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ENUMERATING NECKLACES WITH TRANSITIONS

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Abstract

Necklaces are the equivalence classes of words under the action of the cyclic group. Let a transition in a word be any change between two adjacent letters modulo the word's length. We present a closed-form solution for the enumeration of necklaces in n beads, k colours and t transitions. We show that our result provides a more general solution to the problem of counting alternating (proper) colourings of the vertices of a regular n-gon.

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Keywords and phrases: colouring, composition, necklace, transition.

1. Introduction

Let $W(n, k)$ be the set of all the words of length n over an alphabet of k symbols and denote by $w = \{w_1, \ldots, w_n\}$ any such word. Necklaces are the orbits of $W(n, k)$ under the action of the cyclic group $\mathbb{C}_n$, that is, the subsets of $W(n, k)$ that are each composed of those words that can be transformed into one another by a circular shift. In the literature, n and k are usually referred to, respectively, as the number of beads and colours of the necklace. We shall depict words as circularly arranged chains of circular markers (beads) and use different colours to denote different symbols (see Figure 1). By convention, we shall take the minimal word in lexicographic order as the representative of a necklace.

EXAMPLE 1.1. The words $w_1 = \{a, b, a, c\}$, $w_2 = \{c, a, b, a\}$, $w_3 = \{a, c, a, b\}$ and $w_4 = \{b, a, c, a\}$ are all equivalent under a circular shift and therefore account for one necklace. The representative of the necklace is w_1.

Enumerating necklaces is a classic problem in combinatorics. If we denote by $N(n, k)$ the set of necklaces in n beads and k colours, it is known [9, Corollary 2]

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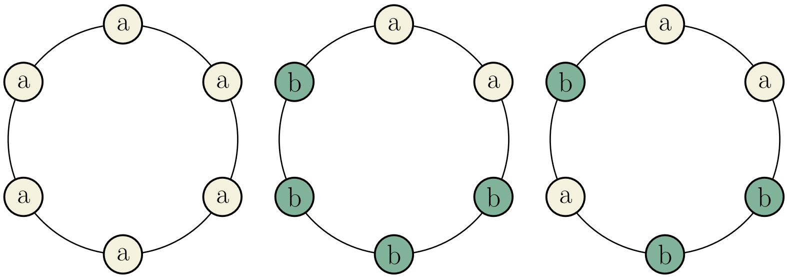

$\{a, a, a, a, a, a\}$ $\{a, a, b, b, b, b\}$ $\{a, a, b, b, a, b\}$

FIGURE 1. From left to right: samples of words in 6 beads, 2 colours and with 0, 2 and 4 transitions, respectively. The words are represented both in linear form (as strings of characters) and as circularly arranged sequences of beads. By convention, the beads are arranged in clockwise order starting from the topmost slot.

that

$$|N(n, k)| = \frac{1}{n} \sum_{d|n} \varphi\Big(\frac{n}{d}\Big) k^d, \tag{1.1}$$

where $d \mid n$ indicates that the sum extends to all the divisors of n (1 and n included). In (1.1), $\varphi(u)$ denotes Euler's totient function (that is, the number of positive integers up to u that are prime to u) and $|X|$ is the cardinality of X. In this article we are concerned with a variation of this problem: enumerating necklaces with transitions.

DEFINITION 1.2 (Transition). Given a word $w = \{w_1 \ldots, w_n\}$, we say that there is a transition whenever $w_i \neq w_{i+1 \bmod n}$.

DEFINITION 1.3 (Number of transitions). The number of transitions t of a word w is

$$t = \sum_{i=1}^{n} \delta(w_i, w_{i+1 \bmod n}),$$

where

$$\delta(u, v) = \begin{cases} 1 & \text{if } u \neq v, \\ 0 & \text{otherwise.} \end{cases}$$

QUESTION 1.4 (Counting necklaces with transitions). Let $N(n, k, t)$ denote the set of necklaces in n beads, k colours and with t transitions. Determine $|N(n, k, t)|$.

2. Background and preliminaries

Ojala *et al.* [11] used transitions to characterise binary necklaces ($k = 2$), which, in their work, they refer to as local binary patterns (LBP). In particular, they introduced the term 'uniform' to denote any such necklace with at most two transitions. These structures were later enumerated by Bianconi and González [6] for a generic number

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of colours. To the best of our knowledge, however, there are no known results for the general case $t \neq 2$ and computer-generated sequences inserted into the On-line Encyclopedia of Integer Sequences [14] produce no results.

2.1. Periodic and aperiodic words. We define the product of two words and denote this as '×', the concatenation of the words. Therefore, given $w_a = \{a_1, \dots, a_i\}$ and $w_b = \{b_1, \dots, b_j\}$, we shall write $w_a \times w_b = \{a_1, \dots, a_i, b_1, \dots, b_j\}$. Likewise, we define the rth power of a word and indicate this by $w^{\wedge r}$, the concatenation of w with itself r times.

We say that a word is *aperiodic* (or *primitive* [2]) if it cannot be written as the power of a word of a smaller length, and that it is *periodic* otherwise. If w is a periodic word and $w_c = \{a_1, \dots, a_q\}$ is the shortest word such that $w = w_c^{\wedge r}$ for some r, we say that w has period q or, equivalently, that w is q-periodic. In this case, if w has length n, then $q \mid n$ and $r = n/q$. Also, note that an aperiodic word of length n can be regarded as an n-periodic one.

EXAMPLE 2.1 (Periodic and aperiodic words). The words $\{a, b, a, b\}$ and $\{a, b, c, a, b, c\}$ are, respectively, 2- and 3-periodic; the word $\{a, b, c, d, e\}$ is aperiodic (or, equivalently, 5-periodic).

2.2. Compositions. A composition of a positive integer n into m parts is any sequence of positive integers $s = \{\sigma_1, \dots, \sigma_m\}$ such that $\sum_{i=1}^{m} \sigma_i = n$ [8, Definition 1.1], where the σ_i are referred to as the parts of the composition. Of course, any composition of n into m parts can also be regarded as a word of length m with symbols in $\{1, \dots, n\}$, and the concepts of periodic and aperiodic words translate seamlessly to compositions [7]. In particular, if a composition has period p and m parts, then $p \mid m$. We shall denote compositions as comma-separated sequences of their parts enclosed in square brackets. For instance, we shall write $[1, 2, 2]$, $[1, 2, 1, 2]$, $[2, 3]$ and $[3, 2]$, respectively, to denote the compositions $1 + 2 + 2$, $1 + 2 + 1 + 2$, $2 + 3$ and $3 + 2$.

2.3. Coloured compositions. A g-coloured composition of a positive integer n is a composition of n in which each part can take any among g colours [1]. Following the same convention as that adopted with words, we denote colours by letters and coloured compositions by compositions with subscripts that indicate the colours. We therefore write $s^g = [\sigma_{1_{c_1}}, \dots, \sigma_{m_{c_m}}]$ to indicate a coloured composition with colours $c_i \in \{1, \dots, g\}$, and $S^g(n, m)$ to denote the set of g-coloured compositions of n into m parts. We say that a coloured composition is *alternating* (or *proper*) if no two adjacent parts (modulo the composition's length) have the same colour.

EXAMPLE 2.2. Let $C = \{a, b, c\}$ be a set of colours. The following are coloured *(non-alternating)* compositions of 6 into 4 parts with colours in C: $[1_a, 2_b, 1_c, 2_c]$, $[1_b, 1_b, 2_c, 2_c]$.

EXAMPLE 2.3. Let $C = \{a, b, c\}$ be a set of colours. The following are *alternating* coloured compositions of 6 into 4 parts with colours in C: $[1_a, 2_b, 1_a, 2_b]$, $[1_a, 1_b, 2_a, 2_c]$.

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2.4. Cyclic compositions and binary necklaces. Cyclic compositions are the orbits of compositions under a circular shift of their parts [10]. For instance, [1, 1, 2, 3] and [3, 1, 1, 2] account for two compositions, but only one cyclic composition. Following the same convention as that adopted with necklaces, we take the least sequence of parts under lexicographical order as the representative of a cyclic composition. Therefore, in the case of the above example, the representative will be [1, 1, 2, 3], but, of course, the choice is arbitrary. We shall denote the set of cyclic compositions of n into m parts by $\tilde{S}(n, m)$.

PROPOSITION 2.4 (Bijection between binary necklaces and cyclic compositions). *Let $N^m(n, 2)$ denote the set of necklaces in n beads and two colours (binary necklaces) with m black beads. A bijection exists between $\tilde{S}(n, m)$ and $N^m(n, 2)$ mapping p-periodic compositions into q-periodic necklaces with $q/p = n/m$.*

PROOF. We retrace the argument proposed in [10]. Consider a necklace as a cyclic graph of which the vertices are the beads. Then, starting from any black bead, e_1 denotes the number of edges we need to traverse to get to the second black bead, e_2 denotes the number of edges to traverse to get to the third black bead and so on (Figure 2). Eventually, we get $n = e_1 + \cdots + e_m$, that is, a representative of the cyclic composition of n into m parts.

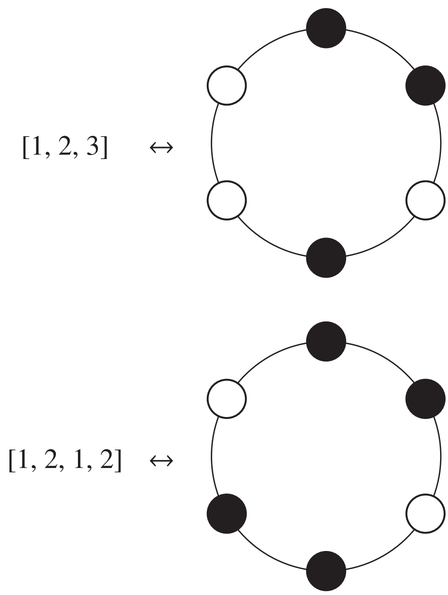

FIGURE 2. Bijection between cyclic compositions of n into m parts and binary necklaces of n beads, of which m are black. The first row shows an aperiodic case and the second shows a periodic one with $p = 2$ (period of the cyclic composition) and $q = np/m = 3$ (period of the binary necklace). By convention, the traversal starts with the top bead and proceeds clockwise, but the choice is arbitrary (taking another black bead as the starting point would result in a different representative of the same cyclic composition).

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Now, let p be the period of a composition $s = \{e_1, \ldots, e_m\} \in \tilde{S}(n, m)$. This means that s can be obtained by repeating its first p parts m/p times. By starting with a black bead, adding $e_1 - 1$ white beads, then a second black bead followed by $e_2 - 1$ white beads and so on until there are $e_q - 1$ white beads separating the qth and first black beads, we obtain the first $q = \sum_{i=1}^{p} e_i$ beads of the corresponding binary necklace $\theta \in N^m(n, 2)$ and, clearly, $q \mid n$. Hence, this construction repeated m/p times fills θ as a concatenation of its first q beads m/p times. The existence of a $q' < q$, $q' \mid n$, satisfying the same property would contradict the hypothesis that s is p-periodic. Thus, θ is q-periodic and the proportionality between p and q follows from $n = qm/p$. □

2.5. Periodic necklaces and Lyndon words. A Lyndon word of n letters over an alphabet of k symbols is a word that is strictly less (in lexicographical order) than any of its circularly shifted versions [12]. For instance, $\{a, a, b\}$ is a Lyndon word, whereas $\{a, a, a\}$ and $\{a, b, a\}$ are not. Let $L(n, k)$ indicate the set of all Lyndon words of length n in k symbols, and denote by $N_p(n, k)$ the set of p-periodic necklaces in n beads and k colours. We prove that there is a bijection between $N_p(n, k)$ and $L(p, k)$ as depicted in Figure 3.

THEOREM 2.5. *The function* $f : L(p, k) \to N_p(n, k);\ x \mapsto x^{\wedge(n/p)}$ *is bijective.*

PROOF. We start by proving that f is injective. Let $l_1, l_2 \in L(p, k)$. It is easy to see that $l_1 = l_2 \to f(l_1) = f(l_2)$, as words obtained by concatenating equal subwords

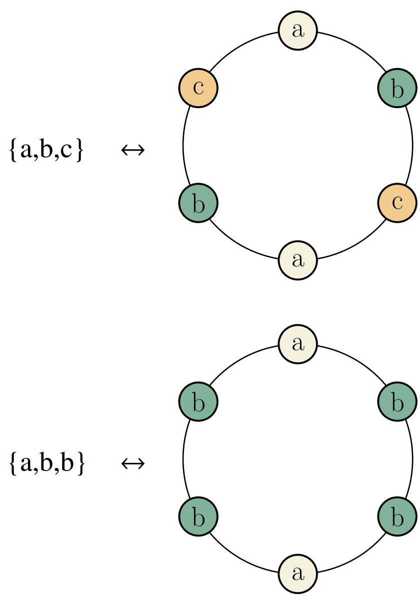

FIGURE 3. Samples of the bijection between p-periodic necklaces in n beads and k colours (right) and Lyndon words in p letters over an alphabet of k colours (left). First row: $n = 6$, $k = p = 3$; second row: $n = 6$, $p = 3$ and $k = 2$.

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are necessarily equal. Conversely, for any $r \in \mathbb{N}$, if $l_1^{\wedge r}$ and $l_2^{\wedge r}$ are equal, then their homologous subwords are also equal; hence $f(l_1) = f(l_2) \rightarrow l_1 = l_2$.

We now prove that f is surjective, that is, for every $y_p \in N_p(n, k)$, there exists an $l \in L(p, k)$ such that $f(l) = y_p$. Start by observing that, by definition, a p-periodic necklace y_p can be decomposed into n/p equal words of length p, and let w indicate any such word, that is, $y_p = w^{\wedge(n/p)}$. We now show that $w \in L(p, k)$. From our choice of representatives of the necklaces, y_p is less than or equal to any of its circularly shifted versions, and this holds for any subword w into which y_p can be decomposed. Therefore, either (a) w is strictly less than all its circularly shifted version (that is, w is a Lyndon word); or (b) w is equal to its circularly shifted version by q positions for some $q < p$. But condition (b) implies that w (and therefore y_p) is q-periodic with $q < p$, which contradicts the hypothesis that y_p is p-periodic. □

REMARK 2.6. Theorem 2.5 implies, for $n = p$, the well-known bijection between aperiodic necklaces and Lyndon words [2, 5].

THEOREM 2.7 (Number of p-periodic cyclic compositions). *Let $\tilde{S}_p(n, m)$ be the set of p-periodic cyclic compositions of n into m parts. Then*

$$|\tilde{S}_p(n, m)| = \begin{cases} 0 & \text{if } p \nmid m, \\ \dfrac{m}{np} \displaystyle\sum_{d \mid \gcd(np/m, p)} \mu(d) \binom{np/md}{p/d} & \text{otherwise,} \end{cases}$$

where gcd(u,v) *indicates the greatest common divisor between u and v and $\mu(x)$ denotes the Möbius function.*

PROOF. The case $p \nmid m$ (p does not divide m) is trivial: there cannot possibly be any periodic composition if the number of parts is not a multiple of the period. Otherwise, when $p \mid m$, observe from Proposition 2.4 that if a cyclic composition of n into m parts is p-periodic, then the equivalent binary necklace in n beads, of which m are black, will be q-periodic with $q = np/m$ (Figure 2). Hence

$$|\tilde{S}_p(n, m)| = |N_q^m(n, 2)|.$$

Also note that Theorem 2.5 holds (by restriction) between q-periodic binary necklaces in n beads, of which m are black, and binary Lyndon words of length q with $mq/n = p$ ones, with the convention (again arbitrary), that white = 0 and black = 1. Therefore, denoting the set of such Lyndon words by $L^p(q, 2)$,

$$|N_q^m(n, 2)| = |L^p(q, 2)|.$$

But, for binary Lyndon words of length u having exactly v ones, it is known [2] that

$$|L^v(u, 2)| = \frac{1}{u} \sum_{d \mid \gcd(u,v)} \mu(d) \binom{u/d}{v/d}. \tag{2.1}$$

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Substituting $u \leftarrow q$ and $v \leftarrow p$ into (2.1) gives

$$|L^p(q,2)| = \frac{1}{q} \sum_{d|\gcd(q,p)} \mu(d) \binom{q/d}{p/d} \tag{2.2}$$

and, eventually, substituting $q \leftarrow np/m$ into (2.2) gives the result. □

Corollary 2.8 (Number of p-periodic compositions). *Let $S_p(n,m)$ indicate the set of p-periodic compositions of n into m parts. Then*

$$|S_p(n,m)| = p|\tilde{S}_p(n,m)|.$$

Proof. This follows immediately because $\tilde{S}_p(n,m)$ is a partition of $S_p(n,m)$ and any p-periodic cyclic composition generates p different p-periodic compositions via a circular shift by ρ positions with $\rho \in \{1, \ldots, p\}$. □

3. Counting necklaces with transitions

3.1. Preliminary considerations. Let $w = \{w_1, \ldots, w_n\}$ be the representative of a necklace in n beads, k colours and t transitions. In general, either $t = 0$ or $t \in \{2, \ldots, n\}$, as we can have at least zero transitions and at most as many as the number of beads. The case $t = 1$ would lead to the contradiction that $w_i \neq w_i$ for some $i \in \{1, \ldots, n\}$, and is therefore impossible.

3.2. Known and trivial cases. The case $t = 0$ occurs when all the beads have the same colour and is trivial: that is,

$$N(n,k,0) = k.$$

The case $t = 2$ was addressed in [6]. Starting from [6, Equation (15,18)], after some manipulations,

$$N(n,k,2) = \frac{(n-1)k(k-1)}{2}. \tag{3.1}$$

The case $t = n$ (*colourful* necklaces [4]) is tantamount to counting the number of alternating colourings of the vertices of a regular polygon, two colourings being equivalent if they can be obtained from one another by a rotation of the polygon (Figure 4). The problem was recently investigated by Singh and Zelenyuk [13] who obtained

$$N(n,k,n) = \frac{1}{n} \sum_{d|n} \varphi(d)[(k-1)^{n/d} + (-1)^{n/d}(k-1)]. \tag{3.2}$$

Note that (3.2) can be rewritten as

$$N(n,k,n) = \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) P_d(k), \tag{3.3}$$

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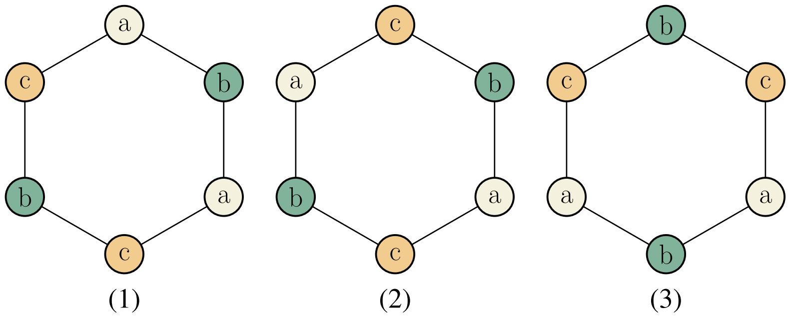

FIGURE 4. Alternating (proper) 3-colourings of a regular hexagon. Colourings (1) and (3) are equivalent under the action of $\mathbb{C}_6$.

where $P_d(k)$ is the chromatic polynomial (number of proper colourings) of a cyclic graph on d vertices and k colours,

$$P_d(k) = (k-1)^d + (-1)^d(k-1). \tag{3.4}$$

3.3. Main result.

THEOREM 3.1 (Number of necklaces in n beads, k colours and t transitions).

$$|N(n,k,t)| = \begin{cases} k & \text{if } t = 0, \\ 0 & \text{if } t = 1, \\ \dfrac{1}{t} \sum_{d|t} \varphi\Big(\dfrac{t}{d}\Big) P_d(k) \sum_{p|d} p|\tilde{S}_p(n,t)| & \text{if } 2 \le t \le n. \end{cases} \tag{3.5}$$

PROOF. The cases $t = 0$ and $t = 1$ have been already discussed in Section 3.2. For the general case ($2 \le t \le n$), start by observing that a necklace in n beads, k colours and t transitions can be encoded as a k-coloured alternating composition of n into t parts in a unique way as follows (see also Example 3.2 and Figure 5): (1) parse the necklace's representative from left to right until the first transition is reached and take this as the origin; (2) from there, count the number of beads we have to traverse until the next transition and keep track of the colours of the beads crossed; and (3) repeat until the origin is reached.

EXAMPLE 3.2 (Encoding necklaces as coloured compositions). Consider the representatives $w_1 = \{a,b,b,a,c,c,c\}$ and $w_2 = \{c,c,a,b\}$ of two necklaces. Their corresponding (alternating) coloured compositions are $[1_a, 2_b, 1_a, 3_c]$ and $[2_c, 1_a, 1_b]$, respectively.

We can therefore reduce the problem of counting necklaces in n beads, k colours and t transitions to that of counting proper k-coloured compositions of n into t parts. Any two such compositions are equivalent if they can be transformed into one another under the action of $\mathbb{C}_t$. Let $\mathcal{A}[S^k(n,t)]$ denote the set of alternating k-coloured compositions

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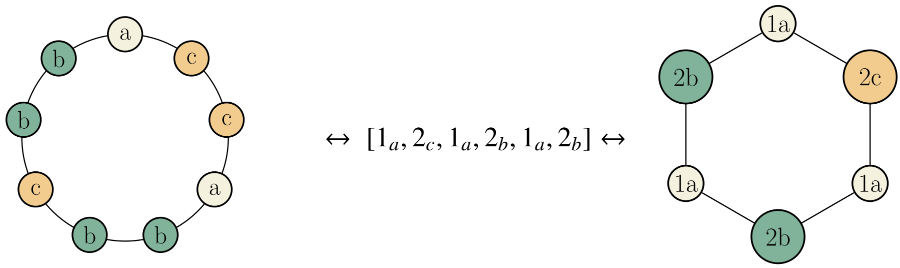

FIGURE 5. Sample of a necklace in 9 beads, 3 colours and 6 transitions (left); the equivalent 3-coloured composition of 9 into 6 parts (centre) and the representation of the latter as a cyclic graph (right).

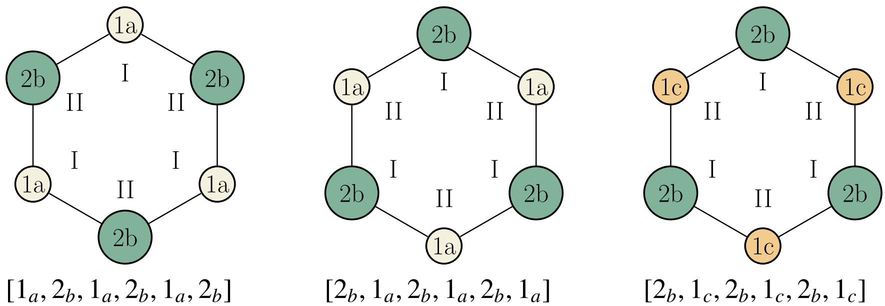

FIGURE 6. Three samples of alternating 3-coloured compositions of 9 into 6 parts (also represented as cyclic graphs) that are left invariant by a permutation with cycle structure x_3^2. Note that the permutation generates two cycles (I and II), each of which is of length 3. For a colouring to be proper, the two cycles must have different colours; for it to be fixed, the vertices on the same cycle must have the same colour. Therefore the colourings fixed by this permutation are all the proper colourings of a cyclic graph on 2 vertices (cycles I and II) and 3 colours.

of n into t parts and let $\tilde{\mathcal{A}}[S^k(n,t)]$ denote the orbits of $\mathcal{A}[S^k(n,t)]$ under the action of $\mathbb{C}_t$. We proceed by Burnside's lemma [3, Theorem 8.4.3], extending the approach described in [13]. Start by recalling that the cycle index of the cyclic group $\mathbb{C}_t$ is

$$Z(\mathbb{C}_t) = \frac{1}{t}\sum_{d|t}\varphi\Big(\frac{t}{d}\Big)x_{t/d}^d.$$

Next we determine how many elements of $\mathcal{A}[S^k(n,t)]$ are fixed by a permutation with cycle structure $x_{t/d}^d$. Denote this number by $|\mathrm{fix}(x_{t/d}^d, \mathcal{A}[S^k(n,t)])|$. In the remainder of this proof, for convenience, we shall also represent compositions of n into t parts as cyclic graphs on t vertices, with each vertex corresponding to one part of the composition (see Figure 6).

When $d = 1$, each part of the composition is mapped into the adjacent one; therefore no alternating colourings can be left unchanged in this case. When $d \geq 2$, the permutation has d cycles, each of length t/d. In this case, there cannot be any fixed elements when $p \nmid d$, where p is the period of the composition. Otherwise, when $p \mid d$,

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for a proper coloured composition to be fixed, all the parts belonging to the same cycle need to have the same size and colour, and the colours of two adjacent cycles need to be different (Figure 6). Observe that, for any p with $p \mid d$, we can generate all the elements of $\mathcal{A}[S^k(n,t)]$ fixed by $x^d_{t/d}$ by taking all the p-periodic compositions of n into t parts and by proper colouring each of them as a cyclic graph on d vertices and k colours. Therefore,

$$|\mathrm{fix}(x^d_{t/d}, \mathcal{A}[S^k(n,t)])| = P_d(k) \sum_{p|d} |S_p(n,t)|,$$

which, by Corollary 2.8, becomes

$$|\mathrm{fix}(x^d_{t/d}, \mathcal{A}[S^k(n,t)])| = P_d(k) \sum_{p|d} p|\tilde{S}_p(n,t)|. \qquad \square$$

We now show how the previously known solutions for $t = 2$ and $t = n$ (Section 3.2) can be easily obtained as special cases of Theorem 3.1.

When $t = 2$, the outer sum in (3.5) extends to $d \in \{1, 2\}$; also, $\varphi(1) = \varphi(2) = 1$ and $P_1(k) = 0$. Therefore $d = 1$ gives no contribution to the count. Since $P_2(k) = k(k-1)$, by substituting into (3.5),

$$\begin{aligned} |N(n,k,2)| &= \tfrac{1}{2}\{P_d(k)[|\tilde{S}_1(n,2)| + 2|\tilde{S}_2(n,2)|]\} \\ &= \tfrac{1}{2}\{k(k-1)[|\tilde{S}_1(n,2)| + 2|\tilde{S}_2(n,2)|]\}. \end{aligned} \tag{3.6}$$

Observe that $|\tilde{S}_1(n,2)|$ counts the ways in which we can divide n into two equal parts. Hence

$$|\tilde{S}_1(n,2)| = \begin{cases} 0 & \text{for } n \text{ even}, \\ 1 & \text{for } n \text{ odd}, \end{cases} \tag{3.7}$$

whereas, for $|\tilde{S}_2(n,2)|$, it is easy to see that

$$|\tilde{S}_2(n,2)| = \begin{cases} \dfrac{n-2}{2} & \text{for } n \text{ even}, \\ \dfrac{n-1}{2} & \text{for } n \text{ odd}. \end{cases} \tag{3.8}$$

Eventually, by substituting (3.7) and (3.8) into (3.6), we obtain (3.1).

The case $t = n$ corresponds to counting the number of necklaces in n beads and k colours with no adjacent beads having the same colours, or, equivalently, the number of alternating colourings of the vertices of a regular n-agon, which was recently treated in [13]. By the definition of a p-periodic cyclic composition,

$$|\tilde{S}_p(n,n)| = \begin{cases} 1 & \text{if } p = 1, \\ 0 & \text{otherwise} \end{cases}$$

which, substituted into (3.5) along with $t \leftarrow n$, gives (3.3).

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