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# Ground states for the pseudo-relativistic Hartree equation with external potential

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We prove the existence of positive ground state solutions to the pseudo-relativistic Schrödinger equation

$$\sqrt{-\Delta + m^2}u + Vu = (W * |u|^\theta)|u|^{\theta-2}u \text{ in } \mathbb{R}^N, \quad u \in H^{1/2}(\mathbb{R}^N),$$

where $N \geqslant 3$, $m > 0$, $V$ is a bounded external scalar potential and $W$ is a radially symmetric convolution potential satisfying suitable assumptions. We also provide some asymptotic decay estimates of the found solutions.

## 1. Introduction

The mean field limit of a quantum system describing many self-gravitating relativistic bosons with rest mass $m > 0$ leads to the time-dependent pseudo-relativistic Hartree equation

$$\mathrm{i}\frac{\partial\psi}{\partial t} = (\sqrt{-\Delta + m^2} - m)\psi - \left(\frac{1}{|x|} * |\psi|^2\right)\psi, \quad x \in \mathbb{R}^3, \tag{1.1}$$

where $\psi\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ is the wave field. Such a physical system is often referred to as a boson star in astrophysics (see [12–14]). Solitary wave solutions $\psi(t, x) = \mathrm{e}^{-\mathrm{i}t\lambda}\phi$, $\lambda \in \mathbb{R}$, to (1.1) satisfy the equation

$$(\sqrt{-\Delta + m^2} - m)\phi - \left(\frac{1}{|x|} * |\phi|^2\right)\phi = \lambda\phi. \tag{1.2}$$

For the non-relativistic Hartree equation, the existence and uniqueness (modulo translations) of a minimizer was proved by Lieb [17] by using symmetric decreasing rearrangement inequalities. Within the same setting, always for the negative Laplacian, Lions [21] proved the existence of infinitely many spherically symmetric solutions by application of abstract critical point theory both without the constraint





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and with the constraint for a more general radially symmetric convolution potential. The non-relativistic Hartree equation is also known as the Choquard–Pekard or Schrödinger–Newton equation, and recently a large number of papers have been devoted to the study of solitary states and its semiclassical limit (see [1, 6–9, 13, 19, 22, 24, 25, 27–29, 31, 32] and references therein).

In [20], Lieb and Yau solved the pseudo-relativistic Hartree equation (1.2) by minimization on the sphere $\{\phi \in L^2(\mathbb{R}^3) \mid \int_{\mathbb{R}^3}|\phi|^2 = M\}$, and they proved that a radially symmetric ground state exists in $H^{1/2}(\mathbb{R}^3)$ whenever $M < M_c$, the so-called Chandrasekhar mass. These results have been generalized in [11]. Later, Lenzmann proved in [16] that this ground state is unique (up to translations and phase change) provided that the mass $M$ is sufficiently small; some results about the non-degeneracy of the ground state solution are also given.

Quite recently, Coti Zelati and Nolasco (see [10]) studied the equation

$$\sqrt{-\Delta + m^2}u = \mu u + \nu|u|^{p-2}u + \sigma(W * u^2)u$$

under the assumptions that $p \in (2, 2N/(N-1))$, $N \geqslant 3$, $\mu < m$, $m > 0$, $\nu \geqslant 0$ and $\sigma \geqslant 0$ but not both 0, $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $W \geqslant 0$, $r > N/2$, $W$ is radially symmetric and decays to 0 at $\infty$. They proved the existence of a positive radial solution that decays to 0 at $\infty$ exponentially fast. For the case $\sigma < 0$, $\mu < m$, we also refer the reader to [26] where a more general nonlinear term is considered.

In the present work we consider a generalized pseudo-relativistic Hartree equation

$$\sqrt{-\Delta + m^2}u + Vu = (W * |u|^\theta)|u|^{\theta-2}u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $N \geqslant 3$, $m > 0$, $V$ is an external potential and $W \geqslant 0$ is a radially symmetric convolution kernel such that $\lim_{|x|\to+\infty} W(|x|) = 0$.

In [23] Melgaard and Zongo proved that (1.3) has a sequence of radially symmetric solutions of higher and higher energy, assuming that $V$ is a radially symmetric potential, $\theta = 2$, and under some restrictive assumptions on the structure of the kernel $W$.

Here we are interested in finding positive ground state solutions for the pseudo-relativistic Hartree equation (1.3) when $V$ is not symmetric. In such a case, the non-locality of $\sqrt{-\Delta + m^2}$ and the presence of the external potential $V$ (not symmetric) complicate the analysis of the pseudo-relativistic Hartree equation in a substantial way. The main difficulty is, as usual, the lack of compactness.

In what follows, we make the following assumptions.

(V1) $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous and bounded function, and $V(y) + V_0 \geqslant 0$ for every $y \in \mathbb{R}^N$ and for some $V_0 \in (0, m)$.

(V2) There exist $R > 0$ and $k \in (0, 2m)$ such that

$$V(x) \leqslant V_\infty - \mathrm{e}^{-k|x|} \quad \text{for all } |x| \geqslant R, \tag{1.4}$$

where $V_\infty = \liminf_{|x|\to+\infty} V(x) > 0$.

(W) $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some

$$r > \max\left\{1, \frac{N}{N(2-\theta)+\theta}\right\} \quad \text{and} \quad 2 \leqslant \theta < \frac{2N}{N-1}.$$

Our main result is the following.

Theorem 1.1. *Retain assumptions (V1), (V2) and (W). Equation (1.3) then has at least a positive solution $u \in H^{1/2}(\mathbb{R}^N)$.*

We remark that theorem 1.1 applies for a large class of bounded electric potentials without symmetric constraints and covers the physically relevant cases of Newton- or Yukawa-type two-body interactions, i.e. $W(x) = 1/|x|^\lambda$ with $0 < \lambda < 2$, $W(x) = \mathrm{e}^{-|x|}/|x|$.

The theorem is proved using variational methods. Firstly we transform the problem into an elliptic equation with nonlinear Neumann boundary conditions, using a *local* realization of the pseudo-differential operator $\sqrt{-\Delta + m^2}$ as in [3, 4, 10, 11]. The corresponding solutions are found as critical points of an Euler functional defined in $H^1(\mathbb{R}^{N+1}_+)$. We show that such a functional satisfies the Palais–Smale condition below some energy level determined by the value of $V$ at $\infty$, and we prove the existence of a mountain pass solution under the level where the Palais–Smale condition holds. Finally, we also provide some asymptotic decay estimates of the found solution.

Local and global well-posedness results for pseudo-relativistic Hartree equations with external potential were proved by Lenzmann in [15].

## 2. The variational framework

Before we state our main result, we recall a few basic facts about the functional setting of our problem. The operator $\sqrt{-\Delta + m^2}$ can be defined by Fourier analysis: given any $\phi \in L^2(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} (m^2 + |k|^2)|\mathcal{F}\phi|^2 \,\mathrm{d}k < +\infty,$$

we define $\sqrt{-\Delta + m^2}\phi$ via the identity

$$\mathcal{F}(\sqrt{-\Delta + m^2}\phi) = \sqrt{m^2 + |k|^2}\mathcal{F}\phi,$$

$\mathcal{F}$ being the usual Fourier transform. The condition

$$\int_{\mathbb{R}^3} \sqrt{m^2 + |k|^2}|\mathcal{F}\phi|^2 \,\mathrm{d}k < +\infty$$

is known to be equivalent to $\phi \in H^{1/2}(\mathbb{R}^3)$. In this sense, the fractional Sobolev space $H^{1/2}(\mathbb{R}^3)$ is the natural space to work in. However, this definition is not particularly convenient for variational methods, and we prefer a *local* realization of the operator in the augmented half-space, originally inspired by the paper [5] for the fractional Laplacian.

Given $u \in \mathcal{S}(\mathbb{R}^N)$, the Schwarz space of rapidly decaying smooth functions defined on $\mathbb{R}^N$, there exists one and only one function $v \in \mathcal{S}(\mathbb{R}^{N+1}_+)$ (where $\mathbb{R}^{N+1}_+ = (0, +\infty) \times \mathbb{R}^N$) such that

$$\begin{aligned} -\Delta v + m^2 v = 0 \quad &\text{in } \mathbb{R}^{N+1}_+, \\ v(0, y) = u(y) \quad &\text{for } y \in \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{aligned}$$

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Setting

$$Tu(y) = -\frac{\partial v}{\partial x}(0, y),$$

we easily see that the problem

$$-\Delta w + m^2 w = 0 \quad \text{in } \mathbb{R}^{N+1}_+,$$
$$w(0, y) = Tu(y) \quad \text{for } y \in \partial\mathbb{R}^{N+1}_+ = \{0\} \times \mathbb{R}^N \simeq \mathbb{R}^N$$

is solved by $w(x, y) = -(\partial v/\partial x)(x, y)$. From this we deduce that

$$T(Tu)(y) = -\frac{\partial w}{\partial x}(0, y) = \frac{\partial^2 v}{\partial x^2}(0, y) = (-\Delta_y v + m^2 v)(0, y),$$

and hence $T \circ T = (-\Delta_y + m^2)$, namely, $T$ is a square root of the Schrödinger operator $-\Delta + m^2$ on $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$.

In the following, we write $|\cdot|_p$ for the norm in $L^p(\mathbb{R}^N)$ and $\|\cdot\|_p$ for the norm in $L^p(\mathbb{R}^{N+1}_+)$. The symbol $\|\cdot\|$ is reserved for the usual norm of $H^1(\mathbb{R}^{N+1}_+)$.

The theory of traces for Sobolev spaces ensures that every function $v \in H^1(\mathbb{R}^{N+1}_+)$ possesses a *trace* $\gamma(v) \in H^{1/2}(\mathbb{R}^N)$ that satisfies the inequality (see [30, lemma 13.1])

$$|\gamma(v)|_p^p \leqslant p\|v\|_{2(p-1)}^{p-1} \left\|\frac{\partial v}{\partial x}\right\|_2 \tag{2.1}$$

whenever $2 \leqslant p \leqslant 2N/(N-1)$. This also implies that, for every $\lambda > 0$,

$$\int_{\mathbb{R}^N} \gamma(v)^2 \leqslant \lambda \int_{\mathbb{R}^{N+1}_+} |v|^2 + \frac{1}{\lambda} \int_{\mathbb{R}^{N+1}_+} \left|\frac{\partial v}{\partial x}\right|^2. \tag{2.2}$$

As a particular case, we record

$$\int_{\mathbb{R}^N} \gamma(v)^2 \leqslant m \int_{\mathbb{R}^{N+1}_+} |v|^2 + \frac{1}{m} \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2. \tag{2.3}$$

It is also known (see [30, lemma 16.1]) that any element of $H^{1/2}(\mathbb{R}^N)$ is the trace of some function in $H^1(\mathbb{R}^{N+1}_+)$.

From the previous construction and following [3, 4, 10, 11], we can replace the *non-local* problem (1.3) with the local Neumann problem

$$\left.\begin{aligned} -\Delta v + m^2 v = 0 \quad &\text{in } \mathbb{R}^{N+1}_+, \\ -\frac{\partial v}{\partial x} = -V(y)v + (W * |v|^\theta)|v|^{\theta-2} v \quad &\text{in } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{aligned}\right\} \tag{2.4}$$

We are looking for solutions to (2.4) as critical points of the Euler functional $I\colon H^1(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ defined by

$$\begin{aligned} I(v) = \frac{1}{2} \iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2)\,\mathrm{d}x\,\mathrm{d}y \\ + \frac{1}{2} \int_{\mathbb{R}^N} V(y)\gamma(v)^2\,\mathrm{d}y - \frac{1}{2\theta} \int_{\mathbb{R}^N} (W * |\gamma(v)|^\theta)|\gamma(v)|^\theta\,\mathrm{d}y. \end{aligned} \tag{2.5}$$

We recall the well-known Young inequality.

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Proposition 2.1. *Assume that* $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$ *with* $1 \leqslant p,\ q, r \leqslant \infty$, $p^{-1} + q^{-1} = 1 + r^{-1}$. *Then*

$$|f * g|_r \leqslant |f|_p |g|_q. \tag{2.6}$$

A major tool for our analysis is the generalized Hardy–Littlewood–Sobolev inequality. We recall that $L^q_w(\mathbb{R}^N)$ is the *weak* $L^q$ space; see [18] for a definition. We denote by $|\cdot|_{q,w}$ the usual norm in $L^q_w(\mathbb{R}^N)$.

Proposition 2.2 (Lieb [18]). *Assume that* $p$, $q$ *and* $t$ *lie in* $(1, +\infty)$ *and that* $p^{-1} + q^{-1} + t^{-1} = 2$. *Then, for some constant* $N_{p,q,t} > 0$ *and for any* $f \in L^p(\mathbb{R}^N)$, $g \in L^t(\mathbb{R}^N)$ *and* $h \in L^q_w(\mathbb{R}^N)$, *we have the inequality*

$$\left| \int f(x) h(x-y) g(y)\,\mathrm{d}x\,\mathrm{d}y \right| \leqslant N_{p,q,t} |f|_p |g|_t |h|_{q,w}. \tag{2.7}$$

For the sake of completeness, we check that the functional $I$ is well defined. As a consequence of (2.1), for every $p \in [2, 2N/(N-1)]$ we deduce that

$$|\gamma(v)|_p \leqslant \frac{p-1}{p} \|v\|_{2(p-1)} + \|\nabla v\|_2 \leqslant C_p \|v\|, \tag{2.8}$$

and the term $\int_{\mathbb{R}^N} V(y)\gamma(v)^2\,\mathrm{d}y$ in the expression of $I(v)$ is finite because of the boundedness of $V$. Writing $W = W_1 + W_2 \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ and using (2.7) we can estimate the convolution term as follows:

$$\begin{aligned} \int_{\mathbb{R}^N} (W * |\gamma(v)|^\theta)|\gamma(v)|^\theta &= \int_{\mathbb{R}^N} (W_1 * |\gamma(v)|^\theta)|\gamma(v)|^\theta + \int_{\mathbb{R}^N} (W_2 * |\gamma(v)|^\theta)|\gamma(v)|^\theta \\ &\leqslant |W_1|_r |\gamma(v)|^{2\theta}_{2r\theta/(2r-1)} + |W_2|_\infty |\gamma(v)|^{2\theta}_\theta \\ &\leqslant |W_1|_r \|v\|^{2\theta} + |W_2|_\infty \|v\|^{2\theta}. \end{aligned} \tag{2.9}$$

Since

$$r > \frac{N}{N(2-\theta)+\theta} \quad \text{and} \quad 2 \leqslant \theta < \frac{2N}{N-1},$$

we have that

$$\frac{2r-1}{2\theta r} = \frac{1}{\theta} - \frac{1}{2\theta r} > \frac{1}{\theta} - \frac{N(2-\theta)+\theta}{2\theta N} = \frac{N-1}{2N},$$

and thus

$$\frac{2\theta r}{2r-1} < \frac{2N}{N-1},$$

and from (2.9) we see that the convolution term in $I$ is finite. It is easy to check, by the same token, that $I \in C^1(H^1(\mathbb{R}^{N+1}_+))$.

Remark 2.3. As we have just seen, estimates involving the kernel $W$ always split into two parts. As a rule, those with the *bounded* kernel $W_2 \in L^\infty(\mathbb{R}^N)$ are straightforward. In the following, we often focus on the contribution of $W_1 \in L^r(\mathbb{R}^N)$ and drop the easy computation with $W_2$.

## 3. The limit problem

We consider the space of the symmetric functions

$$H^\sharp = \{u \in H^1(\mathbb{R}^{N+1}_+) \mid u(x, Ry) = u(x,y) \text{ for all } R \in O(N)\}.$$

We consider the functional $J_\alpha\colon H^\sharp \to \mathbb{R}$ defined by setting

$$J_\alpha(v) = \frac{1}{2}\iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^N} \alpha\gamma(v)^2\,\mathrm{d}y - \frac{1}{2\theta}\int_{\mathbb{R}^N} (W * |\gamma(v)|^\theta)|\gamma(v)|^\theta\,\mathrm{d}y, \tag{3.1}$$

where $W \geqslant 0$ is radially symmetric, $\lim_{|x|\to+\infty} W(|x|) = 0$ and assumption (W) holds. If $\alpha > -m$, we can extend the arguments in [10, theorem 4.3], for the case $\theta = 2$, and prove that the functional $J_\alpha$ has a mountain pass critical point $v_\alpha \in H^\sharp$, namely,

$$J_\alpha(v_\alpha) = E_\alpha = \inf_{g\in\Gamma_\sharp}\max_{t\in[0,1]} J_\alpha(g(t)), \tag{3.2}$$

where $\Gamma_\sharp = \{g \in C([0,1]; H^\sharp) \mid g(0) = 0,\ J_\alpha(g(1)) < 0\}$. The critical point $v_\alpha$ corresponds to a weak solution of

$$\left.\begin{aligned} -\Delta v + m^2 v = 0 &\quad \text{in } \mathbb{R}^{N+1}_+, \\ -\frac{\partial v}{\partial x} = -\alpha v + (W * |v|^\theta)|v|^{\theta-2}v &\quad \text{in } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{aligned}\right\} \tag{3.3}$$

In the following, we need a standard characterization of the mountain pass level $E_\alpha$. We define the *Nehari manifold* $\mathcal{N}_\alpha$ associated with the functional $J_\alpha$ as

$$\mathcal{N}_\alpha = \left\{ v \in H^\sharp \;\middle|\; \iint_{\mathbb{R}^N\times\mathbb{R}^N} |\nabla v|^2 + m^2 v^2\,\mathrm{d}x\,\mathrm{d}y = -\alpha\int_{\mathbb{R}^N}\gamma(v)^2\,\mathrm{d}y + \int_{\mathbb{R}^N}(W * |\gamma(v)|^\theta)|\gamma(v)|^\theta\,\mathrm{d}y \right\}. \tag{3.4}$$

Lemma 3.1. *The following identities hold true:*

$$\inf_{v\in\mathcal{N}_\alpha} J_\alpha(v) = \inf_{v\in H^\sharp}\max_{t>0} J_\alpha(tv) = E_\alpha. \tag{3.5}$$

*Proof.* The proof is straightforward, since $J_\alpha$ is the sum of homogeneous terms; we follow the method of [33]. First of all, for $v \in H^\sharp$ we compute

$$J_\alpha(tv) = \frac{t^2}{2}\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} |\nabla v|^2 + m^2 v^2\,\mathrm{d}x\,\mathrm{d}y + \alpha\int_{\mathbb{R}^N}\gamma(v)^2\,\mathrm{d}y\right) - \frac{t^{2\theta}}{2\theta}\int_{\mathbb{R}^N}(W * |\gamma(v)|^\theta)|\gamma(v)|^\theta\,\mathrm{d}y. \tag{3.6}$$

Since $\theta \geqslant 2$, and by using (2.3), it is easy to check that $t \in (0,+\infty) \mapsto J(tv)$ possesses a unique critical point $t = t(v) > 0$ such that $t(v)v \in \mathcal{N}_\alpha$. Moreover, since $J_\alpha$ has the mountain pass geometry, $t = t(v)$ is a maximum point. It follows that

$$\inf_{v\in\mathcal{N}_\alpha} J_\alpha(v) = \inf_{v\in H^\sharp}\max_{t>0} J_\alpha(tv).$$

The manifold $\mathcal{N}_\alpha$ splits $H^\sharp$ into two connected components, and the component containing 0 is open. In addition, $J_\alpha$ is non-negative on this component, because $\langle J'_\alpha(tv), v\rangle \geqslant 0$ when $0 < t \leqslant t(v)$. It follows immediately that any path $\gamma\colon [0,1] \to H^\sharp$ with $\gamma(0) = 0$ and $J_\alpha(\gamma(1)) < 0$ must cross $\mathcal{N}_\alpha$, so

$$E_\alpha \geqslant \inf_{v\in\mathcal{N}_\alpha} J_\alpha(v).$$

The proof of (3.5) is complete. □

Following a completely analogous argument in [10, theorems 3.14 and 5.1], we can state the following result.

THEOREM 3.2. *Let $\alpha + m > 0$ and let (W) hold. Then $v_\alpha \in C^\infty([0,+\infty) \times \mathbb{R}^N)$, $v_\alpha(x,y) > 0$ in $[0,\infty) \times \mathbb{R}^N$ and for any $0 \leqslant \sigma \in (-\alpha, m)$ there exists $C > 0$ such that*

$$0 < v_\alpha(x,y) \leqslant C\mathrm{e}^{-(m-\sigma)\sqrt{x^2+|y|^2}}\mathrm{e}^{-\sigma x}$$

*for all $(x,y) \in [0,+\infty) \times \mathbb{R}^N$. In particular,*

$$0 < v_\alpha(0,y) \leqslant C\mathrm{e}^{-\delta|y|} \quad \text{for every } y \in \mathbb{R}^N,$$

*where $0 < \delta < m + \alpha$ if $\alpha \leqslant 0$, and $\delta = m$ if $\alpha > 0$.*

## 4. The Palais–Smale condition

For any $v \in H^1(\mathbb{R}^{N+1}_+)$ we define

$$\mathbb{D}(v) = \iint_{\mathbb{R}^N\times\mathbb{R}^N} W(x-y)|\gamma(v)(x)|^\theta|\gamma(v)(y)|^\theta \,\mathrm{d}x\,\mathrm{d}y.$$

Inequality (2.9) immediately yields that

$$\mathbb{D}(v) \leqslant K\|v\|^{2\theta} \tag{4.1}$$

for every $v \in H^1(\mathbb{R}^{N+1}_+)$.

LEMMA 4.1. *Let $\{v_n\}_n$ be a sequence in $H^1(\mathbb{R}^{N+1}_+)$ such that $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^{N+1}_+)$,*

$$I(v_n) \to c < E_{V_\infty} \quad \text{and} \quad I'(v_n) \to 0,$$

*where $V_\infty := \liminf_{|x|\to\infty} V(x) > 0$. A subsequence of $\{v_n\}_n$ then converges strongly to 0 in $H^1(\mathbb{R}^{N+1}_+)$.*

*Proof.* First, we recall (2.3), and we rewrite $I(v)$ as

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 + m^2|v|^2 + \frac{1}{2}\int_{\mathbb{R}^N}(V+V_0)\gamma(v)^2 - \frac{V_0}{2}\int_{\mathbb{R}^N}\gamma(v)^2 - \frac{1}{2\theta}\mathbb{D}(v),$$

so that $V + V_0 \geqslant 0$ everywhere. Now,

$$c + 1 + \|v_n\| \geqslant I(v_n) - \frac{1}{2}\langle I'(v_n), v_n\rangle = \left(\frac{1}{2} - \frac{1}{2\theta}\right)\mathbb{D}(v_n), \tag{4.2}$$

which implies that, for some constants $C_1$ and $C_2$,

$$\frac{1}{2\theta}\mathbb{D}(v_n) \leqslant C_1\|v_n\| + C_2.$$

But then, using (2.3),

$$\begin{aligned} c+1 &\geqslant I(v_n) \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla v_n|^2 + \frac{m^2}{2}\int_{\mathbb{R}^{N+1}_+} |v_n|^2 \\ &\quad - \frac{V_0}{2}\left(m\int_{\mathbb{R}^{N+1}_+} |v_n|^2 + \frac{1}{m}\int_{\mathbb{R}^{N+1}_+} |\nabla v_n|^2\right) - C_1\|v_n\| - C_2 \\ &= \frac{1}{2}\left(1 - \frac{V_0}{m}\right)\int_{\mathbb{R}^{N+1}_+} |\nabla v_n|^2 + \frac{m(m-V_0)}{2}\int_{\mathbb{R}^{N+1}_+} |v_n|^2 - C_1\|v_n\| - C_2, \end{aligned}$$

and since $m - V_0 > 0$ we deduce that $\{v_n\}$ is a bounded sequence in $H^1(\mathbb{R}^{N+1}_+)$.

A standard argument shows that $\|v_n\|$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$,

$$\frac{\theta-1}{2\theta}\left(\|v_n\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(v_n)^2\,\mathrm{d}y\right) \to c \quad \text{and} \quad \frac{\theta-1}{2\theta}\mathbb{D}(v_n) \to c.$$

Therefore, $c \geqslant 0$. If $c = 0$, then

$$\begin{aligned} o(1) &= \left(\|v_n\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(v_n)^2\,\mathrm{d}y\right) \\ &\geqslant \left(1 - \frac{V_0}{m}\right)\int_{\mathbb{R}^{N+1}_+} |\nabla v_n|^2 + m(m-V_0)\int_{\mathbb{R}^{N+1}_+} |v_n|^2, \end{aligned}$$

and $m - V_0 > 0$ yields that $v_n \to 0$ strongly in $H^1(\mathbb{R}^{N+1}_+)$.

Assume, therefore, that $c > 0$. Fix $\alpha < V_\infty$ such that $c < E_\alpha$, and fix $R_0 > 0$ such that $V(x) \geqslant \alpha$ if $|x| \geqslant R_0$. Let $\varepsilon \in (0,1)$. Since $\{v_n\}_n$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$, there exists $R_\varepsilon > R_0$ such that $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and, after passing to a subsequence,

$$\iint_{S_{R_\epsilon}} (|\nabla v_n|^2 + m^2 v_n^2)\,\mathrm{d}x\,\mathrm{d}y + \int_{A_{R_\epsilon}} V(y)\gamma(v_n)^2\,\mathrm{d}y < \varepsilon \quad \text{for all } n \in \mathbb{N}, \tag{4.3}$$

where

$$\begin{aligned} S_{R_\epsilon} &= \{z = (x,y) \in \mathbb{R}^{N+1}_+ \mid R_\varepsilon < |z| < R_\varepsilon + 1\}, \\ A_{R_\epsilon} &= \{y \in \mathbb{R}^N \mid R_\varepsilon < |y| < R_\varepsilon + 1\}. \end{aligned}$$

If this is not the case, for any $m \in \mathbb{N}$, $m \geqslant R_0$, there exists $\nu(m) \in \mathbb{N}$ such that

$$\iint_{S_m} (|\nabla v_n|^2 + m^2 v_n^2)\,\mathrm{d}x\,\mathrm{d}y + \int_{A_m} V(y)\gamma(v_n)^2\,\mathrm{d}y \geqslant \varepsilon \tag{4.4}$$

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for any $n \in \mathbb{N}$, $n \geqslant \nu(m)$. We can assume that $\nu(m)$ is non-decreasing. Therefore, for any integer $m \geqslant R_0$ there exists an integer $\nu(m)$ such that

$$\|v_n\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(v_n)^2 \,\mathrm{d}y \geqslant \iint_{T_m} (|\nabla v_n|^2 + m^2 v_n^2) \,\mathrm{d}x \,\mathrm{d}y + \int_{B_m} V(y)\gamma(v_n)^2 \,\mathrm{d}y$$
$$\geqslant (m - R_0)\varepsilon \tag{4.5}$$

for any $n \geqslant \nu(m)$, where $T_m = \{z = (x, y) \in \mathbb{R}^{N+1}_+ \mid R_0 < |z| < m\}$ and $B_m = \{y \in \mathbb{R}^N \mid R_0 < |y| < m\}$, which contradicts the fact that $\|v_n\|$ is bounded.

We may assume that $|v_n| \to 0$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ with $p < 2N/(N-1)$, and thus $|\gamma(v_n)| \to 0$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

Let $\xi_\varepsilon \in C^\infty(\mathbb{R}^{N+1}_+)$ be a symmetric function, namely, $\xi_\varepsilon(x, gy) = \xi_\varepsilon(x, y)$ for all $g \in O(N)$, $x > 0$, $y \in \mathbb{R}^N$. Moreover, assume that $\xi_\varepsilon(z) = 0$ if $|z| \leqslant R_\varepsilon$, $\xi_\varepsilon(z) = 1$ if $|z| \geqslant R_\varepsilon + 1$ and $\xi(z) \in [0, 1]$ for all $z \in \mathbb{R}^{N+1}_+$. Set $w_n = \xi_\varepsilon v_n$. We now apply Young's inequality (2.7) with $p = q = 2r/(2r-1)$ and

$$h = W, \qquad f = |\gamma(v_n)|^\theta, \qquad g = |\gamma(v_n)|^\theta - |\gamma(w_n)|^\theta$$

to obtain

$$\begin{aligned}
&|\mathbb{D}(v_n) - \mathbb{D}(w_n)| \\
&\quad \leqslant \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)||\gamma(v_n)(x)|^\theta |\gamma(v_n)(y)|^\theta - |\gamma(w_n)(x)|^\theta |\gamma(w_n)(y)|^\theta| \,\mathrm{d}x \,\mathrm{d}y \\
&\quad = \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)||\gamma(v_n)(x)|^\theta |\gamma(v_n)(y)|^\theta - |\gamma(v_n)(x)|^\theta |\gamma(w_n)(y)|^\theta \\
&\qquad\qquad + |\gamma(v_n)(x)|^\theta |\gamma(w_n)(y)|^\theta - |\gamma(w_n)(x)|^\theta |\gamma(w_n)(y)|^\theta| \,\mathrm{d}x \,\mathrm{d}y \\
&\quad \leqslant \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)|\gamma(v_n)(x)|^\theta ||\gamma(v_n)(y)|^\theta - |\gamma(w_n)(y)|^\theta| \,\mathrm{d}x \,\mathrm{d}y \\
&\qquad + \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)|\gamma(w_n)(y)|^\theta ||\gamma(v_n)(x)|^\theta - |\gamma(w_n)(x)|^\theta| \,\mathrm{d}x \,\mathrm{d}y \\
&\quad \leqslant 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)|\gamma(v_n)(x)|^\theta ||\gamma(v_n)(y)|^\theta - |\gamma(w_n)(y)|^\theta| \,\mathrm{d}x \,\mathrm{d}y \\
&\quad \leqslant 2C|W|_r |\gamma(v_n)|^\theta_{2r\theta/(2r-1)} ||\gamma(v_n)|^\theta - |\gamma(w_n)|^\theta|_{2r/(2r-1)} = o(1),
\end{aligned} \tag{4.6}$$

since $|\gamma(v_n)|^\theta - |\gamma(w_n)|^\theta \to 0$ strongly in $L^{2r/(2r-1)}_{\mathrm{loc}}(\mathbb{R}^N)$. Here and in the following $C$ denotes some positive constant independent of $n$, not necessarily the same one each time. Similarly,

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^N} (W * |\gamma(v_n)|^\theta)|\gamma(v_n)|^{\theta-2}\gamma(v_n)\gamma(w_n) \\
&\qquad - \int_{\mathbb{R}^N} (W * |\gamma(w_n)|^\theta)|\gamma(w_n)|^{\theta-2}\gamma(w_n)\gamma(w_n) \bigg| \\
&\qquad\qquad \leqslant 2C|W|_r |\gamma(v_n)|^\theta_{2r\theta/(2r-1)} ||\gamma(v_n)|^\theta - |\gamma(w_n)|^\theta|_{2r/(2r-1)} \\
&\qquad\qquad = o(1).
\end{aligned}$$

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Therefore,

$$|I'(v_n)w_n - I'(w_n)w_n| \leqslant C \iint_{S_\epsilon} (|\nabla v_n|^2 + m^2 v_n^2)\,\mathrm{d}x\,\mathrm{d}y + \int_{A_\epsilon} V(y)\gamma(v_n)^2\,\mathrm{d}y + o(1).$$

Set $u_n = (1-\xi)v_n$. Analogously, we have

$$|I'(v_n)u_n - I'(u_n)u_n| \leqslant C \iint_{S_\epsilon} (|\nabla u_n|^2 + m^2 u_n^2)\,\mathrm{d}x\,\mathrm{d}y + \int_{A_\epsilon} V(y)\gamma(u_n)^2\,\mathrm{d}y + o(1).$$

Therefore,

$$I'(u_n)u_n = O(\epsilon) + o(1) \tag{4.7}$$

and

$$I'(w_n)w_n = O(\epsilon) + o(1). \tag{4.8}$$

From (4.7), we derive that

$$I(u_n) = \frac{\theta-1}{2\theta}\mathbb{D}(u_n) + O(\epsilon) + o(1) \geqslant O(\epsilon) + o(1).$$

Consider $t_n > 0$ such that $I'(t_n w_n)(t_n w_n) = 0$ for any $n$, namely,

$$t_n^{2(\theta-1)} = \frac{\|w_n\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(w_n)^2\,\mathrm{d}y}{\mathbb{D}(w_n)}.$$

From (4.8), we have that $t_n = 1 + O(\epsilon) + o(1)$. Therefore, from the characterization of $E_\alpha$ we have

$$\begin{aligned} c + o(1) = I(v_n) &= I(u_n) + I(w_n) + O(\epsilon) \\ &\geqslant I(w_n) + O(\epsilon) + o(1) \\ &\geqslant I(t_n w_n) + O(\epsilon) + o(1) \\ &\geqslant E_\alpha + O(\epsilon) + o(1). \end{aligned}$$

As $n \to +\infty$, $\epsilon \to 0$, we derive that $c \geqslant E_\alpha$, which is a contradiction. Hence, $c = 0$ and $v_n \to 0$ strongly in $H^1(\mathbb{R}^{N+1}_+)$. □

LEMMA 4.2. *Let $\{v_n\}_n$ be a sequence in $H^1(\mathbb{R}^{N+1}_+)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^{N+1}_+)$. The following hold.*

(i) *$\mathbb{D}'(v_n)u \to \mathbb{D}'(v)u$ for all $u \in H^1(\mathbb{R}^{N+1}_+)$.*

(ii) *After passing to a subsequence, there exists a sequence $\{\tilde{v}_n\}_n$ in $H^1(\mathbb{R}^{N+1}_+)$ such that $\tilde{v}_n \to v$ strongly in $H^1(\mathbb{R}^{N+1}_+)$,*

$$\begin{aligned} \mathbb{D}(v_n) - \mathbb{D}(v_n - \tilde{v}_n) &\to \mathbb{D}(v) \quad \text{in } \mathbb{R}, \\ \mathbb{D}'(v_n) - \mathbb{D}'(v_n - \tilde{v}_n) &\to \mathbb{D}'(v) \quad \text{in } H^{-1}(\mathbb{R}^{N+1}_+). \end{aligned}$$

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*Proof.* The proof is completely analogous to that of [1, lemma 3.5]. The function $\tilde{v}_n$ is the product of $v_n$ with a smooth cut-off function, so $\tilde{v}_n$ belongs to $H^1(\mathbb{R}_+^{N+1})$ if $v_n$ does. We omit the details. □

PROPOSITION 4.3. *The functional* $I\colon H^1(\mathbb{R}_+^{N+1}) \to \mathbb{R}$ *satisfies the Palais–Smale condition* $(\mathrm{PS})_c$ *at each* $c < E_{V_\infty}$, *where* $V_\infty := \liminf_{|x|\to\infty} V(x)$.

*Proof.* Let $v_n \in H^1(\mathbb{R}_+^{N+1})$ satisfy

$$I(v_n) \to c < E_{V_\infty} \quad \text{and} \quad I'(v_n) \to 0$$

strongly in the dual space $H^{-1}(\mathbb{R}_+^{N+1})$. Since $\{v_n\}_n$ is bounded in $H^1(\mathbb{R}_+^{N+1})$, it contains a subsequence such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}_+^{N+1})$ and $\gamma(v_n) \rightharpoonup \gamma(v)$ in $L^p(\mathbb{R}^N)$ for any $p \in [2, 2N/(N-1)]$.

By lemma 4.2, $v$ solves (1.3) and, after passing to a subsequence, there exists a sequence $\{\tilde{v}_n\}_n$ in $H^1(\mathbb{R}_+^{N+1})$ such that $u_n := v_n - \tilde{v}_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}_+^{N+1})$,

$$\begin{aligned} I(v_n) - I(u_n) &\to I(v) && \text{in } \mathbb{R}, \\ I'(v_n) - I'(u_n) &\to 0 && \text{strongly in } H^{-1}(\mathbb{R}_+^{N+1}). \end{aligned}$$

Hence, $I(v) = ((\theta - 2)/2\theta)\mathbb{D}(v) \geqslant 0$,

$$I(u_n) \to c - I(v) \leqslant c \quad \text{and} \quad I'(u_n) \to 0$$

strongly in $H^{-1}(\mathbb{R}_+^{N+1})$. By lemma 4.1 a subsequence of $\{u_n\}_n$ converges strongly to 0 in $H^1(\mathbb{R}_+^{N+1})$. This implies that a subsequence of $\{v_n\}_n$ converges strongly to $v$ in $H^1(\mathbb{R}_+^{N+1})$. □

## 5. Mountain pass geometry

We consider the limit problem

$$\left.\begin{aligned} -\Delta v + m^2 v = 0 &\quad \text{in } \mathbb{R}_+^{N+1}, \\ -\frac{\partial v}{\partial x} = -V_\infty v + (W * |v|^\theta)|v|^{\theta-2} v &\quad \text{in } \mathbb{R}^N = \partial\mathbb{R}_+^{N+1}, \end{aligned}\right\} \tag{5.1}$$

where $V_\infty := \liminf_{|x|\to\infty} V(x) > 0$. By theorem 3.2, the first mountain pass value $E_{V_\infty}$ of the functional $J_{V_\infty}$ associated with (5.1) is attained at a positive function $\omega_\infty \in H^1(\mathbb{R}_+^{N+1})$, which is symmetric, $\omega_\infty(x, gy) = \omega_\infty(x, y)$, for all $g \in O(N)$, $x > 0$, $y \in \mathbb{R}^N$. Moreover, since $V_\infty > 0$, we are allowed to choose $\sigma = 0$, and there exists $C > 0$ such that

$$0 < \omega_\infty(x, y) \leqslant C\mathrm{e}^{-m\sqrt{x^2+|y|^2}} \tag{5.2}$$

for all $(x, y) \in [0, +\infty) \times \mathbb{R}^N$. In particular, $\gamma(\omega_\infty)$ is radially symmetric in $\mathbb{R}^N$ and

$$0 < \gamma(\omega_\infty)(y) \leqslant C\mathrm{e}^{-m|y|}$$

for any $y \in \mathbb{R}^N$. As in theorem 3.2, a bootstrap procedure shows that

$$\omega_\infty \in C^\infty([0, +\infty) \times \mathbb{R}^N).$$

Lemma 5.1. *We have*

$$|\nabla \omega_\infty(z)| = O(\mathrm{e}^{-m|z|}) \quad as\ |z| \to \infty. \tag{5.3}$$

*Proof.* We consider the equation

$$\sqrt{-\Delta + m^2} u + V_\infty u = (W * |u|^\theta)|u|^{\theta-2}u \quad \text{in } \mathbb{R}^N$$

satisfied by $\omega_\infty$. For any index $i = 1, 2, \ldots, N$ we write $v_i = \partial\omega_\infty/\partial y_i$ and observe that $v_i$ satisfies

$$\sqrt{-\Delta + m^2} v_i + V_\infty v_i = \theta(W * \omega_\infty^{\theta-1} v_i)\omega_\infty^{\theta-1} + (\theta - 1)(W * \omega_\infty^\theta)\omega_\infty^{\theta-2} v_i \tag{5.4}$$

or, equivalently,

$$-\Delta v_i + m^2 v_i = 0 \quad \text{in } \mathbb{R}_+^{N+1},$$
$$-\frac{\partial v_i}{\partial x} = -V_\infty v_i + \theta(W * \omega_\infty^{\theta-1} v_i)\omega_\infty^{\theta-1} + (\theta - 1)(W * \omega_\infty^\theta)\omega_\infty^{\theta-2} v_i \quad \text{in } \mathbb{R}^N.$$

The differentiation of the equation is allowed by the regularity of the solution $\omega_\infty$ (see [10, theorem 3.14]). Moreover, $\omega_\infty \in L^p(\mathbb{R}_+^{N+1})$ for any $p > 1$, because it is bounded and decays exponentially fast at $\infty$. By elliptic regularity, $\omega_\infty \in W^{2,p}(\mathbb{R}_+^{N+1})$ for any $p > 1$, and, in particular, $v_i \in L^p(\mathbb{R}_+^{N+1})$ for any $p > 1$. An interpolation estimate shows that $\omega_\infty^{\theta-1} v_i \in L^p(\mathbb{R}_+^{N+1})$ for any $p > 1$. Then the convolution $W * (\omega_\infty^{\theta-1} v_i) \in L^\infty(\mathbb{R}_+^{N+1})$, and the term

$$(W * (\omega_\infty^{\theta-1} v_i))\omega_\infty^{\theta-1} \in L^2(\mathbb{R}_+^{N+1})$$

by the summability properties of $\omega_\infty$. The term

$$(W * \omega_\infty^\theta)\omega_\infty^{\theta-2} v_i \in L^2(\mathbb{R}_+^{N+1})$$

trivially.

The proof of [10, theorem 3.14] now shows that $v_i(x, y) \to 0$ as $x + |y| \to +\infty$. A comparison with the function $\mathrm{e}^{-m\sqrt{x^2+|y|^2}}$ as in [10, theorem 5.1] shows the validity of (5.3). □

Fix $\varepsilon \in (0, (2m - k)/(2m + k))$. For $R > 0$, we consider a symmetric cut-off function $\xi_R \in C^\infty(\mathbb{R}_+^{N+1})$, namely, $\xi_R(x, gy) = \xi_R(x, y)$ for all $g \in O(N)$, $x > 0$, $y \in \mathbb{R}^N$ such that $\xi_R(z) = 0$ if $|z| \geqslant R$ and $\xi_R(z) = 1$ if $|z| \leqslant R(1 - \epsilon)$ and $\xi_R(z) \in [0, 1]$ for all $z \in \mathbb{R}_+^{N+1}$.

We define $\omega^R(z) := \omega_\infty(z)\xi_R(z)$ for any $z \in \mathbb{R}_+^{N+1}$.

Lemma 5.2. *As* $R \to \infty$,

$$\left| \iint_{\mathbb{R}_+^{N+1}} |\nabla\omega_\infty|^2 - |\nabla\omega^R|^2 \right| = O(R^{N-1}\mathrm{e}^{-2m(1-\varepsilon)R}), \tag{5.5}$$
$$|\mathbb{D}(\omega_\infty) - \mathbb{D}(\omega^R)| = O(R^{N-1}\mathrm{e}^{-\theta m(1-\varepsilon)R}). \tag{5.6}$$

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*Proof.* The proof of (5.5) is standard. Indeed, using (5.3) and cylindrical coordinates in $\mathbb{R}^{N+1}_+$,

$$\begin{aligned}\left|\iint_{\mathbb{R}^{N+1}_+} |\nabla\omega^R|^2 - |\nabla\omega_\infty|^2\right| &\leqslant C\iint_{\{z\in\mathbb{R}^{N+1}_+ \mid (1-\varepsilon)R<|z|\}} |\nabla\omega_\infty|^2 \\ &\leqslant C_1\iint_{\{z\in\mathbb{R}^{N+1}_+ \mid (1-\varepsilon)R<|z|\}} \mathrm{e}^{-2m|z|}\,\mathrm{d}z \\ &\leqslant C_1 R^{N-1}\mathrm{e}^{-2m(1-\varepsilon)R}.\end{aligned}$$

To prove (5.6), we recall that $W = W_1 + W_2 \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. The difference $\mathbb{D}(\gamma(\omega_\infty)) - \mathbb{D}(\gamma(\omega^R))$ can be split into two parts, the one with $W_1$ and the one with $W_2$. The former can be estimated as

$$\begin{aligned}&|\mathbb{D}(\gamma(\omega_\infty)) - \mathbb{D}(\gamma(\omega^R))| \\ &\quad\leqslant \int_{\mathbb{R}^N\times\mathbb{R}^N} |\gamma(\omega_\infty)(x)|^\theta|\gamma(\omega_\infty)(y)|^\theta - |\gamma(\omega^R)(x)|^\theta|\gamma(\omega^R)(y)|^\theta|W_1(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\leqslant 2\int_{\mathbb{R}^N\times\mathbb{R}^N} W_1(x-y)|\gamma(\omega_\infty)(x)|^\theta||\gamma(\omega_\infty)(y)|^\theta - |\gamma(\omega^R)(y)|^\theta|\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\leqslant 2\||\gamma(\omega_\infty)|^\theta - |\gamma(\omega^R)|^\theta\|_{2r/(2r-1)}\|\gamma(\omega_\infty)\|^\theta_{2r\theta/(2r-1)}\|W_1\|_r \\ &\quad\leqslant C\biggl(\int_{(1-\varepsilon)R}^\infty t^{N-1}\mathrm{e}^{-m(2r\theta/(2r-1))t}\,\mathrm{d}t\biggr)^{(2r-1)/2r} = C_2R^{N-1}\mathrm{e}^{-\theta m(1-\varepsilon)R}.\end{aligned}$$

The latter is simpler, since we directly use the $L^\infty$-norm of $W_2$. □

For $s\in\mathbb{R}^N$, set $R_s := ((k+2m)/4m)|s|$. Since $k\in(0,2m)$, it results that $R_s\in(0,|s|)$. Hence, $|s| - R_s\to+\infty$, as $|s|\to+\infty$. With this notation, we define the function

$$\omega_s^{R_s}(z) := \omega_\infty(x, y-s)\xi_{R_s}(x, y-s),$$

where $z = (x,y)\in\mathbb{R}^{N+1}$.

LEMMA 5.3. *There exist $\varrho_0$, $d_0\in(0,\infty)$ such that*

$$I(t(\omega_s^{R_s})) \leqslant E_{V_\infty} - d_0\mathrm{e}^{-k|y|} \quad \textit{for all } t\geqslant 0,$$

*provided that $|s|\geqslant\varrho_0$.*

*Proof.* For $u\in H^1(\mathbb{R}^{N+1}_+)$ we have by (2.3) that $\max_{t\geqslant 0} I(tu) = I(t_u u)$ if and only if

$$t_u = \biggl(\frac{\|u\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(u)^2\,\mathrm{d}y}{\mathbb{D}(u)}\biggr)^{1/(2\theta-2)}.$$

Indeed,

$$\|u\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(u)^2\,\mathrm{d}y \geqslant \biggl(1-\frac{V_0}{m}\biggr)\int_{\mathbb{R}^{N+1}_+}|\nabla u|^2 + m(m-V_0)\int_{\mathbb{R}^{N+1}_+}|u|^2 > 0. \tag{5.7}$$

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So, since $\omega_\infty^{R_s} \to \omega_\infty$ in $H^1(\mathbb{R}^{N+1}_+)$ as $|s| \to \infty$, and taking into account that $I_{V_\infty}(\omega_\infty) = \max_{t\geqslant 0} I_{V_\infty}(t(\omega_\infty))$, there exist $0 < t_1 < t_2 < +\infty$ such that

$$\max_{t\geqslant 0} I(t(\omega_s^{R_s})) = \max_{t_1\leqslant t\leqslant t_2} I(t(\omega_s^{R_s}))$$

for all large enough $|s|$.

Let $t \in [t_1, t_2]$. Write $V = V^+ - V^-$, where $V^+(x) = \max\{V(x), 0\}$ and $V^-(x) = \max\{-V(x), 0\}$, and remark that the assumption $V_\infty > 0$ implies that $V(x) = V^+(x)$ whenever $|x|$ is sufficiently large. Assumption ($\mathrm{V}_2$) therefore yields that

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(y)(t\gamma(\omega_s^{R_s}))^2(y)\,\mathrm{d}y \\
&\qquad\leqslant t^2 \int_{|y|\leqslant R_s} V^+(y+s)(\gamma(\omega^{R_s}))^2(y)\,\mathrm{d}y \\
&\qquad\leqslant t^2 \int_{|y|\leqslant R_s} (V_\infty - c_0\mathrm{e}^{-k|y+s|})(\gamma(\omega_\infty))^2(y)\,\mathrm{d}x \\
&\qquad\leqslant \int_{\mathbb{R}^N} V_\infty(t\gamma(\omega_\infty))^2 - \left(c_0 t_1^2 \int_{|y|\leqslant 1} \mathrm{e}^{-k|y|}(\gamma(\omega_\infty))^2(y)\,\mathrm{d}y\right)\mathrm{e}^{-k|s|}
\end{aligned}$$

for $|s|$ large enough.

Therefore, using lemma 5.2, we get that

$$\begin{aligned}
I(t(\omega^{R_s})_s) &= \frac{1}{2}\|t(\omega^{R_s})_s\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(y)(t\gamma(\omega_\infty))^2\,\mathrm{d}y - \frac{1}{2\theta}\mathbb{D}(t\omega_s^{R_s}) \\
&\leqslant \frac{1}{2}\|t\omega_\infty\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty(t\gamma(\omega_\infty))^2\,\mathrm{d}y - \frac{1}{2\theta}\mathbb{D}(t\omega_\infty) \\
&\qquad - C\mathrm{e}^{-k|s|} + O(R_s^{N-1}\mathrm{e}^{-2m(1-\varepsilon)R_s}) \\
&\leqslant \max_{t\geqslant 0} I_{V_\infty}(t\omega_\infty) - d_0\mathrm{e}^{-\kappa|s|} \\
&= E_{V_\infty} - d_0\mathrm{e}^{-k|s|}
\end{aligned}$$

for sufficiently large $|s|$, as our choices of $\varepsilon$ and $R_s$ ensure that $2m(1-\varepsilon)R_s > k|s|$. □

## 6. Proof of theorem 1.1

The proof of theorem 1.1 is now immediate. The Euler functional $I$ satisfies the geometric assumptions of the mountain pass theorem (see [2]) on $H^1(\mathbb{R}^{N+1}_+)$. Since it also satisfies the Palais–Smale condition, as we showed in the previous sections, we conclude that $I$ possesses at least a critical point $v \in H^1(\mathbb{R}^{N+1}_+)$. In addition,

$$I(v) = c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^{N+1}_+)) \mid \gamma(0) < 0,\ I(\gamma(1)) < 0\}$.

To prove that $v \geqslant 0$, we note that, reasoning as in (5.7), the map $t \mapsto I(tw)$ has one and only one strict maximum point at $t = 1$ whenever $w \in H^1(\mathbb{R}^{N+1}_+)$ is a critical point of $I$. Since $I(|w|) \leqslant I(w)$ for all $w \in H^1(\mathbb{R}^{N+1}_+)$, and

$$I(t|w|) \leqslant I(tw) < I(w) \quad \text{for every } t > 0,\ t \neq 1,$$

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we conclude that

$$c \leqslant \sup_{t \geqslant 0} I(t|v|) \leqslant I(v) = c.$$

We claim that $|v|$ is also a critical point of $I$. Indeed, otherwise, we could deform the path $t \mapsto t|v|$ into a path $\gamma \in \Gamma$ such that $I(\gamma(t)) < c$ for every $t \geqslant 0$, a contradiction with the definition of $c$.

## 7. Further properties of the solution

In the next statement we collect some additional features of the weak solution found above.

THEOREM 7.1. *Let $u$ be the solution to (1.3) provided by theorem 1.1. Then $u \in C^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for every $q \geqslant 2$. Moreover,*

$$0 < u(y) \leqslant C\mathrm{e}^{-m|y|}. \tag{7.1}$$

*Proof.* The regularity of $u$ can be established by mimicking the proofs in [10, § 3]. The potential function $V$ is harmless, being bounded from above and below.

To prove the exponential decay at $\infty$, we introduce a comparison function

$$W_R(x,y) = C_R \mathrm{e}^{-m\sqrt{x^2+|y|^2}} \quad \text{for every } (x,y) \in \mathbb{R}^{N+1}_+,$$

and we fix $R > 0$ and $C_R > 0$ in a suitable manner. We also introduce the notation

$$\begin{aligned} B_R^+ &= \{(x,y) \in \mathbb{R}^{N+1}_+ \mid \sqrt{x^2+|y|^2} < R\}, \\ \Omega_R^+ &= \{(x,y) \in \mathbb{R}^{N+1}_+ \mid \sqrt{x^2+|y|^2} > R\}, \\ \Gamma_R &= \{(0,y) \in \partial\mathbb{R}^{N+1}_+ \mid |y| \geqslant R\}. \end{aligned}$$

It is easily seen that

$$\begin{aligned} -\Delta W_R + m^2 W_R \geqslant 0 \quad &\text{in } \Omega_R^+, \\ -\frac{\partial W_R}{\partial x} = 0 \quad &\text{on } \Gamma_R^+. \end{aligned}$$

Set $w(x,y) = W_R(x,y) - v(x,y)$, and remark that $-\Delta w + m^2 w \geqslant 0$ in $\Omega_R^+$. If $C_R = \mathrm{e}^{mR} \max_{\partial B_R^+} v$, then $w \geqslant 0$ on $\partial B_R^+$ and $\lim_{x+|y|\to+\infty} w(x,y) = 0$. We claim that $w \geqslant 0$ in the closure $\bar{\Omega}_R^+$.

If not, then $\inf_{\bar{\Omega}_R^+} w < 0$, and the strong maximum principle provides a point $(0,y_0) \in \Gamma_R$ such that

$$w(0,y_0) = \inf_{\bar{\Omega}_R^+} w < w(x,y) \quad \text{for every } (x,y) \in \Omega_R^+.$$

For some $0 < \lambda < m$, we introduce $z(x,y) = w(x,y)\mathrm{e}^{\lambda x}$. As before,

$$\lim_{x+|y|\to+\infty} z(x,y) = 0$$

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and $z \geqslant 0$ on $\partial B_R^+$. Since

$$0 \leqslant -\Delta w + m^2 w = \mathrm{e}^{-\lambda x}\left(-\Delta z + 2\lambda \frac{\partial z}{\partial x} + (m^2 - \lambda^2) z\right),$$

the strong maximum principle applies and yields that $\inf_{\Gamma_R} z = \inf_{\bar{\Omega}_R^+} z < z(x, y)$ for every $(x, y) \in \Omega_R^+$. Therefore, $z(0, y_0) = \inf_{\Gamma_R} z = \inf_{\Gamma_R} w < 0$. Hopf's lemma now gives

$$-\frac{\partial w}{\partial x}(0, y_0) - \lambda w(0, y_0) < 0.$$

But this is impossible. Indeed,

$$-\frac{\partial w}{\partial x}(0, y_0) = -V(y_0) v(0, y_0) - (W * |v|^\theta) |v(0, y_0)|^{\theta - 2} v(0, y_0),$$

and hence

$$-\frac{\partial w}{\partial x}(0, y_0) - \lambda v(0, y_0) = -\lambda v(0, y_0) - V(y_0) v(0, y_0) - (W * |v|^\theta) |v(0, y_0)|^{\theta - 2} v(0, y_0).$$

Recall that $v(0, y_0) < 0$ and $\lambda > 0$; if we can show that

$$-V(y_0) v(0, y_0) - (W * |v|^\theta) |v(0, y_0)|^{\theta - 2} v(0, y_0) \geqslant 0,$$

we will be done. First of all, we recall (see [10, p. 70] and also [7, lemma 2.3]) that

$$\lim_{|y| \to +\infty} (W * |v|^\theta) |v(0, y)|^{\theta - 2} v(0, y) = 0,$$

since $\lim_{|y| \to +\infty} W(y) = 0$. So, we choose $R > 0$ large enough that

$$|(W * |v|^\theta) |v(0, y_0)|^{\theta - 2} v(0, y_0)|$$

is very small. Choosing $R$ even larger, we can also assume that $V(y_0) > 0$, since $V_\infty > 0$. Hence, $-V(y_0) v(0, y_0) - (W * |v|^\theta) |v(0, y_0)|^{\theta - 2} v(0, y_0) \geqslant 0$, and the proof is finished.

To summarize, we have proved that, whenever $x + |y|$ is sufficiently large,

$$v(x, y) \leqslant W_R(x, y),$$

and hence the validity of (7.1) follows. □

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