*Adv. Appl. Prob.* **48,** 1161–1182 (2016)
*doi:*10.1017/apr.2016.69


# A STOCHASTIC DIFFERENTIAL GAME FOR QUADRATIC-LINEAR DIFFUSION PROCESSES

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**Abstract**

In this paper we study a stochastic differential game between two insurers whose surplus processes are modelled by quadratic-linear diffusion processes. We consider an exit probability game. One insurer controls its risk process to minimize the probability that the surplus difference reaches a low level (indicating a disadvantaged surplus position of the insurer) before reaching a high level, while the other insurer aims to maximize the probability. We solve the game by finding the value function and the Nash equilibrium strategy in explicit forms.

*Keywords:* Stochastic differential game; Nash equilibrium; Fleming–Bellman–Isaacs equations; quadratic-linear diffusion process

2010 Mathematics Subject Classification: Primary 60G40
Secondary 93E20

## 1. Introduction

Stochastic differential games are used to model dynamic competitions or cooperations. In [2] and [3], some noncooperative stochastic differential games were solved and explicit optimal plays given. It is shown that values of games exist if the Isaacs' condition holds. In [5], sup-value and sub-value functions of a finite-horizon game were defined, which were shown to be the unique viscosity solutions of the Bellman–Isaacs equations. In [1], games between two investors were considered using a general payoff function. Conditions under which a game has an achievable value were provided. In an exit probability game (and some other specific games), the value function and resulting equilibrium portfolio strategies are found explicitly. In [14], analytically tractable solutions of cooperative stochastic differential games with subgame consistency were derived. In [10], a theorem giving the Hamilton–Jacobi–Bellman–Isaacs conditions for a two-player game in a jump diffusion setting was proved. The result was then used to study risk minimization problems. In [16], an exit probability game between two insurers was considered for the first time with proportional reinsurance control under a linear diffusion model, where the Nash equilibrium of the game was given in explicit form. In [13], a nonproportional zero-sum game for insurers was studied. In [9], a proportional reinsurance game with a win scenario of absolute dominance was considered. Parameter conditions under which the game is solvable were given. In each solvable case, the value function and the Nash equilibrium strategy were found explicitly.

In this paper we study a competitive game between two insurance companies. Related risk models and game problems can be found in [1], [6], [7], [11], and [16]. For each insurance company, the surplus process is modelled by a diffusion process with one controllable variable, where the diffusion term is proportional to the variable (linear) and the drift term is a quadratic

Received 26 March 2015; revision received 28 October 2015.
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function of the variable. This *quadratic-linear* diffusion process has been considered in [6], [7], [11], and others, which can be applied in a reinsurance model with a varying reinsurance premium rate, or in a capital model with a friction factor of productivity. As in [1] and [16], we define and consider an exit probability game. We note that the controlled risk process (in quadratic-linear form) in this paper is different from the ones in [1] and [16], where the drift and diffusion terms are both linear functions of the controllable variable. We first define a *winning* event (for insurer two) that the difference in surplus levels (between insurer one and insurer two) hits a given low level $a$ before it hits a given high level $b$. In the game, insurer one tries to minimize the probability of the winning event of insurer two, while insurer two tries to maximize the probability. In other words, insurer one controls its risk to push the surplus difference to exit the interval $(a, b)$ through the high level $b$, while insurer two controls to pull the difference to exit through $a$. Using a min-max criterion, a value function is defined. With appropriate regularity, the value function can be characterized by the so-called Fleming–Bellman–Isaacs (FBI) equations (see [13], [15], and [16]) which involve supremum and infimum operations. Using special parameter classifications, the FBI equations are solved and a classical $C^2(a, b)$ solution is given explicitly in each parameter case. By a verification result, the solution is shown to be the value function. When solving the FBI equations (with supremum and infimum operations), a saddle point is found explicitly, which yields an optimal strategy of the players, i.e. the *Nash equilibrium strategy*, in a feedback form.

The rest of the paper is organized as follows. In Section 2 we introduce the mathematical model and formulate the game problem. In Section 3 we present the FBI equations and prove a verification theorem. We solve the FBI equations explicitly in Sections 4 and 5 via different parameter cases. In Section 6 we give examples and discussions.

## 2. Quadratic-linear risk processes and game problem

In this section we formulate the game problem and define the value function. We begin with the surplus processes $R_i$ for company $i$, $i = 1, 2$, which are described by the following diffusion processes:

$$\mathrm{d}R_i = \mu_i(u_i)\,\mathrm{d}t + \sigma_i(u_i)\,\mathrm{d}w_i,$$

where $\{w_i\}_{t\geq 0}$, $i = 1, 2$, are uncorrelated standard Brownian motions adapted to *information filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the drift and diffusion terms are given by

$$\mu_i(u_i) = \alpha_i u_i^2 + \beta_i u_i + \gamma_i, \qquad \sigma_i(u_i) = \sigma_i u_i, \tag{1}$$

which are in a quadratic-linear form (see, e.g. [6] and [7]), where $u_i$ $(0 \leq u_i \leq 1)$, $i = 1, 2$, are levels of risk exposure and $\sigma_i$ $(> 0)$, $i = 1, 2$, are insurance volatilities that represent the risk levels of the insurance companies. This particular form (of the linear-quadratic surplus process) can be viewed as a result of a varying reinsurance premium. For example, suppose that without reinsurance the surplus for company $i$ is approximated by $\mathrm{d}R_i = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}w_i$, $i = 1, 2$ (see, e.g. [4]). If reinsurance is bought with risk exposure level $u_i$ at a constant reinsurance premium level $\lambda_i$, then the surplus is approximated by $\mathrm{d}R_i = [\mu_i - \lambda_i(1 - u_i)]\,\mathrm{d}t + \sigma_i u_i\,\mathrm{d}w_i$ (see, e.g. [12]). However, if we suppose that the reinsurance becomes cheaper when more reinsurance is bought; specifically, if we assume that the reinsurance has a decreasing premium rate of the form $\lambda_i[1 - k_i(1 - u_i)]$ with some constant $0 < k_i < 1$ (i.e. the reinsurance premium decreases linearly when more reinsurance is bought), then the insurance surplus becomes $\mathrm{d}R_i = \{\mu_i - \lambda_i(1 - u_i)[1 - k_i(1 - u_i)]\}\,\mathrm{d}t + \sigma_i u_i\,\mathrm{d}w_i$, which follows the quadratic-linear form of (1) with $\alpha_i = \lambda_i k_i$, $\beta_i = \lambda_i(1 - 2k_i)$, and $\gamma_i = \mu_i - \lambda_i(1 - k_i)$. We note that, from this reinsurance formulation, the parameters $\alpha_i$, $i = 1, 2$, are positive. We also note that, in a model of [6] on the capital of a company (with $u_i$ representing the size of the company), the parameter $\alpha_i$ is assumed to be negative. The parameter is called the internal competition factor which reflects a friction (counter-productivity) phenomenon of the company when over-hiring. However, from the mathematical point of view in this paper, we do not need to require any restrictions on the parameters $\alpha_i$, $i = 1, 2$ (i.e. they can be either positive or negative).

Now we suppose that the insurers earn interest at a constant rate. So for each $i = 1, 2$, the surplus process is governed by the following stochastic differential equation (SDE):

$$\mathrm{d}R_i = [rR_i + \mu_i(u_i)]\,\mathrm{d}t + \sigma_i(u_i)\,\mathrm{d}w_i,$$

where $r$ is the risk-free rate.

Next we consider *dynamic* reinsurance control, i.e. the risk exposures $u_i$, $i = 1, 2$, can be changed over time. We denote by $R_i^{U_i}$ the controlled surplus process of insurer $i$ under a dynamic reinsurance control policy with risk exposure process $U_i := \{u_i(t)\}_{t\geq 0}$. Write the difference of the surplus processes by $X^{U_1,U_2} := R_1^{U_1} - R_2^{U_2}$, which is then governed by the following SDE:

$$\begin{aligned} \mathrm{d}X_t^{U_1,U_2} &= (rX_t^{U_1,U_2} + \alpha_1 u_1^2 + \beta_1 u_1 - \alpha_2 u_2^2 - \beta_2 u_2 + \delta)\,\mathrm{d}t + \sigma_1 u_1\,\mathrm{d}w_1 - \sigma_2 u_2\,\mathrm{d}w_2, \\ X_0^{U_1,U_2} &= x, \end{aligned} \tag{2}$$

where $x$ is the initial surplus difference and $\delta = \gamma_1 - \gamma_2$.

A reinsurance control policy with risk exposure process $U := \{u(t)\}_{t\geq 0}$ is said to be *admissible* if

(i) $0 \leq u(t) \leq 1$;

(ii) $u(t) \in \mathcal{F}_t$ for all $t > 0$; and

(iii) $u(t)$ is square integrable over $[0, T]$ for all $T > 0$ almost surely ($\int_0^T u^2(t)\,\mathrm{d}t$ exists).

We denote by $\Pi$ the set of admissible controls.

For an initial difference $x$ in interval $(a, b)$, where $a$ is the lower boundary and $b$ is the upper boundary, we define a hitting time of the controlled surplus difference process under a paired admissible policy $(U_1, U_2)$, i.e.

$$\tau_y^{U_1,U_2} = \inf\{t : X_t^{U_1,U_2} = y\} \quad \text{for any } a \leq y \leq b.$$

Now we give the definition of a *performance function* under $(U_1, U_2)$, i.e.

$$V^{U_1,U_2}(x) := \mathbb{P}_x(\tau_a^{U_1,U_2} < \infty, \tau_a^{U_1,U_2} < \tau_b^{U_1,U_2}), \tag{3}$$

where $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot \mid X_0^{U_1,U_2} = x)$. The performance function can be viewed as the probability that insurer two *wins* the game, i.e. the difference of the surplus levels (between insurer one and insurer two) reaches the low target level $a$ (indicating a relatively high surplus level of insurer two) at a finite time (before reaching the high target level $b$). Using the performance function, we formulate a noncooperative game. In this game, insurer two controls its risk to maximize the performance function, while insurer one tries to minimize it. Now we use a *min-max* criterion to define the value function of the game. We first define *sub-value* and *sup-value* functions

$$\underline{V}(x) := \sup_{U_2\in\Pi} \inf_{U_1\in\Pi} V^{U_1,U_2}(x), \qquad \bar{V}(x) := \inf_{U_1\in\Pi} \sup_{U_2\in\Pi} V^{U_1,U_2}(x). \tag{4}$$

Obviously, $\underline{V}(x) \le \bar{V}(x)$ for all $x \in (a, b)$. In the case $\underline{V}(x) = \bar{V}(x)$ for all $x \in (a, b)$, we define the function as the *value function*, denoted by $V$, i.e.

$$V(x) = \underline{V}(x) = \bar{V}(x).$$

In this paper we show that the value function always exists in explicit form, and so does the *Nash equilibrium* strategy $(U_1^*, U_2^*)$, which satisfies

$$V^{U_1^*, U_2}(x) \le V^{U_1^*, U_2^*}(x) \le V^{U_1, U_2^*}(x) \tag{5}$$

for any admissible controls $U_1$, $U_2$, and

$$V(x) = V^{U_1^*, U_2^*}(x) \quad \text{on } (a, b). \tag{6}$$

## 3. FBI equations and verification theorem

In this section we give the FBI equations that govern the value function and prove the verification theorem that a classical solution to the FBI equations satisfying appropriate boundary conditions is the value function.

Suppose that the value function of the game exists and is a $C^2(a, b)$ function. Furthermore, suppose that there exists an admissible Nash equilibrium or *saddle point* strategy $(U_1^*, U_2^*)$, satisfying (5) and (6), and that the strategy is a *feedback* strategy, determined by a pair of risk exposure functions $(u_1^*(\cdot), u_2^*(\cdot))$ (i.e. the risk exposure levels at any time are the function values of $u_1^*(x)$ and $u_2^*(x)$ at the then-current difference level $x$). One can show that the value function $V$ solves the following FBI equations (see, e.g. [13], [15], and [16]):

$$\sup_{u_2 \in [0,1]} L^{u_1^*(x), u_2} V(x) = 0, \qquad \inf_{u_1 \in [0,1]} L^{u_1, u_2^*(x)} V(x) = 0, \tag{7}$$

where the operator $L$ is defined by

$$L^{u_1, u_2} V(x) = (rx + \alpha_1 u_1^2 + \beta_1 u_1 - \alpha_2 u_2^2 - \beta_2 u_2 + \delta) V'(x) + \tfrac{1}{2}(\sigma_1^2 u_1^2 + \sigma_2^2 u_2^2) V''(x),$$

and functions $u_1^*(x)$ and $u_2^*(x)$ satisfy

$$u_1^*(x) = \arg \inf_{u_1 \in [0,1]} L^{u_1, u_2^*(x)} V(x), \qquad u_2^*(x) = \arg \sup_{u_2 \in [0,1]} L^{u_1^*(x), u_2} V(x). \tag{8}$$

That is, the pair $(u_1^*(x), u_2^*(x))$ is a saddle point of $L^{u_1, u_2} V(x)$. From (7) and (8), it holds that

$$L^{u_1^*(x), u_2^*(x)} V(x) = 0. \tag{9}$$

We note that the FBI equations are simplified Bellman–Isaacs equations when both the value function and the Nash equilibrium strategy exist (see [5] and [15]). In the following, we prove the verification theorem that if a decreasing $C^2(a, b)$ solution to (7) with boundary conditions

$$V(a) = 1 \quad \text{and} \quad V(b) = 0 \tag{10}$$

exists, then the value function exists and coincides with the solution.

Below we give a result that under certain admissible controls, the surplus difference process exits the interval $(a, b)$ almost surely.

**Lemma 1.** *For admissible controls $U_i = \{u_i(t)\}_{t\geq 0}$, $i = 1, 2$, satisfying $\sigma_1^2 u_1^2(t) + \sigma_2^2 u_2^2(t) > \varepsilon$, for all $t > 0$ and a given positive $\varepsilon$, it holds that $\mathbb{P}_x(\tau_a^{U_1,U_2} \wedge \tau_b^{U_1,U_2} < \infty) = 1$, where $a < x < b$.*

*Proof.* Write $\tau = \tau_a^{U_1,U_2} \wedge \tau_b^{U_1,U_2}$. Applying Itô's formula, we obtain

$$\mathrm{e}^{KX_{\tau\wedge T}^{U_1,U_2}} - \mathrm{e}^{Kx} = \int_0^{\tau\wedge T} K\mathrm{e}^{KX_t^{U_1,U_2}} M_t\,\mathrm{d}t + \sigma_1 K \int_0^{\tau\wedge T} u_1(t)\mathrm{e}^{KX_t^{U_1,U_2}}\,\mathrm{d}w_1 - \sigma_2 K \int_0^{\tau\wedge T} u_2(t)\mathrm{e}^{KX_t^{U_1,U_2}}\,\mathrm{d}w_2 \quad \text{for any } T > 0 \text{ and any } K,$$

where

$$M_t = rX_t^{U_1,U_2} + \alpha_1 u_1(t)^2 + \beta_1 u_1(t) - \alpha_2 u_2(t)^2 - \beta_2 u_2(t) + \delta + \tfrac{1}{2}K[\sigma_1^2 u_1(t)^2 + \sigma_2^2 u_2(t)^2].$$

Taking the expectation on both sides, we obtain

$$\mathbb{E}_x[\mathrm{e}^{KX_{\tau\wedge T}^{U_1,U_2}} - \mathrm{e}^{Kx}] = \mathbb{E}_x\bigg[\int_0^{\tau\wedge T} K\mathrm{e}^{KX_t^{U_1,U_2}} M_t\,\mathrm{d}t\bigg]. \tag{11}$$

Choosing a large $K$ $(> 0)$ such that $\frac{1}{2}K\varepsilon > \varepsilon + r|a| + |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + |\delta|$, it holds that $M_t \geq \varepsilon$ for $0 < t < \tau$. From (11), we have

$$\mathrm{e}^{Kb} - \mathrm{e}^{Kx} \geq K\mathbb{E}_x\bigg[\mathbf{1}_{\{\tau > T\}} \int_0^T \mathrm{e}^{KX_t^{U_1,U_2}} \varepsilon\,\mathrm{d}t\bigg] \geq K\varepsilon\mathrm{e}^{Ka} T\mathbb{P}_x(\tau > T),$$

where $\mathbf{1}$ is the indicator function.

Letting $T \to \infty$, we obtain $\lim_{T\to\infty} \mathbb{P}_x(\tau > T) = 0$. The lemma is proved. □

In the following, we prove the verification theorem.

**Theorem 1.** *Suppose that*

(i) *$W$ is a decreasing $C^2(a, b)$ solution to the FBI equations (7) subject to the boundary conditions (10) with the saddle point $(u_1^*(x), u_2^*(x))$ determined by (8);*

(ii) *$W'$ is bounded on $[a, b]$;*

(iii) *there exists $\varepsilon > 0$ such that $u_2^*(x) > \varepsilon$ for $x \in [a, b]$.*

*Then the value function $V$ exists and it holds that $W(x) = V(x)$ for $x \in [a, b]$. Furthermore, the feedback control strategy $(U_1^*, U_2^*)$ given by*

$$U_1^* = \{u_1^*(X_t^{U_1^*,U_2^*})\}_{t\geq 0} \quad \text{and} \quad U_2^* = \{u_2^*(X_t^{U_1^*,U_2^*})\}_{t\geq 0}$$

*is the Nash equilibrium control such that $V^{U_1^*,U_2^*}(x) = V(x)$, where $\{X_t^{U_1^*,U_2^*}\}_{t\geq 0}$ is the surplus difference process that solves SDE (2) under policy $(U_1^*, U_2^*)$.*

*Proof.* For convenience, write hitting time $\tau^{U_1^*,U_2} = \tau_a^{U_1^*,U_2} \wedge \tau_b^{U_1^*,U_2}$. For any time $T > 0$ and admissible controls $U_i = \{u_i(t)\}_{t\geq 0}$, $i = 1, 2$, applying Itô's formula, we obtain

$$\begin{aligned} W(X^{U_1^*,U_2}_{\tau^{U_1^*,U_2}\wedge T}) - W(x) &= \int_0^{\tau^{U_1^*,U_2}\wedge T} L^{u_1^*(X_t^{U_1^*,U_2}),u_2(t)} W(X_t^{U_1^*,U_2})\,\mathrm{d}t \\ &\quad + \sigma_1 \int_0^{\tau^{U_1^*,U_2}\wedge T} W'(X_t^{U_1^*,U_2}) u_1^*(X_t^{U_1^*,U_2})\,\mathrm{d}w_1 \\ &\quad - \sigma_2 \int_0^{\tau^{U_1^*,U_2}\wedge T} W'(X_t^{U_1^*,U_2}) u_2(t)\,\mathrm{d}w_2 W'(X_t^{U_1^*,U_2})\,\mathrm{d}Z_t \\ &\leq \sigma_1 \int_0^{\tau^{U_1^*,U_2}\wedge T} W'(X_t^{U_1^*,U_2}) u_1^*(X_t^{U_1^*,U_2})\,\mathrm{d}w_1 \\ &\quad - \sigma_2 \int_0^{\tau^{U_1^*,U_2}\wedge T} W'(X_t^{U_1^*,U_2}) u_2(t)\,\mathrm{d}w_2, \end{aligned}$$

where the inequality is because $W$ solves the first equation in (7). Taking the expectation on both sides, we have

$$\begin{aligned} W(x) &\geq \mathbb{E}_x[W(X^{U_1^*,U_2}_{\tau^{U_1^*,U_2}\wedge T})] \\ &= \mathbb{P}_x(X^{U_1^*,U_2}_{\tau^{U_1^*,U_2}} = a,\ \tau^{U_1^*,U_2} < T) W(a) + \mathbb{P}_x(X^{U_1^*,U_2}_{\tau^{U_1^*,U_2}} = b,\ \tau^{U_1^*,U_2} < T) W(b) \\ &\quad + \mathbb{E}_x[W(X_T^{U_1^*,U_2})\,\mathbf{1}_{\{\tau^{U_1^*,U_2} > T\}}] \\ &\geq \mathbb{P}_x(X^{U_1^*,U_2}_{\tau^{U_1^*,U_2}} = a,\ \tau^{U_1^*,U_2} < T), \end{aligned}$$

where the last line is due to the boundary conditions and the fact that the function $W$ is nonnegative on $[a, b]$. Letting $T \to \infty$, we have

$$W(x) \geq \mathbb{P}_x(X^{U_1^*,U_2}_{\tau^{U_1^*,U_2}} = a,\ \tau^{U_1^*,U_2} < \infty) = \mathbb{P}_x(\tau_a^{U_1^*,U_2} < \tau_b^{U_1^*,U_2},\ \tau_a^{U_1^*,U_2} < \infty),$$

which gives

$$W(x) \geq V^{U_1^*,U_2}(x). \tag{12}$$

On the other hand, write $\tau^{U_1,U_2^*} = \tau_a^{U_1,U_2^*} \wedge \tau_b^{U_1,U_2^*}$, from the second equation in (7) we obtain

$$\begin{aligned} W(x) &\leq \mathbb{E}_x[W(X^{U_1,U_2^*}_{\tau^{U_1,U_2^*}\wedge T})] \\ &= \mathbb{P}_x(X^{U_1,U_2^*}_{\tau^{U_1,U_2^*}} = a,\ \tau^{U_1,U_2^*} < T) W(a) + \mathbb{P}_x(X^{U_1,U_2^*}_{\tau^{U_1^*,U_2^*}} = b,\ \tau^{U_1,U_2^*} < T) W(b) \\ &\quad + \mathbb{E}_x[W(X_T^{U_1,U_2^*})\,\mathbf{1}_{\{\tau^{U_1,U_2^*} > T\}}]. \end{aligned} \tag{13}$$

From $u_2^*(x) > \varepsilon$ on $[a, b]$, using Lemma 1, we have $\lim_{T\to\infty} \mathbb{P}(\tau^{U_1,U_2^*} > T) = 0$. Passing $T \to \infty$ in (13) and using the boundary conditions, we have

$$W(x) \leq \mathbb{P}_x(X^{U_1,U_2^*}_{\tau^{U_1,U_2^*}} = a,\ \tau^{U_1,U_2^*} < \infty) = \mathbb{P}_x(\tau_a^{U_1,U_2^*} < \tau_b^{U_1,U_2^*},\ \tau_a^{U_1,U_2^*} < \infty),$$

wherefrom

$$W(x) \le V^{U_1,U_2^*}(x). \tag{14}$$

Thus,

$$\underline{V}(x) \ge \inf_{U_1\in\Pi} V^{U_1,U_2^*}(x) \ge W(x) \ge \sup_{U_2\in\Pi} V^{U_1^*,U_2}(x) \ge \bar{V}(x).$$

From $\underline{V}(x) \le \bar{V}(x)$, we conclude that $\underline{V}(x) = W(x) = \bar{V}(x) = V(x)$.

Furthermore, if we replace $U_2$ by $U_2^*$ in (12) and $U_1$ by $U_1^*$ in (14), we then obtain

$$W(x) = V^{U_1^*,U_2^*}(x).$$

□

**Remark 1.** We observe that the hitting times $\tau_a^{U_1,U_2}$ and $\tau_b^{U_1,U_2}$ under an arbitrary admissible control $(U_1, U_2)$ can be infinite with a positive probability. For example, under an admissible control $(U_1, U_2)$ with $u_1(t) = u_2(t) \equiv 0$, the process $X_t^{U_1,U_2}$ can stay at level 0 forever when $\delta = 0$. In Lemma 1, the condition $\sigma_1^2 u_1^2(t) + \sigma_2^2 u_2^2(t) > \varepsilon$, i.e. the volatility of the controlled process $X_t^{U_1,U_2}$ being uniformly bounded above 0 (which is called the uniform parabolicity condition; see, e.g. [8]), guarantees that the minimum of the two hitting times is finite almost surely.

**Remark 2.** In the event that the hitting times $\tau_a^{U_1,U_2}$ and $\tau_b^{U_1,U_2}$ under an admissible control $(U_1, U_2)$ are infinite, i.e. the controlled process $X_t^{U_1,U_2}$ stays in the interval $(a, b)$ forever, it indicates that insurer two does not win, by the definition of the performance function (3). To prevent this event from occurring, insurer two can take a strategy with the risk exposure level uniformly bounded above 0 (so that the uniform parabolicity condition holds). In the verification theorem, we assume that the Nash equilibrium reinsurance strategy of insurer two satisfies this condition in order to prove the optimality.

In the next two sections, we solve the game problem. To do that, we solve the FBI equations (7) for an explicit solution with boundary conditions (10). By the verification theorem, the solution coincides with the value function.

For any $C^2(a, b)$ function $W$, write

$$\hat{u}_{1,W}(x) = -\frac{\beta_1 W'(x)}{\sigma_1^2 W''(x) + 2\alpha_1 W'(x)}, \qquad \hat{u}_{2,W}(x) = \frac{\beta_2 W'(x)}{\sigma_2^2 W''(x) - 2\alpha_2 W'(x)}, \tag{15}$$

which satisfy

$$\left.\frac{\mathrm{d}L^{u_1,u_2}W(x)}{\mathrm{d}u_1}\right|_{u_1=\hat{u}_{1,W}(x)} = 0, \qquad \left.\frac{\mathrm{d}L^{u_1,u_2}W(x)}{\mathrm{d}u_2}\right|_{u_2=\hat{u}_{2,W}(x)} = 0.$$

Noting that $L^{u_1,u_2}W$ is a quadratic function of $u_1$ and $u_2$, the expressions in (15) are used to determine the minimizer or the maximizer in the equations (7), and, hence, the saddle point of $L^{u_1,u_2}W$.

Suppose that the value function $V$ is $C^2(a, b)$ and solves the FBI equations (7) with $V'(x) < 0$ on $(a, b)$. In what follows, we proceed to identify $V$. In the following two sections, we consider two parameter conditions:

(i) $\beta_1 > 0$ and $\beta_2 > 0$ (symmetric case); and

(ii) $\beta_1 < 0$ and $\beta_2 > 0$ (asymmetric case).

The symmetric parameter condition is more reasonable in the practical world; however, the asymmetric parameter case is mathematically interesting. The other parameter cases can be treated similarly and we omit them.

https://doi.org/10.1017/apr.2016.69 Published online by Cambridge University Press

## 4. Explicit solutions – symmetric case

In this section we solve the FBI equations (7) in order to find the value function and Nash equilibrium strategy explicitly in a symmetric parameter case: $\beta_i > 0$, $i = 1, 2$.

Write, for $i = 1, 2$,

$$\xi_i = \frac{2\alpha_i + \beta_i}{\sigma_i^2}.$$

The parameters $\xi_i$, $i = 1, 2$, play a key role in classifying the solution of the FBI equations. We consider three parameter cases in the following subsections:

(i) $\xi_1 + \xi_2 < 0$;

(ii) $\xi_1 + \xi_2 > 0$; and

(iii) $\xi_1 + \xi_2 = 0$.

### 4.1. The case with $\xi_1 + \xi_2 < 0$

Note that $\xi_2 < -\xi_1$ in this case. Define the following sets:

$$\mathcal{X}_1 = \left\{x \in [a, b] \colon \xi_2 < \frac{V''}{V'} < -\xi_1\right\},$$

$$\mathcal{X}_2 = \left\{x \in [a, b] \colon \frac{V''}{V'} \le \xi_2\right\}, \qquad \mathcal{X}_3 = \left\{x \in [a, b] \colon -\xi_1 \le \frac{V''}{V'}\right\}.$$

We find these sets explicitly. To do that, we first give a simplified equation that governs the value function on each set. We then determine the end point(s) of each set using the definition condition(s) of the set.

On set $\mathcal{X}_1$, from the assumption that $\beta_1 > 0$, we have $V''/V' < -\xi_1 < -2\alpha_1/\sigma_1^2$, which implies that $\frac{1}{2}\sigma_1^2 V'' + \alpha_1 V' > 0$. So $\hat{u}_{1,V} > 0$. In addition, from $V''/V' < -\xi_1$, we have

$$\hat{u}_{1,V} = -\frac{\beta_1}{\sigma_1^2(V''/V') + 2\alpha_1} < -\frac{\beta_1}{\sigma_1^2(-\xi_1) + 2\alpha_1} = 1.$$

Thus, the minimizer in $u_1 \in [0, 1]$ of $L^{u_1,u_2}V$ is $\hat{u}_{1,V}$ and it holds that

$$\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{\hat{u}_{1,V},u_2}V.$$

On the other hand, from $V''/V' > \xi_2 > 2\alpha_2/\sigma_2^2$, we have $\frac{1}{2}\sigma_2^2 V'' - \alpha_2 V' < 0$. So $\hat{u}_{2,V} > 0$. Also, from $V''/V' > \xi_2$, we have

$$\hat{u}_{2,V} = \frac{\beta_2}{\sigma_2^2(V''/V') - 2\alpha_2} < \frac{\beta_2}{\sigma_2^2\xi_2 - 2\alpha_2} = 1.$$

We then conclude that the maximizer in $u_2 \in [0, 1]$ of $L^{u_1,u_2}V$ is $\hat{u}_{2,V}$ and it holds that

$$\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,\hat{u}_{2,V}}V.$$

Hence, the pair $(\hat{u}_{1,V}, \hat{u}_{2,V})$ is the saddle point of $L^{u_1,u_2}V$, and from (9), $V$ solves

$$L^{\hat{u}_{1,V}(x),\hat{u}_{2,V}(x)}V(x) = 0 \quad \text{on } \mathcal{X}_1. \tag{16}$$

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Equation (16) can be simplified to

$$(rx+\delta)V' - \frac{\beta_1^2 V'^2/2}{\sigma_1^2 V'' + 2\alpha_1 V'} - \frac{\beta_2^2 V'^2/2}{\sigma_2^2 V'' - 2\alpha_2 V'} = 0,$$

or

$$(rx+\delta)\left(\frac{V''}{V'}\right)^2 + f_1(x)\left(\frac{V''}{V'}\right) - f_2(x) = 0,$$

where

$$f_1(x) = 2(rx+\delta)(\bar{\alpha}_1 - \bar{\alpha}_2) - \tfrac{1}{2}(\beta_1\bar{\beta}_1 + \beta_2\bar{\beta}_2),$$
$$f_2(x) = 4\bar{\alpha}_1\bar{\alpha}_2(rx+\delta) + \bar{\alpha}_1\beta_2\bar{\beta}_2 - \bar{\alpha}_2\beta_1\bar{\beta}_1,$$

and

$$\bar{\alpha}_i = \frac{\alpha_i}{\sigma_i^2}, \qquad \bar{\beta}_i = \frac{\beta_i}{\sigma_i^2}, \qquad \text{for } i = 1, 2.$$

Thus, we conjecture that

$$\frac{V''}{V'} = f(x) := -\frac{f_1(x) + \sqrt{f_1^2(x) + 4(rx+\delta) f_2(x)}}{2(rx+\delta)} \quad \text{on } \mathcal{X}_1. \tag{17}$$

Note that

$$f_1^2(x) + 4(rx+\delta) f_2(x) = [2(rx+\delta)(\bar{\alpha}_1 + \bar{\alpha}_2) - \tfrac{1}{2}(\beta_1\bar{\beta}_1 - \beta_2\bar{\beta}_2)]^2 + \beta_1\bar{\beta}_1\beta_2\bar{\beta}_2 \geq 0,$$

wherefrom the function $f$ is well defined. From

$$\lim_{x\to-\delta/r} f(x) = \frac{2(\bar{\alpha}_2\beta_1\bar{\beta}_1 - \bar{\alpha}_1\beta_2\bar{\beta}_2)}{\beta_1\bar{\beta}_1 + \beta_2\bar{\beta}_2},$$

we see that $f$ is continuous on $(-\infty, \infty)$. Furthermore, we can check that $f$ is a decreasing function on $(-\infty, \infty)$ by differentiation. Noting that $\bar{\alpha}_1 + \bar{\alpha}_2 < 0$, we have

$$f(x) = -\bar{\alpha}_1 + \bar{\alpha}_2 - |\bar{\alpha}_1 + \bar{\alpha}_2| = 2\bar{\alpha}_2 \quad \text{as } x \to \infty,$$
$$f(x) = -\bar{\alpha}_1 + \bar{\alpha}_2 + |\bar{\alpha}_1 + \bar{\alpha}_2| = -2\bar{\alpha}_1 \quad \text{as } x \to -\infty.$$

Note that $2\bar{\alpha}_2 < \xi_2 < -\xi_1 < -2\bar{\alpha}_1$. So inequality $f(x) < -\xi_1$ gives $x > x_1$, where

$$x_1 = -\frac{\beta_1(\bar{\alpha}_1 + \bar{\alpha}_2) + (\beta_1\bar{\beta}_1 + \beta_2\bar{\beta}_2)/2}{r(\xi_1 + 2\bar{\alpha}_2)} - \frac{\delta}{r}, \tag{18}$$

and $f(x) > \xi_2$ gives $x < x_2$, where

$$x_2 = \frac{\beta_2(\bar{\alpha}_1 + \bar{\alpha}_2) + (\beta_1\bar{\beta}_1 + \beta_2\bar{\beta}_2)/2}{r(\xi_2 + 2\bar{\alpha}_1)} - \frac{\delta}{r}. \tag{19}$$

Hence,

$$\mathcal{X}_1 = (x_1, x_2) \cap [a, b].$$

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On $\mathcal{X}_2$, from $V''/V' \le \xi_2 < -\xi_1$ and the discussions on the set $\mathcal{X}_1$, we see that the minimizer in $u_1 \in [0,1]$ of $L^{u_1,u_2}V$ is $\hat{u}_{1,V}$ and we have $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{\hat{u}_{1,V},u_2}V$ on $\mathcal{X}_2$. Next, we consider three cases in order to identify the maximizer in $u_2$. If $2\alpha_2/\sigma_2^2 < V''/V' \le \xi_2$ then $\frac{1}{2}\sigma_2^2 V'' - \alpha_2 V' < 0$ and $\hat{u}_{2,V} \ge \beta_2/(\sigma_2^2\xi_2 - 2\alpha_2) = 1$; hence, the maximizer in $u_2 \in [0,1]$ of $L^{u_1,u_2}V$ is 1. If $V''/V' < 2\alpha_2/\sigma_2^2$ then $\frac{1}{2}\sigma_2^2 V'' - \alpha_2 V' > 0$ and $\hat{u}_{2,V} < 0$; hence, the maximizer in $u_2 \in [0,1]$ is also 1. If $V''/V' = 2\alpha_2/\sigma_2^2$ then $\frac{1}{2}\sigma_2^2 V'' - \alpha_2 V' = 0$; from $-\beta_2 V' \ge 0$, the maximizer in $u_2 \in [0,1]$ is still 1. To summarize, it holds that $\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,1}V$ when $V''/V' \le \xi_2$. Thus, the pair $(\hat{u}_{1,V}, 1)$ is the saddle point of $L^{u_1,u_2}V$ and we have

$$L^{\hat{u}_{1,V}(x),1}V(x) = 0 \quad \text{on } \mathcal{X}_2. \tag{20}$$

Equation (20) is equivalent to

$$(rx + \delta - \alpha_2 - \beta_2)V' + \frac{1}{2}\sigma_2^2 V'' - \frac{\beta_1^2 V'^2/2}{\sigma_1^2 V'' + 2\alpha_1 V'} = 0,$$

which can be simplified to

$$\frac{1}{2}\left(\frac{V''}{V'}\right)^2 + g_1(x)\left(\frac{V''}{V'}\right) - g_2(x) = 0,$$

where

$$g_1(x) = \frac{rx+\delta}{\sigma_2^2} - \bar{\alpha}_2 - \bar{\beta}_2 + \bar{\alpha}_1, \qquad g_2(x) = -2\bar{\alpha}_1\left(\frac{rx+\delta}{\sigma_2^2} - \bar{\alpha}_2 - \bar{\beta}_2\right) + \frac{1}{2}\frac{\bar{\beta}_1\beta_1}{\sigma_2^2}.$$

Now, we conjecture that

$$\frac{V''}{V'} = g(x) := -g_1(x) - \sqrt{g_1^2(x) + 2g_2(x)} \quad \text{on } \mathcal{X}_2. \tag{21}$$

Note that

$$g_1^2(x) + 2g_2(x) = \left(\frac{rx+\delta}{\sigma_2^2} - \bar{\alpha}_2 - \bar{\beta}_2 - \bar{\alpha}_1\right)^2 + \frac{\bar{\beta}_1\beta_1}{\sigma_2^2} \ge 0,$$

and we see that the function $g$ is well-defined. We also note that $g$ is a decreasing function in $x \in (-\infty, \infty)$ and

$$\lim_{x\to\infty} g(x) = -\infty, \qquad \lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{2g_2(x)}{g_1(x) - \sqrt{g_1^2(x) + 2g_2(x)}} = -2\bar{\alpha}_1.$$

From $\xi_2 < -2\bar{\alpha}_1$ (which is equivalent to $\xi_1 + \xi_2 < \bar{\beta}_1$), inequality $g(x) \le \xi_2$ gives $x \ge x_2$, where $x_2$ is defined in (19). Thus, we conclude that

$$\mathcal{X}_2 = [x_2, \infty) \cap [a, b].$$

On set $\mathcal{X}_3$, if $-\xi_1 \le V''/V' < -2\alpha_1/\sigma_1^2$, we have $\frac{1}{2}\sigma_1^2 V'' + \alpha_1 V' > 0$ and $\hat{u}_{1,V} \ge -\beta_1/(\sigma_1^2(-\xi_1) + 2\alpha_1) = 1$. So the minimizer in $u_1 \in [0,1]$ of $L^{u_1,u_2}V$ is 1. If $-2\alpha_1/\sigma_1^2 < V''/V'$, we have $\frac{1}{2}\sigma_1^2 V'' + \alpha_1 V' < 0$ and $\hat{u}_{1,V} < 0$; hence, the minimizer in $u_1 \in [0,1]$ is 1. If $-2\alpha_1/\sigma_1^2 = V''/V'$, we have $\frac{1}{2}\sigma_1^2 V'' + \alpha_1 V' = 0$ and $\beta_2 V' < 0$, so the minimizer in $u_1 \in [0,1]$ is still 1. We conclude that $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{1,u_2}V$ on $\mathcal{X}_3$. Also, from $\xi_2 < -\xi_1 \le V''/V'$ and the discussions on $\mathcal{X}_1$, we see that the maximizer in $u_2 \in [0,1]$ of $L^{u_1,u_2}V$ is $\hat{u}_{2,V}$. Thus, we conclude that the pair $(1, \hat{u}_{2,V})$ is the saddle point of $L^{u_1,u_2}V$ and it holds that

$$L^{1,\hat{u}_{2,V}}V(x) = 0 \quad \text{on } \mathcal{X}_3. \tag{22}$$

The equation (22) is equivalent to

$$(rx+\delta+\alpha_1+\beta_1)V' + \frac{1}{2}\sigma_1^2 V'' - \frac{\beta_2^2 V'^2/2}{\sigma_2^2 V'' - 2\alpha_2 V'} = 0,$$

which simplifies to

$$\frac{1}{2}\bigg(\frac{V''}{V'}\bigg)^2 + h_1(x)\bigg(\frac{V''}{V'}\bigg) - h_2(x) = 0,$$

where

$$h_1(x) = \frac{rx+\delta}{\sigma_1^2} + \bar{\alpha}_1 + \bar{\beta}_1 - \bar{\alpha}_2, \qquad h_2(x) = 2\bar{\alpha}_2\bigg(\frac{rx+\delta}{\sigma_1^2} + \bar{\alpha}_1 + \bar{\beta}_1\bigg) + \frac{1}{2}\frac{\bar{\beta}_2\beta_2}{\sigma_1^2}.$$

We let

$$\frac{V''}{V'} = h(x) := -h_1(x) + \sqrt{h_1^2(x) + 2h_2(x)} \quad \text{on } \mathcal{X}_3. \tag{23}$$

From

$$h_1^2(x) + 2h_2(x) = \bigg(\frac{rx+\delta}{\sigma_1^2} + \bar{\alpha}_1 + \bar{\beta}_1 + \bar{\alpha}_2\bigg)^2 + \frac{\bar{\beta}_2\beta_2}{\sigma_1^2} \ge 0,$$

the function $h$ is well-defined. Note that $h$ is a decreasing function in $x \in (-\infty, \infty)$ and

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{2h_2(x)}{h_1(x) + \sqrt{h_1^2(x) + 2h_2(x)}} = 2\bar{\alpha}_2, \qquad \lim_{x\to-\infty} h(x) = \infty.$$

From $2\bar{\alpha}_2 < -\xi_1$ (which is equivalent to $\xi_1 + \xi_2 < \bar{\beta}_2$), inequality $h(x) \ge -\xi_1$ is equivalent to $x \le x_1$, where $x_1$ is defined in (18). So we conclude that

$$\mathcal{X}_3 = (-\infty, x_1] \cap [a,b].$$

To summarize, we have

$$\frac{V''(x)}{V'(x)} = \phi(x) := \begin{cases} h(x), & x \in \mathcal{X}_3 = (-\infty, x_1] \cap [a,b], \\ f(x), & x \in \mathcal{X}_1 = (x_1, x_2) \cap [a,b], \\ g(x), & x \in \mathcal{X}_2 = [x_2, \infty) \cap [a,b]. \end{cases} \tag{24}$$

Note that $f(x_1) = h(x_1) = -\xi_1$ and $f(x_2) = g(x_2) = \xi_2$. So $\phi$ is a continuous function on $[a,b]$. Since functions $h$, $f$, and $g$ are decreasing in $x$, function $\phi$ is a decreasing function. From (24) and the boundary conditions (10), the value function $V$ can be determined. Hence, we obtain the following theorem.

**Theorem 2.** *If $\xi_1 + \xi_2 < 0$ and $\beta_i > 0$, $i = 1, 2$, then the value function of the game is a decreasing $C^2$ function given by*

$$V(x) = \frac{\int_x^b \exp\{\int_a^u \phi(v)\,\mathrm{d}v\}\,\mathrm{d}u}{\int_a^b \exp\{\int_a^u \phi(v)\,\mathrm{d}v\}\,\mathrm{d}u} \quad \textit{for } x \in [a, b], \tag{25}$$

*and the Nash equilibrium strategy is a feedback control associated with the risk exposure functions given by*

$$(u_1^*(x), u_2^*(x)) = \begin{cases} \left(1, \dfrac{\beta_2}{\sigma_2^2 h(x) - 2\alpha_2}\right), & x \in \mathcal{X}_3 = (-\infty, x_1] \cap [a, b], \\ \left(-\dfrac{\beta_1}{\sigma_1^2 f(x) + 2\alpha_1}, \dfrac{\beta_2}{\sigma_2^2 f(x) - 2\alpha_2}\right), & x \in \mathcal{X}_1 = (x_1, x_2) \cap [a, b], \\ \left(-\dfrac{\beta_1}{\sigma_1^2 g(x) + 2\alpha_1}, 1\right), & x \in \mathcal{X}_2 = [x_2, \infty) \cap [a, b], \end{cases} \tag{26}$$

*where functions $\phi$, $f$, $g$, and $h$ are given in (24), (17), (21), and (23), respectively, and threshold points $x_1$ and $x_2$ are given in (18) and (19), respectively.*

*Proof.* From the discussions in this subsection, we see that $V$ defined in (25) is a $C^2(a, b)$ function and solves the FBI equations (7) subject to the boundary conditions (10) with the saddle point given in (26). Its derivative $V'$ is continuous on $[a, b]$ and, hence, bounded. Furthermore, from $f(x_1) = h(x_1)$ and $\beta_2/(\sigma_2^2 f(x_2) - 2\alpha_2) = \beta_2/(\sigma_2^2 \xi_2 - 2\alpha_2) = 1$, the risk exposure function $u_2^*$ is continuous on $[a, b]$. So $u_2^*$ is uniformly bounded above 0. The results of the theorem immediately follow from the verification theorem. □

**Remark 3.** In Theorem 2 for the Nash equilibrium strategy, the risk exposure function $u_1^*$ is a decreasing function in surplus difference $x$, and $u_2^*$ is an increasing function. This implies that if any insurer is into a better surplus position (e.g. insurer one holds a better surplus position when the surplus difference $x$ is higher), then the insurer takes a lower level of risk exposure and buys more reinsurance.

**Remark 4.** In [9] and [16] under the linear model, the Nash equilibrium strategy $(u_1^*, u_2^*)$ occurs on only the boundary of the control region $[0, 1] \times [0, 1]$, i.e. at a given surplus difference level, at least one insurer must take a trivial risk exposure strategy 0 or 1. However, under the quadratic-linear model, the Nash equilibrium strategy can occur inside the control region $[0, 1] \times [0, 1]$, e.g. in Theorem 2, it holds that $0 < u_i^* < 1$, $i = 1, 2$, on $\mathcal{X}_1$.

### 4.2. The case with $\xi_1 + \xi_2 > 0$

We have $-\xi_1 < \xi_2$ in this case. Define sets

$$\mathcal{X}_4 = \left\{x \in [a, b] \colon -\xi_1 < \frac{V''}{V'} < \xi_2\right\},$$

$$\mathcal{X}_5 = \left\{x \in [a, b] \colon \xi_2 \le \frac{V''}{V'}\right\}, \qquad \mathcal{X}_6 = \left\{x \in [a, b] \colon \frac{V''}{V'} \le -\xi_1\right\}.$$

Similar to the previous subsection, now we find these sets explicitly.

On set $\mathcal{X}_4$, we have $-\xi_1 < V''/V'$; from the discussions on $\mathcal{X}_3$ in the previous subsection, we see that the minimizer in $u_1 \in [0, 1]$ of $L^{u_1,u_2}V$ is 1 and it holds that $\inf_{u_1 \in [0,1]} L^{u_1,u_2}V = L^{1,u_2}V$. On the other hand, from $V''/V' < \xi_2$ and the discussions on $\mathcal{X}_2$, we see that the

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maximizer in $u_2 \in [0,1]$ is 1 and conclude that $\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,1}V$. Hence, the pair $(1,1)$ is the saddle point of $L^{u_1,u_2}V$. So it holds that

$$L^{1,1}V(x) = 0 \quad \text{on } \mathcal{X}_4, \tag{27}$$

which gives

$$(rx + \delta + \alpha_1 + \beta_1 - \alpha_2 - \beta_2)V' + \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2)V'' = 0.$$

Thus, we have

$$\frac{V''(x)}{V'(x)} = k(x) := -\frac{2(rx + \delta + \alpha_1 + \beta_1 - \alpha_2 - \beta_2)}{\sigma_1^2 + \sigma_2^2}, \tag{28}$$

which is a decreasing function. So $k(x) < \xi_2$ gives $x > x_3$, where

$$x_3 = \frac{\beta_2/2 - \alpha_1 - \beta_1}{r} - \left(\bar{\alpha}_2 + \frac{1}{2}\bar{\beta}_2\right)\frac{\sigma_1^2}{r} - \frac{\delta}{r}. \tag{29}$$

Also $-\xi_1 < k(x)$ gives $x < x_4$, where

$$x_4 = -\frac{\beta_1/2 - \alpha_2 - \beta_2}{r} + \left(\bar{\alpha}_1 + \frac{1}{2}\bar{\beta}_1\right)\frac{\sigma_2^2}{r} - \frac{\delta}{r}. \tag{30}$$

Hence,

$$\mathcal{X}_4 = (x_3, x_4) \cap [a,b].$$

On $\mathcal{X}_5$, from $\xi_2 \le V''/V'$ and the discussions on $\mathcal{X}_1$ in the previous subsection, we have $\frac{1}{2}\sigma_2^2 V'' - \alpha_2 V' < 0$ and $\hat{u}_{2,V} \le \beta_2/(\sigma_2^2\xi_2 - 2\alpha_2) = 1$. Thus, the maximizer in $u_2 \in [0,1]$ of $L^{u_1,u_2}V$ is $\hat{u}_{2,V}$ and it holds that $\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,\hat{u}_{2,V}}V$. Furthermore, from $-\xi_1 < \xi_2 \le V''/V'$ and the discussions on $\mathcal{X}_3$ in the previous subsection, we see that the minimizer in $u_1 \in [0,1]$ is 1 and it holds that $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{1,u_2}V$. So we conclude that $(1, \hat{u}_{2,V})$ is the saddle point of $L^{u_1,u_2}V$ and $V$ solves $L^{1,\hat{u}_{2,V}}V = 0$ on $\mathcal{X}_5$. We then conjecture that

$$\frac{V''(x)}{V'(x)} = h(x) \quad \text{on } \mathcal{X}_5,$$

where $h$ is given in (23). Note that inequality $\xi_2 \le h(x)$ gives $x \le x_3$. So we come to

$$\mathcal{X}_5 = (-\infty, x_3] \cap [a,b],$$

where $x_3$ is given in (29).

On $\mathcal{X}_6$, from $V''/V' \le -\xi_1$ and the discussions on $\mathcal{X}_1$ in the previous subsection, we see that the minimizer in $u_1 \in [0,1]$ is $\hat{u}_{1,V}$ and it holds that $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{\hat{u}_{1,V},u_2}V$. Furthermore, from $V''/V' \le -\xi_1 < \xi_2$ and the discussions on $\mathcal{X}_2$ in the previous subsection, we see that the maximizer in $u_2 \in [0,1]$ is 1 and it holds that $\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,1}V$. So we conclude that $(\hat{u}_{1,V}, 1)$ is the saddle point of $L^{u_1,u_2}V$ and $V$ solves $L^{\hat{u}_{1,V},1}V = 0$ on $\mathcal{X}_6$. Hence, we conjecture that

$$\frac{V''(x)}{V'(x)} = g(x) \quad \text{on } \mathcal{X}_6,$$

where $g$ is given in (21). Also inequality $g(x) \le -\xi_1$ gives $x \ge x_4$. So it holds that

$$\mathcal{X}_6 = [x_4, \infty) \cap [a,b],$$

where $x_4$ is given in (30).

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Summarizing the discussions, we conjecture that

$$\frac{V''(x)}{V'(x)} = \varphi(x) := \begin{cases} h(x), & x \in \mathcal{X}_5 = (-\infty, x_3] \cap [a, b], \\ k(x), & x \in \mathcal{X}_4 = (x_3, x_4) \cap [a, b], \\ g(x), & x \in \mathcal{X}_6 = [x_4, \infty) \cap [a, b], \end{cases} \tag{31}$$

where functions $h$, $k$, and $g$ are given in (23), (28), and (21). Noting that $k(x_3) = h(x_3) = \xi_2$ and $k(x_4) = g(x_4) = -\xi_1$, we see that $\varphi$ is a continuous function on $[a, b]$. Since $h$, $k$, and $g$ are decreasing functions, so is $\varphi$.

**Theorem 3.** *If $\xi_1 + \xi_2 > 0$ and $\beta_i > 0$, $i = 1, 2$, then the value function of the game is a decreasing $C^2$ function given by*

$$V(x) = \frac{\int_x^b \exp\{\int_a^u \varphi(v)\,\mathrm{d}v\}\,\mathrm{d}u}{\int_a^b \exp\{\int_a^u \varphi(v)\,\mathrm{d}v\}\,\mathrm{d}u} \quad \textit{for } x \in [a, b],$$

*and the Nash equilibrium strategy is a feedback control associated with the risk exposure functions given by*

$$(u_1^*(x), u_2^*(x)) = \begin{cases} \left(1, \dfrac{\beta_2}{\sigma_2^2 h(x) - 2\alpha_2}\right), & x \in \mathcal{X}_5 = (-\infty, x_3] \cap [a, b], \\ (1, 1), & x \in \mathcal{X}_4 = (x_3, x_4) \cap [a, b], \\ \left(\dfrac{\beta_2}{\sigma_2^1 g(x) + 2\alpha_1}, 1\right), & x \in \mathcal{X}_6 = [x_4, \infty) \cap [a, b], \end{cases}$$

*where functions $\varphi$, $g$, and $h$ are given in (31), (21), and (23), respectively, and threshold points $x_3$ and $x_4$ are given in (29) and (30), respectively.*

### 4.3. The case with $\xi_1 + \xi_2 = 0$

In this case, we observe that

$$-\xi_1 = \xi_2 = f(x_1) = f(x_2) = h(x_1) = h(x_3) = k(x_3) = k(x_4) = g(x_4) = g(x_2),$$

and it holds that $x_1 = x_2 = x_3 = x_4$. So for the Nash equilibrium strategy, the two insurers now share a common threshold point. We then let

$$\frac{V''(x)}{V'(x)} = \psi(x) := \begin{cases} h(x), & x \in (-\infty, x_1] \cap [a, b], \\ g(x), & x \in (x_1, \infty) \cap [a, b], \end{cases} \tag{32}$$

and obtain the following theorem.

**Theorem 4.** *If $\xi_1 + \xi_2 = 0$ and $\beta_i > 0$, $i = 1, 2$, then the value function of the game is a decreasing $C^2$ function given by*

$$V(x) = \frac{\int_x^b \exp\{\int_a^u \psi(v)\,\mathrm{d}v\}\,\mathrm{d}u}{\int_a^b \exp\{\int_a^u \psi(v)\,\mathrm{d}v\}\,\mathrm{d}u} \quad \textit{for } x \in [a, b],$$

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*and the Nash equilibrium strategy is a feedback control associated with the risk exposure functions given by*

$$(u_1^*(x), u_2^*(x)) = \begin{cases} \left(1, \dfrac{\beta_2}{\sigma_2^2 h(x) - 2\alpha_2}\right), & x \in (-\infty, x_1] \cap [a, b], \\ \left(-\dfrac{\beta_1}{\sigma_1^2 g(x) + 2\alpha_1}, 1\right), & x \in (x_1, \infty) \cap [a, b], \end{cases}$$

*where the functions $\psi$, $g$, and $h$ are given in (32), (21), and (23), respectively, and $x_1$ is given in (18).*

## 5. Explicit solutions – asymmetric case

In this section we solve the game under an asymmetric parameter case: $\beta_1 < 0$ and $\beta_2 > 0$. Define

$$\theta_1 = \frac{2(\alpha_1 + \beta_1)}{\sigma_1^2}.$$

The parameters $\theta_1$ and $\xi_2$ play a key role in classifying the solutions. We consider three cases:

(i) $\theta_1 + \xi_2 < 0$;

(ii) $\theta_1 + \xi_2 > 0$; and

(iii) $\theta_1 + \xi_2 = 0$.

### 5.1. The case with $\theta_1 + \xi_2 < 0$

In this case, we have $\xi_2 < -\theta_1$. Define the following sets:

$$\mathcal{Y}_1 = \left\{x \in [a, b] \colon \xi_2 < \frac{V''}{V'} < -\theta_1\right\},$$

$$\mathcal{Y}_2 = \left\{x \in [a, b] \colon \frac{V''}{V'} \le \xi_2\right\}, \qquad \mathcal{Y}_3 = \left\{x \in [a, b] \colon -\theta_1 \le \frac{V''}{V'}\right\}.$$

Now we identify these sets.

On set $\mathcal{Y}_1$, from $\beta_1 < 0$, we have $-2\alpha_1/\sigma_1^2 < -\theta_1$. If $V''/V' < -2\alpha_1/\sigma_1^2$ then $\frac{1}{2}\sigma_1^2 V'' + \alpha_1 V' > 0$ and $\hat{u}_{1,V} < 0$; hence, the minimizer in $u_1 \in [0, 1]$ of $L^{u_1,u_2}V$ is 0. If $-2\alpha_1/\sigma_1^2 < V''/V' \le -\theta_1$ then $\frac{1}{2}\sigma_1^2 V'' + \alpha_1 V' < 0$ and

$$\hat{u}_{1,V} = -\frac{\beta_1}{\sigma_1^2(V''/V') + 2\alpha_1} \ge -\frac{\beta_1}{\sigma_1^2(-\theta_1) + 2\alpha_1} = \frac{1}{2}.$$

So the minimizer in $u_1 \in [0, 1]$ of $L^{u_1,u_2}V$ is also 0. If $V''/V' = -2\alpha_1/\sigma_1^2$ then $\frac{1}{2}\sigma_1^2 V'' + \alpha_1 V' = 0$; from $\beta_1 V' > 0$, the minimizer in $u_1 \in [0, 1]$ is still 0. Hence, we see that the minimizer in $u_1 \in [0, 1]$ of $L^{u_1,u_2}V$ is 0 and it holds that

$$\inf_{u_1 \in [0,1]} L^{u_1,u_2}V = L^{0,u_2}V \quad \text{on } \mathcal{Y}_1.$$

On the other hand, from $V''/V' > \xi_2$ and the discussions on $\mathcal{X}_1$ in the previous section, we see that the maximizer in $u_2 \in [0, 1]$ of $L^{u_1,u_2}V$ is $\hat{u}_{2,V}$ and it holds that

$$\sup_{u_2 \in [0,1]} L^{u_1,u_2}V = L^{u_1,\hat{u}_{2,V}}V \quad \text{on } \mathcal{Y}_1.$$

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Hence, the pair $(0, \hat{u}_{2,V})$ is the saddle point of $L^{u_1,u_2}V$. Thus, $V$ solves

$$L^{0,\hat{u}_{2,V}(x)}V(x) = 0 \quad \text{on } \mathcal{Y}_1. \tag{33}$$

Then (33) is equivalent to

$$(rx+\delta)V' - \frac{\beta_2^2 V'^2/2}{\sigma_2^2 V'' - 2\alpha_2 V'} = 0,$$

or

$$\frac{V''}{V'} = m(x) := \frac{\beta_2\bar{\beta}_2}{2(rx+\delta)} + 2\bar{\alpha}_2 \quad \text{on } \mathcal{Y}_1. \tag{34}$$

Note that $m$ is a decreasing continuous function on $(-\delta/r, \infty)$. Also, note that the range of function $m$ is $(2\bar{\alpha}_2, \infty)$ on $(-\delta/r, \infty)$, and $2\bar{\alpha}_2 < \xi_2 < -\theta_1$. So $m(x) < -\theta_1$ gives $x > x_5$, where

$$x_5 = -\frac{\beta_2\bar{\beta}_2}{2r(\theta_1 + 2\bar{\alpha}_2)} - \frac{\delta}{r}. \tag{35}$$

Also $m(x) > \xi_2$ gives $x < x_6$, where

$$x_6 = \frac{\beta_2}{2r} - \frac{\delta}{r}. \tag{36}$$

Hence, we conclude that

$$\mathcal{Y}_1 = (x_5, x_6) \cap [a, b].$$

On $\mathcal{Y}_2$, from $V''/V' \le \xi_2 < -\theta_1$ and the discussions on the set $\mathcal{Y}_1$, we see that the minimizer in $u_1 \in [0,1]$ of $L^{u_1,u_2}V$ is 0 and it holds that $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{0,u_2}V$. From $V''/V' \le \xi_2$ and the discussions on $\mathcal{X}_2$ in the previous section, we see that the maximizer in $u_2$ is 1 and it holds that $\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,1}V$. Thus, the pair $(0,1)$ is the saddle point of $L^{u_1,u_2}V$ and we have

$$L^{0,1}V(x) = 0 \quad \text{on } \mathcal{Y}_2. \tag{37}$$

Then (37) is equivalent to

$$(rx + \delta - \alpha_2 - \beta_2)V' + \tfrac{1}{2}\sigma_2^2 V'' = 0,$$

which gives

$$\frac{V''}{V'} = n(x) := -\frac{2(rx+\delta-\alpha_2-\beta_2)}{\sigma_2^2} \quad \text{on } \mathcal{Y}_2. \tag{38}$$

Note that $n(x) \le \xi_2$ gives $x \ge x_6$, where $x_6$ is given in (36). Thus, we conclude that

$$\mathcal{Y}_2 = [x_6, \infty) \cap [a, b].$$

On set $\mathcal{Y}_3$, from $-2\alpha_1/\sigma_1^2 < -\theta_1 \le V''/V'$, we have $\frac{1}{2}\sigma_1^2 V'' + \alpha_1 V' < 0$ and

$$\hat{u}_{1,V} = -\frac{\beta_1}{\sigma_1^2(V''/V') + 2\alpha_1} \le -\frac{\beta_1}{\sigma_1^2(-\theta_1) + 2\alpha_1} = \frac{1}{2}.$$

So the minimizer in $u_1 \in [0,1]$ of $L^{u_1,u_2}V$ is 1 and we conclude that $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{1,u_2}V$ on $\mathcal{Y}_3$. Also, from $\xi_2 < -\theta_1 \le V''/V'$ and the discussions on $\mathcal{X}_1$ in the previous section, we see that the maximizer in $u_2 \in [0, 1]$ of $L^{u_1,u_2}V$ is $\hat{u}_{2,V}$. Thus, we conclude that the pair $(1, \hat{u}_{2,V})$ is the saddle point of $L^{u_1,u_2}V$ and $V$ solves $L^{1,\hat{u}_{2,V}}V(x) = 0$, i.e. (22) on $\mathcal{Y}_3$. So

$$\frac{V''}{V'} = h(x) \quad \text{on } \mathcal{Y}_3,$$

where $h$ is given in (23). The inequality $h(x) \geq -\theta_1$ is equivalent to $x \leq x_5$, where $x_5$ is defined in (35). Then we conclude that

$$\mathcal{Y}_3 = (-\infty, x_5] \cap [a, b].$$

To summarize this subsection, we have

$$\frac{V''(x)}{V'(x)} = \zeta(x) := \begin{cases} h(x), & x \in \mathcal{Y}_3 = (-\infty, x_5] \cap [a, b], \\ m(x), & x \in \mathcal{Y}_1 = (x_5, x_6) \cap [a, b], \\ n(x), & x \in \mathcal{Y}_2 = [x_6, \infty) \cap [a, b], \end{cases} \tag{39}$$

where $n$, $m$, and $h$ are given in (38), (34), and (23). Note that $h(x_5) = m(x_5) = -\theta_1$ and $m(x_6) = n(x_6) = \xi_2$. So $\zeta$ is a continuous function on $[a, b]$. Obviously, $\zeta$ is a decreasing function.

**Theorem 5.** *If $\theta_1 + \xi_2 < 0$, $\beta_1 < 0$, and $\beta_2 > 0$, then the value function of the game is a decreasing $C^2$ function given by*

$$V(x) = \frac{\int_x^b \exp\{\int_a^u \zeta(v)\,\mathrm{d}v\}\,\mathrm{d}u}{\int_a^b \exp\{\int_a^u \zeta(v)\,\mathrm{d}v\}\,\mathrm{d}u} \quad \textit{for } x \in [a, b],$$

*and the Nash equilibrium strategy is a feedback control associated with the risk exposure functions given by*

$$(u_1^*(x), u_2^*(x)) = \begin{cases} \left(1, \dfrac{\beta_2}{\sigma_2^2 h(x) - 2\alpha_2}\right), & x \in \mathcal{Y}_3 = (-\infty, x_5] \cap [a, b], \\ \left(0, \dfrac{\beta_2}{\sigma_2^2 m(x) - 2\alpha_2}\right), & x \in \mathcal{Y}_1 = (x_5, x_6) \cap [a, b], \\ (0, 1), & x \in \mathcal{Y}_2 = [x_6, \infty) \cap [a, b], \end{cases}$$

*where functions $\zeta$, $h$, and $m$ are given in (39), (23), and (34), respectively, and threshold points $x_5$ and $x_6$ are given in (35) and (36), respectively.*

**Remark 5.** The Nash equilibrium strategy for insurer one in Theorem 5 shows a bang-bang behaviour in that it takes only the extreme values 0 (full reinsurance) and 1 (no reinsurance).

### 5.2. The case with $\theta_1 + \xi_2 > 0$

It holds that $-\theta_1 < \xi_2$ in this case. Define the following sets:

$$\mathcal{Y}_4 = \left\{x \in [a, b] \colon -\theta_1 < \frac{V''}{V'} < \xi_2\right\},$$

$$\mathcal{Y}_5 = \left\{x \in [a, b] \colon \xi_2 \leq \frac{V''}{V'}\right\}, \qquad \mathcal{Y}_6 = \left\{x \in [a, b] \colon \frac{V''}{V'} \leq -\theta_1\right\}.$$

We find these sets as follows.

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On set $\mathcal{Y}_4$, from $-\theta_1 < V''/V'$ and the discussions on $\mathcal{Y}_3$ in the previous subsection, the minimizer in $u_1 \in [0, 1]$ of $L^{u_1,u_2}V$ is 1 and it holds that $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{1,u_2}V$. From $V''/V' < \xi_2$ and the discussions on $\mathcal{Y}_2$, the maximizer in $u_2 \in [0, 1]$ is 1 and

$$\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,1}V.$$

So the pair (1, 1) is the saddle point of $L^{u_1,u_2}V$. Also $V$ solves $L^{1,1}V(x) = 0$, i.e. (27) on $\mathcal{Y}_4$. Thus,

$$\frac{V''(x)}{V'(x)} = k(x),$$

where $k$ is given in (28). Since inequality $k(x) < \xi_2$ gives $x > x_3$, where $x_3$ is given in (29), and $-\theta_1 < k(x)$ gives $x < x_7$, where

$$x_7 = \frac{\sigma_2^2\theta_1}{2r} + \frac{\alpha_2 + \beta_2}{r} - \frac{\delta}{r}, \tag{40}$$

so we obtain

$$\mathcal{Y}_4 = (x_3, x_7) \cap [a, b].$$

On $\mathcal{Y}_5$, from $\xi_2 \le V''/V'$ and the discussions on $\mathcal{X}_1$ in the previous section, the maximizer in $u_2 \in [0, 1]$ of $L^{u_1,u_2}V$ is $\hat{u}_{2,V}$ and it holds that $\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,\hat{u}_{2,V}}V$. From $-\theta_1 < \xi_2 \le V''/V'$ and the discussions on $\mathcal{Y}_3$ in the previous subsection, we see the minimizer in $u_1 \in [0, 1]$ is 1 and it holds that $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{1,u_2}V$. So we conclude that $(1, \hat{u}_{2,V})$ is the saddle point of $L^{u_1,u_2}V$ and $V$ solves $L^{1,\hat{u}_{2,V}}V = 0$, i.e. (22) on $\mathcal{Y}_5$. Thus,

$$\frac{V''(x)}{V'(x)} = h(x) \quad \text{on } \mathcal{Y}_5,$$

where $h$ is given in (23). Note that inequality $\xi_2 \le h(x)$ gives $x \le x_3$, where $x_3$ is given in (29). Thus,

$$\mathcal{Y}_5 = (-\infty, x_3] \cap [a, b].$$

On $\mathcal{Y}_6$, from $V''/V' \le -\theta_1$ and the discussions on $\mathcal{Y}_1$ in the previous subsection, the minimizer in $u_1 \in [0, 1]$ is 0 and it holds that $\inf_{u_1\in[0,1]} L^{u_1,u_2}V = L^{0,u_2}V$. From $V''/V' \le -\theta_1 < \xi_2$ and the discussions on $\mathcal{X}_2$ in the previous section, the maximizer in $u_2 \in [0, 1]$ is 1 and it holds that $\sup_{u_2\in[0,1]} L^{u_1,u_2}V = L^{u_1,1}V$. So we conclude that (0, 1) is the saddle point of $L^{u_1,u_2}V$ and $V$ solves $L^{0,1}V = 0$, i.e. (37) on $\mathcal{Y}_6$. Then it holds that

$$\frac{V''(x)}{V'(x)} = n(x) \quad \text{on } \mathcal{Y}_6,$$

where $n$ is given in (38). Inequality $n(x) \le -\theta_1$ gives $x \ge x_7$, where $x_7$ is given in (40). Thus,

$$\mathcal{Y}_6 = [x_7, \infty) \cap [a, b].$$

So we have

$$\frac{V''(x)}{V'(x)} = \vartheta(x) := \begin{cases} h(x), & x \in \mathcal{Y}_5 = (-\infty, x_3] \cap [a, b], \\ k(x), & x \in \mathcal{Y}_4 = (x_3, x_7) \cap [a, b], \\ m(x), & x \in \mathcal{Y}_6 = [x_7, \infty) \cap [a, b], \end{cases} \tag{41}$$

where functions $h$, $k$, and $n$ are given in (23), (28), and (38). Noting that $k(x_3) = h(x_3) = \xi_2$ and $k(x_7) = n(x_7) = -\theta_1$, we see that $\vartheta$ is a continuous and decreasing function on $[a, b]$.

**Theorem 6.** *If $\theta_1 + \xi_2 > 0$, $\beta_1 < 0$, and $\beta_2 > 0$, then the value function of the game is a decreasing $C^2$ function given by*

$$V(x) = \frac{\int_x^b \exp\{\int_a^u \vartheta(v)\,\mathrm{d}v\}\,\mathrm{d}u}{\int_a^b \exp\{\int_a^u \vartheta(v)\,\mathrm{d}v\}\,\mathrm{d}u} \quad \text{for } x \in [a, b],$$

*and the Nash equilibrium strategy is a feedback control associated with the risk exposure functions given by*

$$(u_1^*(x), u_2^*(x)) = \begin{cases} \left(1, \dfrac{\beta_2}{\sigma_2^2 h(x) - 2\alpha_2}\right), & x \in \mathcal{Y}_5 = (-\infty, x_3] \cap [a, b], \\ (1, 1), & x \in \mathcal{Y}_4 = (x_3, x_7) \cap [a, b], \\ (0, 1), & x \in \mathcal{Y}_6 = [x_7, \infty) \cap [a, b], \end{cases}$$

*where functions $\vartheta$ and $h$ are given in (41) and (23), and threshold points $x_3$ and $x_7$ are given in (29) and (40), respectively.*

### 5.3. The case with $\xi_1 + \xi_2 = 0$

In this case, we observe that

$$-\theta_1 = \xi_2 = n(x_7) = k(x_7) = k(x_3) = h(x_3) = h(x_5) = m(x_5) = m(x_6) = n(x_6),$$

and it holds that $x_3 = x_5 = x_6 = x_7$. So the two insurers share a common threshold point. We then let

$$\frac{V''(x)}{V'(x)} = \eta(x) := \begin{cases} h(x), & x \in (-\infty, x_3] \cap [a, b], \\ n(x), & x \in (x_3, \infty) \cap [a, b], \end{cases} \tag{42}$$

where functions $h$ and $n$ are given in (23) and (38), and obtain the following theorem.

**Theorem 7.** *If $\theta_1 + \xi_2 = 0$, $\beta_1 < 0$, and $\beta_2 > 0$, then the value function of the game is a decreasing $C^2$ function given by*

$$V(x) = \frac{\int_x^b \exp\{\int_a^u \eta(v)\,\mathrm{d}v\}\,\mathrm{d}u}{\int_a^b \exp\{\int_a^u \eta(v)\,\mathrm{d}v\}\,\mathrm{d}u} \quad \text{for } x \in [a, b],$$

*and the Nash equilibrium strategy is a feedback control associated with the risk exposure functions given by*

$$(u_1^*(x), u_2^*(x)) = \begin{cases} \left(1, \dfrac{\beta_2}{\sigma_2^2 h(x) - 2\alpha_2}\right), & x \in (-\infty, x_3] \cap [a, b], \\ (0, 1), & x \in (x_3, \infty) \cap [a, b], \end{cases}$$

*where the functions $\eta$ and $h$ are given in (42) and (23), and $x_3$ is given in (29).*

## 6. Examples and discussions

In this section we give three numerical examples and some discussions. We also present concluding remarks.

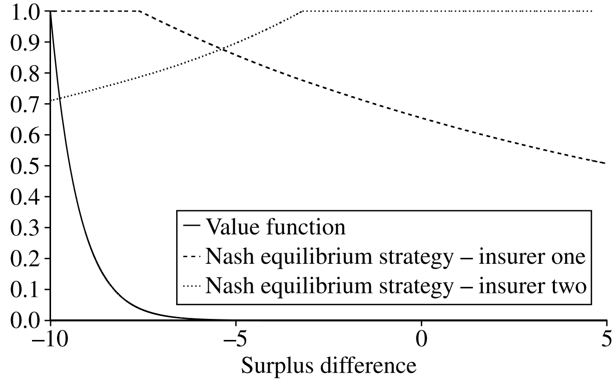


FIGURE 1: The case with $\xi_1 + \xi_2 < 0$, $\beta_1 > 0$, and $\beta_2 > 0$ in Theorem 2.

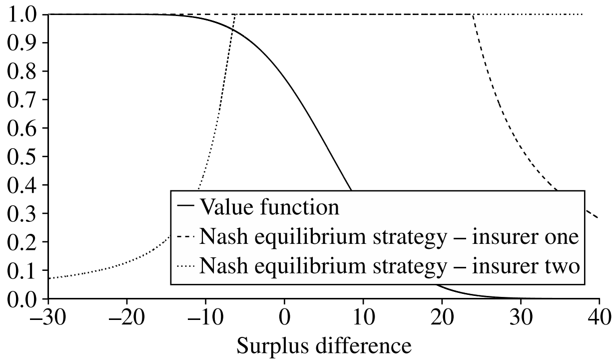


FIGURE 2: The case with $\xi_1 + \xi_2 > 0$, $\beta_1 > 0$, and $\beta_2 > 0$ in Theorem 3.

**Example 1.** In this example, we set the parameters as follows: $a = -10$, $b = 5$, $r = 0.1$, $\alpha_1 = 1.5$, $\alpha_2 = -2.5$, $\beta_1 = 2.5$, $\beta_2 = 1.5$, $\delta = 0.1$, $\sigma_1 = 2$, and $\sigma_2 = 1.5$. So $\xi_1 + \xi_2 = -0.1806$, $x_1 = -7.5984$, and $x_2 = -3.1983$. The value function and the Nash equilibrium strategy given in Figure 1 are calculated using the results in Theorem 2. We see that the parameter settings in this example seem to be more advantageous for insurer one. The value function in Figure 1 reflects this observation. In fact, one can see that the value function, i.e. the probability for insurer two to win under the Nash equilibrium strategy, is low on a large sub-interval of $(a, b)$.

**Example 2.** In this example, we set the parameters as follows: $a = -30$, $b = 40$, $r = 0.1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.5$, $\beta_1 = 0.5$, $\beta_2 = 0.7$, $\delta = -0.2$, $\sigma_1 = 2$, and $\sigma_2 = 3$. So $\xi_1 + \xi_2 = 0.4639$, $x_3 = -6.2778$, and $x_4 = 23.8750$. The value function and the Nash equilibrium strategy are given in Figure 2 and they are calculated using Theorem 3. The parameter settings in this example seem to be fairly even for each insurer. The value function exhibits a somewhat symmetric pattern in the interval $(a, b)$. The value function has one reflection point at $x_0 = (\alpha_2 + \beta_2 - \alpha_1 - \beta_1 - \delta)/r = 6$. The function is concave below the reflection point and convex above it.

**Example 3.** In this example, we set the parameters as follows: $a = -5$, $b = 10$, $r = 0.1$, $\alpha_1 = 1.5$, $\alpha_2 = -2.5$, $\beta_1 = -2.5$, $\beta_2 = 1.5$, $\delta = 0.1$, $\sigma_1 = 2$, and $\sigma_2 = 1.5$. So $\theta_1 + \xi_2 = -2.0556$, $x_5 = 0.8367$, and $x_6 = 6.5$. The value function and the Nash equilibrium

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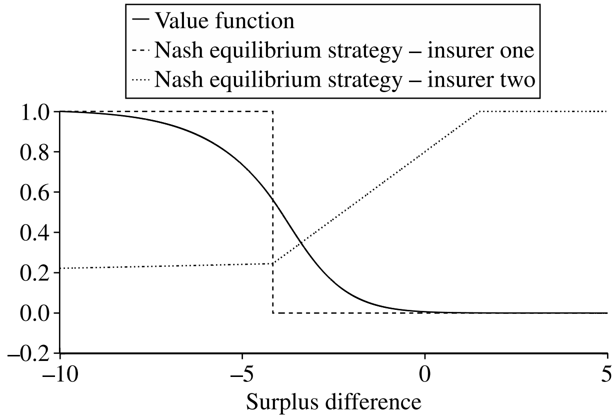


FIGURE 3: The case with $\theta_1 + \xi_2 < 0$, $\beta_1 < 0$, and $\beta_2 > 0$ in Theorem 5.

strategy are given in Figure 3 which are calculated using Theorem 5. The risk exposure function of insurer one has a jump at $x_5$. The value function has one reflection point at

$$x_0 = -\frac{\beta_2 \bar{\beta}_2}{4r\bar{\alpha}_2} - \frac{\delta}{r} = 1.25.$$

From the explicit results, we observe that for each insurer the Nash equilibrium strategy has a unique reinsurance threshold point (provided that the threshold is within the target levels $a$ and $b$). The threshold point separates the interval $(a, b)$ into a no-reinsurance region and a reinsurance region. For example, in the symmetric parameter case with $\xi_1 + \xi_2 < 0$, insurer one has a unique threshold at $x_1$. That is, when the surplus difference is below this level, the risk exposure is equal to 1 and no reinsurance is bought; and when the surplus difference is above the level, the risk exposure level is less than 1 and some reinsurance is bought. We also observe that the Nash equilibrium strategy for each insurer is monotone in the surplus difference. The value function has at most one reflection point (where concavity changes), and the function can be convex (see Figure 1), concave or $S$-shaped (see Figures 2 and 3) on the interval $(a, b)$. Another interesting observation is that in the asymmetric parameter case ($\beta_1 < 0$ and $\beta_2 > 0$), the Nash equilibrium strategy for insurer one shows a bang-bang property, i.e. the risk exposure function takes only the values 0 and 1, while the risk exposure function of insurer two is always continuous; and in the symmetric parameter case ($\beta_1 > 0$ and $\beta_2 > 0$), all risk exposure functions are continuous.

In this paper we focused on two parameter cases – a symmetric case and an asymmetric case. Using parameter classifications (depending on the signs of $\beta_1$, $\beta_2$, $\xi_1 + \xi_2$, and $\theta_1 + \xi_2$), the game is solved with explicit solutions given in each case. We note that under other parameter cases, e.g. $\beta_1 < 0$ and $\beta_2 < 0$, the Nash equilibrium strategy for insurer two may take the value 0 on some interval and a modified verification theorem is needed. However, the main methodology in this paper is still applicable for these parameter cases and we omit them. We also note that the two insurance surplus processes are uncorrelated in our model. In the model with correlation, the game problem becomes more difficult. In some parameter cases the game may not be solvable and the value function may not exist. We leave these problems for future research.

## Acknowledgement

The author would like to thank an anonymous referee whose suggestions have helped improve this paper. The author also acknowledges the support of a summer research fellowship from the University of Northern Iowa.

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