

Does von Neumann Entropy Correspond to Thermodynamic Entropy?

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Conventional wisdom holds that the von Neumann entropy corresponds to thermodynamic entropy, but Meir Hemmo and Orly Shenker have recently argued against this view by attacking von Neumann's argument. I argue that Hemmo and Shenker's arguments fail because of several misunderstandings about statistical-mechanical and thermodynamic domains of applicability, about the nature of mixed states, and about the role of approximations in physics. As a result, their arguments fail in all cases: in the single-particle case, the finite-particles case, and the infinite-particles case.

1. Introduction. According to conventional wisdom in physics, von Neumann entropy corresponds to phenomenological thermodynamic entropy. The origin of this claim is von Neumann's (1955) argument that his proposed entropy corresponds to the thermodynamic entropy, which appears to be the only explicit argument for the equivalence of the two entropies. However, Hemmo and Shenker (2006)—and earlier, Shenker (1999)—have argued that this correspondence fails, contrary to von Neumann. If so, this leaves conventional wisdom without explicit justification.

Correspondence can be understood, at the very least, as a numerical consistency check: in this context, this means that the von Neumann entropy has to be included in calculating thermodynamic entropy to ensure consistent accounting in contexts in which both thermodynamic and von Neumann entropy are physically relevant. Successful correspondence provides strong

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evidence of equivalence. While it does not guarantee equivalence, it seems to be at least a necessary condition for equivalence. If thermodynamic entropy and von Neumann entropy correspond, then we have reason to think that von Neumann entropy is rightfully thermodynamic in nature, since proper accounting of thermodynamic entropy would demand von Neumann entropy. By contrast, a failure of correspondence seems to entail that the von Neumann entropy is not thermodynamical in nature, since it is irrelevant to thermodynamic calculations in contexts in which both entropies are physically significant (e.g., when a system both has quantum degrees of freedom and is sufficiently large to warrant thermodynamical considerations).

Although Henderson (2003), in my view, has successfully criticized Shenker's earlier argument, little has been done in the philosophical literature to evaluate Hemmo and Shenker's more recent arguments.¹ This lacuna is striking because, as I mentioned, von Neumann appears to offer the only explicit argument for correspondence for the two entropies.

My goal in this article is to fill this lacuna by providing a novel set of criticisms to Hemmo and Shenker. Here is the plan. I introduce key terms (sec. 2) and then present von Neumann's thought experiment, which aims to establish the correspondence between thermodynamic entropy and von Neumann entropy; along the way, a novel counterpart to the usual argument for correspondence is discussed (sec. 3). I then present and criticize Hemmo and Shenker's arguments for the single-particle case in the context of thermodynamics (sec. 4.1) and in the context of statistical mechanics (sec. 4.2), the N -particles case (sec. 4.3), and the infinite-particles case (sec. 4.4). I conclude that their argument fails in all cases—in turn, we have good reasons to reject their claim that the von Neumann entropy fails to correspond to thermodynamic entropy and, hence, the claim that von Neumann entropy is not thermodynamic in nature.

2. Key Terms. Let me first define the notions of thermodynamic entropy and von Neumann entropy. Following Hemmo and Shenker (2006), I define the *change in thermodynamic entropy* S_{TD} between two thermodynamic states in an *isothermal quasi-static* process as follows:²

1. It is only slightly better in the physics literature: Deville and Deville's (2013) paper appears to be the only one to critique Hemmo and Shenker (2006). On the philosophical side, one (very recent) exception is Prunkl (2020), although she restricts discussion to the single-particle case and appears to conflate information entropy with thermodynamic entropy. See secs. 4.1 and 4.2 for why this is not obviously right.

2. There is no change in temperature in an isothermal quasi-static process, which is why T is taken to be constant. As a matter of historical note, von Neumann uses an isothermal setup in his argument, with a box containing a quantum ideal gas coupled to a (much larger) heat sink ensuring constant temperature over time (1955, 361–71).

$$\Delta S_{\text{TD}} = \frac{1}{T} \int P dV. \quad (1)$$

We will restrict our discussion to ideal gases in equilibrium (i.e., systems where pressure P , volume V , and temperature T remain constant).

Next, the *von Neumann entropy* S_{VN} , for any pure or mixed quantum system, is defined as

$$S_{\text{VN}} = -k \text{Tr}(\rho \log \rho), \quad (2)$$

where k is the Boltzmann constant and $\text{Tr}(\cdot)$ is the trace function. Generally, the density matrix ρ is such that

$$\rho = \sum_{n=1}^i p_i |\psi_i\rangle \langle \psi_i|, \quad (3)$$

where $\psi_1, \psi_2, \dots, \psi_n$ correspond to the number of pure states in a statistical mixture represented by ρ , with p_1, p_2, \dots, p_n being their associated classical probabilities (which must sum to unity). In the case in which there is only one pure state possible for a system (e.g., when we are absolutely certain about its quantum state), then $n = 1$, with probability 1, so the appropriate density matrix is $\rho = |\psi\rangle \langle \psi|$. For such a system in a pure state (i.e., represented by a single-state vector in Hilbert space), $S_{\text{VN}} = 0$. For mixed states (i.e., states that cannot be represented by a single-state vector in Hilbert space, hence *mixture* of pure states or a *mixed state*), $\text{Tr}(\rho \log \rho) < 1$ and $S_{\text{VN}} > 0$ in general. A mixed state is often said to represent our *ignorance* about a system—this will suffice as a first approximation (more on how to interpret this ignorance in sec. 4.2).

Prima facie, S_{VN} and S_{TD} appear to share nothing in common, apart from the word ‘entropy’. However, von Neumann claims that there are important correlations between the two, which suggests a correspondence between S_{TD} and S_{VN} .

3. Von Neumann’s Thought Experiment. For the sake of parity, I adopt Hemmo and Shenker’s (2006) presentation of von Neumann’s thought experiment, which aims to show that changes in thermodynamic entropy can be made consistent with the laws of thermodynamics only if we considered the von Neumann entropy as contributing to the calculation of the thermodynamic entropy.³ Figure 1 depicts the stages of the thought experiment.

We begin, in stage 1, with a box with a partition in the middle. On one side of the partition there is a gas at volume V , constant temperature T , and

3. It is not clear to me that von Neumann’s original (1955) argument is exactly the same as the argument Hemmo and Shenker (2006) reproduce. However, for the sake of argument, I refer to Hemmo and Shenker’s version as von Neumann’s argument in this article.

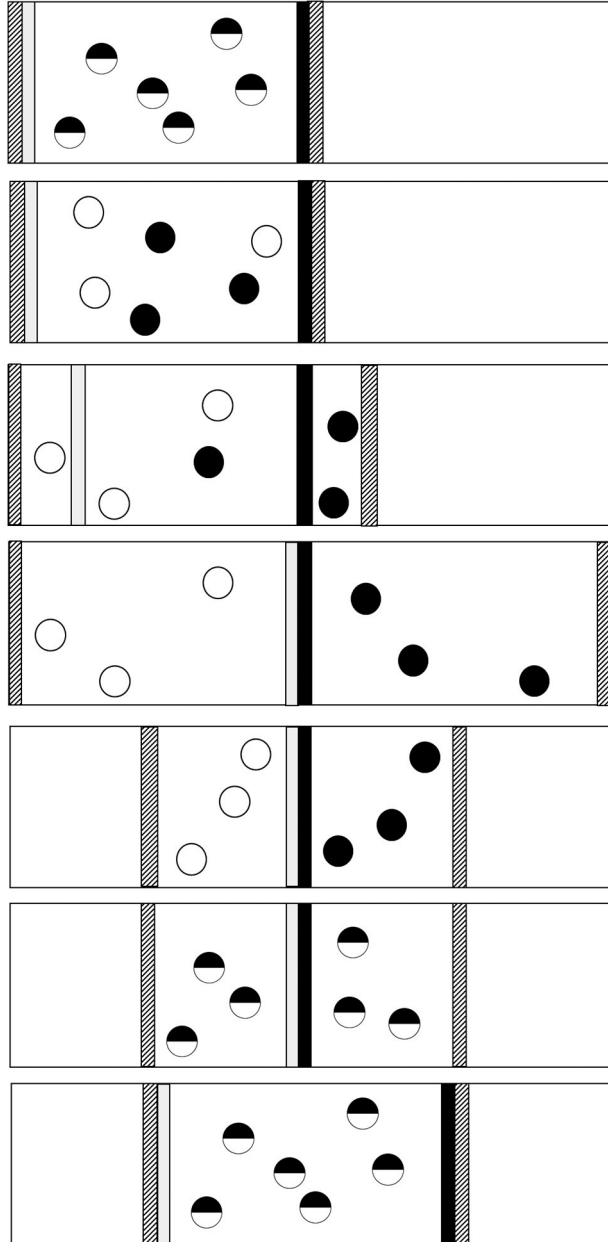


Figure 1. *Top to bottom*, stages 1–7, as described by Hemmo and Shenker (2006).

constant pressure P . Each gas particle starts off having the pure state spin-up along the x-direction $|\psi_x^\uparrow\rangle$, which is equivalent to a superposition of spin-up and spin-down pure states along the z-direction, labeled $|\psi_z^\uparrow\rangle$ and $|\psi_z^\downarrow\rangle$ respectively. According to standard quantum mechanics, the state of each particle is thus $(1/\sqrt{2})(|\psi_z^\uparrow\rangle + |\psi_z^\downarrow\rangle)$.

In this context, particles with quantum behavior may be taken to be *ideal gases*, that is, sets of particles each of which does not interact with other particles and takes up infinitesimal space. Following von Neumann's (1955, 361) assumptions, each gas particle is understood as a quantum particle with a spin degree of freedom contained inside a large impenetrable box, and each gas particle is put inside an even larger container isolated from the environment (i.e., the box we began with).⁴ This ensures that each spin degree of freedom is incapable of interacting with other particles. These boxes' sizes also ensure that the positions of these boxes (and hence of the particles) can be approximately classical. Since the container is much larger than each gas particle, this ensures that the gas particles take up negligible space relative to the massive container. Accepting these assumptions, we may then take these quantum particles to behave like an ideal gas.⁵ Following Hemmo and Shenker, we further assume that the position degrees of freedom of the gas particles have no interaction with the spin degrees of freedom at this point, and "due to the large mass of the boxes, the position degrees of freedom of the gas may be taken to be classical and represented by a quantum mechanical mixture" (2006, 155).

Moving on, stage 2 involves a spin measurement along the Z-axis on all the particles in the container, with a result being an equally weighted statistical mixture of particles with either $|\psi_z^\uparrow\rangle$ or $|\psi_z^\downarrow\rangle$ states. As a result, the spin state of each particle is represented instead by a density matrix ρ_{spin} , such that

$$\rho_{\text{spin}} = \frac{1}{2} (|\psi_z^\uparrow\rangle \langle \psi_z^\uparrow| + |\psi_z^\downarrow\rangle \langle \psi_z^\downarrow|). \quad (4)$$

More precisely, there should be terms for the measurement device too, when truly considering the entire system. Matrix ρ_{spin} describes only the subsystem (i.e., the quantum ideal gas) sans measurement device (i.e., a state with the measurement device traced out)—this is in line with von Neumann's focus on the entropy changes due to changes in the subsystem (1955, 358–79). I follow Henderson (2003) and Hemmo and Shenker (2006) in talking about

4. Von Neumann's assumptions are borrowed from Einstein (1914/1997). For more, see Peres (2002, 271).

5. I follow everyone in this debate in assuming that the above setup is physically possible.

the system's state as though I have already traced the measurement device out whenever measurement is involved.

Stages 3 and 4 are where the particles are (reversibly) separated according to their spin states by a semipermeable wall into two sides of the box, each with volume V .⁶ As a result of this separation, we in effect double the mixture's volume. The gas expands to fill up volume V on each side.

Stage 5 involves an isothermal and quasi-static compression of the mixture so that we return to a total volume V (effectively halving the volume on each side of the box), while pressure on both sides becomes equal. Importantly, because of this compression, S_{TD} decreases because of the decrease in volume.

Stage 6 brings all the particles into the pure spin state $|\psi_x^\uparrow\rangle$ quasi-statically and without work done, while stage 7 removes the semipermeable wall, such that the system returns to its original state.

Now consider how S_{VN} and S_{TD} change across the various stages. Stage 7 ends with the body of gas having the same thermodynamic state (same V , same P , and constant T) as stage 1. Furthermore, all the thermodynamic transformations performed were reversible, and removing the wall alone does no additional work. Thus, the system at stage 1 must have the same thermodynamic entropy as stage 7 (i.e., $\Delta S_{TD} = 0$) since S_{TD} depends only on the initial and final state of the system. From stages 1 to 7, $\Delta S_{VN} = 0$ too, since the system is in the same state in both the first and seventh stages.

Since stage 6 does not involve thermodynamic transformations, there is no change in S_{TD} . Likewise, the transformation of ρ_{spin} to $|\psi_x^\uparrow\rangle$ here does not change S_{VN} , as the transformation can be performed unitarily. This is possible as a result of our separation of the gases to different sides of the box according to their spin eigenstates—given this, we can perform unitary operations on each side of the box (or perform the more general measurement procedure recommended by von Neumann [1955, 365–67]), to transform them into the same state as stage 1. Both unitary transformations and von Neumann's procedure do not increase S_{VN} , and so there is no change in S_{VN} at stage 6 as a result.

There are no changes in S_{TD} or S_{VN} in stages 3 and 4. While there is an increase in the gas's volume, as noted above, from V to $2V$, and hence an accompanying increase in S_{TD} by $n.R.\log 2$,⁷ there is also a compensating change in the thermodynamic entropy of mixing, by $-n.R.\log 2$, which exactly compensates this increase in S_{TD} (see Hemmo and Shenker 2006,

6. This semipermeable wall can be assumed to be a black box that reversibly separates particles to different sides because of their different orthogonal/disjoint states; see von Neumann (1955, 367–70) for discussion. I follow everyone in the debate in accepting this assumption.

7. Here, N refers to the number of moles of gas in the system, and R is the gas constant.

157 n. 4).⁸ Since the particles are in orthogonal spin states at this stage, there are no quantum effects (e.g., ‘collapse’ effects) from simply filtering the gases with the semipermeable walls, and hence S_{VN} does not change either.⁹

However, importantly, there is a decrease in S_{TD} in stage 5, of $-n.R.\log 2$, because of the isothermal compression and decrease in volume. Yet, nowhere else is there any further change in S_{TD} . We have to account for why the overall change in S_{TD} from the first to the seventh stages is 0.

As von Neumann argues, only one possibility remains. While S_{TD} remains constant in stage 2, notice that there was an increase in S_{VN} , of $-N.k. -\log 2 = (N.R)/N_A.\log 2 = n.R.\log 2$, as a result of the spin measurement.¹⁰ This is equivalent to the change of S_{TD} in stage 5. The state of each particle changes from a pure state $(1/\sqrt{2})(|\psi_z^{\uparrow}\rangle + |\psi_z^{\downarrow}\rangle)$ to a mixed state represented by ρ_{spin} , and hence S_{VN} for the gas increases on the whole. In order to ensure that entropic changes are consistent, von Neumann thinks that we should accept S_{VN} 's contribution to S_{TD} in this context, where both quantum effects and thermodynamical considerations are at play. Without accepting S_{VN} in our entropic accounting, we end up with a violation of thermodynamics since we have a reversible thermodynamic cycle with nonzero change in S_{TD} , contra the Second Law. In other words, we should accept that S_{VN} corresponds to S_{TD} .

The correspondence of S_{VN} and S_{TD} in this context can be defended from another perspective, apart from considerations about consistency from the

8. Henderson explains the mixing entropy, describing the mixing of different gases, crisply: “After separation, each separated gas occupies the original volume V alone. To return to the mixture, each gas is compressed to a volume $c_i V$ (where c_i is the concentration of the i th gas). The compression requires work $W = -nkT \sum_i c_i \log c_i$ to be invested, and the entropy of the gas is reduced by $\Delta S = -nk \sum_i c_i \log c_i$. An increase in entropy of the same amount must then be associated with the mixing step of removing the partitions. This is the ‘mixing entropy’” (2003, 292). Separation simply results in a decrease in entropy of the same amount. Tim Maudlin raised the following objection to the applicability of the entropy of mixing in this context when a version of the current article was presented at a summer school. Mixing should have a thermodynamic effect only when differences between the gases are already assumed to be thermodynamically relevant: e.g., mixing differently colored gases should not have a thermodynamic effect unless the difference in color is thermodynamically relevant. It is, however, not clear whether the difference in spin is a thermodynamically relevant one and might amount to begging the question. This is a good point, but one that I am setting aside for now, since everyone in the debate accepts the assumption that separating the gases here decreases the entropy of mixing. As we will see later, a more fundamental issue arises with using the entropy of mixing in the ‘single particle’ case.

9. This is argued for in von Neumann (1955, 370–76).

10. The total number of particles is N : since each particle is assumed to be noninteracting and independent from others under the ideal gas assumption, their entropies are additive. Avogadro’s number is N_A .

thermodynamic perspective: consistent accounting from the perspective of *quantum mechanics* also demands correspondence. This is simply a change in perspective with regard to the thought experiment, but, to my knowledge, this argument has not been explicitly made in the literature, thus underselling the case for correspondence in von Neumann's thought experiment.

Instead of arguing for correspondence by considering thermodynamic consistency, that is, ensuring that $\Delta S_{\text{TD}} = 0$ throughout the cycle, we can consider consistency from the quantum mechanical perspective. We started and ended with the same spin state, and so it should be the case that $\Delta S_{\text{VN}} = 0$ throughout the cycle. Yet, there is an inconsistency: if we only consider the increase of S_{VN} in stage 2 as a result of measurement, we should end in stage 7 with an increase in S_{VN} , not $\Delta S_{\text{VN}} = 0$. As described, there is nowhere else in the thought experiment where S_{VN} changes. However, there is a decrease in S_{TD} in stage 5 because of the *thermodynamic process* of isothermal compression, exactly balancing out the increase in S_{VN} . Hence, we can ensure consistency (i.e., that $\Delta S_{\text{VN}} = 0$), only by taking S_{VN} to correspond to S_{TD} . In other words, just as the thermodynamic accounting of S_{TD} is consistent only if we consider S_{VN} , the quantum entropic accounting of S_{VN} is also consistent only if we consider S_{TD} . Consistency from a quantum mechanical perspective also demands correspondence between S_{VN} and S_{TD} .

Although the debate has largely focused only on how the thought experiment demonstrates one direction of correspondence, of S_{VN} to S_{TD} as a result of thermodynamical considerations, the correspondence demonstrated by this thought experiment in fact goes both ways. Of course, since von Neumann was focused on demonstrating the thermodynamic nature of S_{VN} (specifically the irreversibility of measurement), rather than the quantum nature of S_{TD} , it was natural that he chose to approach it the way he did.

4. Hemmo and Shenker's Arguments. Hemmo and Shenker disagree with von Neumann's argument and criticize it by considering three cases: the single-particle case, the finite but large N particles case, and the infinite-particles case.

4.1. Single-Particle Case: Thermodynamics. Hemmo and Shenker first consider von Neumann's argument in the single-particle case (see fig. 2). They claim that the argument does not go through here, since S_{TD} actually remains constant, contrary to our thought experiment's description. In other words, using thermodynamical considerations, they find that S_{VN} should not be included in our accounting for S_{TD} .

Here is their argument. Consider the stages where there are entropic changes. In stage 2 when the spin measurement was performed, S_{VN} increases as before, since it tracks the change of the particle's spin state from pure to mixed.

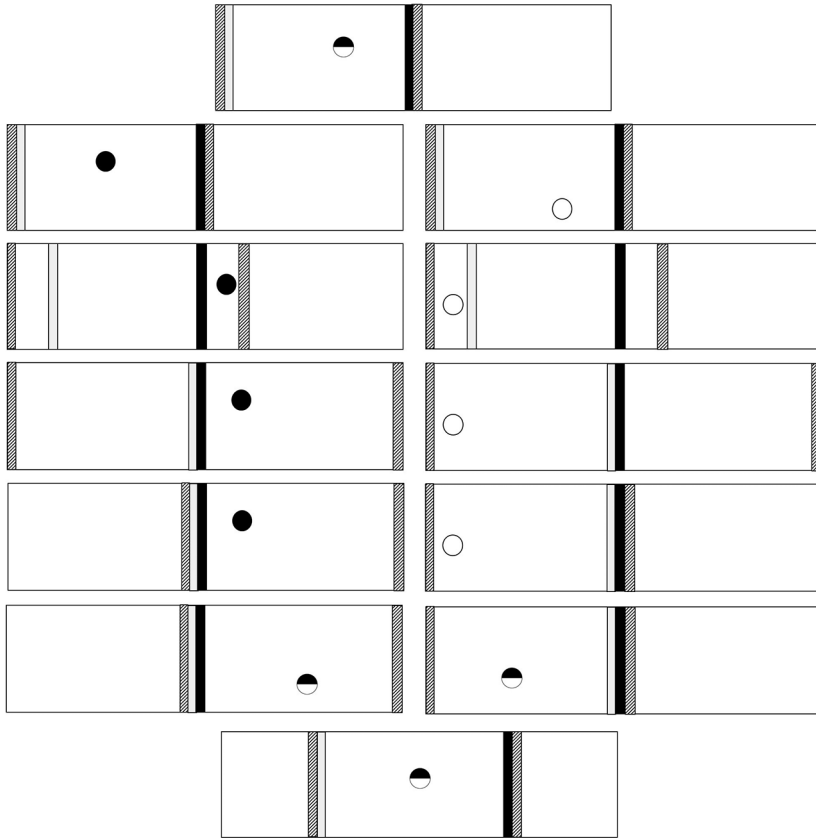


Figure 2. *Top to bottom*, stages 1–7 for the single-particle case, as described by Hemmo and Shenker (2006).

Contrariwise, S_{TD} does not change in stage 5 (isothermal quasi-static compression) or anywhere else (this will be important later). After stage 2, the single particle is in either the $|\psi_z^\uparrow\rangle$ state or the $|\psi_z^\downarrow\rangle$ state. After stages 3 and 4, with the expansion and separation via a semipermeable wall, there is a particle in one side of the box only and not the other. We make an S_{TD} -conserving location measurement to figure out which side of the box is empty and which side the particle is at, so as to compress the box against the empty side.¹¹ The compression is then performed as before. However,

11. Prunkl (2020) claims that the location measurement leads to a violation of the Second Law. If true, this makes Hemmo and Shenker’s argument even more problematic. Here, for the sake of argument, I assume that the location measurement is unproblematic.

this compression does not decrease S_{TD} .¹² To restore the volume of the ‘gas’ to V , no work needs to be done, since we are compressing against a vacuum. Since there is a change in S_{VN} in this cycle, but no change in S_{TD} , the apparent answer, in order to do our entropic accounting, is to ignore, not incorporate, S_{VN} into S_{TD} . Hence, S_{VN} does not correspond to S_{TD} .

Their analysis is problematic. Although their ultimate point in this analysis—that S_{TD} fails to correspond to S_{VN} —still holds, it does not hold in the way they claim. In fact, the way it fails suggests to us that we should disregard the single-particle case.

For the single-particle case, they claim that “[S_{TD}] is null throughout the experiment” (Hemmo and Shenker 2006, 162). This then allows them to claim that thermodynamic accounting for S_{TD} is consistent only if we did not consider S_{VN} . This then supports their claim that S_{VN} does not correspond to S_{TD} since adding S_{VN} into the thermodynamic accounting actually renders the otherwise consistent calculations inconsistent.

They are right to say that the stage 5 compression (after location measurement) has no thermodynamic effect because we are compressing against a vacuum: no work needs to be done, and so $\Delta S_{TD} = 0$ for stage 5. However, I claim that $\Delta S_{TD} \neq 0$ for the single-particle case overall, because $\Delta S_{TD} \neq 0$ in stages 3 and 4 in this context.

As far as I can tell, Hemmo and Shenker did not analyze stages 3 and 4 (i.e., the isothermal expansion and separation) in terms of the single-particle case at all. Rather, they seem to have assumed that $\Delta S_{TD} = 0$ in these stages as with the original case of the macroscopic gas.¹³ However, this assumes that there is both a change in entropy of $n.R.\log 2$ because of isothermal expansion and a change in the entropy of mixing of $-n.R.\log 2$ because of separation, as they say so themselves for the original case: “The increase of thermodynamic entropy due to the volume increase $\Delta S = (1/T)\int PdV$ is exactly compensated by the decrease of thermodynamic mixing entropy $\Delta S = \sum w_k \ln w_k$ (where w_k is *the relative frequency of molecules of type k*) due to the separation” (Hemmo and Shenker 2006, 157 n. 4, emphasis mine).

In the single-particle case, it makes sense that isothermal expansion should still increase S_{TD} , since the single-particle ‘gas’ is expanding against

12. As an anonymous reviewer rightfully notes, the location measurement is important for ensuring $\Delta S_{TD} = 0$ here. Without the location measurement, we might end up compressing in the wrong direction against the side with the gas, rather than the empty vacuum—this will have thermodynamic effects since we are doing work on the gas. However, the Hemmo and Shenker setup emphasizes the location measurement, and I will play along for the sake of argument.

13. Prunkl (2020) appears to do the same.

a piston and doing work. However, it does not make physical sense to speak of the entropy of mixing here at all, since there is no separation of gases in the single-particle case. The entropy of mixing is explicitly defined for systems where different gases are separated from/mixed with one another via semipermeable walls, but a single particle cannot be separated from/mixed with itself. The quote above makes this conceptual point explicit: by Hemmo and Shenker's own lights, the relative frequency of a single particle is simply unity (and null for particles of other types), so the entropy of mixing is $1 \ln 1 = 0$. There is no thermodynamic entropy of mixing in the single-particle case.

Discounting the entropy of mixing, however, we find that $\Delta S_{TD} = n.R.\log 2 \neq 0$ for stages 3 and 4, and hence for the entire process, contrary to Hemmo and Shenker's claim. Interestingly, correspondence does fail to obtain between S_{TD} and S_{VN} , since $\Delta S_{TD} + \Delta S_{VN} = 2n.R.\log 2 \neq 0$, despite the process being reversible ex hypothesi: incorporating S_{VN} into thermodynamic accounting violates the Second Law.

However, on this new analysis, we gain some clarity as to why the single-particle case is problematic. While it is true that incorporating S_{VN} into the thermodynamic accounting violates the Second Law, S_{TD} accounting by itself violates the Second Law (contrary to Hemmo and Shenker 2006). Even without considering S_{VN} , $\Delta S_{TD} \neq 0$ despite the process being reversible. Thermodynamic accounting is inconsistent here no matter what we do, which suggests that the reversible process they described for the single-particle case is thermodynamically unsound: if so, any argument Hemmo and Shenker make in this context may be disregarded.

The upshot: I agree with Hemmo and Shenker that correspondence fails for the single-particle case, but I do not agree about why it fails. It is not because the process they described is already thermodynamically consistent without taking S_{VN} into account. Rather, it is because the process is already thermodynamically inconsistent anyway.

In recent work, John Norton argued that thermodynamically reversible processes for single-particle systems are impossible in principle, which might explain why the process described by Hemmo and Shenker (2006) is thermodynamically unsound: it was not justified to assume the process was reversible for a single-particle system. For Norton, a reversible process is "loosely speaking, one whose driving forces are so delicately balanced around equilibrium that only a very slight disturbance to them can lead the process to reverse direction. Because such processes are arbitrarily close to a perfect balance of driving forces, they proceed arbitrarily slowly while their states remain arbitrarily close to equilibrium states" (2017, 135). Norton notes that these thermodynamic equilibrium states are balanced not because there are no fluctuations but because these fluctuations are negligible for macroscopic systems. However, fluctuations relative to single-particle

systems are large and generally prevent these systems from being in equilibrium states at any point of the process, rendering reversible processes impossible in the single-particle case (135). If reversible processes are impossible for single-particle systems in general, then it should come as no surprise that the particular single-particle reversible process used by Hemmo and Shenker is likewise thermodynamically unsound, as my analysis above suggests. If so, their claim that correspondence fails in this process is simply beside the point, since this process is not thermodynamic at all.

Since any reversible process cannot be realized for single-particle systems in general, the issue seems not to be with any particular process per se but with the single-particle case simpliciter. To my knowledge, no one before Hemmo and Shenker (2006) discussed von Neumann's experiment in terms of a single particle; von Neumann (1955), Peres (1990, 2002), Shenker (1999), and Henderson (2003) all explicitly or implicitly assume a large (or infinite) number of particles. This is for good reason. As Hemmo and Shenker acknowledge, and as we have seen: "The case of a single particle is known to be problematic as far as arguments in thermodynamics are concerned" (2006, 158). Matter in phenomenological thermodynamics is assumed to be continuous.¹⁴ A 'gas' composed of one particle can be many things, but it is surely not continuous in any commonly accepted sense. In other words, it is just not clear whether the domain of thermodynamics should apply to the single-particle case at all.

As Myrvold (2011) notes, Maxwell also made a similar claim with regard to phenomenological thermodynamics in general; it does not and should not hold in the single-particle case. On his view, the laws of phenomenological thermodynamics, notably the Second Law, must be continually violated on small scales:

If we restrict our attention to any one molecule of the system, we shall find its motion changing at every encounter in a most irregular manner. . . . If we go on to consider a finite number of molecules, even if the system to which they belong contains an infinite number, the average properties of this group, though subject to smaller variations than those of a single molecule, are still every now and then deviating very considerably from the theoretical mean of the whole system, because the molecules which form the group do not submit their procedure as individuals to the laws which prescribe the behaviour of the average or mean molecule. . . . Hence the second law of thermodynamics is continually being violated, and that to a considerable extent, in any sufficiently small group of molecules belonging to a

14. See Compagner (1989) for a discussion of the so-called continuum limit as a counterpart to the thermodynamic limit in phenomenological thermodynamics.

real body. As the number of molecules in the group is increased, the deviations from the mean of the whole become smaller and less frequent. (Maxwell 1878, 280)

The Second Law, and hence phenomenological thermodynamics, should not be expected to hold true universally in small-scale cases, especially not in the single-particle case. Von Neumann and everyone else in the debate should have recognized this point. Why, then, should it matter that the thought experiment succeeds or fails in this case? Phenomenological thermodynamics does not apply to single-particle cases. There is thus no profit in trying to establish correspondence between S_{VN} and S_{TD} in this case. Indeed, if we took seriously Maxwell's claim that the Second Law fails at small scales, a failure of thermodynamic entropic accounting might even be expected; it does not rule out the possible thermodynamic nature of S_{VN} even though the sum of S_{VN} and S_{TD} might be inconsistent with the Second Law. In short, it is not clear why the single-particle case is relevant to the discussion at hand.

Hemmo and Shenker's reasoning is untenable, because they fail to respect the context of phenomenological thermodynamics by bringing it into a context in which it is not expected to hold. Instead, it seems more appropriate that the single-particle case is precisely beyond the purview of classical thermodynamics, requiring an analogue that only corresponds to classical thermodynamics at the appropriate scales and limits. We may then take S_{VN} to be the analogue of S_{TD} in this case, only approximating S_{TD} as the system in question approaches the context suitable for traditional thermodynamic analysis. If so, we may see von Neumann as merely demonstrating that S_{VN} corresponds, not at all domains but in the domain where thermodynamics is taken to hold, to S_{TD} .

4.2. Single-Particle Case Redux: Statistical Mechanics and Information. Given the foregoing discussion, Hemmo and Shenker might insist that S_{VN} fails to correspond to S_{TD} even when taking into account a more relevant domain for single particles—statistical mechanics. After directly arguing that S_{VN} does not correspond to S_{TD} (Hemmo and Shenker 2006, 162–65), they further argue that S_{VN} does not correspond to information entropy (more on this below) in the single-particle case. *Prima facie*, this should seem irrelevant to von Neumann's argument, which was to establish the correspondence of the thermodynamic S_{TD} and quantum S_{VN} : Why should information entropy's failure to correspond with S_{VN} be a worry at all?

Here is one plausible worry, on a charitable reading. If information entropy corresponds to S_{TD} , and Hemmo and Shenker show that S_{VN} fails to correspond to information entropy, then we might conclude, indirectly, that

S_{VN} does not correspond to S_{TD} after all.¹⁵ This argument assumes that information entropy does correspond to S_{TD} , an assumption Hemmo and Shenker seem to hold as well: this is in line with the so-called subjectivist view of statistical mechanics (notably, see Jaynes 1957). Furthermore, my above argument against the misapplication of phenomenological thermodynamics does not seem to apply here, since this argument is being made in the context of statistical mechanics and its particle picture, with no commitment to phenomenological thermodynamics.

However, Hemmo and Shenker do not do much to motivate the linkage between information entropy and S_{TD} ; indeed, in their words, “a linkage between the Shannon information and thermodynamic entropy has not been established” (2006, 164). Without this link, the failure of correspondence between the information entropy and S_{VN} appears, at best, irrelevant to the correspondence between S_{TD} and S_{VN} . Nevertheless, I will take a charitable view here and assume that there is a correspondence between information entropy and S_{TD} , for the sake of assessing their argument. Here is a plausible (if arguable) sketch: if one were a subjectivist like Jaynes (1957), one might take the Gibbs entropy in statistical mechanics to be a special case of the information entropy. After all, both have the following form:

$$-\sum_i p_i \ln p_i, \quad (5)$$

with i being the number of possible states with associated probabilities of occurring p_i and the Gibbs entropy being multiplied by an additional Boltzmann’s constant k .¹⁶ We know that statistical mechanics corresponds to phenomenological thermodynamics at the thermodynamic limit, so we can think of the Gibbs entropy, and hence information entropy, as corresponding to S_{TD} . I take this to be in line with what Hemmo and Shenker have in mind: “to the extent that the Shannon information underwrites the thermodynamic entropy, it does so via statistical mechanics” (2006, 165). Assuming that the above picture is plausible, a failure of correspondence between S_{VN} and the information entropy provides evidence against the correspondence between S_{VN} and S_{TD} .

Their argument comes into two parts. Ignoring S_{TD} for the time being (which does not change throughout the cycle for the single-particle case—see sec. 4.1), they claim that we can consider the stage 5 location measurement to be a decrease in information entropy of $\ln 2$, as a result of learning

15. Caveat: I am not committed to the information entropy’s relationship to thermodynamics. One may, like Earman and Norton (1998, 1999), be skeptical that information entropy is related to S_{TD} at all, in which case Hemmo and Shenker’s argument here is simply irrelevant.

16. Using the so-called Planck units, where $k = 1$, Gibbs entropy and information entropy are then formally equivalent.

information about which one of two parts of the box the particle is in. On first glance, this seems to resolve the arithmetic inconsistency in entropic accounting: $\ln 2$ is exactly the increase in S_{VN} as a result of the spin state changing from a pure state $|\psi_x^\uparrow\rangle$ to the mixed state ρ_{spin} in stage 2. In other words, for both the information and von Neumann entropies' accounting to be correct (i.e., net change of zero across the cycle), we must consider S_{VN} as corresponding to information entropy. Now, since information entropy also corresponds, *ex hypothesi*, to S_{TD} , we have an indirect argument for the correspondence of S_{VN} to S_{TD} .

However, Hemmo and Shenker claim that this argument fails for *collapse* interpretations, that is, interpretations of quantum mechanics on which a superposed quantum state ontologically collapses into a pure state upon measurement (either precisely or approximately).¹⁷ They allow that, on *no-collapse* interpretations (e.g., Bohmian or many-worlds interpretations), the location measurement in stage 5 does not decrease S_{VN} , since the state of the system never changes in light of measurements, and so the above argument goes through.

Let us see what they could mean by this claim by following the state of the particle through the cycle. At stage 2, everyone agrees that the state of the system is ρ_{spin} following the z-spin measurement; S_{VN} increases by $\ln 2$. At this point, the particle's position degrees of freedom remain independent from its spin degrees of freedom, as per our ideal gas assumption, although we might assume the particle starts out on the left half of the box, with the mixture of position states $\rho_{\text{pos}}(L)$ with 'L' representing the left side. (Consider fig. 1 but with only one particle.) Following the semipermeable wall's filtering at stages 3 and 4, the location of the particle becomes classically correlated with the spin. Let us say that the semipermeable wall sends $|\psi_z^\uparrow\rangle$ particles to the left, represented by $\rho_{\text{pos}}(L)$, and $|\psi_z^\downarrow\rangle$ particles to the right, represented by $\rho_{\text{pos}}(R)$. As such, the (mixed) state of the particle is now

$$\rho_{\text{particle}} = \frac{1}{2} (|\psi_z^\uparrow\rangle \langle \psi_z^\uparrow| \otimes \rho_{\text{pos}}(L) + |\psi_z^\downarrow\rangle \langle \psi_z^\downarrow| \otimes \rho_{\text{pos}}(R)). \quad (6)$$

For no-collapse interpretations, Hemmo and Shenker agree that the state of the particle stays the same as above after the location measurement in stage 5. We perform the compression in stage 5 and remove the partition at the end of stage 6, thereby removing the classical correlations between position and spin. No further change in either information entropy or S_{VN} occurs, and hence the correspondence goes through (Hemmo and Shenker 2006, 164). The spin state remains mixed until unitarily transformed into a pure state and completing the cycle.

17. On Ghirardi-Rimini-Weber-type approaches, though, collapse occurs with or without measurement, but measurement increases the likelihood of collapse, roughly speaking.

For collapse interpretations, they claim that the location measurement decreases S_{VN} by $\ln 2$, because, on collapse interpretations, the state of the particle upon the measurement, depending on which side the particle is found, becomes

$$\rho_{\text{particle}} = \begin{cases} |\psi_z^\uparrow\rangle \langle \psi_z^\uparrow| \otimes \rho_{\text{pos}}(L) \\ |\psi_z^\downarrow\rangle \langle \psi_z^\downarrow| \otimes \rho_{\text{pos}}(R) \end{cases}. \quad (7)$$

The spin state of the system here effectively goes from being a mixed state to a pure state as a result of this measurement: S_{VN} decreases by $\ln 2$. Summing up the entropy changes, there was a decrease of $\ln 2$ in information entropy and a net change of zero for S_{VN} as a result of the increase in stage 2 and the decrease in stage 5. Overall, then, the change is not zero but $-\ln 2$; our accounting has gone awry, and there is a failure of correspondence between S_{VN} and information entropy. If this is right, S_{VN} does not correspond to S_{TD} .

However, I think that Hemmo and Shenker are wrong to claim that S_{VN} decreases following the location measurement for collapse interpretations. As Prunkl (2020, 272) notes, there is an inconsistency here. Everyone, including Hemmo and Shenker, agrees that the spin state of the particle is mixed—not pure—after stage 2’s spin measurement, even on collapse interpretations (2006, 160). In that case, why does the particle’s spin become pure after the location measurement?

I think this results from a confusion over the nature of mixed states. In particular, they seem to have adopted what Hughes (1992, secs. 5.4, 5.8) calls the “ignorance interpretation” of mixed states, confusing what I call *classical* and *quantum* ignorance. They seem to be assuming that mixed states simply represents classical ignorance (i.e., the lack of knowledge about a particular system): a system represented by a mixed state really is in a pure state, but we know not which. This is why the location measurement is supposed to reveal to us the pure state of this system (by revealing which side it is on and hence the correlated spin state) and hence ‘wash away’ our classical ignorance of the real state of the system. Postmeasurement, we know exactly which pure state this system is in, unlike premeasurement; hence, S_{VN} decreases.

However, as Hughes (1992, 144–50) argues, this interpretation of mixed states—as representing classical ignorance about which pure state a particular system is in—cannot be the right interpretation of all mixed states. To begin, a mixed state can be decomposed in nonunique ways in general. Here is a simple example: a mixed state representing a mixture of $|\psi_z^\uparrow\rangle$ and $|\psi_z^\downarrow\rangle$ can also represent a mixture of $|\psi_x^\uparrow\rangle$ and $|\psi_x^\downarrow\rangle$, and so on. If we insist that a mixed state represent our classical ignorance about the real state of a particular system, then we end up having to say that a system’s state is really both either $|\psi_z^\uparrow\rangle$ or $|\psi_z^\downarrow\rangle$ and either $|\psi_x^\uparrow\rangle$ or $|\psi_x^\downarrow\rangle$. Of course, this is impossible given quantum mechanics. The defender of the classical ignorance interpretation might

insist that we simply pick one pair of possible pure states but not both at once. In general, however, there is no way to do that nonarbitrarily given some density matrix. Furthermore, this problem only worsens when we consider that there are usually more than just two ways to decompose a density matrix—a principled choice based on the mixed state alone is not feasible. The mixed state cannot be a representation of classical ignorance.

Instead, to paraphrase Hughes (1992, 144–45), mixed states should be (minimally) interpreted as such. If we prepared in the same way an ensemble of systems, each described with the same mixed state (i.e., a mixture of pure states with certain weights), then the relative frequency of any given measurement outcome from the ensemble is exactly what we would get if the ensemble were composed of various ‘subensembles’ each in one of the pure states in the mixture, with the relative frequency of each subensemble in the ensemble given by the corresponding weights.

In other words, the sort of quantum ignorance relevant in the right interpretation of mixed states is not whether we are ignorant about the real state of this particular system but whether we are ignorant about the measured states of an ensemble of identically prepared systems like this one. If this is right, quantum ignorance cannot be ‘washed away’ upon measurement of a single system unlike the sort of ignorance Hemmo and Shenker were implicitly assuming, and it seems like this quantum ignorance is precisely what remains after the location measurement.

This was roughly Henderson’s (2003) criticism against Shenker (1999), which is why it is puzzling that Hemmo and Shenker (2006) commit the same mistake:

This preparation produces the pure states $[|\psi_z^\uparrow\rangle]$ and $[|\psi_z^\downarrow\rangle]$ with equal probabilities. In a particular trial, the observer may take note of the measurement result, and he therefore discovers that he has say a $[|\psi_z^\uparrow\rangle]$. If he applies a projective measurement in the $\{|\psi_z^\uparrow\rangle, |\psi_z^\downarrow\rangle\}$ basis, he could predict that he will measure $[|\psi_z^\uparrow\rangle]$. However, this does not mean that, if someone handed him another state prepared in the same way, he could again predict that the outcome of his measurement would be $[|\psi_z^\uparrow\rangle]$. In this sense the observer does not know the state of the system which is being prepared, and it is because of this ignorance that the state is mixed. Looking at the measurement result does not remove the fact that there is a probability distribution over the possible outcomes. (Henderson 2003, 294)

This applies to the location measurement in stage 5 too: measuring the location of the particle in this case does not change the state of the particle from a mixed one to a pure one even on collapse interpretations. First, it seems quite irrelevant whether we adopt a collapse or no-collapse interpretation, because the collapse mechanism applies to superposed pure states,

not statistical mixtures. If anything, collapse had already happened in the stage 2 measurement procedure, yet everyone including Hemmo and Shenker (2006, 160) accept that the system is in a mixed state after stage 2 even for collapse interpretations. More importantly, there remains a probability distribution over the states of the particle as a result of stage 2's spin measurement, even after the location measurement. Given an ensemble of particles prepared from stages 1 to 5 in the same way, we are still not be able to predict with certainty whether an ensemble of particles would all be measured on the left or right sides of the box (and hence all spin-up or spin-down) as a result of the mixed state resulting from stage 2, only that half of the ensembles will be on the left and the other half will be on the right. Quantum ignorance remains—the system remains in a mixed state even after the location measurement, as

$$\rho_{\text{particle}} = \frac{1}{2}(|\psi_z^\uparrow\rangle\langle\psi_z^\uparrow| \otimes \rho_{\text{pos}}(L) + |\psi_z^\downarrow\rangle\langle\psi_z^\downarrow| \otimes \rho_{\text{pos}}(R)). \quad (8)$$

This is exactly the state of the system in no-collapse interpretations; that is, quantum ignorance does not discern between collapse and no-collapse interpretations. What has gone away is the classical ignorance that Hemmo and Shenker (mistakenly) assumed was relevant for mixed states, ignorance about this particular system's state. By measuring the system's location, we come to learn of the correlations between location measurement and the particle's spin. This ignorance does not change the mixed state to a pure state: instead, this loss of classical ignorance—gain in information—is represented as a decrease in information entropy just as before, and this information is what we use to perform the compression in stage 5.

As a result, there is no additional decrease in S_{VN} in stage 5 for collapse interpretations; the entropy accounting lines up after all, as with no-collapse interpretations. The decrease in information entropy does correspond to the increase in S_{VN} , and so information entropy does correspond to S_{VN} after all. Hemmo and Shenker's argument does not establish the failure of correspondence between S_{VN} and S_{TD} via the failure of S_{VN} and information entropy to correspond.

To sum up, their arguments in the single-particle case are either ill motivated and irrelevant to von Neumann and our discussion of correspondence when considered in terms of phenomenological thermodynamics or outright fail when considered in the more relevant domain of (informational approaches to) statistical mechanics. Either way, their argument does not support the failure of correspondence between S_{VN} and S_{TD} .¹⁸

18. Let me briefly note that their argument in the two-particles case fails for similar reasons. On the one hand, from the perspective of phenomenological thermodynamics, their argument is irrelevant: following Maxwell and others, two particles do not a

4.3. *Finitely Many Particles.* Hemmo and Shenker's argument in the case of finitely many particles rests on the assumption of *equidistribution*; that is, that the particles will be equally distributed across the left and right sides of the box after separation by the semipermeable wall. Assuming equidistribution, the increase in S_{VN} given the spin measurement in stage 2 is $N \ln 2$ (Hemmo and Shenker 2006, 169). Furthermore, the decrease in thermodynamic entropy in the fourth stage is $N \ln 2$ as well. The entropic accounting therefore seems to work out.

However, Hemmo and Shenker press further on the 'rough' nature of equidistribution when N is large but finite: they claim that the change in S_{VN} will only be $N \ln 2$ when N is infinite, since equidistribution only truly holds when $N \rightarrow \infty$. In other cases, S_{VN} will strictly only approximate S_{TD} , and hence S_{VN} and S_{TD} combined will never be exactly zero. Hence, "Von Neumann's argument goes through as an approximation" (Hemmo and Shenker 2006, 169). However, they claim that this state of affairs suggests, instead, that von Neumann's argument strictly fails: "since Von Neumann's argument is meant to establish a conceptual identity between the quantum mechanical entropy and thermodynamic entropy, we think that such an implication is mistaken. . . . No matter how large N may be, as long as it is finite, the net change of entropy throughout the experiment will not be exactly zero" (169). As I have already discussed in section 4.1, it is not clear to me that von Neumann's goal really was to establish *strict identity* (what they call "conceptual identity"), that is, correspondence between S_{VN} and S_{TD} in all domains. Rather, it seems to be the establishing of correspondence only in domains where S_{TD} is taken to hold. If so, their argument here simply misses the point.

Furthermore, as is well known, the particle analogue of thermodynamics, statistical mechanics, becomes equivalent to phenomenological thermodynamics only when $N = \infty$, that is, when N arrives at the thermodynamic limit. As such, to complain that S_{VN} does not match up to S_{TD} outside of this domain is to demand the unreasonable, since it is not clear that even statistical mechanics, the bona fide particle analogue of thermodynamics, can satisfy this demand. Since S_{VN} approximates S_{TD} the same way statistical mechanical entropies approximate S_{TD} (and becomes equivalent at $N = \infty$), and physicists generally accept that statistical mechanics corresponds to thermodynamics nevertheless, why should this approximation be particularly problematic

thermodynamic system make. On the other hand, in the domain of statistical mechanics, the analysis in terms of information entropy is irrelevant from noninformational views of statistical mechanics. From an informational perspective, however, their argument rests again on the supposed difference between collapse and no-collapse interpretations of mixed states. Since this difference is nonexistent, their argument likewise falls apart in that case.

for S_{VN} ? I think Hemmo and Shenker take too seriously the notion of conceptual identity involved in von Neumann's thought experiment to be strict equality, although I suspect a better way to understand von Neumann's strategy is to understand S_{VN} as an approximation to S_{TD} that is more fundamental than S_{TD} in small N cases but becomes part of the S_{TD} calculus in domains where S_{TD} applies.

To have a case against S_{VN} as a quantum analogue of S_{TD} in the case of finitely many particles, Hemmo and Shenker must explain what exactly the problem is with approximations in this case, if it has worked out so well for the case of statistical mechanics and thermodynamics. If not, they might just be "taking thermodynamics too seriously" (see Callender 2001).

One might say something stronger: unless they can justify why we cannot use approximations at all in science, they do not have a case at all. As they note themselves, S_{TD} is itself only on average approximately $-N \ln 2$ (Hemmo and Shenker 2006, 169), only being equal to $-N \ln 2$ when $N = \infty$. So, in fact, the approximate quantity of S_{VN} , $\sim N \ln 2$, exactly matches the approximate quantity of S_{TD} , $\sim -N \ln 2$, in the case of finitely many particles. Unless there is something wrong with approximations in physics in general, this, then, is in fact a case of S_{VN} corresponding to S_{TD} , contrary to their argument.

4.4. Infinitely Many Particles. Hemmo and Shenker consider von Neumann's argument in the infinite-particles case in two different ways: one as $N \rightarrow \infty$ and one as $N = \infty$. As they rightly point out, the two cases are very different for calculations of physical quantities.

Consider stages 2 and 5 in this context. Hemmo and Shenker emphasize that a spin measurement is "a physical operation which takes place in time" (2006, 170), which constrains what is physically possible. For the case in which $N \rightarrow \infty$, stage 2 is to be understood as a succession of physical measurements in which "we measure individual quantities of each of the particles separately and only then count the relative frequencies" (170), before coming up with a density matrix describing this state. In this case, as with the case described in section 4.3, S_{VN} approaches $N \ln 2$ as $N \rightarrow \infty$. Their complaint here consists of two premises: one, that, as with section 4.3, S_{VN} never reaches $N \ln 2$ unless $N = \infty$ and, two, that since measurements are physical measurements, we can never perform an infinite series of these measurements, and so we can never measure infinite particles. A fortiori the measurable S_{VN} can never arrive at $N \ln 2$, and so the entropic accounting is again supposed to be inconsistent if we consider both S_{VN} and S_{TD} .

However, it is clear that their argument is moot given an understanding of the sort of thermodynamics we are interested in (see sec. 4.3). While it is true that S_{VN} will never reach $N \ln 2$, recall that S_{TD} (or, more likely, one of its statistical mechanical analogues, given the domain of finitely many

particles merely approaching ∞ rather than $N = \infty$) will likewise never reach $N \ln 2$. In other words, it does not matter that we can never perform an infinite series of these measurements and, hence, never come to know of S_{VN} at the thermodynamic limit, since we can likewise never have a thermodynamic entropy equivalent to $N \ln 2$ unless we are at the thermodynamic limit. The two entropies, then, in fact correspond in this case.

What of the second case? Here, Hemmo and Shenker concede that “arithmetically Von Neumann’s argument goes through at the infinite limit” (2006, 172), which makes sense because, as I have insisted so far, von Neumann’s strategy was never to demonstrate the strict identity of S_{VN} and S_{TD} , that is, the correspondence of S_{VN} and S_{TD} in all domains. Instead, it was to show that S_{VN} corresponds to S_{TD} only in the domain where phenomenological thermodynamics hold, in all other cases merely approximating S_{TD} in large N cases or replacing it altogether (in, e.g., single-particle cases). I maintain that Hemmo and Shenker’s main mistake was to confuse the domain where phenomenological thermodynamics hold with domains where they do not hold.

Hemmo and Shenker complain that “real physical systems are finite. This means that Von Neumann’s argument does not establish a conceptual identity between the Von Neumann entropy and thermodynamic entropy of physical systems. Identities of physical properties mean that the two quantities refer to the same magnitude in the world” (2006, 172). In line with what I have said in section 4.1, it seems that there was no physically meaningful theoretical term in phenomenological thermodynamics that could refer to some quantity in the single-particle case, which was why von Neumann needed to come up with a new measure of entropy to begin with. Furthermore, extending a concept to a new domain does not require strict identity, as we have seen and understood for a long time in the case of statistical mechanics and phenomenological thermodynamics.

As Peres summarizes: “There should be no doubt that von Neumann’s entropy. . . is equivalent to the entropy of classical thermodynamics. (This statement must be understood with the same vague meaning as when we say that the quantum notions of energy, momentum, angular momentum, etc., are equivalent to the classical notions bearing the same names)” (2002, 174). ‘Equivalence’ here should not be understood in terms of strict (or conceptual) identity, that is, correspondence at all domains. Rather, we should understand equivalence loosely as correspondence in the suitable domains of application and successful extension of old concepts in these domains to new domains. As Peres noted above, ‘equivalence’ should be understood in the context of discovery, where one is trying to develop new concepts that are analogous to old ones in different domains. For von Neumann, we have a theory (phenomenological thermodynamics) that is well understood but also another theory (quantum mechanics) that we want to understand in light of the former theory. Finding correspondence provides us with ways

to extend concepts from the original theory to the new theory: for example, with S_{VN} we may now define ‘something like’ S_{TD} , whereas before there was no way to talk about these cases. The same goes for statistical mechanics: by finding a correspondence between, for example, temperature and mean kinetic energy in the thermodynamic limit, we can extend the notion of ‘something like’ temperature beyond its original domain into systems with small numbers of particles, whereas before there was, again, no way to talk about these cases.

I see nothing wrong in these cases in the context of discovery. We should give up a strong and untenable notion of conceptual identity in this context. If so, Hemmo and Shenker’s objection loses much bite.

They further claim that “the fact that the behavior of the two quantities coincides approximately for a very large number of particles is not enough, because in any ensemble of finite gases there are systems in which the identity will not be true. This means that in a real experiment the Von Neumann entropy is not identical with the thermodynamic entropy” (Hemmo and Shenker 2006, 172). This again reveals a confusion between phenomenological and statistical thermodynamics. If they want to talk about particles at all, it seems they must adopt some form of statistical mechanical picture with microscopic variables, given phenomenological thermodynamics’ emphasis on purely macroscopic variables like volume or temperature. Yet, if so, they must recognize that thermodynamic entropy S_{TD} is in general not strictly identical to statistical mechanical entropy, for example, the Gibbs entropy or information entropy (briefly discussed in sec. 4.2) either. Their complaint about approximate coincidences not being enough for (the relevant sort of) equivalence thus weakens significantly, especially since they must assume some such equivalence (which cannot be strict identity) to even talk about particles within the context of phenomenological thermodynamics to begin with. Furthermore, statistical mechanics is evidently empirically successful in explaining and predicting traditionally thermodynamic phenomena despite this ‘nonequivalence’—it is not clear why this ‘nonequivalence’ should matter if, for all practical purposes, statistical mechanics is the conceptual successor of thermodynamics. Of course, if they could come up with a principled reason why approximations should not be allowed period, while accounting for statistical mechanics’ empirical success in accounting for thermodynamic behavior, then this could change. As of now, I see no such argument forthcoming.

5. Conclusion and Some Open Questions. I hope to have shown that Hemmo and Shenker’s (2006) argument against the correspondence of S_{VN} and S_{TD} —to my knowledge the only one in the philosophical literature—fails to hold in all three cases considered (secs. 4.1–4.4), as a result of their misunderstanding about domains where phenomenological thermodynamics

should hold and domains where it should not. This is compounded with misunderstandings about the role of approximations and the relevant interpretation of density matrices and ignorance in quantum mechanics. I conclude that their argument fails on the whole; the correspondence holds for now.

Of course, even if Hemmo and Shenker's claims were debunked, this does not yet amount to a positive argument for the equivalence between von Neumann entropy and thermodynamic entropy. Even assuming correspondence, correspondence does not entail equivalence. However, the former does provide good *prima facie* reasons to believe the latter, especially given the novel take on correspondence I provided in the end of section 3: we can accept the correspondence because of thermodynamic considerations about the Second Law and S_{TD} accounting but also because of quantum mechanical considerations about S_{VN} accounting. The correspondence supports a 'two-way street'—equivalence—between S_{TD} and S_{VN} .

While I hope to have conclusively refuted Hemmo and Shenker's argument, this is but the beginning of further inquiry into questions arising from this supposed correspondence. Amid the tangle of entropies, there remains much more housekeeping to be done for philosophers of physics.

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