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LOCAL GROUP RINGS

BY
W. K. NICHOLSON

The purpose of this note is to generalize a result of Gulliksen, Ribenboim and Viswanathan which characterized local group rings when both the ring and the group are commutative.

We assume throughout that all rings are associative with identity. If R is a ring we call R *local* if $R/J(R)$ is a division ring where $J(R)$ denotes the Jacobson radical of R. It is well known that R is local if and only if each element of $R\backslash J(R)$ is a unit. We need the following.

LEMMA. *Let $f: R \to A$ be a nonzero epimorphism of rings. Then R is local if and only if A is local and* $\ker(f) \subset J(R)$.

Proof. If $\ker(f) \subset J(R)$ then f establishes a one-to-one correspondence between the maximal left ideals of A and the maximal left ideals of R which contain $\ker(f)$. Hence, if A is local, R has exactly one maximal left ideal so $J(R)$ is maximal as a left ideal. It follows that $R/J(R)$ is a division ring.

Conversely: If R is local then $\ker(f) \subset J(R)$ since otherwise $f(R)=0$. Then it follows that A has exactly one maximal left ideal so A is local.

If A is a ring and G a group let AG denote the group ring. If $H \lhd G$ is a normal subgroup let $w(H)$ be the left (and also right) ideal of AG generated by $\{1-h: h \in H\}$ The *augmentation ideal* of AG is $\Delta(AG)=w(G)$.

COROLLARY. *If A is a ring and G a group, and if $H \lhd G$ is a normal subgroup, then AG is local$\Leftrightarrow A(G/H)$ is local and $w(H) \subset J(AG)$. In particular AG is local$\Leftrightarrow A$ is local and $\Delta(AG) \subset J(AG)$.*

Proof. There is an epimorphism $AG \to A(G/H)$ with kernel $w(H)$.

PROPOSITION. *Suppose AG is local and $H < G$ is a subgroup. Then AH is local and $J(AH)=J(AG) \cap AH$. In particular $J(A)=J(AG) \cap A$.*

Proof. It is always the case that $J(AG) \cap AH \subset J(AH)$. (See [2, Proposition 9].) Suppose $r \in \Delta(AH)$. Then $\sum_h r(h)=0$ so $r \in \Delta(AG) \subset J(AG)$. Thus $r \in J(AG) \cap AH \subset J(AH)$; i.e. $\Delta(AH) \subset J(AH)$. Since A is local it follows from the corollary that AH is local.

But then if $r \in AH\backslash J(AG)$ we have $r \notin J(AG)$ so r is a unit in AG. Since $r \in AH$, we have that r is a unit in AH (see [1]) so $r \notin J(AH)$. Thus $J(AH) \subset J(AG) \cap AH$.

COROLLARY. *AG is local if and only if AH is local for every finitely generated subgroup $H < G$.*

Proof. If AG is local so is each AH by the proposition. To prove the converse let $r \in \Delta(AG)$. If H is the subgroup generated by the support of r then $r \in \Delta(AH) \subset J(AH)$. Thus r is quasi-regular so $\Delta(AG)$ is a quasi-regular ideal. Hence $\Delta(AG) \subset J(AG)$.

We can now prove our main theorem.

THEOREM. *Let A be a ring and G a group.*

(1) *If AG is local then A is local, G is a p-group and $p \in J(A)$.*

(2) (*Partial converse.*) *If A is local, G is a locally finite p-group and $p \in J(A)$ then AG is local.*

(3) *If G is abelian then AG is local if and only if A is local, G is a p-group and $p \in J(A)$.*

Proof. (1) This follows from the above and the result of Connell ([2, Proposition 15]) that if $\Delta(AG) \subset J(AG)$ then G is a p-group and $p \in J(A)$.

(2) By the above corollary we may assume G is finitely generated and so, since G is locally finite, that G is finite. Let $\bar{A}$ denote the division ring $A/J(A)$. There is an epimorphism $f\colon AG \to \bar{A}G$ with $\ker(f) = \{r \in AG\colon r(g) \in J(A) \text{ for all } g \in G\}$. Since G is finite $\ker(f) \subset J(AG)$ by another result of Connell ([2, Proposition 9]). Hence by the lemma, AG is local if $\bar{A}G$ is. But $\bar{A}$ is a division ring of characteristic p and G is a finite p-group so, by [2, Theorem 9], $\Delta(\bar{A}G)$ is nilpotent. Hence $\Delta(\bar{A}G) \subset J(\bar{A}G)$ and so $\bar{A}G$ is local.

(3) This is clear by (1) and (2).

Remark 1. The proof of (2) goes through if we replace the condition that G is locally finite by

(i) $J(A) \subset J(AG)$ and
(ii) $\bar{A}G$ is local.

Remark 2. The result in (3) generalizes a result of Gulliksen, Ribenboim and Viswanathan ([3, p. 153]) who proved it assuming A is commutative.

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UNIVERSITY OF CALGARY,
CALGARY, ALBERTA