Combinatorics, Probability and Computing (2019) 28, 465–472.
doi:10.1017/S0963548318000421

Decomposing Graphs into Edges and Triangles†

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Received 22 October 2017; revised 6 June 2018; first published online 13 March 2019

We prove the following 30 year-old conjecture of Győri and Tuza: the edges of every n-vertex graph G can be decomposed into complete graphs $C_1,\dots,C_\ell$ of orders two and three such that $|C_1|+\cdots+|C_\ell| \leqslant (1/2+o(1))n^2$. This result implies the asymptotic version of the old result of Erdős, Goodman and Pósa that asserts the existence of such a decomposition with $\ell \leqslant n^2/4$.

2010 *Mathematics subject classification*: 05C70

1. Introduction

Results on the existence of edge-disjoint copies of specific subgraphs in graphs is a classical theme in extremal graph theory. Motivated by the following result of Erdős, Goodman and Pósa [11], we study the problem of covering edges of a given graph by edge-disjoint complete graphs.

† This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement 648509). This publication reflects only its authors' view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains.
§ The first author was also supported by the Engineering and Physical Sciences Research Council (EPSRC) Standard Grant EP/M025365/1.
¶ This author was supported in part by NSF grant DMS-1600390.
‖ This author was also supported by the CNPq Science Without Borders grant 200932/2014-4.

Theorem 1.1 (Erdős, Goodman and Pósa [11]). *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ complete graphs.*

In fact, they proved the following stronger statement.

Theorem 1.2 (Erdős, Goodman and Pósa [11]). *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ copies of K_2 and K_3.*

The bounds given in Theorems 1.1 and 1.2 are best possible as witnessed by complete bipartite graphs with parts of equal sizes.

Theorem 1.1 actually holds in a stronger form that we now present. Chung [7], Győri and Kostochka [19] and Kahn [24], independently, proved a conjecture of Katona and Tarján asserting that the edges of every n-vertex graph can be covered with complete graphs $C_1, \ldots, C_\ell$ such that the sum of their orders is at most $n^2/2$. In fact, the first two proofs yield a stronger statement, which implies Theorem 1.1 and which we next state as a separate theorem. To state the theorem, we define $\pi_k(G)$ for a graph G to be the minimum integer m such that the edges of G can be decomposed into complete graphs $C_1, \ldots, C_\ell$ of order at most k with $|C_1| + \cdots + |C_\ell| = m$, and we let $\pi(G) = \min_{k \in \mathbb{N}} \pi_k(G)$.

Theorem 1.3 (Chung [7], Győri and Kostochka [19]). *For every n-vertex graph G we have $\pi(G) \leqslant n^2/2$.*

Observe that Theorem 1.3 indeed implies the existence of a decomposition into at most $\lfloor n^2/4 \rfloor$ complete graphs. McGuinness [31, 32] extended these results by showing that decompositions from Theorems 1.1 and 1.3 can be constructed in the greedy way, which confirmed a conjecture of Winkler of this being the case in the setting of Theorem 1.1.

In view of Theorem 1.2, it is natural to ask whether Theorem 1.3 holds under the additional assumption that all complete graphs in the decomposition are copies of K_2 and K_3, that is, whether $\pi_3(G) \leqslant n^2/2$. Győri and Tuza [20] provided a partial answer by proving that $\pi_3(G) \leqslant 9n^2/16$ and conjectured the following.

Conjecture 1.4 (Győri and Tuza [34, Problem 40]). *Every n-vertex graph G satisfies $\pi_3(G) \leqslant (1/2 + o(1))n^2$.*

We prove this conjecture. Our result also solves [34, Problem 41], which we state as Corollary 3.4. We remark that we stated the conjecture in the version given by Győri in several of his talks and by Tuza in [34, Problem 40]; the paper [20] contains a version with a different lower-order term.

We would also like to mention a closely related variant of the problem suggested by Erdős, where the cliques in the decomposition have weights one less than their orders. Formally, define $\pi^-(G)$ for a graph to be the minimum m such that the edges of a graph G can be decomposed into complete graphs $C_1, \ldots, C_\ell$ with $(|C_1| - 1) + \cdots + (|C_\ell| - 1) = m$. Erdős (see [34, Problem 43]) asked whether $\pi^-(G) \leqslant n^2/4$ for every n-vertex graph G.

This problem remains open and was proved for K_4-free graphs only recently by Győri and Keszegh [17, 18]. Namely, they proved that every K_4-free graph with n vertices and $\lfloor n^2/4 \rfloor + k$ edges contains k edge-disjoint triangles.

2. Preliminaries

We follow the standard graph theory terminology; we review here some less standard notation and briefly introduce the flag algebra method. If G is a graph, then $|G|$ denotes the order of G, *i.e.* the number of vertices of G. Further, if W is a subset of vertices of G, then $G[W]$ is the subgraph of G induced by W.

In our arguments, we also consider fractional decompositions. A fractional k-decomposition of a graph G is an assignment of non-negative real weights to complete subgraphs of order at most k such that the sum of the weights of the complete subgraphs containing any edge e is equal to one. The weight of a fractional k-decomposition is the sum of the weights of the complete subgraphs multiplied by their orders, and the minimum weight of a fractional k-decomposition of a graph G is denoted by $\pi_{k,f}(G)$. Observe that $\pi_{k,f}(G) \leqslant \pi_k(G)$ for every graph G.

2.1. Flag algebra method

The flag algebra method introduced by Razborov [33] has changed the landscape of extremal combinatorics. It has been applied to many long-standing open problems, for example [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 21, 23, 27, 28, 29, 30]. The method is designed to analyse asymptotic behaviour of substructure densities and we now briefly describe it.

We start by introducing some necessary notation. The family of all finite graphs is denoted by $\mathcal{F}$ and the family of graphs with ℓ vertices by $\mathcal{F}_\ell$. If F and G are two graphs, then $p(F,G)$ is the probability that $|F|$ distinct vertices chosen uniformly at random among the vertices of G induce a graph isomorphic to F; if $|F| > |G|$, we set $p(F,G) = 0$. A type is a graph with its vertices labelled with $1,\ldots,|\sigma|$ and a σ-flag is a graph with $|\sigma|$ vertices labelled by $1,\ldots,|\sigma|$ such that the labelled vertices induce a copy of σ preserving the vertex labels. In analogy with the notation for ordinary graphs, the set of all σ-flags is denoted by $\mathcal{F}^\sigma$ and the set of all σ-flags with exactly ℓ vertices by $\mathcal{F}^\sigma_\ell$.

We next extend the definition of $p(F,G)$ to σ-flags and generalize it to pairs of graphs. If F and G are two σ-flags, then $p(F,G)$ is the probability that $|F| - |\sigma|$ distinct vertices chosen uniformly at random among the unlabelled vertices of G induce a copy of the σ-flag F; if $|F| > |G|$, we again set $p(F,G) = 0$. Let F and F' be two σ-flags and G a σ-flag with at least $|F| + |F'| - |\sigma|$ vertices. The quantity $p(F,F';G)$ is the probability that two disjoint $|F| - |\sigma|$ and $|F'| - |\sigma|$ subsets of unlabelled vertices of G induce together with the labelled vertices of G the σ-flags F and F', respectively. It holds [33, Lemma 2.3] that

$$p(F,F';G) = p(F,G) \cdot p(F',G) + o(1), \tag{2.1}$$

where $o(1)$ tends to zero with $|G|$ tending to infinity.

https://doi.org/10.1017/S0963548318000421 Published online by Cambridge University Press

Let $\vec{F} = [F_1, \dots, F_t]$ be a vector of σ-flags, *i.e.* $F_i \in \mathcal{F}^\sigma$. If M is a $t \times t$ positive semidefinite matrix, it follows from (2.1) (see [33]) that

$$0 \leqslant \sum_{i,j=1}^{t} M_{ij} p(F_i, G) p(F_j, G) = \sum_{i,j=1}^{t} M_{ij} p(F_i, F_j; G) + o(1). \tag{2.2}$$

Inequality (2.2) is usually applied to a large graph G with a randomly chosen labelled vertices in a way that we now describe. Fix σ-flags F and F' and a graph G. We now define a random variable $p(F, F'; G^\sigma)$ as follows: label $|\sigma|$ vertices of G with $1, \dots, |\sigma|$ and if the resulting graph G' is a σ-flag, then $p(F', F'; G^\sigma) = p(F, F'; G')$; if G' is not a σ-flag, then $p(F_i, F_j; G^\sigma) = 0$. The expected value of $p(F, F'; G^\sigma)$ can be expressed as a linear combination of densities of $(|F| + |F'| - |\sigma|)$-vertex subgraphs of G [33], that is, there exist coefficients α_H, $H \in \mathcal{F}_{|F|+|F'|-|\sigma|}$, such that

$$\mathbb{E}\, p(F, F'; G^\sigma) = \sum_{H \in \mathcal{F}_{|F|+|F'|-|\sigma|}} \alpha_H \cdot p(H, G) \tag{2.3}$$

for every graph G. It can be shown that $\alpha_H = \mathbb{E}\, p(F, F'; H^\sigma)$.

Let $\vec{F} = [F_1, \dots, F_t]$ be a vector of ℓ-vertex σ-flags and let M be a $t \times t$ positive semidefinite matrix. Equality (2.3) yields that there exist coefficients α_H such that

$$\mathbb{E} \sum_{i,j=1}^{t} M_{ij} p(F_i, F_j; G^\sigma) = \sum_{H \in \mathcal{F}_{2\ell-|\sigma|}} \alpha_H \cdot p(H, G) \tag{2.4}$$

for every graph G, which combines with (2.2) to

$$0 \leqslant \sum_{H \in \mathcal{F}_{2\ell-|\sigma|}} \alpha_H \cdot p(H, G) + o(1) \tag{2.5}$$

for every graph G, where

$$\alpha_H = \sum_{i,j=1}^{t} M_{ij} \cdot \mathbb{E}\, p(F_i, F_j; H^\sigma).$$

In particular, the coefficients α_H depend only on the choice of $\vec{F}$ and M.

3. Main result

We start by proving the following lemma using the flag algebra method.

Lemma 3.1. *Let G be a weighted graph with all edges of weight one. It holds that*

$$\mathbb{E}_W \pi_{3,f}(G[W]) \leqslant 21 + o(1),$$

where W is a uniformly chosen random subset of seven vertices of G.

Proof. We use the flag algebra method to find coefficients c_U, $U \in \mathcal{F}_7$, such that

$$0 \leqslant \sum_{U \in \mathcal{F}_7} c_U \cdot p(U, G) + o(1) \tag{3.1}$$

https://doi.org/10.1017/S0963548318000421 Published online by Cambridge University Press

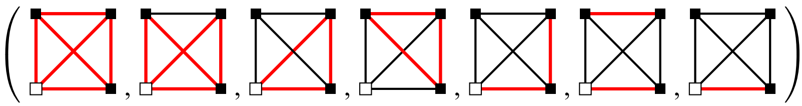
and

$$\pi_{3,f}(U) + c_U \leqslant 21 \tag{3.2}$$

for every $U \in \mathcal{F}_7$. The statement of the lemma would then follow from (3.1) and (3.2) using $\sum_{U\in\mathcal{F}_7} p(U,G) = 1$, as we next show:

$$\begin{aligned}\mathbb{E}_W \pi_{3,f}(G[W]) &= \sum_{U\in\mathcal{F}_7} \pi_{3,f}(U)\cdot p(U,G)\\ &\leqslant \sum_{U\in\mathcal{F}_7} (\pi_{3,f}(U)+c_U)\cdot p(U,G) + o(1)\\ &\leqslant \sum_{U\in\mathcal{F}_7} 21\cdot p(U,G) + o(1) = 21 + o(1).\end{aligned}$$

We now focus on finding the coefficients c_U, $U \in \mathcal{F}_7$, satisfying (3.1) and (3.2). Let σ_1 be a flag consisting of a single vertex labelled with 1 and consider the following vector $\vec{F} = (F_1,\dots,F_7)$ of σ_1-flags from $\mathcal{F}_4^{\sigma_1}$ (the single labelled vertex is depicted by a white square and the remaining vertices by black circles):

$\vec{F} =$ 

Let M be the following 7×7-matrix:

$$M = \frac{1}{12\cdot 10^9}\begin{pmatrix} 1800000000 & 2444365956 & 640188285 & -1524146769 & 1386815580 & -732139362 & -129387078\\ 2444365956 & 4759879134 & 1177441152 & -1783771230 & 2546923788 & -1397639394 & -143552208\\ 640188285 & 1177441152 & 484273772 & -317303211 & 1038156300 & -591902130 & -6783162\\ -1524146769 & -1783771230 & -317303211 & 1558870290 & -651906630 & 305728704 & 154602378\\ 1386815580 & 2546923788 & 1038156300 & -651906630 & 2285399634 & -1283125950 & -10755036\\ -732139362 & -1397639394 & -591902130 & 305728704 & -1283125950 & 734039016 & -1621938\\ -129387078 & -143552208 & -6783162 & 154602378 & -10755036 & -1621938 & 23860164 \end{pmatrix}.$$

The matrix M is a positive semidefinite matrix with rank six; the eigenvector corresponding to the zero eigenvalue is $(1,0,3,1,0,3,0)$. Let

$$c_U = \sum_{i,j=1}^{7} M_{ij}\mathbb{E}\, p(F_i,F_j;U^{\sigma_1}).$$

Inequality (2.5) implies that

$$0 \leqslant \sum_{U\in\mathcal{F}_7} c_U\cdot p(U,G) + o(1),$$

which establishes (3.1). Inequality (3.2) is verified with computer assistance by evaluating the coefficient c_U and the quantity $\pi_{3,f}(U)$ for each $U \in \mathcal{F}_7$. Since $|\mathcal{F}_7| = 1044$, we do not list c_U and $\pi_{3,f}(U)$ here. The computer programs used and their outputs have been made available on arXiv as ancillary files and are also available at http://orion.math.iastate.edu/lidicky/pub/tile23. □

The following lemma can be derived from the result of Haxell and Rödl [22] on fractional triangle decompositions or from a more general result of Yuster [35].

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Lemma 3.2. *Let G be a graph with n vertices. It holds that $\pi_3(G) \leqslant \pi_{3,f}(G) + o(n^2)$.*

We now use Lemmas 3.1 and 3.2 to prove our main result.

Theorem 3.3. *Every n-vertex graph G satisfies $\pi_3(G) \leqslant (1/2 + o(1))n^2$.*

Proof. Fix an n-vertex graph G. By Lemma 3.2, it is enough to show that $\pi_{3,f}(G) \leqslant (1/2 + o(1))n^2$.

Fix an optimal fractional 3-decomposition of $G[W]$ for every 7-vertex subset $W \subseteq V(G)$, and set the weight $w(e)$ of an edge e to the sum of its weights in the optimal fractional 3-decomposition of $G[W]$ with $e \subseteq W$ multiplied by $\binom{n-2}{5}^{-1}$, and the weight $w(t)$ of a triangle t to the sum of its weights in the optimal fractional 3-decomposition of $G[W]$ with $t \subseteq W$ also multiplied by $\binom{n-2}{5}^{-1}$. Since each edge e of G is contained in $\binom{n-2}{5}$ subsets W, we have obtained a fractional 3-decomposition of G. The weight of this decomposition is equal to

$$\frac{1}{\binom{n-2}{5}} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) \leqslant \frac{\binom{n}{7}}{\binom{n-2}{5}}(21 + o(1)) = n^2/2 + o(n^2),$$

where the inequality follows from Lemma 3.1. We conclude that $\pi_{3,f}(G) \leqslant n^2/2 + o(n^2)$, which completes the proof. □

The next corollary follows directly from Theorem 3.3.

Corollary 3.4. *Every n-vertex graph with $n^2/4 + k$ edges contains $2k/3 - o(n^2)$ edge-disjoint triangles.*

4. Concluding remarks

Our first proof of this result, which can be found in [26], combined the flag algebra method and regularity method arguments. In particular, we proved the fractional relaxation of Conjecture 1.4 in the setting of weighted graphs and with an additional restriction on its support; this statement was then combined with a blow-up lemma for edge-decompositions recently proved by Kim, Kühn, Osthus and Tyomkyn [25]. It was then brought to our attention that the results from [22, 35] allow us to obtain our main result directly from the fractional relaxation, which is the proof that we present here. We believe that the argument using combinatorial designs that we applied in [26] to combine the flag algebra method and the blow-up lemma of Kim, Kühn, Osthus and Tyomkyn [25] can be of independent interest, and so we wanted to mention the original proof of our result and its idea here.

We also tried to prove Lemma 3.1 in the non-fractional setting, *i.e.* to show that $\mathbb{E}_W \pi_3(G[W]) \leqslant 21 + o(1)$. Unfortunately, the computation with 7-vertex flags yields only that $\mathbb{E}_W \pi_3(G[W]) \leqslant 21.588 + o(1)$. We would like to remark that if it were possible to prove Lemma 3.1 in the non-fractional setting, we would be able to prove Theorem 3.3

without using additional results as a black box: we would consider a random $(n,7,2,1)$-design on the vertex set of an n-vertex graph G as in [26] and apply the non-fractional version of Lemma 3.1 to this design.

Finally, we would also like to mention two open problems related to our main result. Theorem 3.3 asserts that $\pi_3(G) \leqslant n^2/2 + o(n^2)$ for every n-vertex graph G. However, it could be true (*cf.* the remark after Problem 41 in [34]) that $\pi_3(G) \leqslant n^2/2 + 2$ for every n-vertex graph G. The second problem that we would like to mention is a possible generalization of Corollary 3.4, which is stated in [34] as Problem 42. Fix $r \geqslant 4$. Does every n-vertex graph with $((r-2)/(2r-2))n^2 + k$ edges contain $(2/r)k - o(n^2)$ edge-disjoint complete graphs of order r?

Acknowledgements

The authors would like to thank Ervin Győri and Katherine Staden for their comments on the problems considered in this paper. The authors would also like to thank Allan Lo for drawing their attention to the paper [22].

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