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THE FIBRE OF THE DEGREE 3 MAP, ANICK SPACES AND THE DOUBLE SUSPENSION

STEVEN AMELOTTE

Department of Mathematics, University of Rochester, 915 Hylan Building, Rochester, NY 14627, USA (steven.amelotte@rochester.edu)

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Abstract Let $S^{2n+1}\{p\}$ denote the homotopy fibre of the degree p self map of S^{2n+1}. For primes $p \geq 5$, work by Selick shows that $S^{2n+1}\{p\}$ admits a non-trivial loop space decomposition if and only if $n = 1$ or p. Indecomposability in all but these dimensions was obtained by showing that a non-trivial decomposition of $\Omega S^{2n+1}\{p\}$ implies the existence of a p-primary Kervaire invariant one element of order p in $\pi^S_{2n(p-1)-2}$. We prove the converse of this last implication and observe that the homotopy decomposition problem for $\Omega S^{2n+1}\{p\}$ is equivalent to the strong p-primary Kervaire invariant problem for all odd primes. For $p = 3$, we use the 3-primary Kervaire invariant element θ_3 to give a new decomposition of $\Omega S^{55}\{3\}$ analogous to Selick's decomposition of $\Omega S^{2p+1}\{p\}$ and as an application prove two new cases of a long-standing conjecture stating that the fibre of the double suspension $S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ is homotopy equivalent to the double loop space of Anick's space.

Keywords: loop space decomposition; double suspension; Anick space; Kervaire invariant

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1. Introduction

Localize all spaces and maps at an odd prime p. Let $S^{2n+1}\{p\}$ denote the homotopy fibre of the degree p map on S^{2n+1} and let W_n denote the homotopy fibre of the double suspension $E^2\colon S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$. In [20] and [21], Selick showed that there is a homotopy decomposition

$$\Omega^2 S^{2p+1}\{p\} \simeq \Omega^2 S^3\langle 3\rangle \times W_p, \tag{1}$$

where $S^3\langle 3\rangle$ is the 3-connected cover of S^3, and obtained as an immediate corollary that p annihilates all p-torsion in $\pi_*(S^3)$. This exponent result is generalized by the exponent theorem of Cohen, Moore and Neisendorfer [6, 7, 16], who used different loop space decompositions to construct a map $\varphi\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ with the property that the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the pth power map on $\Omega^2 S^{2n+1}$ and proved by induction on n that p^n annihilates the p-torsion in $\pi_*(S^{2n+1})$. By a result of Gray [**10**], if p is an odd prime, then $\pi_*(S^{2n+1})$ contains infinitely many elements of order p^n, so this is the best possible odd primary homotopy exponent for spheres. The work of Cohen, Moore and Neisendorfer suggested that there should exist a space $T^{2n+1}(p)$ fitting in a fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \longrightarrow T^{2n+1}(p) \longrightarrow \Omega S^{2n+1}$$

in which their map φ occurs as the connecting map. The existence of such a fibration was first proved by Anick for $p \geq 5$ in [**2**]. A much simpler construction, valid for all odd primes, was later given by Gray and Theriault in [**15**], in which they also show that Anick's space $T^{2n+1}(p)$ has the structure of an H-space and that all maps in the fibration above can be chosen to be H-maps.

A well-known conjecture in unstable homotopy theory states that the fibre W_n of the double suspension $E^2\colon S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ is a double loop space. Anick's space represents a potential candidate for a double classifying space of W_n, and one of Cohen, Moore and Neisendorfer's remaining open conjectures in [**8**] states that there should be a p-local homotopy equivalence $W_n \simeq \Omega^2 T^{2np+1}(p)$. A stronger form of the conjecture (see e.g. [**3**, **12**, **26**]) states that

$$BW_n \simeq \Omega T^{2np+1}(p),$$

where BW_n is the classifying space of W_n first constructed by Gray [**11**]. Such equivalences have only been shown to exist for $n = 1$ and $n = p$. In the former case, both BW_1 and $\Omega T^{2p+1}(p)$ are known to be homotopy equivalent to $\Omega^2 S^3\langle 3\rangle$. Using Anick's fibration, Selick showed in [**23**] that $T^{2p+1}(p) \simeq \Omega S^3\langle 3\rangle$ and that the decomposition (1) can be delooped to a homotopy equivalence

$$\Omega S^{2p+1}\{p\} \simeq \Omega S^3\langle 3\rangle \times BW_p.$$

The $n = p$ case was proved in the strong form $BW_p \simeq \Omega T^{2p^2+1}(p)$ by Theriault [**26**] using the above decomposition in an essential way. Under these identifications, he further showed that $\Omega S^{2p+1}\{p\}$ and $T^{2p+1}(p) \times \Omega T^{2p^2+1}(p)$ are equivalent as H-spaces.

For primes $p \geq 5$, similar decompositions of $\Omega S^{2n+1}\{p\}$ are not possible if $n \neq 1$ or p. This result was obtained in [**22**] by first showing that for $n > 1$ the existence of a certain spherical homology class imposed by a non-trivial homotopy decomposition of $\Omega S^{2n+1}\{p\}$ implies the existence of an element of p-primary Kervaire invariant one in $\pi^S_{2n(p-1)-2}$, and then appealing to Ravenel's [**18**] result on the non-existence of such elements when $p \geq 5$ and $n \neq p$. For $p = 3$, the question of whether $\Omega S^{2n+1}\{3\}$ admits a non-trivial decomposition for $n = 3^j$ with $j > 1$ was left open. In this short note, we prove that the strong odd primary Kervaire invariant problem is in fact equivalent to the problem of decomposing the loop space $\Omega S^{2n+1}\{p\}$. When $p = 3$, this equivalence can be used to import results from stable homotopy theory to obtain new results concerning the unstable homotopy type of $\Omega S^{2n+1}\{3\}$, as well as some cases of the conjecture that W_n is a double loop space.

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Theorem 1.1. *Let p be an odd prime. The following conditions are equivalent.*

(a) *There exists a p-primary Kervaire invariant one element $\theta_j \in \pi^S_{2p^j(p-1)-2}$ of order p.*

(b) *There is a homotopy decomposition of H-spaces*

$$\Omega S^{2p^j+1}\{p\} \simeq T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p).$$

Furthermore, if the above conditions hold, then there are homotopy equivalences of H-spaces

$$BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p) \quad \text{and} \quad BW_{p^j} \simeq \Omega T^{2p^{j+1}+1}(p).$$

From this point of view, Selick's decomposition of $\Omega S^{2p+1}\{p\}$ and the previously known equivalences $BW_1 \simeq \Omega T^{2p+1}(p)$ and $BW_p \simeq \Omega T^{2p^2+1}(p)$ correspond to the existence (at all odd primes) of the Kervaire invariant class $\theta_1 = \beta_1 \in \pi^S_{2p^2-2p-2}$ given by the first element of the periodic beta family in the stable homotopy groups of spheres. By Ravenel's negative solution to the Kervaire invariant problem for primes $p \geq 5$, Theorem 1.1 has new content only at the prime $p = 3$. For example, in addition to the 3-primary Kervaire invariant element $\theta_1 \in \pi^S_{10}$ for $p = 3$ and $j = 1$ corresponding to the decomposition of $\Omega S^7\{3\}$, it is known that there exists a 3-primary Kervaire invariant element $\theta_3 \in \pi^S_{106}$ (see [**18**, **19**]), which we use to obtain the following decomposition of $\Omega S^{55}\{3\}$ and prove the $n = p^2$ and $n = p^3$ cases of the $BW_n \simeq \Omega T^{2np+1}(p)$ conjecture at $p = 3$.

Corollary 1.2. *There are 3-local homotopy equivalences of H-spaces*

(a) $\Omega S^{55}\{3\} \simeq T^{55}(3) \times \Omega T^{163}(3)$;

(b) $BW_9 \simeq \Omega T^{55}(3)$;

(c) $BW_{27} \simeq \Omega T^{163}(3)$.

Remark 1.3. The equivalence of conditions (a) and (b) in Theorem 1.1 does not hold for $p = 2$. In [**4**], Campbell, Cohen, Peterson and Selick showed that for $n > 1$ a non-trivial decomposition of the fibre $\Omega^2 S^{2n+1}\{2\}$ of the squaring map implies the existence of an element $\theta \in \pi^S_{2n-2}$ of Kervaire invariant one such that $\theta\eta$ is divisible by 2. Since such elements are well known to exist only for $n = 2, 4$ or 8, these are the only dimensions for which $\Omega^2 S^{2n+1}\{2\}$ can decompose non-trivially. Explicit decompositions of $\Omega^2 S^5\{2\}$, $\Omega^2 S^9\{2\}$ and $\Omega^3 S^{17}\{2\}$ corresponding to the first three 2-primary Kervaire invariant classes $\theta_1 = \eta^2$, $\theta_2 = \nu^2$ and $\theta_3 = \sigma^2$ are given in [**5**], [**9**] and [**1**].

A further consequence of Theorem 1.1 unique to the $p = 3$ case concerns the associativity of Anick spaces. Unlike when $p \geq 5$, in which case $T^{2n+1}(p)$ is a homotopy commutative and homotopy associative H-space for all $n \geq 1$, counterexamples to the homotopy associativity of $T^{2n+1}(3)$ have been observed in [**13**] and [**24**]. In particular, in [**13**], it was shown that if $T^{2n+1}(3)$ is homotopy associative, then $n = 3^j$ for some $j \geq 0$. The proof given there also shows that a homotopy associative H-space structure on $T^{2n+1}(3)$ implies the existence of a three-cell complex

$$S^{2n+1} \cup_3 e^{2n+2} \cup e^{6n+1}$$

with non-trivial mod 3 Steenrod operation $\mathcal{P}^n$, which in turn implies (by Spanier–Whitehead duality and the Liulevicius–Shimada–Yamanoshita factorization of $\mathcal{P}^{p^j}$ by secondary cohomology operations) the existence of an element of strong Kervaire invariant one. Using Theorem 1.1, we observe that the converse is also true to obtain the following.

Theorem 1.4. *Let $n > 1$. Then the mod 3 Anick space $T^{2n+1}(3)$ is homotopy associative if and only if there exists a 3-primary Kervaire invariant one element of order 3 in π^S_{4n-2}.*

2. Proof of Theorem 1.1

The bulk of the proof of Theorem 1.1 will consist of a slight generalization of the argument given in [**26**, Theorem 1.2], which we briefly describe below. As in [**26**], the following extension lemma, originally proved in [**3**] for $p \geq 5$ and later extended to include the $p = 3$ case in [**15**], will be crucial. We write $P^n(p^r)$ for the mod p^r Moore space $S^{n-1} \cup_{p^r} e^n$ and for a space X define homotopy groups with $\mathbb{Z}/p^r\mathbb{Z}$ coefficients by $\pi_n(X; \mathbb{Z}/p^r\mathbb{Z}) = [P^n(p^r), X]$.

Lemma 2.1. *Let p be an odd prime. Let X be an H-space such that $p^k \cdot \pi_{2np^k-1}(X; \mathbb{Z}/p^{k+1}\mathbb{Z}) = 0$ for $k \geq 1$. Then any map $P^{2n}(p) \to X$ extends to a map $T^{2n+1}(p) \to X$.*

In [**15**], Anick's space is constructed as the homotopy fibre in a secondary EHP fibration

$$T^{2n+1}(p) \xrightarrow{E} \Omega S^{2n+1}\{p\} \xrightarrow{H} BW_n, \tag{2}$$

where E is an H-map which induces in mod p homology the inclusion of

$$H_*(T^{2n+1}(p)) \cong \Lambda(a_{2n-1}) \otimes \mathbb{Z}/p\mathbb{Z}[c_{2n}]$$

into

$$H_*(\Omega S^{2n+1}\{p\}) \cong \left(\bigotimes_{i=0}^{\infty} \Lambda(a_{2np^i-1})\right) \otimes \left(\bigotimes_{i=1}^{\infty} \mathbb{Z}/p\mathbb{Z}[b_{2np^i-2}]\right) \otimes \mathbb{Z}/p\mathbb{Z}[c_{2n}],$$

and H induces the projection onto

$$H_*(BW_n) \cong \left(\bigotimes_{i=1}^{\infty} \Lambda(a_{2np^i-1})\right) \otimes \left(\bigotimes_{i=1}^{\infty} \mathbb{Z}/p\mathbb{Z}[b_{2np^i-2}]\right).$$

When $n = p$, it follows from Selick's decomposition of $\Omega S^{2p+1}\{p\}$ that H admits a right homotopy inverse $s\colon BW_p \to \Omega S^{2p+1}\{p\}$, splitting the homotopy fibration (2) in this case.

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Restricting to the bottom cell of BW_p, Theriault [26] extended the composite

$$S^{2p^2-2} \hookrightarrow BW_p \xrightarrow{s} \Omega S^{2p+1}\{p\}$$

to a map $P^{2p^2-1}(p) \to \Omega S^{2p+1}\{p\}$ and then applied Lemma 2.1 to the adjoint map $P^{2p^2}(p) \to S^{2p+1}\{p\}$ to obtain an extension $T^{2p^2+1}(p) \to S^{2p+1}\{p\}$. Finally, looping this last map, he showed that the composite

$$\Omega T^{2p^2+1}(p) \longrightarrow \Omega S^{2p+1}\{p\} \xrightarrow{H} BW_p$$

is a homotopy equivalence, thus proving the $n = p$ case of the conjecture that $BW_n \simeq \Omega T^{2np+1}(p)$.

In our case, we will use Lemma 2.1 to first construct a right homotopy inverse of $H\colon \Omega S^{2n+1}\{p\} \to BW_n$ in dimensions $n = p^j$ for which there exists an element $\theta_j \in \pi^S_{2p^j(p-1)-2}$ of strong Kervaire invariant one, and then follow the same strategy as above to obtain both a homotopy decomposition of $\Omega S^{2p^j+1}\{p\}$ and a homotopy equivalence $BW_{p^j} \simeq \Omega T^{2p^{j+1}+1}(p)$. These equivalences can then be used to compare the loops on (2) with the $n = p^{j-1}$ case of a homotopy fibration

$$BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\} \longrightarrow W_{np}$$

to further obtain a homotopy equivalence of fibres $BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p)$.

Proof of Theorem 1.1. We first show that condition (b) implies condition (a). Given any homotopy equivalence

$$\psi\colon T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p) \xrightarrow{\sim} \Omega S^{2p^j+1}\{p\},$$

set $n = p^j$ and let f denote the composite

$$f\colon S^{2np-2} \hookrightarrow \Omega T^{2np+1}(p) \xrightarrow{i_2} T^{2n+1}(p) \times \Omega T^{2np+1}(p) \xrightarrow{\psi} \Omega S^{2n+1}\{p\},$$

where the first map is the inclusion of the bottom cell of $\Omega T^{2np+1}(p)$ and the second map i_2 is the inclusion of the second factor. Then

$$f_*(\iota) = b_{2np-2} \in H_{2np-2}(\Omega S^{2n+1}\{p\}),$$

where ι is the generator of $H_{2np-2}(S^{2np-2})$. Since the homology class b_{2np-2} is spherical if and only if there exists a stable map $g\colon P^{2n(p-1)-1}(p) \to S^0$ such that the Steenrod operation $\mathcal{P}^n$ acts non-trivially on $H^*(C_g)$ by [22], it follows that $\pi^S_{2n(p-1)-2}$ contains an element of p-primary Kervaire invariant one and order p.

Conversely, suppose there exists a p-primary Kervaire invariant one element $\theta_j \in \pi^S_{2p^j(p-1)-2}$ of order p. Then, by [22], the homology class $b_{2p^{j+1}-2} \in H_{2p^{j+1}-2}(\Omega S^{2p^j+1}\{p\})$ is spherical, so there exists a map $f\colon S^{2p^{j+1}-2} \to \Omega S^{2p^j+1}\{p\}$ with

Hurewicz image $b_{2p^{j+1}-2}$. Now, following the proof of [**26**, Theorem 1.2], since $\Omega S^{2p^j+1}\{p\}$ has H-space exponent p, it follows that f has order p and hence extends to a map

$$e\colon P^{2p^{j+1}-1}(p) \longrightarrow \Omega S^{2p^j+1}\{p\}.$$

Let $\hat{e}\colon P^{2p^{j+1}}(p) \to S^{2p^j+1}\{p\}$ denote the adjoint of e. Again, because $\Omega S^{2p^j+1}\{p\}$ has H-space exponent p, we have that

$$p \cdot \pi_*(S^{2p^j+1}\{p\}; \mathbb{Z}/p^k\mathbb{Z}) = 0$$

for all $k \geq 1$, and since $S^{2p^j+1}\{p\}$ is an H-space [**17**], the map $\hat{e}$ satisfies the hypotheses of Lemma 2.1 and therefore admits an extension

$$s\colon T^{2p^{j+1}+1}(p) \longrightarrow S^{2p^j+1}\{p\}.$$

Note that this factorization of $\hat{e}$ through s implies that the adjoint map e factors through Ωs, so we have a commutative diagram

$$\begin{array}{ccccc} S^{2p^{j+1}-2} & \longrightarrow & P^{2p^{j+1}-1}(p) & \longrightarrow & \Omega T^{2p^{j+1}+1}(p) \\ & {\scriptstyle f}\searrow & {\scriptstyle e}\searrow & & \downarrow{\scriptstyle \Omega s} \\ & & & & \Omega S^{2p^j+1}\{p\} \end{array}$$

where the maps along the top row are skeletal inclusions; hence, $(\Omega s)_*$ is an isomorphism on $H_{2p^{j+1}-2}(\ )$ since f_* is. Now, since $H\colon \Omega S^{2p^j+1}\{p\} \to BW_{p^j}$ induces an epimorphism in homology, the composite

$$\Omega T^{2p^{j+1}+1}(p) \xrightarrow{\Omega s} \Omega S^{2p^j+1}\{p\} \xrightarrow{H} BW_{p^j}$$

induces an isomorphism of the lowest non-vanishing reduced homology group

$$H_{2p^{j+1}-2}(\Omega T^{2p^{j+1}+1}(p)) \cong H_{2p^{j+1}-2}(BW_{p^j}) \cong \mathbb{Z}/p\mathbb{Z}.$$

By [**14**], any map $\Omega T^{2np+1}(p) \to BW_n$ which is degree one on the bottom cell must be a homotopy equivalence, and thus $H \circ \Omega s$ is a homotopy equivalence. Composing a homotopy inverse of $H \circ \Omega s$ with Ωs, we obtain a right homotopy inverse of H, which shows that the homotopy fibration

$$T^{2p^j+1}(p) \xrightarrow{E} \Omega S^{2p^j+1}\{p\} \xrightarrow{H} BW_{p^j}$$

splits. Moreover, letting m denote the loop multiplication on $\Omega S^{2p^j+1}\{p\}$, the composite

$$T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p) \xrightarrow{E \times \Omega s} \Omega S^{2p^j+1}\{p\} \times \Omega S^{2p^j+1}\{p\} \xrightarrow{m} \Omega S^{2p^j+1}\{p\}$$

defines an equivalence of H-spaces, since E and Ωs are H-maps and m is homotopic to the loops on the H-space multiplication on $S^{2p^j+1}\{p\}$.

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The homotopy equivalence $H \circ \Omega s \colon \Omega T^{2p^{j+1}+1}(p) \to BW_{p^j}$ is not necessarily multiplicative, but the H-space decomposition of $\Omega S^{2p^j+1}\{p\}$ constructed above can now be used exactly as in the proof of [26, Theorem 1.1] to produce an H-map $BW_{p^j} \to \Omega T^{2p^{j+1}+1}(p)$ which is also a homotopy equivalence.

It remains to show that there is an equivalence of H-spaces $BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p)$. In his construction of a classifying space of W_n, Gray [11] introduced a p-local homotopy fibration

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1},$$

where the map j has order p and hence lifts to a map $j' \colon BW_n \to \Omega^2 S^{2np+1}\{p\}$. By [25], j' can be chosen to be an H-map when $p \geq 3$. Since j_* is an isomorphism in degree $2np-1$, it follows by commutativity with the Bockstein that j'_* is an isomorphism in degree $2np-2$. Let γ denote the equivalence of H-spaces $T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p) \xrightarrow{\sim} \Omega S^{2p^j+1}\{p\}$ constructed above. As $\Omega\gamma$ is also an equivalence of H-spaces, it has a homotopy inverse $(\Omega\gamma)^{-1}$ which is also an H-map. Consider the composite

$$BW_{p^{j-1}} \xrightarrow{j'} \Omega^2 S^{2p^j+1}\{p\} \xrightarrow{(\Omega\gamma)^{-1}} \Omega T^{2p^j+1}(p) \times \Omega^2 T^{2p^{j+1}+1}(p) \xrightarrow{\pi_1} \Omega T^{2p^j+1}(p),$$

where j' is the lift of j with $n = p^{j-1}$ and π_1 is the projection onto the first factor. Since all three maps in this composition induce isomorphisms on $H_{2p^j-2}(\ )$, it again follows from the atomicity result in [14] that the composite defines a homotopy equivalence $BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p)$, which is an equivalence of H-spaces since each map above is an H-map. □

3. Applications

In this section we derive Corollary 1.2 and Theorem 1.4 from Theorem 1.1 and discuss some other consequences in the $p = 3$ case.

3.1. The homotopy decomposition of $\Omega S^{55}\{3\}$

Since, by [22], $\Omega S^{2n+1}\{p\}$ is atomic for all n such that $\pi^S_{2n(p-1)-2}$ contains no element of p-primary Kervaire invariant one, $\Omega S^{2n+1}\{p\}$ is indecomposable for $n \neq p^j$ and it follows from Theorem 1.1 that the decomposition problem for $\Omega S^{2n+1}\{p\}$ is equivalent to the strong p-primary Kervaire invariant problem for odd primes p. The 3-primary Kervaire invariant problem is open, but the elements $b_{j-1} \in \mathrm{Ext}^{2,2p^j(p-1)}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ in the E_2-term of the Adams spectral sequence which potentially detect elements of odd primary Kervaire invariant one are known to behave differently for $p = 3$ than for primes $p \geq 5$.

While b_0 is a permanent cycle representing $\theta_1 \in \pi^S_{2p(p-1)-2}$ at all odd primes, Ravenel showed in [18] that for $j > 1$ and $p \geq 5$ the elements b_{j-1} support non-trivial differentials in the Adams spectral sequence and hence that none of the θ_j exist for $j > 1$ and $p \geq 5$. For $p = 3$, however, it is known (see [18, 19]) that although b_1 supports a non-trivial differential, b_2 is a permanent cycle representing a 3-primary Kervaire invariant class $\theta_3 \in \pi^S_{106}$.

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Proof of Corollary 1.2. According to [**19**], $\pi^S_{106} \cong \mathbb{Z}/3\mathbb{Z}$ after localizing at $p = 3$, so θ_3 has order 3 and the result follows from Theorem 1.1. □

Remark 3.1. We note that the non-existence of θ_2 at $p = 3$ implies that $\Omega S^{2p^2+1}\{p\} = \Omega S^{19}\{3\}$ is atomic and hence indecomposable by the result in [**22**] mentioned above.

Observe that since the mod p Moore space $P^2(p)$ is the homotopy cofibre of the degree p self map $p\colon S^1 \to S^1$, by applying the functor $\mathrm{Map}_*(-, S^{2n+1})$ to the homotopy cofibration

$$S^1 \xrightarrow{p} S^1 \longrightarrow P^2(p)$$

we obtain a homotopy fibration

$$\mathrm{Map}_*(P^2(p), S^{2n+1}) \longrightarrow \Omega S^{2n+1} \xrightarrow{p} \Omega S^{2n+1},$$

which identifies the mapping space $\mathrm{Map}_*(P^2(p), S^{2n+1})$ with the homotopy fibre $\Omega S^{2n+1}\{p\}$ of the pth power map on the loop space ΩS^{2n+1}. The decomposition of $\Omega S^{55}\{3\}$ in Corollary 1.2 therefore induces the following splitting of homotopy groups with $\mathbb{Z}/3\mathbb{Z}$ coefficients, analogous to Selick's [**21**] splitting of $\pi_*(S^{2p+1}; \mathbb{Z}/p\mathbb{Z})$.

Corollary 3.2. *For $k \geq 4$, there are isomorphisms*

$$\begin{aligned}\pi_k(S^{55}; \mathbb{Z}/3\mathbb{Z}) &\cong \pi_{k-2}(T^{55}(3)) \oplus \pi_{k-1}(T^{163}(3))\\ &\cong \pi_{k-4}(W_9) \oplus \pi_{k-3}(W_{27}).\end{aligned}$$

3.2. Homotopy associativity and exponents for mod 3 Anick spaces

The following two useful properties of $T^{2n+1}(p)$ were conjectured by Anick and Gray [**2**, **3**].

(a) $T^{2n+1}(p)$ is a homotopy commutative and homotopy associative H-space.

(b) $T^{2n+1}(p)$ has homotopy exponent p.

Both properties have been established for all $p \geq 5$ and $n \geq 1$, but only partial results have been obtained in the $p = 3$ case. For example, it was found in [**24**] that $T^7(3)$ is both homotopy commutative and homotopy associative but that homotopy associativity fails for $T^{11}(3)$. More generally, Gray showed in [**13**] that if $T^{2n+1}(3)$ is homotopy associative, then $n = 3^j$ for some $j \geq 0$ and, moreover, property (i) implies property (ii).

Concerning property (ii), in general $T^{2n+1}(3)$ is only known to have homotopy exponent bounded above by 9. (This can be seen using fibration (2) and the fact that BW_n has 3-primary exponent 3, for example.) Since $T^{2n+1}(p)$ is an H-space for all $p \geq 3$, one could also ask for the stronger property that $T^{2n+1}(p)$ has H-space exponent p, i.e. that its pth power map is null homotopic. We note that decompositions of $\Omega S^{2n+1}\{3\}$, when they occur, give some evidence for (ii).

Corollary 3.3. *The following hold.*

(a) $T^7(3)$ *and* $T^{55}(3)$ *are homotopy commutative and homotopy associative* H*-spaces.*

(b) $T^7(3)$, $T^{55}(3)$, $\Omega T^{19}(3)$ *and* $\Omega T^{163}(3)$ *each have* H*-space exponent* 3.

Proof. Since the homotopy equivalences $\Omega S^7\{3\} \simeq T^7(3) \times \Omega T^{19}(3)$ and $\Omega S^{55}\{3\} \simeq T^{55}(3) \times \Omega T^{163}(3)$ which follow from Theorem 1.1 are equivalences of H-spaces, part (b) follows immediately from the fact that $\Omega S^{2n+1}\{3\}$ has H-space exponent 3 [**17**], and part (a) follows from the fact that $\Omega S^{2n+1}\{3\}$ is homotopy associative and homotopy commutative as it is the loop space of an H-space. □

Proof of Theorem 1.4. Let $n > 1$ and suppose $T^{2n+1}(3)$ is homotopy associative. Then the proof of [**13**, Theorem A.2] shows that there exists a three-cell complex

$$X = S^{2n+1} \cup_3 e^{2n+2} \cup e^{6n+1}$$

with non-trivial mod 3 Steenrod operation $\mathcal{P}^n\colon H^{2n+1}(X) \to H^{6n+1}(X)$. The attaching map of the middle cell of a Spanier–Whitehead dual of X then defines an element of Kervaire invariant one in π^S_{4n-2}, which has order 3 since it extends over a mod 3 Moore space. Alternatively, by [**24**, Proposition 7.1], the homotopy associativity of $T^{2n+1}(3)$ implies that a certain composite

$$S^{6n-3} \xrightarrow{[\iota,[\iota,\iota]]} \Omega S^{2n} \xrightarrow{r} S^{2n-1}$$

is divisible by 3, where $[\iota,[\iota,\iota]]$ denotes the triple Samelson product of the generator of $\pi_{2n-1}(\Omega S^{2n})$ and r is a left homotopy inverse of the suspension $E\colon S^{2n-1} \to \Omega S^{2n}$. It is easy to check that the composite above coincides with the image of the generator of the lowest non-vanishing 3-local homotopy group $\pi_{6n-3}(W_n) \cong \mathbb{Z}/3\mathbb{Z}$ under the homotopy fibre map $W_n \to S^{2n-1}$, and the divisibility of this element is a well-known equivalent formulation of the strong Kervaire invariant problem.

Conversely, if there exists a 3-primary Kervaire invariant element of order 3 in π^S_{4n-2}, then $n = 3^j$ for some $j \geq 1$, and it follows from Theorem 1.1 that $T^{2n+1}(3)$ is a homotopy associative H-space as it is an H-space retract of a loop space. □

4. A stable splitting of $\Omega S^{2n+1}\{p\}$

It is well known that $S^{2n+1}\{p\}$ splits as a wedge of mod p Moore spaces after suspending once. In this section, we determine the stable homotopy type of the loop space $\Omega S^{2n+1}\{p\}$ by observing that although the homotopy fibration

$$T^{2n+1}(p) \xrightarrow{E} \Omega S^{2n+1}\{p\} \xrightarrow{H} BW_n$$

only splits in Kervaire invariant one dimensions, it splits for all n after suspending twice. As in the previous sections, p denotes an odd prime and all spaces and maps are assumed to be localized at p.

Proposition 4.1. *For all $n \geq 1$, there is a homotopy equivalence*

$$\Sigma^2 \Omega S^{2n+1}\{p\} \simeq \Sigma^2 (T^{2n+1}(p) \times BW_n).$$

Proof. In [**11**], Gray showed that the classifying space BW_n of the fibre of the double suspension fits in a homotopy fibration

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$$

and that there is a homotopy equivalence $\Sigma^2 \Omega^2 S^{2n+1} \simeq \Sigma^2 (S^{2n-1} \times BW_n)$. Let $s\colon \Sigma^2 BW_n \to \Sigma^2 \Omega^2 S^{2n+1}$ be a right homotopy inverse of $\Sigma^2 \nu$. In the construction of Anick's fibration in [**15**], $T^{2n+1}(p)$ is defined as the homotopy fibre of the map H, where H is constructed as an extension

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\partial} & \Omega S^{2n+1}\{p\} \\ \downarrow \nu & \swarrow H & \\ BW_n & & \end{array}$$

of ν through the connecting map of the homotopy fibration $\Omega S^{2n+1}\{p\} \longrightarrow \Omega S^{2n+1} \xrightarrow{p} \Omega S^{2n+1}$. Therefore, by composing s with $\Sigma^2 \partial$, we obtain a right homotopy inverse $s'\colon \Sigma^2 BW_n \to \Sigma^2 \Omega S^{2n+1}\{p\}$ of $\Sigma^2 H$. Next, consider the composite map f defined by

$$f\colon T^{2n+1}(p) \wedge \Sigma^2 BW_n \xrightarrow{E \wedge s'} \Omega S^{2n+1}\{p\} \wedge \Sigma^2 \Omega S^{2n+1}\{p\} \longrightarrow \Sigma^2 \Omega S^{2n+1}\{p\},$$

where the second map is obtained by suspending the Hopf construction $\Sigma \Omega S^{2n+1}\{p\} \wedge \Omega S^{2n+1}\{p\} \to \Sigma \Omega S^{2n+1}\{p\}$ on $\Omega S^{2n+1}\{p\}$. Finally, since $\Sigma^2 E$, s' and f each induce monomorphisms in mod p homology, it follows that the map

$$\Sigma^2 (T^{2n+1}(p) \times BW_n) \simeq \Sigma^2 T^{2n+1}(p) \vee \Sigma^2 BW_n \vee (\Sigma^2 T^{2n+1}(p) \wedge BW_n) \longrightarrow \Sigma^2 S^{2n+1}\{p\}$$

defined by their wedge sum is a homology isomorphism and hence a homotopy equivalence. □

Since $T^{2n+1}(p)$ stably splits as a wedge of mod p Moore spaces [**3**], it follows from Proposition 4.1 that $\Omega S^{2n+1}\{p\}$ has the stable homotopy type of a wedge of Moore spaces, Snaith summands $D_{2,k}(S^{2n-1})$ of the stable splitting of $\Omega^2 S^{2n+1}$, and their smash products.

A similar argument can be used to give a stable splitting of the homotopy fibre E^{2n+1} of the natural inclusion $i\colon P^{2n+1}(p) \to S^{2n+1}\{p\}$, where BW_n is a stable retract. More precisely, it follows from [**15**] that the extension H of ν appearing in the proof of Proposition 4.1 can be chosen to factor through a map $\delta\colon \Omega S^{2n+1}\{p\} \to E^{2n+1}$, and thus BW_n

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also retracts off E^{2n+1} after suspending twice. The space E^{2n+1}, along with a homotopy pullback diagram

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \longrightarrow & E^{2n+1} & \longrightarrow & F^{2n+1} & \longrightarrow & \Omega S^{2n+1} \\
\big\downarrow & & \big\| & & \big\downarrow & & \big\downarrow \\
\Omega S^{2n+1}\{p\} & \xrightarrow{\delta} & E^{2n+1} & \longrightarrow & P^{2n+1}(p) & \xrightarrow{i} & S^{2n+1}\{p\} \\
 & & & & \big\downarrow{\scriptstyle q} & & \big\downarrow \\
 & & & & S^{2n+1} & = & S^{2n+1}
\end{array} \tag{3}$$

determined by the factorization of the pinch map $q\colon P^{2n+1}(p) \to S^{2n+1}$ in the bottom right square, was thoroughly analysed in Cohen, Moore and Neisendorfer's study of the homotopy theory of Moore spaces [**6**, **7**], where decompositions of ΩE^{2n+1}, ΩF^{2n+1} and $\Omega P^{2n+1}(p)$ were used to determine the homotopy exponents of spheres and Moore spaces. The double suspension fibration $W_n \longrightarrow S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$ was shown to retract off the homotopy fibration along the top row of the loops on (3), with W_n and S^{2n-1} appearing as factors containing the bottom cells in product decompositions of ΩE^{2n+1} and ΩF^{2n+1}, respectively.

Consider the morphism of homotopy fibrations

$$\begin{array}{ccc}
T^{2n+1}(p) & \longrightarrow & X \\
\big\downarrow{\scriptstyle E} & & \big\downarrow \\
\Omega S^{2n+1}\{p\} & \xrightarrow{\delta} & E^{2n+1} \\
\big\downarrow{\scriptstyle H} & & \big\downarrow \\
BW_n & = & BW_n
\end{array} \tag{4}$$

determined by the factorization of H through δ. As with the splitting in Proposition 4.1, the retraction of $\Sigma^2 BW_n$ off $\Sigma^2 E^{2n+1}$ can be desuspended in Kervaire invariant one dimensions. One difference, however, is that since W_n is always a retract of ΩE^{2n+1} by Cohen, Moore and Neisendorfer's decomposition, the image of the homology class $b_{2np-2} \in H_{2np-2}(\Omega S^{2n+1}\{p\})$ under δ_* is spherical for all n (as opposed to just those $n = p^j$ for which $\pi^S_{2n(p-1)-2}$ contains an element of strong Kervaire invariant one), and thus the non-existence of Kervaire invariant elements does not obstruct the possibility of an unstable decomposition of E^{2n+1} as it does for $\Omega S^{2n+1}\{p\}$. It would therefore be interesting to know whether a homotopy class $S^{2np-2} \to E^{2n+1}$ with Hurewicz image $\delta_*(b_{2np-2})$ could be extended to a map $\Omega T^{2np+1}(p) \to E^{2n+1}$, as in the proof of Theorem 1.1, to prove the conjectured homotopy equivalence $BW_n \simeq \Omega T^{2np+1}(p)$ for all n and split the homotopy fibration in the second column of the diagram above. We show below that BW_n is in fact a retract of E^{2n+1} for all n, delooping the result of Cohen, Moore and Neisendorfer (see [**7**, Theorem 3.2]).

Proposition 4.2. *For all $n \geq 1$, BW_n is a retract of E^{2n+1}.*

Proof. By the construction of BW_n in [**11**], there is a homotopy fibration sequence

$$\Omega^2 S^{2n+1} \longrightarrow BW_n \times S^{4n-1} \longrightarrow S^{2n} \xrightarrow{E} \Omega S^{2n+1},$$

where the connecting map factors as $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \xrightarrow{i_1} BW_n \times S^{4n-1}$. Since the composite $S^{2n} \xrightarrow{E} \Omega S^{2n+1} \longrightarrow S^{2n+1}\{p\}$ is just the inclusion of the bottom cell of $S^{2n+1}\{p\}$, there is a homotopy commutative diagram

$$\begin{array}{ccc}
BW_n \times S^{4n-1} & \longrightarrow & E^{2n+1} \\
\downarrow & & \downarrow \\
S^{2n} & \longrightarrow & P^{2n+1}(p) \\
\downarrow{\scriptstyle E} & & \downarrow{\scriptstyle i} \\
\Omega S^{2n+1} & \longrightarrow & S^{2n+1}\{p\}
\end{array}$$

where the induced map of fibres determines a map $g\colon BW_n \to E^{2n+1}$. Observe that the connecting map $\Omega^2 S^{2n+1} \to BW_n \times S^{4n-1}$ of the first column induces an isomorphism on $H_{2np-2}(\ )$ since ν does, and the connecting map $\delta\colon \Omega S^{2n+1}\{p\} \to E^{2n+1}$ of the second column induces an isomorphism on $H_{2np-2}(\ )$ by the commutativity of (4). Therefore, since the map $\Omega^2 S^{2n+1} \to \Omega S^{2n+1}\{p\}$ given by the loops on the bottom horizontal map induces a monomorphism in homology by a Serre spectral sequence argument, we conclude that $g\colon BW_n \to E^{2n+1}$ induces an isomorphism on $H_{2np-2}(\ )$ so that the composition $BW_n \xrightarrow{g} E^{2n+1} \to BW_n$ with the extension in (4) is degree one on the bottom cell and thus a homotopy equivalence. □

In cases of strong Kervaire invariant one, Proposition 4.2 can be improved to a product decomposition of E^{2n+1}.

Corollary 4.3. *If $\pi^S_{2n(p-1)-2}$ contains a p-primary Kervaire invariant one element of order p, then there is a homotopy equivalence*

$$E^{2n+1} \simeq BW_n \times X.$$

Proof. Let $a\colon \Omega S^{2n+1}\{p\} \times E \longrightarrow E$ be the homotopy action of the fibre on the total space of the principal homotopy fibration $\Omega S^{2n+1}\{p\} \xrightarrow{\delta} E \longrightarrow P^{2n+1}(p)$. Under the given assumption, it follows from the proof of Theorem 1.1 that $H\colon \Omega S^{2n+1}\{p\} \to BW_n$ has a right homotopy inverse s. Let j denote the fibre inclusion in (4) and consider the composite

$$BW_n \times X \xrightarrow{s \times j} \Omega S^{2n+1}\{p\} \times E \xrightarrow{a} E.$$

Since the restriction to each factor induces a monomorphism in homology, it follows that $a \circ (s \times j)$ is a homotopy equivalence. □

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