be a generic extension of M by the forcing $\langle X, \leq^* \rangle$. Then M and N have the same sets of ordinals; moreover, every sequence in N of elements of M lies in M.*

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Proof. Recall that $\mathbb{P}^*$ being an EC poset means that $\langle X, \leq\rangle$, $\langle X, \leq^*\rangle$, and $\langle X^*, \leq^*\rangle$ have isomorphic separative quotients. Formally, we shall force with $\langle X^*, \leq^*\rangle$, and the forcing relation $\Vdash$ refers to this quasi-order.

It suffices to show that given any fixed $p_0 \in X^*$, $\dot{f}$, and ordinal λ satisfying $p_0 \Vdash \dot{f} : \check{\lambda} \to \check{M}$, there is some $q \in X^*[p_0]^*$ satisfying $q \Vdash \dot{f} \in \check{M}$. We will in fact find such a q in X. Assume towards a contradiction that for some such $p_0 \in X^*$ with $p_0 \Vdash \dot{f} : \check{\lambda} \to \check{M}$, there is no $q \in X^*[p_0]^*$ such that $q \Vdash \dot{f} \in \check{M}$. Then for each $p \in X^*[p_0]^*$, there is a least ordinal $\varphi(p) < \lambda$ such that p does not decide $\dot{f}(\check{\varphi}(p))$; that is, $(\forall u \in M)\, p \nVdash \dot{f}(\check{\varphi}(p)) = \check{u}$. Notice that φ is invariant, meaning that whenever $x, y \in X^*$ satisfy $x =^* y$, then $\varphi(x) = \varphi(y)$. Since $\langle X, \leq^*\rangle$ is dense in $\langle X^*, \leq^*\rangle$, take some $p_1 \in X$ such that $p_1 \leq^* p_0$. Define the coloring $c : [p_1] \to 3$ as follows: For $p \in [p_1]$, let

$$c(p) = \begin{cases} 0 & \text{if } \varphi(\text{Left}(p)) < \varphi(\text{Right}(p)), \\ 1 & \text{if } \varphi(\text{Left}(p)) = \varphi(\text{Right}(p)), \\ 2 & \text{if } \varphi(\text{Left}(p)) > \varphi(\text{Right}(p)). \end{cases}$$

By the hypotheses, there is some $p_2 \in [p_1]$ such that $[p_2]$ is homogeneous for c; that is, $c \restriction [p_2]$ is constant. We claim that $c(p_2) = 1$.

Suppose towards a contradiction that $c(p_2) = 0$. Take $y, z \in [p_2]$ satisfying 2a) and 2b) of the Left-Right Axiom. Since $c(y) = c(p_2) = 0$,

$$\varphi(\text{Left}(y)) < \varphi(\text{Right}(y)).$$

Since φ is invariant under $=^*$, by 2a) and 2b) of the LRA, we have

$$\varphi(\text{Right}(z)) = \varphi(\text{Left}(y)) \text{ and } \varphi(\text{Right}(y)) = \varphi(\text{Left}(z)).$$

Thus,

$$\varphi(\text{Right}(z)) < \varphi(\text{Left}(z)),$$

so $c(z) = 2$, a contradiction to $c \restriction [p_2]$ being constant. A similar argument shows that $c(p_2) \neq 2$.

Since p_2 does not decide the value of $\dot{f}(\varphi(\check{p}_2))$, there are $x^*, y^* \in X^*[p_2]^*$ and $u \neq v$ in M such that $x^* \Vdash \dot{f}(\varphi(\check{p}_2)) = \check{u}$ and $y^* \Vdash \dot{f}(\varphi(\check{p}_2)) = \check{v}$. Since $\langle X, \leq^*\rangle$ is dense in $\langle X^*, \leq^*\rangle$, there are $x', y' \in X$ with $x' \leq^* x^*$ and $y' \leq^* y^*$; in particular, $x', y' \leq^* p_2$. By 2) of the definition of coarsened poset, there are $x, y \in X$ such that $x \leq p_2$ and $x =^* x'$, and $y \leq p_2$ and $y =^* y'$. Thus, we have $x, y \in [p_2]$ such that $x \Vdash \dot{f}(\varphi(\check{p}_2)) = \check{u}$ and $y \Vdash \dot{f}(\varphi(\check{p}_2)) = \check{v}$. Fix some $z \in [p_2]$ satisfying 3) of the LRA with regard to x and y. By 3a) we have that $\text{Left}(z) \leq^* x$, which implies that

$$\varphi(\text{Left}(z)) \geq \varphi(x).$$

At the same time, $\varphi(x) > \varphi(p_2)$, so

$$\varphi(\text{Left}(z)) > \varphi(p_2).$$

Further, $\text{Right}(z) \leq^* z \leq p_2$ implies that $\varphi(\text{Right}(z)) \geq \varphi(p_2)$. At the same time, 3b) and 3c) of LRA imply that $\text{Right}(z)$ is $\leq^*$-compatible with both x and y, so $\text{Right}(z)$ does not determine the value of $\dot{f}(\varphi(p_2))$; hence, $\varphi(\text{Right}(z)) \leq \varphi(p_2)$. Thus,

$$\varphi(\text{Right}(z)) = \varphi(p_2).$$

It follows that

$$\varphi(\text{Left}(z)) > \varphi(p_2) = \varphi(\text{Right}(z)),$$

implying that $c(z) = 2$, contradicting that c has constant value 1 on $[p_2]$. ⊣

At this point, let us explain one of the main ways to show that every subset of a topological Ramsey space is Ramsey. Recall that a subset of a Polish space is Polish if and only if it is G_δ. We will need the following:

DEFINITION 3.5. Let X be a Polish space and $\mathbb{P} = (X, \leq)$ be a poset with the property that the subspace $\{(x,y) \in X \times X : x \leq y\}$ of $X \times X$ is also Polish. We say that this space is *projectively presented* iff there is an injection $\eta : X \to {}^\omega\omega$ such that the following hold:

1) $\text{Im}(\eta)$ is projective.
2) $\{(u,v) \in \text{Im}(\eta) \times \text{Im}(\eta) : \eta^{-1}(u) \leq \eta^{-1}(v)\}$ is projective.
3) Given $p \in X$ and a function $f : [p] \to {}^\omega\omega$ that is continuous with respect to the metric topology on X, then the relation $S \subseteq {}^\omega\omega \times {}^\omega\omega$ is $\mathbf{\Sigma}^1_1$, where

$$S = \{(u, f(\eta^{-1}(u))) : u \in \text{Im}(\eta) \text{ and } \eta^{-1}(u) \leq p\}.$$

4) For every $p \in X$,
   4a) for every continuous $f : [p] \to {}^\omega 2$, the function $f \circ \eta^{-1} : \text{Im}(\eta) \to {}^\omega 2$ is Σ^1_1;
   4b) the following set is projective (uniformly in $\eta(p)$): the set of codes $c \in {}^\omega\omega$ for $\mathbf{\Sigma}^1_1$ functions $g : {}^\omega\omega \to {}^\omega 2$ such that $g \circ \eta : [p] \to {}^\omega 2$ is continuous.

Recalling Definition 2.4, we say that a triple $\langle \mathcal{R}, \leq, r\rangle$ is an *axiomatized topological Ramsey space* if it is closed as a subspace of $\mathcal{AR}^{\mathbb{N}}$ and satisfies axioms **A.1**–**A.4.** In most topological Ramsey spaces, the set $\mathcal{AR}$ of all finite approximations of members of the space is countable. In the unlikely case that it is not, Axiom **A.2** guarantees that relativizing below any member $p \in \mathcal{R}$, the set $\mathcal{AR} \restriction p$ is countable. Thus, without loss of generality, we shall assume that $\mathcal{AR}$ is countable. Assuming countable choice for sets of reals, each axiomatized topological Ramsey space is projectively presented with the following particularly simple form.

LEMMA 3.6. *Assuming countable choice for sets of reals, each axiomatized topological Ramsey $\langle \mathcal{R}, \leq, r\rangle$ is projectively presented. In fact, there is a bijection $\eta : \mathcal{R} \to {}^\omega\omega$ which is continuous with respect to the metric topology on $\mathcal{R}$ so that the set in* 2) *is analytic, and* 3) *and* 4) *are true since η is a continuous bijection.*

PROOF. Either $\mathcal{AR}$ is countable, or else fix any $p \in \mathcal{R}$ and relativize the proof to $\mathcal{R} \restriction p$. For $a \in \mathcal{AR}$, let

$$E(a) = \{b \in \mathcal{AR}_{|a|+1} : r_{|a|}(b) = a\}. \tag{9}$$

By countable choice for reals, there is a set of bijections

$$\eta_a : E(a) \to \omega, \ \ a \in \mathcal{AR}. \tag{10}$$

By definition, $\mathcal{AR}_0 = \{r_0(x) : x \in \mathcal{R}\}$, which has $\emptyset$ as its only member. Define $\eta : \mathcal{R} \to {}^\omega\omega$ as follows: Given $p \in \mathcal{R}$, define

$$\eta(p) = \langle \eta_{r_n(p)}(r_{n+1}(p)) : n < \omega\rangle. \tag{11}$$

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This function η is continuous with respect to the metric topology on $\mathcal{R}$, and it is a bijection onto ${}^{\omega}\omega$; thus, 1) trivially holds. Furthermore, the set of all pairs $(p,q) \in \mathcal{R} \times \mathcal{R}$ with $p < q$ is a Polish subspace of $\mathcal{R} \times \mathcal{R}$, since $\mathcal{R}$ is a closed subspace of $\mathcal{AR}^{\mathbb{N}}$. Thus, the continuous image of this set by η is analytic, showing that 2) holds.

To show 3), let $p \in \mathcal{R}$ be fixed, and let $f : [p] \to {}^{\omega}\omega$ be a continuous function. Then the relation S in 3) is certainly Σ^1_1, since $f \circ \eta^{-1}$ is a continuous function on $[p]$, which is a Polish space since it is a closed subset of $\mathcal{R}$. Condition 4a) is trivial, since η^{-1} is a continuous bijection. Likewise, 4b) holds. $\dashv$

Definition 3.7. Σ^2_1-*reflection* is the statement that given any Σ^2_1 formula φ, if φ is witnessed by some $A \subseteq \mathbb{R}$, then $\varphi(A)$ is witnessed by some $A \subseteq \mathbb{R}$ that is Suslin coSuslin.

$AD_{\mathbb{R}}$ implies that every set of reals is Suslin coSuslin. It is also well known that the axiom $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ implies Σ^2_1-reflection. See, for example, [37] and Theorem 25 in [41]. We will use this shortly.

The following lemma appears in a modified form in [18] in Theorem 2.2, where it is shown that assuming ZFC, every universally Baire set of reals is Ramsey using a countable elementary substructure argument. Instead of a countable model, we use an inner model M such that ω_1 is inaccessible in M. Given a cardinal κ and a tree $T \subseteq {}^{<\omega}(\omega \times \kappa)$, let

$$p[T] := \{x \in {}^{\omega}\omega : (\exists y \in {}^{\omega}\kappa)(\forall n \in \omega)\, \langle (x(0),y(0)),\dots,(x(n),y(n)) \rangle \in T\}.$$

Lemma 3.8. *Assume there is no injection of ω_1 into $\mathbb{R}$. Let $A \subseteq [\omega]^{\omega}$. Let $e : [\omega]^{\omega} \to {}^{\omega}\omega$ be the function that maps each $q \in [\omega]^{\omega}$ to its increasing enumeration. Assume $A' := \{e(q) : q \in A\} \subseteq {}^{\omega}\omega$ is Suslin, meaning there is a cardinal κ and a tree $T \subseteq {}^{<\omega}(\omega \times \kappa)$ such that $A' = p[T]$. Then A is Ramsey.*

Proof. Let $[u,q] \subseteq [\omega]^{\omega}$ be a basic open neighborhood in the Ellentuck topology. We will find a $g \in [u,q]$ such that either $[u,g] \subseteq A$ or $[u,g] \cap A = \emptyset$. Let $M = L[T,q]$ be the inner model generated by T and q. It satisfies the Axiom of Choice (because T and q can be coded by a set of ordinals), and so since there is no injection of ω_1 into $\mathbb{R}$, it must be that ω_1 is inaccessible in M.

By the nature of tree representations, if N is any inner model which contains M, then $A' \cap N = (p[T])^N$. Now let $\mathbb{P} \in M$ be the Mathias forcing of M. We have $[u,q] \in M$. Let $\dot{g} \in M$ be the canonical name for the generic real, so $1 \Vdash \dot{g} \in [\omega]^{\omega}$. Consider the statement "$e(\dot{g}) \in p[\check{T}]$". Since $\mathbb{P}$ has the Prikry property, there is an $s \in [u,q]$ such that the condition $[u,s]$ decides the statement to be either true or false. Assume for now that

$$[u,s] \Vdash e(\dot{g}) \in p[\check{T}].$$

Now since ω_1 is inaccessible in M, fix a real $g \in [\omega]^{\omega}$ that is $\mathbb{P}$-generic over M such that $g \in [u,s]$. But one property of Mathias forcing is "the Mathias property" (see [26, 27]). In this case, it tells us that every $h \in [u,g]$ (in V) is $\mathbb{P}$-generic over M. And so the condition $[u,s]$ forces each $h \in [u,g]$ (in V) to be such that $e(h) \in p[T] = A'$, so $h \in A$. Hence, $[u,g] \subseteq A$.

The proof for the case that $[u,s] \Vdash e(\dot{g}) \notin p[\check{T}]$ is similar. $\dashv$

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If there exists a supercompact cardinal, then in $L(\mathbb{R})$ every subset of a topological Ramsey space is Ramsey. This can be shown by considering a topological Ramsey space $\mathcal{R} \in L(\mathbb{R})$ and a set $S \subseteq \mathcal{R}$ in $L(\mathbb{R})$. Fix a continuous bijection $\eta : \mathcal{R} \to {}^{\omega}\omega$ in $L(\mathbb{R})$. We see that $f(S)$ is 2^{ω}-universally Baire in V (because large cardinals imply every subset of ${}^{\omega}\omega$ in $L(\mathbb{R})$ is 2^{ω}-universally Baire). Next $f^{-1}(f(S)) = S$ has the property of Baire in $\mathcal{R}$. Then we apply the Abstract Ellentuck Theorem 2.3. In the next theorem, we show that this follows directly from either $\mathrm{AD}_{\mathbb{R}}$ or else $\mathrm{AD}^{+} + V = L(\mathcal{P}(\mathbb{R}))$.

Lemma 3.9. *Assume either* 1) $AD_{\mathbb{R}}$ *or* 2) $AD^{+} + V = L(\mathcal{P}(\mathbb{R}))$. *Let* $\langle \mathcal{R}, \leq, r\rangle$ *be an axiomatized topological Ramsey space. Then every* $S \subseteq \mathcal{R}$ *is Ramsey.*

Proof. First, suppose there is a counterexample $(\langle \mathcal{R}, \leq, r\rangle, \mathcal{S})$. Since $\mathcal{R}$ is axiomatized, there is a continuous bijection $\eta : \mathcal{R} \to {}^{\omega}\omega$ as in Lemma 3.6. Then there must be a counterexample of the form $(\langle \mathcal{R}, \leq, r\rangle, S)$ such that there is a bijection $\eta : \mathcal{R} \to {}^{\omega}\omega$ such that

- $\{(x,y) \in {}^{\omega}\omega \times {}^{\omega}\omega : \eta^{-1}(x) = \eta^{-1}(y)\}$ is Suslin,
- $\{(x,y) \in {}^{\omega}\omega \times {}^{\omega}\omega : \eta^{-1}(x) \leq \eta^{-1}(y)\}$ is Suslin,
- the set coding the r function is Suslin, and
- $\{x : \eta^{-1}(x) \in S\}$ is Suslin.

Here is why: if we have 1), then every set of reals is Suslin. If we have 2), then by Σ^2_1 reflection, if there were (a set of reals coding) a counterexample, there would be one that is Suslin coSuslin. We will now show that if $\mathcal{Z} \subseteq {}^{\omega}\omega$ is a set of reals coding a $(\langle \mathcal{R}, \leq, r\rangle, S)$, then in fact $S \subseteq \mathcal{R}$ is Ramsey.

We now argue just as in the previous lemma. Fix $[u,q]$. We will find a $g \in [u,q]$ such that either $[u,g] \subseteq S$ or $[u,g] \cap S = \emptyset$. Let κ be a cardinal and $T \subseteq {}^{<\omega}(\omega \times \kappa)$ be a tree such that $\mathcal{Z} = p[T]$. Let M be the inner model $L[T,g]$. Again T and g can be coded as sets of ordinals. Both 1) and 2) imply there is no injection of ω_1 into $\mathbb{R}$, so ω_1 is inaccessible in M. Note that in any inner model N containing M, T can be used to talk about $\mathcal{R} \cap N$ and $\mathcal{S} \cap N$. For example, given any $s \in \mathcal{R} \cap N$, N knows whether or not $s \in \mathcal{S}$.

Let $\mathbb{P}$ be the Mathias forcing associated with $\mathcal{R}$ in M. Conditions are nonempty basic open sets $[a,q]$ where $a \in \mathcal{AR}$ and $q \in \mathcal{R}$. The ordering is $[a,q] \leq [b,s]$ iff $[a,q] \subseteq [b,s]$. Let $\dot{g}$ be the name for the generic object, so $1 \Vdash \dot{g} \in \mathcal{R}$. The forcing $\mathbb{P}$ has the Prikry property (see Theorem 6.7 in [8]). So fix $s \in [a,q]$ such that either $[a,s] \Vdash \dot{g} \in \mathcal{S}$ or $[a,s] \Vdash \dot{g} \notin \mathcal{S}$. Without loss of generality, assume the former.

Since ω_1 is inaccessible in M, fix a $g \in \mathcal{R}$ that is $\mathbb{P}$-generic over M such that $g \in [a,s]$. But $\mathbb{P}$ also has the Mathias property (see Theorem 6.24 in [8]). So every $h \in [a,g]$ (in V) is $\mathbb{P}$-generic over M. So $[a,s]$ forces each $h \in [a,g]$ (in V) to be such that $h \in \mathcal{S}$. Hence $[a,g] \subseteq \mathcal{S}$. ⊣

In the following theorem, recall the definition at the end of Section 2.2 of the ultrafilter $\mathcal{U}_{\mathcal{R}}$ forced by $\langle \mathcal{R}, \leq^{*}\rangle$.

Theorem 3.10. *Assume that either* 1) $AD_{\mathbb{R}}$ *holds in* V *or* 2) $AD^{+} + V = L(\mathcal{P}(\mathbb{R}))$. *Let* $\langle \mathcal{R}, \leq, r\rangle$ *be an axiomatized topological Ramsey space and* $\leq^{*}$ *be a* σ*-closed coarsening of* $\mathcal{R}$. *Suppose there is some extended coarsening* $\langle \mathcal{R}, \mathcal{R}^{*}, \leq, \leq^{*}\rangle$ *satisfying the Left-Right Axiom, and let* $\mathcal{U}_{\mathcal{R}}$ *be the ultrafilter forced by* $\langle \mathcal{R}, \leq^{*}\rangle$ *over* V. *Then*

V and $V[\mathcal{U}_{\mathcal{R}}]$ have the same sets of ordinals; moreover, every sequence in $V[\mathcal{U}_{\mathcal{R}}]$ of elements of V lies in V.

PROOF. By the previous lemma, all subsets of $\mathcal{R}$ are Ramsey. Hence it holds that for any $x \in \mathcal{R}$, $k \geq 1$ and coloring $c : [x] \to k$, there is some $y \in [x]$ such that $c \restriction [y]$ is constant. The rest follows from Theorem 1.3. ⊣

The first half of Theorem 1.5 follows from Theorem 3.10.

§4. Preservation of strong partition cardinals. In [21], Henle, Mathias, and Woodin proved that $\mathcal{P}(\omega)/\text{fin}$ preserves strong partition cardinals over a model of ZF + EP + LU (Theorem 1.2). In this section, we extend their result to a wide array of forcings, providing conditions which guarantee that a forcing preserves uncountable strong partition cardinals.

Given a coarsened poset $\langle X, \leq, \leq^* \rangle$, a function f from X to some other set Y is called *invariant* if and only if whenever $p,q \in X$ satisfy $p =^* q$, then $f(p) = f(q)$. Similarly, for any subset $S \subseteq X$, a function $f : S \to Y$ is *invariant* if and only if whenever $p,q \in S$ and $p =^* q$, then $f(p) = f(q)$. We call a set $S \subseteq X$ *invariant* if and only if its characteristic function (from X to 2) is invariant. Given a function f whose domain is a subset of X (a partial function), we call f invariant iff for $p,q \in \text{Dom}(f)$ with $p =^* q$, then $f(p) = f(q)$. We call a function $f : X \to Y$ invariant below $p \in X$ iff $f \restriction [p]$ is an invariant partial function. A set $S \subseteq X$ is invariant below p iff $(\forall q_1, q_2 \leq p)$ if $q_1 =^* q_2$, then $q_1 \in S \Leftrightarrow q_2 \in S$.

DEFINITION 4.1. Given $S \subseteq X$ and $p \in X$, define

$$\begin{aligned} S_p^+ &= \{q \leq p : [q] \subseteq S\}, \\ S_p^- &= \{q \leq p : [q] \cap S = \emptyset\}. \end{aligned} \tag{12}$$

We call S *Ramsey below p*, or simply *R below p*, iff $S_p^+ \cup S_p^-$ is $\leq$-dense below p. We say that S is *R^+ below p* iff S_p^+ is $\leq$-dense below p, and S is *R^- below p* iff S_p^- is $\leq$-dense below p.

We shall say that S is *Ramsey* iff S is Ramsey below p for each $p \in X$. Likewise for R^+ and R^-.

REMARK 4.2. Note that the definition of *Ramsey* in Definition 4.1 is weaker than that of Todorcevic in Definition 2.2. As no ambiguity will arise, we use this term rather than defining yet more terminology.

Note that if S is invariant and $[q] \subseteq S$, then $[q]^* \subseteq S$. Likewise, if S is invariant and $[q] \cap S = \emptyset$, then $[q]^* \cap S = \emptyset$.

The following definition of $\text{LU}(\mathbb{P})$ extends the Axiom LU in [21] for $([\omega]^\omega, \subseteq^*)$ to all partial orderings $\mathbb{P}$.

DEFINITION 4.3. Given a poset $\mathbb{P} = \langle X, \leq \rangle$, $\text{LU}(\mathbb{P})$ is the statement that given any relation $R \subseteq X \times {}^\omega 2$ and $p \in X$ such that

$$(\forall x \leq p)(\exists y \in {}^\omega 2)\, R(x,y),$$

there is some $q \leq p$ and some function $f : [q] \to {}^{\omega}2$ such that

$$(\forall r \leq q)\, R(r, f(r)).$$

OBSERVATION 4.4. *Let $\mathbb{P} = \langle X, \leq \rangle$ be a poset for which there is an injection $\eta : X \to {}^{\omega}\omega$. Then the Uniformization Axiom implies $LU(\mathbb{P})$.*

PROOF. Fix $p \in X$ and a relation $R \subseteq X \times {}^{\omega}2$ such that $(\forall x \leq p)(\exists y \in {}^{\omega}2)\, R(x, y)$. Consider the relation $\tilde{R} \subseteq {}^{\omega}\omega \times {}^{\omega}2$ defined by $\tilde{R}(\tilde{x}, y)$ iff either $\tilde{x} \notin \mathrm{Im}(\eta)$, or $R(\eta^{-1}(\tilde{x}), y)$. By the Uniformization Axiom, there is a uniformization $\tilde{f} : {}^{\omega}\omega \to {}^{\omega}2$ of $\tilde{R}$. This induces a uniformization function $f = \tilde{f} \circ \eta$ for R. That is,

$$(\forall x \leq p)\, R(x, f(x)).$$

Thus, $\mathrm{LU}(\mathbb{P})$ holds, as witnessed by $f \restriction [p]$. ⊣

DEFINITION 4.5. Let $\langle \mathcal{R}, \leq, r \rangle$ be a topological Ramsey space and let $\mathbb{P} = \langle \mathcal{R}, \leq \rangle$. Then $\mathrm{LCU}(\mathbb{P})$ is the statement $\mathrm{LU}(\mathbb{P})$, where additionally f is required to be continuous with respect to the metric topology on $\mathcal{R}$. $\mathrm{LCU}^+(\mathbb{P})$ is the same statement as $\mathrm{LCU}(\mathbb{P})$ but replacing ${}^{\omega}2$ with ${}^{\omega}\omega$.

Certainly $\mathrm{LCU}^+(\mathbb{P})$ implies $\mathrm{LCU}(\mathbb{P})$. The other direction holds as well:

PROPOSITION 4.6. *$LCU(\mathbb{P})$ implies $LCU^+(\mathbb{P})$.*

PROOF. Recall that there is an injection $\varphi : {}^{\omega}\omega \to {}^{\omega}2$ such that $\varphi^{-1} : \mathrm{Im}(\varphi) \to {}^{\omega}\omega$ is continuous. For example, the function φ that takes a sequence $\langle a_0, a_1, \ldots \rangle \in {}^{\omega}\omega$ to the sequence

$$\overbrace{0 \ldots 0}^{a_0} 1 \overbrace{0 \ldots 0}^{a_1} 1 \ldots.$$

is such a function. Now consider any $\tilde{R} \subseteq X \times {}^{\omega}\omega$ and $p \in X$ such that $(\forall x \leq p)(\exists y \in {}^{\omega}\omega)\, \tilde{R}(x, y)$. Define $R \subseteq X \times {}^{\omega}2$ by $(x, y) \in R$ iff $y \in \mathrm{Im}(\varphi)$ and $(x, \varphi^{-1}(y)) \in \tilde{R}$. Applying $\mathrm{LCU}(X)$ to R, there is some $q \leq p$ and some continuous $f : [q] \to {}^{\omega}2$ which uniformizes R below q. But then $\varphi^{-1} \circ f$ uniformizes $\tilde{R}$ below q and is continuous.⊣

The following proposition was proved by Mathias [28] for relations of the form $R \subseteq [\omega]^{\omega} \times {}^{\omega}\omega$, assuming $\omega \to (\omega)^{\omega}_2$. Todorcevic extended this to relations of the form $R \subseteq [\omega]^{\omega} \times X$, where R is coanalytic and X is an arbitrary Polish space. This is stated in [39]; a proof appears as Theorem 7 in [9], and we follow the structure of that proof. First we use the hypotheses to find a uniformization f, and then perform a fusion argument to find a set $[q]$ on which f is continuous.

PROPOSITION 4.7. *Let $\langle \mathcal{R}, \leq, r \rangle$ be a closed axiomatized topological Ramsey space, and let $\mathbb{P}$ be the poset $\langle \mathcal{R}, \leq \rangle$. Also assume that either 1) $AD_{\mathbb{R}}$ holds or 2) $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ holds. Then $LCU(\mathbb{P})$ holds.*

PROOF. *Claim.* Suppose $R \subseteq \mathcal{R} \times {}^{\omega}2$ is a relation and $f : [p^*] \to {}^{\omega}2$ is a uniformization for R. Then there is a $q \leq p^*$ for which $f \restriction [q]$ is continuous.

First, let $p_0 \leq p^*$ be such that each $q \leq p_0$ has the same value for $f(q)(0)$. Such a p_0 exists because, by Lemma 3.9, all subsets of $\mathcal{R}$ are Ramsey, including the set

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$\{q \le p : f(q)(0) = 0\}$. Let $s_1 = r_1(p_0)$. The proof proceeds by induction, recalling the definition of *depth* in Section 2 just before Axiom **A.3**.

Let $n \ge 0$ be fixed and suppose that we have chosen p_m for all $m \le n$ and, letting $s_{m+1} = r_{m+1}(p_m)$, the following hold for each $0 \le m \le n-1$:

1. $p_{m+1} \in [s_{m+1}, p_m]$; and
2. For each $t \le_{\text{fin}} s_{m+1}$ with $\text{depth}_{p_m}(t) = \text{depth}_{p_m}(s_{m+1})$, there is a sequence $g_t : m+1 \to \{0,1\}$ such that for each $q \in [t, p_{m+1}]$ and each $k \le m$, $f(q)(k) = g_t(k)$.

Note that this induction hypothesis is satisfied vacuously for $n = 0$, and that for each $m \le n$, $\text{depth}_{p_m}(s_{m+1}) = m+1$.

Given $s_{n+1} = r_{n+1}(p_n)$, let T denote the set of all $t \le_{\text{fin}} s_{n+1}$ for which $\text{depth}_{p_n}(t) = \text{depth}_{p_n}(s_{n+1})$. T is finite, by Axiom **A.2** (1). Let $\lhd$ be the ordering of the members of T induced by η.

Let t be the $\lhd$–least member of T for which p_t has not yet been chosen. If t is not $\lhd$–minimum in T, then let u denote the $\lhd$–predecessor of t in T. If t is $\lhd$–minimal in T, then let p_u denote p_n. For each sequence $g : n+1 \to 2$, define

$$\mathcal{X}^t_g = \{q \in [t, p_u] : \forall k \le n\,(f(q)(k) = g(k))\}. \tag{13}$$

These sets $\mathcal{X}^t_g$, $g \in {}^{n+1}2$, form a partition of the basic open set $[t, p_u]$ into finitely many pieces. Since each piece of the partition is Ramsey, there are $q_t \in [t, p_u]$ and $g_t \in {}^{n+1}2$ for which $[t, q_t] \subseteq \mathcal{X}^t_{g_t}$. By Axiom **A.3** (2), there is some $p_t \in [s_{n+1}, p_u]$ such that $[t, p_t] \subseteq [t, q_t]$. At the end of this induction on $(T, \lhd)$, let $p_{n+1} = p_{t^*}$, where t^* denotes the $\lhd$–maximum member of T. Note that for each $t \in T$, $p_{n+1} \le p_t$, so in particular,

$$[t, p_{n+1}] \subseteq [t, p_t] \subseteq [t, q_t]. \tag{14}$$

Thus, for each $q \in [t, p_{n+1}]$ and $k \le n$, $f(q)(k) = g_t(k)$. Hence, (1) and (2) hold for p_{n+1}.

Let $q = \bigcup_{n \ge 1} s_n$. Since each $s_{n+1} \sqsupset s_n$ and $\mathcal{R}$ is a closed topological Ramsey space, it follows that q is a member of $\mathcal{R}$. We claim that f is continuous on $[q]$. Suppose $q' \le q$ and $n < \omega$. Then $f(q')(n)$ is determined by $r_k(q')$, where k is minimal such that $\text{depth}_q(r_k(q')) > n$. To see this, let $g : n+1 \to 2$ be given, and let N_g denote $\{h \in {}^{\omega}2 : h \restriction (n+1) = g\}$, the basic open set in ${}^{\omega}2$ determined by g. Then

$$\begin{aligned} f^{-1}(N_g) \cap [q] &= \{q' \le q : f(q) \restriction (n+1) = g\} \\ &= \bigcup \{[t, q] : t \in \mathcal{AR}|q,\ \text{depth}_q(t) > n, \text{ and } g_t \restriction (n+1) = g\}, \end{aligned} \tag{15}$$

which is a union of basic open set in the metric topology on $\mathcal{R}$ restricted to $[q]$. This concludes the proof of the Claim.

Supposing $\text{AD}_{\mathbb{R}}$ holds, let p^* in $\mathcal{R}$ be given, and let $R \subseteq \mathcal{R} \times {}^{\omega}2$ be a relation with the property that for each $x \le p^*$ there exists $y \in {}^{\omega}2$ such that $R(x, y)$. By the argument in Observation 4.4, there is a uniformization $f : [p^*] \to {}^{\omega}2$ for R. Then the Claim implies that there is some $q \le p^*$ such that $f \restriction [q]$ is continuous. Thus, $\text{LCU}(\mathbb{P})$ holds.

Now assume $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $\eta : \mathcal{R} \to {}^\omega\omega$ be the continuous bijection defined in Lemma 3.6. Given any relation $\tilde{R} \subseteq {}^\omega\omega \times {}^\omega 2$, let $\varphi(\tilde{R})$ be the conjunction of the following formulas:

- $(\forall x \in {}^\omega\omega)(\exists y \in {}^\omega 2)\, \tilde{R}(x,y)$;
- $(\forall p \in \mathcal{R})(\forall \text{ continuous } f : [p] \to {}^\omega 2)(\exists q \leq p)\neg\tilde{R}(\eta(q), f(q))$.

By part 4) of Definition 3.5, quantifying over continuous functions is equivalent to quantifying over reals. By Lemma 3.6, $\exists \tilde{R}\, \varphi(\tilde{R})$ is Σ^2_1. For any relation $\tilde{R} \subseteq {}^\omega\omega \times {}^\omega 2$, let $N(\tilde{R}) \subseteq \mathcal{R} \times {}^\omega 2$ be the relation

$$N(\tilde{R})(x,y) \Leftrightarrow \tilde{R}(\eta(x),y).$$

Note that for any $\tilde{R}$, $\varphi(\tilde{R})$ holds if and only if $N(\tilde{R})$ witnesses the failure of $\mathrm{LCU}(\mathbb{P})$.

Suppose toward a contradiction that there is some $p^* \in \mathcal{R}$ and some relation $R' \subseteq \mathcal{R} \times {}^\omega 2$ which witnesses the failure of $\mathrm{LCU}(\mathbb{P})$ below p^*. Define $\tilde{R}'$ by

$$\tilde{R}'(\tilde{x},y) \Leftrightarrow R'(\eta^{-1}(\tilde{x}),y).$$

Note that $R' = N(\tilde{R}')$, so $\varphi(\tilde{R}')$ holds. By Σ^2_1-reflection, there is a Suslin, co-Suslin set $\tilde{R}$ such that $\varphi(\tilde{R})$. Since $\tilde{R}$ is Suslin, co-Suslin it has a uniformization. Now $R := N(\tilde{R})$ has a uniformization on $[p^*]$ as well.

By the Claim, there is some $q \leq p^*$ for which $f \restriction [q]$ is continuous, contradicting our assumption that R' witnesses the failure of $\mathrm{LCU}(\mathbb{P})$. Therefore, $\mathrm{LCU}(\mathbb{P})$ holds.⊣

This next definition differs from [21] in that we require the sets to be invariant.

Definition 4.8. Given a coarsened poset $\mathbb{P} = \langle X, \leq, \leq^* \rangle$, $\mathrm{EP}(\mathbb{P})$ is the statement that given any $p \in X$ and well-ordered sequence $\langle C_\alpha \subseteq X : \alpha < \kappa \rangle$ of sets that are invariant and R^+ below p, the intersection of the sequence is also invariant and R^+ below p.

Definition 4.9. We say that $\langle \mathcal{R}, \leq, \leq^*, r \rangle$ is a *coarsened topological Ramsey space* if $\langle \mathcal{R}, \leq, r \rangle$ is an axiomatized topological Ramsey space and the following hold:

1) $\leq^*$ is a σ-closed partial order;
2) $\langle \mathcal{R}, \leq, \leq^*, r \rangle$ is a coarsened partial order in the sense of Definition 3.1;
3) Whenever $p,q \in \mathcal{R}$ and there is an $a \in \mathcal{AR}$ satisfying $\emptyset \neq [a,q] \subseteq [a,p]$, then $q \leq^* p$.

Note that if $\langle \mathcal{R}, \leq, \leq^*, r \rangle$ is a coarsened topological Ramsey space, then whenever $S \subseteq \mathcal{R}$ is invariant, $([p] \subseteq S \to [p]^* \subseteq S)$.

Proposition 4.10. *Suppose $\langle \mathcal{R}, \leq, \leq^*, r \rangle$ is a coarsened topological Ramsey space. Let $\langle C_n \subseteq \mathcal{R} : n < \omega \rangle$ be a sequence of invariant R^+ sets. Then $\bigcap_n C_n$ is invariant R^+.*

Proof. For each $n < \omega$, let $D_n := \{q \in \mathcal{R} : [q] \subseteq C_n\}$. Note that each D_n is dense in $\langle \mathcal{R}, \leq \rangle$ (since C_n is R^+) and is open in the Ellentuck topology. Furthermore, each D_n is R^+ and invariant. Fix $p \in \mathcal{R}$. It suffices to find some $q \leq p$ such that $[q] \subseteq \bigcap_n C_n$.

Since D_0 is dense, take some $p_0 \leq p$ in D_0. Suppose now that $n < \omega$ and p_n has been chosen. Since D_{n+1} is open in the Ellentuck topology, by Theorem 2.3 there is some $p_{n+1} \in [r_n(p_n), p_n]$ such that either $[r_n(p_n), p_{n+1}] \subseteq D_{n+1}$ or else

$[r_n(p_n), p_{n+1}] \cap D_{n+1} = \emptyset$. The second option cannot happen since D_{n+1} is R^+. Hence, $[r_n(p_n), p_{n+1}] \subseteq D_{n+1}$; and in particular, $p_{n+1} \in [r_n(p_n), p_n] \cap D_{n+1}$.

At the end of this process, we have conditions p_n such that $r_0(p_0) \sqsubset r_1(p_1) \sqsubset r_2(p_2) \sqsubset \cdots$. Since the space $\mathcal{R}$ is closed, there is a $q \in \mathcal{R}$ such that for each $n < \omega$, $r_n(q) = r_n(p_n)$. Then for each n, $[r_n(q), q] \subseteq [r_n(q), p_n]$; so by (3) of Definition 4.9, we have $q \leq^* p_n$. Since each D_n is closed downwards in $\langle \mathcal{R}, \leq \rangle$, each C_n is invariant, and $\langle \mathcal{R}, \leq, \leq^* \rangle$ is a coarsened poset, it follows that $[q]^* \subseteq \bigcap_{n<\omega} C_n$. Hence $\bigcap_{n<\omega} C_n$ is R^+; and the intersection of invariant sets is again invariant. ⊣

Corollary 4.11. *Let $\langle \mathcal{R}, \leq, \leq^*, r \rangle$ be a coarsened closed axiomatized topological Ramsey space. Let $\langle C_n \subseteq \mathcal{R} : n < \omega \rangle$ be a sequence of invariant* R^- *sets. Then $\bigcup_n C_n$ is invariant* R^-.

Proof. Apply Proposition 4.10 to the complements of the C_n's. ⊣

The proof of the next proposition will use the Kunen–Martin Theorem, which states that given an infinite cardinal κ and a wellfounded relation $\prec$ on ${}^\omega\omega$ which is κ-Souslin as a subset of ${}^\omega\omega \times {}^\omega\omega$, then $\rho(\prec) < \kappa^+$. (See Theorem 25.43 on p. 503 in [23].)

Proposition 4.12. *Let $\langle \mathcal{R}, \leq, \leq^*, r \rangle$ be a coarsened topological Ramsey space. Assume LCU($\langle \mathcal{R}, \leq \rangle$), countable choice for sets of reals, and every subset of $\mathcal{R}$ is Ramsey. Then EP ($\mathbb{P}$) holds.*

Proof. Towards a contradiction, assume there is a sequence of length θ which witnesses the failure of EP ($\mathbb{P}$), but EP ($\mathbb{P}$) holds for all sequences strictly shorter than θ. By Proposition 4.10, it must be that θ is an uncountable cardinal, and by minimality of θ for the failure of EP ($\mathbb{P}$), θ must be regular. Fix $p \in X$ and a sequence $\langle C_\alpha : \alpha < \theta \rangle$ of invariant subsets of $\mathcal{R}$ that are R^+ below p, such that $\bigcap_{\alpha<\theta} C_\alpha$ is not R^+ below p.

For each $\alpha < \theta$, let

$$D_\alpha := \{p \in X : (\forall \beta < \alpha)\, [p] \subseteq C_\beta\}.$$

Each D_α is downward $\leq$-closed and is a subset of $\bigcap_{\beta<\alpha} C_\beta$. Note that each D_α is invariant, because if $[p] \subseteq C_\beta$, then $[p]^* \subseteq C_\beta$ (by the invariance of C_β and since $\leq^*$ coarsens $\leq$) and so any $p' =^* p$ satisfies $[p'] \subseteq C_\beta$. Next, we claim that each D_α is R^+ below p. This is immediate from the hypothesis that $\bigcap_{\beta<\alpha} C_\beta$ is R^+ below p. The sequence $\langle D_\alpha : \alpha < \theta \rangle$ is decreasing. Let $D_\theta = \bigcap_{\alpha<\theta} D_\alpha$.

Since $\bigcap_{\alpha<\theta} C_\alpha$ is not R^+ below p, but is Ramsey (since we are assuming every subset of $\mathcal{R}$ is Ramsey), we may fix a $p' \leq p$ such that $\bigcap_{\alpha<\theta} C_\alpha$ and therefore D_θ is empty below p'. Now define the function $\chi : [p'] \to \theta$ as follows:

$$\chi(q) = \min\{\alpha < \theta : q \notin D_\alpha\}.$$

Let the continuous bijection $\eta : \mathcal{R} \to {}^\omega\omega$ come from Proposition 3.6. Let $W \subseteq \mathcal{R} \times {}^\omega\omega$ be the relation

$$W(x, y') \iff \eta^{-1}(y') \leq x \text{ and } \chi(\eta^{-1}(y')) > \chi(x).$$

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Since $\mathrm{LCU}(\langle \mathcal{R}, \leq \rangle)$ (and therefore $\mathrm{LCU}^+(\langle \mathcal{R}, \leq \rangle)$ holds, fix $\bar{p} \leq p'$ and a continuous $f : [\bar{p}] \to {}^\omega\omega$ such that

$$(\forall r \leq \bar{p})\, W(r, f(r)).$$

Now define $S \subseteq {}^\omega\omega \times {}^\omega\omega$ by

$$S(x', y') \iff \eta^{-1}(x') \leq \bar{p} \text{ and } f(\eta^{-1}(x')) = y'.$$

Note that

$$S(x', y') \Longrightarrow \eta^{-1}(x') \geq \eta^{-1}(y') \text{ and } \chi(\eta^{-1}(x')) < \chi(\eta^{-1}(y')).$$

So, S is well-founded, meaning there is no sequence $\langle x_0', x_1', \ldots \rangle$ of elements of ${}^\omega\omega$ such that

$$\cdots \wedge S(x_2', x_1') \wedge S(x_1', x_0').$$

Let $D = \{x' \in {}^\omega\omega : \eta^{-1}(x') \leq \bar{p}\}$. Note that for each $x' \in D$, $(\exists y' \in D)\, S(x', y')$. Since S is a well-founded relation, we may assign an S-rank $\rho'(x')$ to each $x' \in D$. Specifically, for $y' \in D$,

$$\rho'(y') := \sup\{\rho(x') + 1 : x' \in D \text{ and } S(x', y')\}.$$

For $x' \in {}^\omega\omega$ not in D, define $\rho'(x') :=-1$. Since f is continuous, by part 3) of the Definition 3.5 of being projectively presented, S is a $\mathbf{\Sigma}^1_1$ relation. Since S is $\mathbf{\Sigma}^1_1$, by the Kunen–Martin theorem, fix an ordinal $\gamma < \omega_1$ such that

$$(\forall x' \in D)\, \rho'(x') < \gamma.$$

For each $0 \leq \alpha < \gamma$, let

$$E_\alpha' := \{x' \in {}^\omega\omega : \rho'(x) = \alpha\}.$$

Let $E_\alpha := \{\eta^{-1}(x') : x' \in E_\alpha'\}$. Note that the E_α's form a partition of $[\bar{p}]$, because the E_α''s form a partition of D. Here is the second place where we use the assumption that every subset of $\mathcal{R}$ is Ramsey. We will show that each E_α is R^-. Fix $0 \leq \alpha < \gamma$. The set E_α is Ramsey, so to show it is R^-, it suffices to show

$$(\forall q \in E_\alpha)(\exists r \leq q)\, q' \notin E_\alpha.$$

This is immediate, because given $q \in E_\alpha$, there is an $r \leq q$ such that $S(\eta(q), \eta(r))$. Thus, by definition of ρ',

$$\alpha = \rho'(\eta(q)) < \rho'(\eta(r)).$$

Hence, $r \notin E_\alpha$.

We now have that the E_α's form a partition of $[\bar{p}]$, and that they are each R^-. By Corollary 4.11, the countable union of all the E_α's is R^-. Hence, $[\bar{p}]$ is R^-, which is impossible. ⊣

PROPOSITION 4.13. *Let $\mathbb{P} = \langle X, \leq, \leq^* \rangle$ be a coarsened poset such that $EP(\mathbb{P})$ holds. Fix $p \in X$. Let $\langle C_\alpha \subseteq X : \alpha < \kappa \rangle$ be a sequence of subsets of X that are Ramsey and invariant below p. Then there is some $q \leq p$ such that for each $\alpha < \kappa$, either $[q]^* \subseteq C_\alpha$ or $[q]^* \cap C_\alpha = \emptyset$.*

PROOF. Fix $\alpha < \kappa$. Let $D_\alpha \subseteq X$ be the set

$$D_\alpha = \{q \leq p : [q] \subseteq C_\alpha \text{ or } [q] \cap C_\alpha = \emptyset\}.$$

Because C_α is Ramsey, the set D_α is R^+. Note also that because C_α is invariant, we have

$$D_\alpha = \{q \leq p : [q]^* \subseteq C_\alpha \text{ or } [q]^* \cap C_\alpha = \emptyset\}.$$

This also establishes that D_α is invariant.

All we need is a q that is in the intersection of the D_α's. This follows from $\mathrm{EP}(\mathbb{P})$, because each D_α is invariant and R^+. ⊣

PROPOSITION 4.14. *Let $\mathbb{P} = \langle X, \leq, \leq^* \rangle$ be a coarsened poset and assume $EP(\mathbb{P})$. Assume every $S \subseteq X$ is Ramsey. Let κ be a cardinal, $p \in X$, and $\Phi : [p] \to [\kappa]^\kappa$ be an invariant function. Then there is a $p' \leq p$ such that $\Phi \restriction [p']$ is constant.*

PROOF. For each $\alpha < \kappa$, put $C_\alpha := \{q \leq p : \alpha \in \Phi(q)\}$. Each C_α is Ramsey and invariant below p. By Proposition 4.13, fix a $q \leq p$ such that for each $\alpha < \kappa$, either $[q]^* \subseteq C_\alpha$ or $[q]^* \cap C_\alpha = \emptyset$. It suffices to show that for each $r \leq q$ that $(\forall \alpha < \kappa)\, \alpha \in \Phi(q) \Leftrightarrow \alpha \in \Phi(r)$. Fix such α and r. We have

$$\alpha \in \Phi(q) \Rightarrow q \in C_\alpha \Rightarrow [q]^* \subseteq C_\alpha \Rightarrow r \in C_\alpha \Rightarrow \alpha \in \Phi(r)$$

and

$$\alpha \notin \Phi(q) \Rightarrow q \notin C_\alpha \Rightarrow [q]^* \cap C_\alpha = \emptyset \Rightarrow r \notin C_\alpha \Rightarrow \alpha \notin \Phi(r). \qquad ⊣$$

PROPOSITION 4.15. *Suppose $\kappa \to (\kappa)^\lambda_\mu$, where κ, λ, μ are nonzero ordinals such that $\lambda = \omega\lambda \leq \kappa$ and $2 \leq \mu < \kappa$. Let $\mathbb{P} = \langle X, \leq, \leq^* \rangle$ be a coarsened poset with the property that each $=^*$ equivalence class is countable, and assume $LU(\mathbb{P})$. Assume there is a surjection ψ from ${}^\omega 2$ onto $[\kappa]^\kappa$. Let $\langle \pi_p : p \in X \rangle$ be a collection of functions $\pi_p : [\kappa]^\lambda \to \mu$. Then below any $p \in X$ there exists $p^* \leq p$ and an invariant function $\Phi : [p^*] \to [\kappa]^\kappa$ such that $(\forall q \in [p^*])\, \Phi(q)$ is homogeneous for π_q.*

PROOF. Given a set of ordinals x in ordertype λ, let $\Omega(x)$ be the set of all limits of the ω-blocks of x. That is, if $\{x_\alpha : \alpha < \lambda\}$ is the increasing enumeration of x, then

$$\Omega(x) = \{\sup_{n \in \omega} x_{\omega\beta+n} : \beta < \lambda\}.$$

Note that since $\omega\lambda = \lambda$, $\Omega(x)$ is in $[\kappa]^\lambda$ whenever x is.

For each $q \leq p$ define $\rho_q : [\kappa]^\lambda \to \mu$ by

$$\rho_q(y) = \pi_q(\Omega(y)).$$

Let $R \subseteq X \times {}^\omega 2$ be the relation

$$R(q, r) \iff \psi(r) \text{ is homogeneous for } \rho_q.$$

By $\mathrm{LU}(\mathbb{P})$, we may fix a $p^* \leq p$ and a function $f : [p^*] \to {}^\omega 2$ such that

$$(\forall q \leq p^*)\, R(q, f(q)).$$

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Thus,

$$(\forall q \in \mathrm{Dom}(f))\, \psi(f(q)) \text{ is homogeneous for } \rho_q.$$

Write $B(q)$ for $\psi(f(q))$.

For each $q \in \mathrm{Dom}(f)$, define $C(q) \in [\kappa]^\kappa$ as follows: $C(q)(0)$ is the least ordinal greater than $B(q')(0)$ for all q' such that $q' =^* q$. Let $C(q)(\nu)$ be the least ordinal ξ such that letting $\eta = \sup\{C(q)(\nu') : \nu' < \nu\}$, the interval $[\eta, \xi)$ contains at least one element of each $B(q')$ for each $q' =^* q$. It is in this definition of $C(q)$ that we use that each $=^*$ equivalence class is countable. Without this assumption, we might have $C(q)(0) = \kappa$. Note that if $q' =^* q$ then $C(q') = C(q)$. Hence, C is invariant.

Now for each $q \in \mathrm{Dom}(f)$, define $\Phi(q) := \Omega(C(q))$. We have that Φ is invariant. We claim that $\Phi(q)$ is homogeneous for π_q. Consider any $x \in [\Phi(q)]^\lambda$. By construction of $C(q)$, there is some $y \in [B(q)]^\lambda$ such that $x = \Omega(y)$. Now $\pi_q(x) = \pi_q(\Omega(y)) = \rho_q(y)$. $B(q)$ is homogeneous for $\rho_q(y)$, so each such value of $\rho_q(y)$ is the same. Hence, each $x \in [\Phi(q)]^\lambda$ has the same $\pi_q(x)$ value, so $\Phi(q)$ is homogeneous for π_q. ⊣

Remark 4.16. It is not known if the previous proposition holds if the $=^*$ equivalence classes are uncountable.

Theorem 1.4. *Suppose $\kappa \to (\kappa)^\lambda_\mu$, where κ, λ, μ are nonzero ordinals such that $\lambda = \omega\lambda \le \kappa$ and $2 \le \mu < \kappa$. Suppose also that there is a surjection from ${}^\omega 2$ to $[\kappa]^\kappa$ (which happens if we assume AD and $\kappa < \Theta$). Let $\mathbb{P} = \langle X, \le, \le^* \rangle$ be a coarsened poset such that $EP(\mathbb{P})$ and each $=^*$-equivalence class is countable. Assume every $S \subseteq X$ is Ramsey. If $LU(\mathbb{P})$ holds and $\langle X, \le \rangle$ adds no new sets of ordinals, then $\langle X, \le \rangle$ forces $\kappa \to (\kappa)^\lambda_\mu$.*

Proof. The relation $\Vdash$ corresponds to the forcing $\langle X, \le \rangle$. Note that the assumption $\kappa \to (\kappa)^\lambda_\mu$ implies that $\kappa \to (\kappa)^\lambda_{\mu+1}$ also holds. Suppose $p_0 \Vdash \dot{f} : [\check{\kappa}]^{\check{\lambda}} \to \check{\mu}$. We will find a $p_2 \le p_0$ in X and some $A \in [\kappa]^\kappa$ such that $p_2 \Vdash \dot{f}$ is constant on $[\check{A}]^{\check{\lambda}}$.

For each $p \le p_0$, define a partition $\pi_p : [\kappa]^\lambda \to \mu + 1$ by

$$\pi_p(A) = \begin{cases} \zeta & \text{if } p \Vdash \dot{f}(\check{A}) = \check{\zeta}, \\ \mu & \text{if there is no such } \zeta. \end{cases}$$

Using Proposition 4.15 and assuming $\mathrm{LU}(\mathbb{P})$, there is a $p_1 \le p_0$ and an invariant function $\Phi : [p_1] \to [\kappa]^\kappa$ such that for each $p \le p_1$, $\Phi(p)$ is homogeneous for π_p. By Proposition 4.14, there is some $p_2 \le p_1$ with Φ constant on $[p_2]$. Set $A = \Phi(p_2)$. We claim that

$$p_2 \Vdash \check{A} \text{ is homogeneous for } \dot{f}.$$

If not, there are $D, E \in [A]^\lambda$, $q \le p_2$ and $\alpha < \beta < \mu$ such that

$$q \Vdash \dot{f}(\check{D}) = \check{\alpha} \text{ and } \dot{f}(\check{E}) = \check{\beta}.$$

So, $\pi_q(D) = \alpha$ and $\pi_q(E) = \beta$. Thus A is not homogeneous for π_q, contradicting that $A = \Phi(q)$, which is homogeneous for π_q. ⊣

Theorem 4.17. *Assume either $AD_\mathbb{R}$ or $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $\mathbb{P} = \langle \mathcal{R}, \le, \le^*, r \rangle$ be a coarsened topological Ramsey space, where the $=^*$-equivalences classes* *are countable. Then forcing with $\langle \mathcal{R}, \leq \rangle$ preserves $\kappa \to (\kappa)^\lambda_\mu$, whenever $\kappa \to (\kappa)^\lambda_\mu$ holds in the ground model, where κ, λ, μ are nonzero ordinals such that $\lambda = \omega\lambda \leq \kappa$ and $2 \leq \mu < \kappa$, and there is a surjection from ${}^\omega 2$ to $[\kappa]^\kappa$.*

PROOF. Assuming AD, countable choice for reals holds, so there is a continuous bijection between $\mathcal{R}$ and ${}^\omega\omega$, by Proposition 3.6. Lemma 3.9 gives us that every subset of $\mathcal{R}$ is Ramsey. Then LCU $(\mathbb{P})$ holds by Proposition 4.7, so Proposition 4.12 implies EP $(\mathbb{P})$ holds. Theorem 1.4 yields the result. ⊣

The second part of Theorem 1.5 follows from Theorem 4.17.

In Section 5 we shall show that three families of topological Ramsey spaces forcing ultrafilters with weak partition properties satisfy the conditions of Theorem 4.17.

§5. Ultrafilters with barren extensions preserving strong partition cardinals.

In this and the next section, we provide examples of many forcings producing barren extensions with ultrafilters satisfying different partition relations. Recall Definition 2.6: Given an ultrafilter $\mathcal{U}$ on a countable base set S, for each $n \geq 2$, $t(\mathcal{U}, n)$ is the least number t, if it exists, such that for any $\ell \geq 2$ and any coloring $c : [S]^n \to \ell$, there is a member $U \in \mathcal{U}$ such that $c \restriction [U]^n$ takes no more than t colors. An ultrafilter $\mathcal{U}$ is *Ramsey* if and only if $t(\mathcal{U}, n) = 1$ for all n.

In this section, we show that three classes of topological Ramsey spaces forcing non-Ramsey ultrafilters have generic extensions with no new sets of ordinals and preserve strong partition cardinals. These are the class of Milliken–Taylor ultrafilters investigated by Mildenberger in [30], a hierarchy of ultrafilters of Laflamme in [25] extending weakly Ramsey ultrafilters, and a class of ultrafilters of Dobrinen, Mijares and Trujillo in [14] which encompass k-arrow, non-$(k+1)$-arrow ultrafilters of Baumgartner and Taylor in [2] as well as n-square forcing of Blass in [3].

5.1. Milliken–Taylor ultrafilters. The first class of coarsened posets that we look at are topological Ramsey spaces of infinite block sequences of vectors. The reader is referred to Section 5.2 in [39] for a thorough presentation of these spaces. The members of $\mathrm{FIN}_k^{[\infty]}$ are infinite sequences $x = \langle x_i : i < \omega \rangle$ such that for each $i < \omega$, the following hold:

1) x_i is a function from ω into $k+1$, and the support of x_i, defined by $\mathrm{supp}(x_i) = \{n \in \omega : x_i(n) \neq 0\}$, is finite;
2) There is some $n \in \mathrm{supp}(x_i)$ such that $x_i(n) = k$;
3) $\max(\mathrm{supp}(x_i)) < \min(\mathrm{supp}(x_{i+1}))$.

The nth approximation to x is $r_n(x) = \langle x_i : i < n \rangle$. For $x, y \in \mathrm{FIN}_k^{[\infty]}$, $y \leq x$ iff y is obtainable from x using the tetris operation. The definition of the tetris operation is not needed for the proof in this section, so we refer the interested reader to [39]. For $x \in \mathrm{FIN}_k^{[\infty]}$ and $n < \omega$, let x/n denote the tail $\langle x_n, x_{n+1}, x_{n+2}, \dots \rangle$. The coarsening $\leq^*$ on $\mathrm{FIN}_k^{[\infty]}$ is defined as follows: $y \leq^* x$ iff there is some n such that $y/n \leq x$. This quasi-order $\leq^*$ is σ-closed and $\langle \mathrm{FIN}_k^{[\infty]}, \leq \rangle$ and $\langle \mathrm{FIN}_k^{[\infty]}, \leq^* \rangle$ are forcing equivalent.

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$\langle \mathrm{FIN}_k^{[\infty]}, \leq^* \rangle$ forces ultrafilters, referred to as Milliken–Taylor ultrafilters in [30]. For $k = 1$, such ultrafilters are called stable ordered union ultrafilters and were first investigated by Blass in [4]. In [30], Mildenberger showed that forcing with $\langle \mathrm{FIN}_k^{[\infty]}, \leq^* \rangle$ produces an ultrafilter, denoted $\mathcal{U}^k$, with at least $k+1$-near coherence classes of ultrafilters Rudin–Keisler below it.

LEMMA 5.1. *The coarsened topological Ramsey space* $\langle \mathrm{FIN}_k^{[\infty]}, \leq, \leq^*, r \rangle$ *satisfies the Left-Right Axiom. Furthermore, each* $=^*$*-class is countable.*

PROOF. In order to show that the Left-Right Axiom is satisfied by the coarsened topological Ramsey space $\langle \mathrm{FIN}_k^{[\infty]}, \leq, \leq^*, r \rangle$, if suffices to know the following simple fact: Given any $x \in \mathrm{FIN}_k^{[\infty]}$, both sequences $\langle x_{2i} \colon i < \rangle$ and $\langle x_{2i+1} : i < \omega \rangle$ are members of $\mathrm{FIN}_k^{[\infty]}$. Define the functions Left and Right on $X \in \mathrm{FIN}_k^{[\infty]}$ as follows: Given $x = \langle x_i : i < \omega \rangle \in X \in \mathrm{FIN}_k^{[\infty]}$, let $\mathrm{Left}(x) = \langle x_{2i} \colon i < \rangle$ and $\mathrm{Right}(x) = \langle x_{2i+1} : i < \omega \rangle$. Then in the Left-Right Axiom, 1) is satisfied, since in fact, $\mathrm{Left}(x) \leq x$ and $\mathrm{Right}(x) \leq x$. To show that 2) holds, given $x \in \mathrm{FIN}_k^{[\infty]}$, let $y = x$ and $z = \langle x_i : i \geq 1 \rangle$. Then both $y, z \leq x$, $\mathrm{Left}(y) =^* \mathrm{Right}(z)$ since $\mathrm{Left}(y)/1 = \mathrm{Right}(z)$, and $\mathrm{Right}(y) = \mathrm{Left}(z)$.

For 3), given $p \in \mathrm{FIN}_k^{[\infty]}$ and $x, y \leq p$, define z as follows: Take $z_i = x_i$ for $i < 3$. Then let $z_3 = y_j$ for j minimal such that $\min(\mathrm{supp}(y_j)) > \max(\mathrm{supp}(x_2))$. Given z_i for $i \equiv 0,1,3 \pmod 4$, let $z_{i+1} = x_j$ for j minimal such that $\min(\mathrm{supp}(x_j)) > \max(\mathrm{supp}(z_i))$. Given z_i for $i \equiv 2 \pmod 4$, let $z_{i+1} = y_j$ for j minimal such that $\min(\mathrm{supp}(y_j)) > \max(\mathrm{supp}(z_i))$. Then this $z = \langle z_i : i < \omega \rangle$ is in $\mathrm{FIN}_k^{[\infty]}$, $z \leq a$, and z satisfies 3) for x and y. Thus, the Left-Right Axiom holds.

By the definition of $\leq^*$, $x =^* y$ if and only if there are m, n such that $x/m = y/n$. Thus, each $=^*$-equivalence class is countable. ⊣

Since $\langle \mathrm{FIN}_k^{[\infty]}, \leq, \leq^*, r \rangle$ is a coarsened closed axiomatized topological Ramsey space, it produces a barren extension.

COROLLARY 5.2. *Assume* M *is a model of ZF + either* 1) $AD_{\mathbb{R}}$ *or* 2) $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Then forcing with* $\langle \mathrm{FIN}_k^{[\infty]}, \leq^* \rangle$ *over* M *adds an ultrafilter* $\mathcal{U}_k$ *such that* $M[\mathcal{U}_k]$ *has the same sets of ordinals as* M. *Furthermore, for all nonzero ordinals* κ, λ, μ *such that* $\lambda = \omega\lambda \leq \kappa$ *and* $2 \leq \mu \leq \kappa < \Theta$, *if* $\kappa \to (\kappa)^\lambda_\mu$ *in* M, *then it also holds in* $M[\mathcal{U}_k]$.

PROOF. This follows from Theorems 3.10 and 4.17 and Lemma 5.1. ⊣

5.2. Extended coarsened posets with independent sequencing. We define a general property called *Independent Sequencing* for partial orders and then for extended coarsened partial orders. We show that when an EC poset has Independent Sequencing, then the Left-Right Axiom is satisfied and hence, Theorem 1.3 holds. If further the $=^*$-equivalence classes are countable, then Theorem 1.4 holds. In the following subsections, we show that the classes of topological Ramsey spaces in the papers [10, 11, 14, 15, 16] have extended coarsenings which have Independent Sequencing.

DEFINITION 5.3 (Independent Sequencing for posets). A poset $\mathbb{P} = \langle X, \leq \rangle$ *has Independent Sequencing* (*IS*) if the following hold:

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1) For each $x \in X$, x can be written as a sequence $\langle x(n) : n < \omega \rangle$. Each $x(n)$ is a countable set, possibly with some structure on it.
2) Given $x, y \in X$, $y \le x$ iff there is a strictly increasing sequence $(i_n)_{n<\omega}$ such that each $y(n) \subseteq x(i_n)$.
3) Given $x, y \in X$ and a partition of ω into three pieces, P_0, P_1, P_2, where at least one of P_0 and P_1 is infinite, there is a $z \in X$ such that for each $n \in P_0$, $z(n) \subseteq x(i)$ for some i, and for each $n \in P_1$, $z(n) \subseteq y(i)$ for some i. Moreover, if $x, y \le p \in X$, then there is such a $z \le p$.

This last property 3) is why we call the sequencing "independent".

Definition 5.4 (Independent Sequencing for EC posets). An extended coarsened poset $\mathbb{P}^* = \langle X, X^*, \le, \le^* \rangle$ *has Independent Sequencing* (*IS*) if $\langle X, \le \rangle$ has IS for posets and additionally,

4) X^* is the collection of all subsequences of members of X. Thus, $a = \langle a(n) : n < \omega \rangle \in X^*$ iff there is an $x \in X$ and a strictly increasing sequence $(i_n)_{n<\omega}$ such that each $a(n) = x(i_n)$.
5) Given $a, b \in X^*$, $b \le a$ iff there is a strictly increasing sequence $(i_n)_{n<\omega}$ such that each $b(n) \subseteq a(i_n)$.
6) The coarsening $\le^*$ has the property that for $a, b \in X^*$, if $\langle b(n) : n \ge m \rangle \le a$ for some m, then $b \le^* a$.

Notice that 4) implies that $X \subseteq X^*$. By 2) and 5), the order $\le$ on X is the restriction to X of the order $\le$ on X^*. By 3) and 4), for each $a \in X^*$ there is an $x \in X$ such that $x \le a$. Thus, $\langle X, \le \rangle$ is dense in $\langle X^*, \le \rangle$ and hence, $\langle X, \le^* \rangle$ is dense in $\langle X^*, \le^* \rangle$.

Lemma 5.5. *Each extended coarsened poset with Independent Sequencing satisfies the Left-Right Axiom.*

Proof. Let $\mathbb{P}^* = \langle X, X^*, \le, \le^* \rangle$ be an extended coarsened poset having Independent Sequencing. For each $x \in X$, define $\text{Left}(x) = \langle x(2n) : n < \omega \rangle$ and $\text{Right}(x) = \langle x(2n+1) : n < \omega \rangle$. Then IS 4) implies that Left and Right are functions from X into X^* and IS 5) implies that $\text{Left}(x), \text{Right}(x) \le x$, so LRA 1) holds. For LRA 2), given $x \in X$, let $y = x$ and $z = \langle x(n) : n \ge 1 \rangle$. Then

$$\text{Left}(y) = \langle x(2n) : n < \omega \rangle =^* \langle x(2n+2) : n < \omega \rangle = \langle z(2n+1) : n < \omega \rangle = \text{Right}(z),$$

where the $=^*$ holds by IS 6), so LRA 2a) holds; and

$$\text{Right}(y) = \langle y(2n+1) : n < \omega \rangle = \langle x(2n+1) : n < \omega \langle z(2n) : n \ge 1 \rangle = \text{Left}(z),$$

so LRA 2b) holds.

Now let $p, x, y \in X$ with $x, y \le p$ be given. By IS 3), there is some $z \le p$ in X such that for each $n \equiv 0, 1, 2 \pmod 4$, $z(n) = x(i)$ for some i, and for each $n \equiv 3 \pmod 4$, $z(n) = y(i)$ for some i. Furthermore, $\text{Left}(z) \le x$, $\text{Left}(\text{Right}(z)) \le x$, and $\text{Right}(\text{Right}(z)) \le y$, so LRA 3) holds. ⊣

Given an extended coarsened topological Ramsey space $\langle \mathcal{R}, \mathcal{R}^*, \le, \le^*, r \rangle$, the forcing $\langle \mathcal{R}, \le^* \rangle$ adds an ultrafilter on the countable base set $\mathcal{AR}_1$ as follows: Letting G be the generic filter on $\langle \mathcal{R}, \le^* \rangle$, define $\mathcal{U}_{\mathcal{R}}$ to be the filter generated by the collection of sets $\mathcal{AR}_1 \restriction X := \{s \in \mathcal{AR}_1 : \exists Y \le X\, (s = r_1(Y))\}$, for $X \in G$. By genericity and

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Theorem 2.3, $\mathcal{U}$ is an ultrafilter. When the space $\langle \mathcal{R}, \leq^* \rangle$ is dense inside some poset $\mathbb{P}$ forcing an ultrafilter, then $\mathcal{U}_{\mathcal{R}}$ is isomorphic to the ultrafilter forced by $\mathbb{P}$. This fact was behind the work in [14, 15, 16], which investigated structural results on known ultrafilters constructed by Laflamme in [25], Baumgartner and Taylor [2], Blass [3], as well as new ultrafilters related to these. In the next several subsections, we will show that these classes of ultrafilters have Independent Sequencing and countable $=^*$-equivalence classes, so generic extensions by these ultrafilters will satisfy the next theorem.

The ultrafilters in Section 6 are forced by Ramsey spaces constructed in [10] and [11], forming a hierarchy over the ultrafilter investigated by Szymanski and Xua [38], forced by $\mathcal{P}(\omega \times \omega)/\mathrm{Fin} \otimes \mathrm{Fin}$. These forcings have Independent Sequencing, but their $=^*$-equivalence classes are uncountable, so the first part of the next theorem holds for them, but we do not know if they preserve strong partition cardinals.

THEOREM 5.6. *Assume M is a model of ZF + either* 1) $AD_{\mathbb{R}}$ *or* 2) $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. *In M, suppose $\langle \mathcal{R}, \mathcal{R}^*, \leq, \leq^*, r \rangle$ is an extended coarsened topological Ramsey space with Independent Sequencing. Then forcing with $(\mathcal{R}, \leq^*)$ adds an ultrafilter $\mathcal{U}_{\mathcal{R}}$ such that $M[\mathcal{U}_{\mathcal{R}}]$ has the same sets of ordinals as M. If furthermore, the $=^*$-equivalence classes are each countable, then $\kappa \to (\kappa)^{\lambda}_{\mu}$ holds in $M[\mathcal{U}_{\mathcal{R}}]$ whenever $\lambda = \omega\lambda \leq \kappa$, $2 \leq \mu < \kappa$, there is a surjection from ${}^{\omega}2$ to $[\kappa]^{\kappa}$, and $\kappa \to (\kappa)^{\lambda}_{\mu}$ holds in M.*

PROOF. This follows from Lemma 5.5 and Theorems 1.3 and 4.17. ⊣

5.3. A hierarchy of rapid p-points above a weakly Ramsey ultrafilter. Laflamme constructed a sequence of forcings $\mathbb{P}_\alpha$, $1 \leq \alpha < \omega_1$, in [25] where $\mathbb{P}_\alpha$ forces an ultrafilter $\mathcal{U}_\alpha$ which is a rapid p-point satisfying some weak partition relation. Moreover, the Rudin–Keisler structure below $\mathcal{U}_\alpha$ contains a decreasing sequence of order-type $(\alpha+1)^*$, where the minimal filter is a Ramsey ultrafilter. For $k \geq 1$ a finite integer, the Ramsey degrees of $\mathcal{U}_k$ are as follows: For each $n \geq 1$, $t(\mathcal{U}_k, n) = (k+1)^{n-1}$. These Ramsey degrees are stated in [25] and succinct proofs using Ramsey theoretic techniques appear in [13]. In particular, $t(\mathcal{U}_1, n) = 2$, so $\mathcal{U}_1$ is a weakly Ramsey ultrafilter.

In [15] and [16], for each $1 \leq \alpha < \omega_1$, a topological Ramsey space $\mathcal{R}_\alpha$ was constructed which is dense in $\mathbb{P}_\alpha$. In those papers, these Ramsey spaces were used to find exact Tukey and Rudin–Keisler structures below each $\mathcal{U}_\alpha$. Recalling Theorem 2.3, each $\mathcal{R}_\alpha$ satisfies the infinite partition relations required in the hypothesis of Theorem 1.3.

The definitions of the spaces $\mathcal{R}_\alpha$ are somewhat involved and the interested reader is referred to the original papers [15] and [16]. What is important here is that each member $x \in \mathcal{R}_\alpha$ is a sequence $\langle x(n) : n < \omega \rangle$ where each $x(n)$ is a finite tree, and that $\langle \mathcal{R}_\alpha, \leq \rangle$ has Independent Sequencing. We define $\mathcal{R}^*_\alpha$ to be the set of all sequences $\langle x(i_n) \colon n < \rangle$ for $x \in \mathcal{R}_\alpha$ and $(i_n)_{n<\omega}$ strictly increasing so that IS 4) holds. Extend $\leq$ to $\mathcal{R}^*_\alpha$ so that the first sentence of IS 5) holds. The σ-closed partial order $\leq^*$ on $\mathcal{R}_\alpha$ is simply mod finite initial segment of the sequence, so IS 6) holds.

Given $y \leq^* x$ in X, letting k be least such that for each $n \geq k$, $y(n) \subseteq x(i_n)$ for some i_n, the sequence $z = r_k(x)^\frown \langle y(n) : n \geq k \rangle$ is a member of $\mathcal{R}_\alpha$. Furthermore, this $z \leq x$ and $z =^* y$. Hence $\langle \mathcal{R}_\alpha, \leq, \leq^* \rangle$ is a coarsened poset. We extend this order

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to $\mathcal{R}_\alpha^*$ to again mean mod finite initial segment. Thus, for $a,b \in \mathcal{R}_\alpha^*$, $b \leq^* a$ iff for some k, for all $n \geq m$, $b(n) \subseteq a(i)$ for some i. This coarsening satisfies IS 6).

Thus, $\langle \mathcal{R}_\alpha, \mathcal{R}_\alpha^*, \leq, \leq^* \rangle$ is an EC poset having IS. Furthermore, since each $a(n)$ is finite, for $a \in \mathcal{R}_\alpha^*$ and $n < \omega$, and since $\leq^*$ is mod finite initial segment, it follows that each $=^*$-equivalence class is finite. Therefore, by Theorem 5.6 Laflamme's forcings produce barren extensions preserving strong partition cardinals, provided that the ground model satisfies $\mathrm{AD}_{\mathbb{R}}$ or $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$.

5.4. *k*-Arrow ultrafilters, *n*-square ultrafilters, and their extended family of rapid p-points. Ultrafilters with asymmetric partition relations were constructed by Baumgartner and Taylor in [2]. For $k \geq 3$, a *k-arrow* ultrafilter is an ultrafilter $\mathcal{U}$ such that for each function $f : [\omega]^2 \to 2$, either there is a set $X \in \mathcal{U}$ such that $f([X]^2) = \{0\}$ or else there is a set $Y \in [\omega]^k$ such that $f([Y]^2) = \{1\}$. This is written as

$$\mathcal{U} \to (\mathcal{U}, k)^2.$$

For each $k \geq 3$, Baumgartner and Taylor constructed a partial order, let's call it $\mathbb{P}_k^{\mathrm{BT}}$ which, by using CH, MA or $\mathfrak{p} = \mathfrak{c}$, constructs a p-point which is k-arrow but not $(k+1)$-arrow. The partial order $\mathbb{P}_k^{\mathrm{BT}}$ used finite ordered k-clique-free graphs and applications of a theorem of Nešetřil and Rödl, that the collection of finite ordered k-clique-free graphs has the Ramsey property [33, 32]. Recall that a k-clique is a complete graph on k vertices, and is denoted by K_k.

In [14], for each $k \geq 3$, a topological Ramsey space $\mathcal{A}_k$ which is dense in Baumgartner and Taylor's partial order $\mathbb{P}_k^{\mathrm{BT}}$ was constructed. Thus, forcing with $\mathcal{A}_k$ produces a p-point which is k-arrow and not $(k+1)$-arrow. Furthermore, since $\mathcal{A}_k$ is a topological Ramsey space, it satisfies Theorem 2.3, so the desired infinite dimensional partition relation holds. To make a space $\mathcal{A}_k$, all that is required is that we fix some sequence $\langle \mathbb{A}_n : n < \omega \rangle$ of finite ordered K_k-free graphs such that each $\mathbb{A}_n$ embeds as an induced subgraph into $\mathbb{A}_{n+1}$ and that each finite ordered K_k-free graph embeds as an induced subgraph into some $\mathbb{A}_n$, and hence into all but finitely many $\mathbb{A}_n$. The members of $\mathcal{A}_k$ are sequences $\langle x(n) : n < \omega \rangle$ where each $x(n)$ is a subgraph of $\mathbb{A}_{i_n}$ for some strictly increasing sequence $(i_n)_{n<\omega}$. In particular, the topological Ramsey space $\langle \mathcal{A}_k, \leq, r \rangle$ has Independent Sequencing.

The extended coarsening of $\mathcal{A}_k$ is obtained by letting $\mathcal{A}_k^*$ be defined as in 2) of IS. The coarsened quasi-order $\leq^*$ is simply mod finite initial segment. This order is σ-closed and $\langle \mathcal{A}_k, \mathcal{A}_k,^*, \leq, \leq^*, r \rangle$ forms an extended coarsened poset with Independent Sequencing. Furthermore, the $=^*$-equivalence classes are countable, since $\leq^*$ is mod finite. Thus, Theorem 5.6 holds for Baumgartner and Taylor's p-points which are k-arrow and not $(k+1)$-arrow.

Another hierarchy of interesting p-points are forced by the hypercube Ramsey spaces (see [14, 40]). The basis for these spaces is the n-square forcing $\mathbb{P}_{n\text{–square}}$ which Blass constructed in [3] in order to show that under MA, there is a p-point which has two Rudin–Keisler incomparable predecessors. Conditions in $\mathbb{P}_{n\text{–square}}$ are subsets $p \subseteq \omega \times \omega$ such that for each $n \geq 1$, there are $K, L \in [\omega]^n$ such that $K \times L \subseteq p$; the partial ordering is inclusion mod finite. A topological Ramsey space forming a dense subset of $\mathbb{P}_{n\text{–square}}$ was constructed in [40]; this was used to show that both the Rudin–Keisler structure and the Tukey structure below this forced p-point is exactly the four element Boolean algebra, that is, a diamond shape. This result was generalized in [14] where it was shown that for each $k \geq 2$, there is a topological Ramsey space $\mathcal{H}^k$ in which each member is a sequence $x = \langle x(n) : n < \omega \rangle$ where each $x(n)$ is a k-dimensional cube with side length n. These were used to show that for each $k \geq 2$, there is a p-point with both the initial Rudin–Keisler structure and the initial Tukey structure being the Boolean algebra on k atoms, that is, of cardinality 2^k. In particular, this answered a question about the initial Tukey structure of $\mathcal{G}_2$, which was left open in [5].

In fact, $\langle \mathcal{H}^k, \leq \rangle$ as defined in [14] has Independent Sequencing. Defining X^* as in 4) of Definition 5.4 and $\leq^*$ to be mod finite initial segment, then $\langle \mathcal{H}^k, X^*, \leq, \leq^* \rangle$ has Independent Sequencing. Furthermore, the $=^*$-equivalence classes are countable, since $\leq^*$ is mod finite. Thus, Theorem 5.6 holds for Blass' n-square forcing as well as the collection of hypercube topological Ramsey spaces $\mathcal{H}^k$, $k \geq 2$.

Combining the approaches for the k-arrow p-points and the hypercube spaces, the authors of [14] formed a general template for constructing topological Ramsey spaces from countable collections of Fraïssé classes. These spaces are formed as follows: For each $n < \omega$, let $J_n \geq 1$ such that either all J_n are equal or else they form an increasing sequence. For each $j < J := \sup_{n<\omega} J_n$, let $\mathcal{K}_j$ be a Fraïssé class of finite structures satisfying the Ramsey property. For each $j < J$, let $\langle \mathbb{A}_{n,j} : n < \omega \rangle$ be a sequence of members of $\mathcal{K}_j$ such that each member of $\mathcal{K}_j$ embeds into all but finitely many $\mathbb{A}_{n,j}$, and each $\mathbb{A}_{0,j}$ has cardinality one. Given a sequence $\bar{\mathbb{A}} = \langle \mathbb{A}_{n,j} : n < \omega \rangle$, a member of the space $\mathcal{R}(\bar{\mathbb{A}})$ is a sequence $x = \langle x(n) : n < \omega \rangle$, where for each $n < \omega$, $x(n)$ is a sequence $\langle \mathbb{B}_{n,j} : j < J_n \rangle$ where each $\mathbb{B}_{n,j}$ is a substructure of some $\mathbb{A}_{m,j}$ for $m \geq n$ which is isomorphic to $\mathbb{A}_{n,j}$. Each such topological Ramsey space $\mathcal{R}(\bar{\mathbb{A}})$ has Independent Sequencing where each $=^*$-class is countable. Hence, Theorem 5.6 holds for Blass' n-square forcing as well as the collection of hypercube topological Ramsey spaces $\mathcal{H}^k$, $k \geq 2$. Moreover, the forced p-points have the following interesting property: Given $\mathcal{U}_{\bar{\mathbb{A}}}$ the forced p-point from $\mathcal{R}(\bar{\mathbb{A}})$, its initial Rudin–Keisler structure is isomorphic to the embedding structure of the sequence of Fraïssé classes, whereas its initial Tukey structure is either a finite Boolean algebra (if J is finite) or else has the structure of $([\omega]^{<\omega}, \subseteq)$ if $J = \omega$.

§6. A hierarchy of non-p-points with barren extensions. The forcing $([\omega]^\omega, \subseteq^*)$ produces a Ramsey ulftrafilter, which we shall denote by $\mathcal{G}_1$ in this section. Recall that the separative quotient of $([\omega]^\omega, \subseteq^*)$ is the collection of nonzero elements of the Boolean algebra $\mathcal{P}(\omega)/\text{Fin}$. There is natural hierarchy of Boolean algebras, $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$, $k \geq 2$, extending $\mathcal{P}(\omega)/\text{Fin}$. In fact, an even more general collection of Boolean algebras $\mathcal{P}(B)/\text{Fin}^{\otimes B}$, where B is a uniform barrier of countable rank, was constructed in [11]; these can be thought of as very precise means of forming $\mathcal{P}(\omega^\alpha)/\text{Fin}^{\otimes\alpha}$, for all $1 \leq \alpha < \omega_1$. This collection of Boolean algebras differs substantially from the hierarchies of forcings described in the previous section: in particular, these forced ultrafilters are not p-points and their $=^*$-equivalence classes are not countable. Nevertheless, we will see that there are extended coarsened partial orders having Independent Sequencing which are forcing equivalent to these Boolean algebras, so they will all produce barren extensions.

The ideal $\text{Fin}^{\otimes 2}$ consists of those sets $A \subseteq \omega \times \omega$ such that for all but finitely many $n < \omega$, the nth fiber $\{j < \omega : (n, j) \in A\}$ is finite. $\text{Fin}^{\otimes 2}$ is a σ-ideal under the quasi-order $\subseteq^{\text{Fin}^{\otimes 2}}$ and the Boolean algebra $\mathcal{P}(\omega \times \omega)/\text{Fin}^{\otimes 2}$ forces an ultrafilter $\mathcal{G}_2$

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which is not a p-point but in the terminology of [5] is the *next best thing to a p-point* in the following sense: $t(\mathcal{G}_2,2)=4$. This is the strongest partition property that a non-p-point can have, since any ultrafilter satisfying $t(\mathcal{U},2)=3$ actually is a p-point. Furthermore, the projection of $\mathcal{P}(\omega\times\omega)/\mathrm{Fin}^{\otimes 2}$ to its first coordinate recovers $\mathcal{P}(\omega)/\mathrm{Fin}$, and this projection of members of $\mathcal{G}_2$ recovers a Ramsey ultrafilter on ω. Properties of this ultrafilter $\mathcal{G}_2$ have been studied in [5, 10, 22, 38. The only nonprincipal ultrafilter Rudin–Keisler strictly below $\mathcal{G}_2$ is exactly the Ramsey ultrafilter $\pi_1(\mathcal{G}_2)$, or any ultrafilter isomorphic to it (Corollary 3.9 in [5]). Thus, $\mathcal{G}_2$ is RK-minimally more complex than a Ramsey ultrafilter.

The construction of $\mathrm{Fin}^{\otimes 2}$ from Fin can be extended recursively to obtain σ-closed ideals on ω^k as well. Given $k\geq 2$ and the σ-closed ideal $\mathrm{Fin}^{\otimes k}$ on ω^k, define $A\subseteq\omega^{k+1}$ to be a member of $\mathrm{Fin}^{\otimes k+1}$ iff for all but finitely many $n<\omega$, the nth fiber $\{\vec{j}\in\omega^k:(n)^\frown\vec{j}\in A\}$ is in $\mathrm{Fin}^{\otimes k}$. This produces a hierarchy of Boolean algebras with the property that for any $1\leq j<k<\omega$, projecting the members of $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ to the first j coordinates recovers $\mathcal{P}(\omega^j)/\mathrm{Fin}^{\otimes j}$. Likewise, given an ultrafilter $\mathcal{G}_k$ forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$, projecting its members to the first j coordinates produces an ultrafilter on ω^j which is generic for $\mathcal{P}(\omega^j)/\mathrm{Fin}^{\otimes j}$. A formula for the Ramsey degrees of these ultrafilters was found by Navarro Flores and appears in [13]: For each $1\leq k<\omega$,

$$t(\mathcal{G}_k,2)=\sum_{i=0}^{k-1}3^i.$$

Moving to the countable transfinite, similarly to countable iterations of Fubini products of ultrafilters, there are choices to be made in deciding how to define $\mathrm{Fin}^{\otimes\alpha}$ for $\omega\leq\alpha<\omega_1$. However, if one works with barriers, the construction is concrete. This paper will not go into the definition and theory of barriers, but refers the interested reader to [1] for an introduction to this area. Suffice it to mention here that given a uniform barrier B on ω, the order type of B with its lexicographic order is some countable ordinal, say α_B, and every countable ordinal is achievable in this way. The recursive construction of the ideals continues by recursion on the rank of the barrier to form $\mathrm{Fin}^{\otimes B}$. Each B produces a different Boolean algebra $\mathcal{P}(B)/\mathrm{Fin}^{\otimes B}$, which force interesting ultrafilters $\mathcal{G}_B$ with the property that if B projects to a barrier C, then $\mathcal{G}_B$ has a copy of $\mathcal{G}_C$ Rudin–Keisler below it. If α_B is infinite, then for any $1\leq k<\omega$, the projection of $\mathcal{G}_B$ to its first k coordinates reproduces $\mathcal{G}_k$. We point out that the forcing properties of a related hierarchy was studied by Kurilić in [24]; his hierarchy agrees with the one here for k finite, but differs for $\alpha\geq\omega$.

In [10, 11], topological Ramsey spaces were constructed forming dense subsets of $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ for $2\leq k<\omega$ and $\mathcal{P}(B)/\mathrm{Fin}^{\otimes B}$ for all uniform barriers B of countable rank, respectively. These Ramsey spaces are denoted $\mathcal{E}_k$ and $\mathcal{E}_B$, respectively; here we shall simply use $\mathcal{E}_B$ as it subsumes the former case. The Ramsey structure was utilized in [10] to prove that both the initial Rudin–Keisler and initial Tukey structures below the forced ultrafilter $\mathcal{G}_k$ is exactly a chain of length k. (The extension of this to the broader collection of $\mathcal{G}_B$ for B a barrier is in preparation.) In particular, it was shown that $\mathcal{G}_2$ is Tukey-minimal above the projected Ramsey ultrafilter $\pi_1(\mathcal{G}_2)$, answering a question that was left open in [5].

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Refraining from going into detail about these spaces here, what is important for this paper is that given a uniform barrier B, each member of the space $\mathcal{E}_B$ may be regarded as an infinite sequence $x = \langle x(n) : n < \omega \rangle$ such that each $x(n)$ is a fiber of B (so infinite) and that $\langle \mathcal{E}_B, \leq \rangle$ has Independent Sequencing. Furthermore, letting $\mathcal{E}_B^*$ be defined as in 4) of IS and $\leq^*$ be defined by $b \leq^* a$ iff for all but finitely many n, $b(n) \subseteq a(i)$ for some i, then $\langle \mathcal{E}_B, \mathcal{E}_B^*, \leq, \leq^* \rangle$ is an extended coarsened poset having Independent Sequencing. The forcings $\langle \mathcal{E}_B, \leq \rangle$, $\langle \mathcal{E}_B^*, \leq^* \rangle$ and $\langle (\mathrm{Fin}^{\otimes B})^+, \subseteq^{\mathrm{Fin}^{\otimes B}} \rangle$ each have separative quotient which is isomorphic to the nonzero members of $\mathcal{P}(B)/\mathrm{Fin}^{\otimes B}$, hence all force the same ultrafilter, $\mathcal{G}_B$. The properties of the spaces that make this true are contained in the work in [10, 11]. Thus, Theorem 5.6 yields that forcing with $\mathcal{P}(B)/\mathrm{Fin}^{\otimes B}$ adds no new sets of ordinals in $M[\mathcal{G}_B]$, assuming M satisfies ZF and either $\mathrm{AD}_{\mathbb{R}}$ or $\mathrm{AD}^+ + V = L(\mathcal{P}(R))$.

§7. Further directions and open problems. As forcings, topological Ramsey spaces have so many characteristics in common with $\mathcal{P}(\omega)/\mathrm{Fin}$ that a natural goal is to find out exactly how far these similarities persist. One direction is preservation of strong partition cardinals. In Section 6, we showed that the forcings $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$, $2 \leq \alpha < \omega_1$, produce barren extensions. However, our methods do not prove that these forcings preserve strong partition cardinals.

QUESTION 7.1. *Does forcing with* $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$ *preserve strong partition cardinals? If so, does the same hold for each* $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$, $2 \leq \alpha < \omega_1$?

If so, then the following question is worth pursuing, as Navarro Flores has shown that each topological Ramsey space adds a new ultrafilter with a Ramsey ultrafilter Rudin–Keisler below it [19].

QUESTION 7.2. *Does forcing with any topological Ramsey space preserve strong partition cardinals?*

Recently, Zheng proved in [42] and [44] that the ultrafilters considered in Sections 5.1 and 5.3 are preserved under side-by-side Sacks forcing with countable support, and it follows from work in [44] that this also holds for the ultrafilters considered in 5.4. In [43], Zheng also proved that ultrafilters forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ are preserved under side-by-side Sacks forcing with countable support, and her methods should also hold for the whole hierarchy into countable ordinals, using work in [11].

QUESTION 7.3. *Is there a connection between an ultrafilter being preserved by side-by-side Sacks forcing and having a barren extensions? Does one imply the other?*

By Corollary 5.3 in [7], if $L(\mathbb{R})$ is a Solovay model, then the $\mathcal{P}(\omega)/\mathrm{Fin}$ extension of $L(\mathbb{R})$ satisfies the perfect set property. So we ask the following:

QUESTION 7.4. *Assume* $AD_{\mathbb{R}}$ *or* $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Which topological Ramsey spaces preserve the perfect set property via forcing?*

In particular, we conjecture that topological Ramsey spaces with Independent Sequencing will preserve the perfect set property. It may well be the case that all topological Ramsey spaces behave like $([\omega]^\omega, \subseteq^*)$ in this and many more respects.

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