Proceedings of the Royal Society of Edinburgh, **142A**, 1131–1177, *2012*

The Lax–Oleinik semi-group: a Hamiltonian point of view

Patrick Bernard*
Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France (patrick.bernard@ens.fr)

(MS received 10 June 2011; accepted 27 March 2012)

The weak KAM theory was developed by Fathi in order to study the dynamics of convex Hamiltonian systems. It somehow makes a bridge between viscosity solutions of the Hamilton–Jacobi equation and Mather invariant sets of Hamiltonian systems, although this was fully understood only *a posteriori*. These theories converge under the hypothesis of convexity, and the richness of applications mostly comes from this remarkable convergence. In this paper, we provide an elementary exposition of some of the basic concepts of weak KAM theory. In a companion paper, Albert Fathi exposed the aspects of his theory which are more directly related to viscosity solutions. Here, on the contrary, we focus on dynamical applications, even if we also discuss some viscosity aspects to underline the connections with Fathi's lecture. The fundamental reference on weak KAM theory is the still unpublished book *Weak KAM theorem in Lagrangian dynamics* by Albert Fathi. Although we do not offer new results, our exposition is original in several aspects. We only work with the Hamiltonian and do not rely on the Lagrangian, even if some proofs are directly inspired by the classical Lagrangian proofs. This approach is made easier by the choice of a somewhat specific setting. We work on $\mathbb{R}^d$ and make uniform hypotheses on the Hamiltonian. This allows us to replace some compactness arguments by explicit estimates. For the most interesting dynamical applications, however, the compactness of the configuration space remains a useful hypothesis and we retrieve it by considering periodic (in space) Hamiltonians. Our exposition is centred on the Cauchy problem for the Hamilton–Jacobi equation and the Lax–Oleinik evolution operators associated to it. Dynamical applications are reached by considering fixed points of these evolution operators, the weak KAM solutions. The evolution operators can also be used for their regularizing properties; this opens an alternative route to dynamical applications.

1. The method of characteristics, existence and uniqueness of regular solutions

We consider a C^2 Hamiltonian

$$H(t, q, p) : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d*} \to \mathbb{R}$$

and study the associated Hamiltonian system

$$\dot{q}(t) = \partial_p H(t, q(t), p(t)), \qquad \dot{p}(t) = -\partial_q H(t, q(t), p(t)), \tag{HS}$$

*This paper is a late addition to the papers surveying active areas in partial differential equations, published in issue 141.2, which were based on a series of mini-courses held in the International Centre for Mathematical Sciences (ICMS) in Edinburgh during 2010.

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

and Hamilton–Jacobi equation

$$\partial_t u + H(t, q, \partial_q u(t, q)) = 0. \tag{HJ}$$

We denote by $X_H(x) = X_H(q, p)$ the Hamiltonian vector field $X_H = J\,\mathrm{d}H$, where J is the matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The Hamiltonian system can be written in condensed terms $\dot{x}(t) = X_H(t, x(t))$. We shall always assume that the solutions extend to $\mathbb{R}$. We denote by

$$\varphi_\tau^t = (Q_\tau^t, P_\tau^t) \colon \mathbb{R}^d \times \mathbb{R}^{d*} \to \mathbb{R}^d \times \mathbb{R}^{d*}$$

the flow map which associate to a point $x \in \boldsymbol{T}^*\mathbb{R}^d$ the value at time t of the solution $x(s)$ of (HS) which satisfies $x(\tau) = x$.

If $u(t, q)$ solves (HJ), and if $q(s)$ is a curve in $\mathbb{R}^d$, then the formula

$$u(t_1, q(t_1)) - u(t_0, q(t_0)) = \int_{t_0}^{t_1} \partial_q u(s, q(s)) \cdot \dot{q}(s) - H(s, q(s), \partial_q u(s, q(s)))\,\mathrm{d}s \tag{1.1}$$

follows from an obvious computation. The integral on the right-hand side is the *Hamiltonian action* of the curve $s \mapsto (q(s), \partial_q u(s, q(s)))$. The *Hamiltonian action* of the curve $(q(s), p(s))$ on the interval $[t_0, t_1]$ is the quantity

$$\int_{t_0}^{t_1} p(s) \cdot \dot{q}(s) - H(s, q(s), p(s))\,\mathrm{d}s.$$

A classical and important property of the Hamiltonian actions is that orbits are critical points of this functional. More precisely, we have the following.

Proposition 1.1. *The C^2 curve $x(t) = (q(t), p(t)) \colon [t_0, t_1] \to \mathbb{R}^d \times \mathbb{R}^{d*}$ solves (HS) if and only if the equality*

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \left(\int_{t_0}^{t_1} p(t, s) \cdot \dot{q}(t, s) - H(t, q(t, s), p(t, s))\,\mathrm{d}t \right) = 0$$

(where the dot is the derivative with respect to t) holds for each C^2 variation $x(t, s) = (q(t, s), p(t, s)) \colon [t_0, t_1] \times \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^{d*}$ fixing the endpoints, which means that $x(t, 0) = x(t)$ for each t and that $q(t_0, s) = q(t_0)$ and $q(t_1, s) = q(t_1)$ for each s.

Proof. We set $\theta(t) = \partial_s q(t, 0)$, $\zeta(t) = \partial_s p(t, 0)$ and compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \left(\int_{t_0}^{t_1} p(t, s)\dot{q}(t, s) - H(t, q(t, s), p(t, s))\,\mathrm{d}t \right) \\
&\quad = \int_{t_0}^{t_1} p(t)\dot{\theta}(t) + \zeta(t)\dot{q}(t) - \partial_q H(t, q(t), p(t))\theta(t) - \partial_p H(t, q(t), p(t))\zeta(t)\,\mathrm{d}t \\
&\quad = p(t_1)\theta(t_1) - p(t_0)\theta(t_0) + \int_{t_0}^{t_1} (\dot{q}(t) - \partial_p H(t, q(t), p(t)))\zeta(t)\,\mathrm{d}t \\
&\qquad - \int_{t_0}^{t_1} (\dot{p}(t) + \partial_q H(t, q(t), p(t)))\theta(t)\,\mathrm{d}t.
\end{aligned}$$

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

As a consequence, the derivative of the action vanishes if $(q(t), p(t))$ is a Hamiltonian trajectory and if the variation $q(t, s)$ fixes the boundaries. Conversely, this computation can be applied to the variation $q(t, s) = q(t) + s\theta(t), p(t, s) = p(t) + s\zeta(t)$, and implies that

$$\int_{t_0}^{t_1} (\dot{q}(t) - \partial_p H(t, q(t), p(t)))\zeta(t)\,\mathrm{d}t - \int_{t_0}^{t_1} (\dot{p}(t) + \partial_q H(t, q(t), p(t)))\theta(t)\,\mathrm{d}t = 0$$

for each C^2 curve $\theta(t)$ vanishing on the boundary and each C^2 curve $\zeta(t)$. This implies that $\dot{q}(t) - \partial_p H(t, q(t), p(t)) \equiv 0$ and $\dot{p}(t) + \partial_q H(t, q(t), p(t)) \equiv 0$. □

We now return to the connections between (HS) and (HJ). A function is said to be of class $C^{1,1}$ if it is differentiable and if its differential is Lipschitz. It is said to be of class $C^{1,1}_{\mathrm{loc}}$ if it is differentiable with a locally Lipschitz differential. Rademacher's theorem states that a locally Lipschitz function is differentiable almost everywhere.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ be an open set, and let $u(t, q)\colon \Omega \to \mathbb{R}$ be a $C^{1,1}_{\mathrm{loc}}$ solution of (HJ). Let $q(t)\colon [t_0, t_1] \to \mathbb{R}^d$ be a C^1 curve such that $(t, q(t)) \in \Omega$ and*

$$\dot{q}(t) = \partial_p H(t, q(t), \partial_q u(t, q(t)))$$

for each $t \in [t_0, t_1]$. Then, setting $p(t) = \partial_q u(t, q(t))$, the curve $(q(t), p(t))$ is C^1 and it solves (HS).

The curves $q(t)$ satisfying the hypothesis of the theorem as well as the associated trajectories $(q(t), p(t))$ are called the *characteristics* of u.

Proof. Let $\theta(t)\colon [t_0, t_1] \to \mathbb{R}^d$ be a smooth curve vanishing on the boundaries. We define $q(t, s) := q(t) + s\theta(t)$ and $p(t, s) := \partial_q u(t, q(t, s))$. Our hypothesis is that $\dot{q}(t) = \partial_p H(t, q(t), p(t))$, which is the first part of (HS). For each s, we have

$$u(t_1, q(t_1)) - u(t_0, q(t_0)) = \int_{t_0}^{t_1} p(t, s) \cdot \dot{q}(t, s) - H(t, q(t, s), p(t, s))\,\mathrm{d}t;$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \biggl(\int_{t_0}^{t_1} p(t, s)\dot{q}(t, s) - H(t, q(t, s), p(t, s)\biggr)\,\mathrm{d}t) = 0.$$

We now claim that

$$\int_{t_0}^{t_1} \partial_q H(t, q(t), p(t)) \cdot \theta(t) - p(t)\dot{\theta}(t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \biggl(\int_{t_0}^{t_1} p(t, s)\dot{q}(t, s) - H(t, q(t, s), p(t, s))\,\mathrm{d}t \biggr).$$

Assuming the claim, we obtain the equality

$$\int_{t_0}^{t_1} \partial_q H(t, q(t), p(t)) \cdot \theta(t) - p(t) \cdot \dot{\theta}(t)\,\mathrm{d}t = 0$$

for each smooth function θ vanishing at the boundary. In other words, we have

$$\dot{p}(t) = -\partial_q H(t, q(t), p(t))$$

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

in the sense of distributions. Since the right-hand side is continuous, this implies that p is C^1 and that the equality holds for each t. We have proved the theorem, assuming the claim.

The claim can be proved by an easy computation in the case where u is C^2. Under the assumption that u is only $C^{1,1}_{\mathrm{loc}}$, the map p is only locally Lipschitz, and some care is necessary. For each fixed θ, we have

$$\begin{aligned}\partial_q H(t,q(t,s),p(t,s))\cdot\theta(t)-p(t,s)\cdot\dot\theta(t)&\\ &= \partial_q H(t,q(t),p(t))\cdot\theta(t)-p(t)\cdot\dot\theta(t)+O(s),\\ \partial_t q(t,s)-\partial_p H(t,q(t,s),p(t,s)) &= \dot q-\partial_p H(t,q(t),p(t))+O(s)\\ &= O(s),\end{aligned}$$

where $O(s)$ is uniform in t. We then have, for small $S>0$,

$$\begin{aligned}\int_{t_0}^{t_1}\partial_q H(t,q(t),p(t))\cdot\theta(t)-p(t)\cdot\dot\theta(t)\,\mathrm{d}t&\\ &= O(S)+\frac{1}{S}\int_{t_0}^{t_1}\int_0^S \partial_q H(t,q(t,s),p(t,s))\cdot\theta(t)-p(t,s)\cdot\dot\theta(t)\,\mathrm{d}s\,\mathrm{d}t\\ &= O(S)+\frac{1}{S}\int_{t_0}^{t_1}\int_0^S \partial_q H\cdot\partial_s q-p\cdot\partial_{st}q+(\partial_t q-\partial_p H)\cdot\partial_s p\,\mathrm{d}s\,\mathrm{d}t\\ &= O(S)+\frac{1}{S}\int_{t_0}^{t_1}[p\cdot\partial_t q-H]_0^S\,\mathrm{d}t\\ &= O(S)+\frac{1}{S}\biggl[\int_{t_0}^{t_1}p\cdot\partial_t q-H\,\mathrm{d}t\biggr]_0^S.\end{aligned}$$

We obtain the claimed equality at the limit $S\to 0$. □

The following restatement of theorem 1.2 has a more geometric flavour.

Corollary 1.3. *Let $\Omega\subset\mathbb{R}\times\mathbb{R}^d$ be an open set, and let $u(t,q)\colon\Omega\to\mathbb{R}$ be a $C^{1,1}_{\mathrm{loc}}$ solution of the Hamilton–Jacobi equation (HJ). Then the extended Hamiltonian vector field $Y_H=(1,X_H)$ is tangent to the graph*

$$G:=\{(t,q,\partial_q u)\colon (t,q)\in\Omega\}.$$

Proof. Let us fix a point (t_0,q_0) in Ω. By the Cauchy–Lipschitz theorem, there exists a solution $q(t)$ of the ordinary differential equation $\dot q=\partial_p H(t,q(t),\partial_q u(t,q(t)))$, defined on an open time interval containing t_0 and such that $q(t_0)=q_0$. Let us, as above, define $p(t):=\partial_q u(t,q(t))$. The curve $(t,q(t),p(t))$ is contained in the graph G, and we deduce from theorem 1.2 that it solves (HS). As a consequence, the derivative Y_H of the curve $(t,q(t),p(t))$ is tangent to G. □

Corollary 1.4. *Let $u(t,q)$ be a $C^{1,1}_{\mathrm{loc}}$ solution of (HJ) defined on the open set $\Omega={]t_0,t_1[}$. Then, for each s and t in ${]t_0,t_1[}$ we have*

$$\Gamma_t=\varphi_s^t(\Gamma_s),$$

where Γ_t is defined by

$$\Gamma_t:=\{(q,\mathrm{d}u_t(q))\colon q\in\mathbb{R}^d\}.$$

Proof. Let (q_s, p_s) be a point in Γ_s. Let us consider the Lipschitz map

$$F(t,q) := \partial_p H(t, q, \partial_q u(t,q)),$$

and consider the differential equation $\dot{q}(t) = F(t, q(t))$. By the Cauchy–Peano theorem, there exists a solution $q(t)$ of this equation, defined on the interval $]t^-, t^+[$, and such that $q(s) = q_s$. Setting $p(t) = \partial_q u(t, q(t))$, theorem 1.2 implies that the curve $(q(t), p(t))$ solves (HS). We can choose t^+ such that either $t^+ = t^1$ or the curve $q(t)$ is unbounded on $[s, t^+[$. The second case is not possible, because $(q(t), p(t))$ is a solution of (HS), which is complete; hence, we can take $t^+ = t_1$. Similarly, we can take $t^- = t_0$. We have proved that $(q(t), p(t))$ is the Hamiltonian orbit of the point (q_s, p_s). Then, for each $t \in]t_0, t_1[$, we have

$$\varphi_s^t(q_s, p_s) = (q(t), p(t)) = (q(t), \partial_q u(t, q(t))) \in \Gamma_t.$$

Since this holds for each $(q_s, p_s) \in \Gamma_s$, we conclude that $\varphi_s^t(\Gamma_s) \subset \Gamma_t$ for each $s, t \in]t_0, t_1[$. By symmetry, this inclusion is an equality. □

Let us now consider an initial condition $u_0(q)$ and study the Cauchy problem consisting of finding a solution $u(t,q)$ of (HJ) such that $u(0,q) = u_0(q)$.

Proposition 1.5. *Given a time interval $]t_0, t_1[$ containing the initial time $t = 0$ and a $C^{1,1}_{\text{loc}}$ initial condition u_0, there is at most one $C^{1,1}_{\text{loc}}$ solution $u(t,q)\colon]t_0, t_1[$ of (HJ) such that $u(0,q) = u_0(q)$ for all $q \in \mathbb{R}^d$.*

Proof. Let u and $\tilde{u}$ be two solutions of this Cauchy problem. Let us associate to them the graphs Γ_t and $\tilde{\Gamma}_t$, $t \in]t_0, t_1[$. Since $\tilde{u}(\tau, q) = u(\tau, q)$, we have $\Gamma_\tau = \tilde{\Gamma}_\tau$; hence, by corollary 1.4,

$$\Gamma_t = \varphi_\tau^t(\Gamma_\tau) = \varphi_\tau^t(\tilde{\Gamma}_\tau) = \tilde{\Gamma}_t.$$

We conclude that $\partial_q u = \partial_q \tilde{u}$, and then, from (HJ), that $\partial_t u = \partial_t \tilde{u}$. The functions u and $\tilde{u}$ thus have the same differential on $]t_0, t_1[$; hence, they differ by a constant. Finally, since these functions have the same value on $\{\tau\} \times \mathbb{R}^d$, they are equal. □

To study the existence problem, we lift the function u_0 to the surface Γ_0 by defining $w_0 = u_0 \circ \pi$, where π is the projection $(q,p) \mapsto q$ (later we shall also use the symbol π to denote the projection $(t,q,p) \mapsto (t,q)$). It is then useful to work in a more general setting.

A *geometric initial condition* is the data of a subset $\Gamma_0 \subset \mathbb{R}^d \times \mathbb{R}^{d*}$ and of a function $w_0\colon \Gamma_0 \to \mathbb{R}$ such that $\mathrm{d}w_0 = p\,\mathrm{d}q$ on Γ_0. More precisely, we require that the equality $\partial_s(w_0(q(s), p(s))) = p(s)\partial_s q(s)$ holds almost everywhere for each Lipschitz curve $(q(s), p(s))$ on Γ_0. We shall consider mainly two types of geometric initial conditions:

(i) the geometric initial condition $(\Gamma_0, w_0 = u_0 \circ \pi)$ associated to the C^1 initial condition u_0;

(ii) the geometric initial condition $(\Gamma_0 = \{q_0\} \times \mathbb{R}^{d*}, w_0 = 0)$, for $q_0 \in \mathbb{R}^d$.

Given the geometric initial condition $(\mathit{\Gamma}_0, w_0)$, we define

$$G := \bigcup_{t \in \,]t_0, t_1[} \{t\} \times \varphi_0^t(\mathit{\Gamma}_0) \tag{G}$$

and, denoting by $\dot{Q}_t^s(x)$ the derivative with respect to s, the function

$$\begin{aligned} &w\colon G \to \mathbb{R}, \\ &(t, x) \mapsto w_0(\varphi_t^0(x)) + \int_0^t P_t^s(x)\dot{Q}_t^s(x) - H(s, \varphi_t^s(x))\,\mathrm{d}s. \end{aligned} \tag{w}$$

The pair (G, w) is called the *geometric solution* emanating from the geometric initial condition $(\mathit{\Gamma}_0, w_0)$.

This definition is motivated by the following observation: assume that a C^2 solution $u(t, q)$ of (HJ) emanating from the genuine initial condition u_0 exists. Let $(\mathit{\Gamma}_0, w_0)$ be the geometric initial condition associated to u_0. Let G be the graph of $\partial_q u$, as defined in corollary 1.4, and let w be the function defined on G by $w := u \circ \pi$. Then, (G, w) is the geometric solution emanating from the geometric initial condition $\mathit{\Gamma}_0$. This follows immediately from corollary 1.4 and equation (1.1). In general, we have the following.

Proposition 1.6. *Let $(\mathit{\Gamma}_0, w_0)$ be a geometric initial condition, and let (G, w) be the geometric solution emanating from $(\mathit{\Gamma}_0, w_0)$. Then, the function w satisfies $\mathrm{d}w = p\,\mathrm{d}q - H\,\mathrm{d}t$ on G. More precisely, for each Lipschitz curve $Y(s) = (T(s), \theta(s), \zeta(s))$ contained in G, then for almost every s,*

$$\frac{\mathrm{d}}{\mathrm{d}s}(w(T(s), \theta(s), \zeta(s))) = \zeta(s)\frac{\mathrm{d}\theta}{\mathrm{d}s} - H(Y(s))\frac{\mathrm{d}T}{\mathrm{d}s}.$$

Proof. Let us first consider a C^2 curve $Y(s) = (T(s), \theta(s), \zeta(s))$ on G. We set $q(t, s) = Q_{T(s)}^t(\theta(s), \zeta(s))$ and $p(t, s) = P_{T(s)}^t(\theta(s), \zeta(s))$ and, finally, $x(t, s) = (q(t, s), p(t, s))$. We have

$$w(T(s), \theta(s), \zeta(s)) = w_0(q(0, s), p(0, s)) + \int_0^{T(s)} p(t, s)\dot{q}(t, s) - H(t, x(t, s))\,\mathrm{d}t.$$

Since $\mathrm{d}w_0 = p\,\mathrm{d}q$ on $\mathit{\Gamma}_0$, the calculations in the proof of proposition 1.1 imply that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(w \circ Y) &= p(0, s) \cdot \partial_s q(0, s) + p(T(s), s) \cdot \partial_s q(T(s), s) - p(0, s) \cdot \partial_s q(0, s) \\ &\qquad + (p(T(s), s) \cdot \partial_t q(T(s), s) - H(T(s), x(T(s), s)))\frac{\mathrm{d}T}{\mathrm{d}s} \\ &= \zeta(s)\left(\partial_s q(T(s), s) + \partial_t q(T(s), s)\frac{\mathrm{d}T}{\mathrm{d}s}\right) + H(Y(s))\frac{\mathrm{d}T}{\mathrm{d}s}. \end{aligned}$$

The desired equality follows from the observation that

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \partial_t q(T(s), s)\left(\frac{\mathrm{d}T}{\mathrm{d}s}\right) + \partial_s q(T(s), s),$$

which can be seen by differentiating the equality $\theta(s) = q(T(s), s)$.

These computations, however, cannot be applied directly in the case where $Y(s)$ is only C^1, or, even worse, Lipschitz. In this case, we shall prove the desired equality in integral form:

$$[w \circ Y]_{S_0}^{S_1} = \int_{S_0}^{S_1} \zeta(s) \cdot \partial_s \theta(s) - H \circ Y(s) \cdot \partial_s T(s) \, \mathrm{d}s$$

for each $S_0 < S_1$. Fixing S_0 and S_1, we can approximate uniformly the curve $Y(s)$ by a sequence $Y_n(s)\colon [S_0, S_1] \to \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d*}$ of equi-Lipschitz smooth curves such that $Y_n(S_0) = Y(S_0)$ and $Y_n(S_1) = Y(S_1)$. To the curves Y_n, we associate $x_n(t,s) = (p_n(t,s), q_n(t,s))$ as above. The functions x_n are equi-Lipschitz and converge uniformly to x. In general, we do not have $Y_n(s) \in G$ on $]S_0, S_1[$; hence, we do not have $x_n(0,s) \in \Gamma_0$, and we cannot express $\partial_s w(x_n(0,s))$ as we did above. Since this is the only part of the above computation which used the inclusion $Y(s) \in G$, we can still get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(w \circ Y_n) = \frac{\mathrm{d}}{\mathrm{d}s}(w_0(x_n(0,s)) - p_n(0,s) \cdot \partial_s q_n(0,s) \\ + \zeta_n(s) \cdot \partial_s \theta_n(s) + H(Y_n(s)) \partial_s T_n(s)). \end{aligned}$$

Noting that $[w \circ Y]_{S_0}^{S_1} = [w \circ Y_n]_{S_0}^{S_1}$ and that $[w_0(x(0,\cdot))]_{S_0}^{S_1} = [w_0(x_n(0,\cdot))]_{S_0}^{S_1}$, we obtain

$$\begin{aligned} [w \circ Y]_{S_0}^{S_1} &= [w_0(x(0,\cdot))]_{S_0}^{S_1} \\ &\quad + \int_{S_0}^{S_1} -p_n(0,s) \cdot \partial_s q_n(0,s) + \zeta_n(s) \partial_s \theta_n(s) + H(Y_n(s)) \partial_s T_n(s) \, \mathrm{d}s \\ &= \int_{S_0}^{S_1} p(0,s) \cdot \partial_s q(0,s) - p_n(0,s) \cdot \partial_s q_n(0,s) \, \mathrm{d}s \\ &\quad + \int_{S_0}^{S_1} \zeta_n(s) \partial_s \theta_n(s) + H(Y_n(s)) \partial_s T_n(s) \, \mathrm{d}s. \end{aligned}$$

We derive the desired formula at the limit $n \to \infty$, along a subsequence such that

$$\partial_s q_n(0,\cdot) \rightharpoonup \partial_s q(0,\cdot), \qquad \partial_s \theta_n \rightharpoonup \partial_s \theta, \qquad \partial_s T_n \rightharpoonup \partial_s T$$

weakly-$*$ in L^∞, taking into account that

$$p_n(0,\cdot) \to p(0,\cdot), \qquad \zeta_n(s) \to \zeta(s), \qquad H(Y_n(s)) \to H(Y(s))$$

uniformly, and hence strongly in L^1. Recall that a sequence of curves $f_n\colon [t_0, t_1] \to \mathbb{R}^d$ is said to converge to f weakly-$*$ in L^∞ if

$$\int_{t_0}^{t_1} f_n g \, \mathrm{d}t \to \int_{t_0}^{t_1} f g \, \mathrm{d}t$$

for each L^1 curve $g\colon [t_0, t_1] \to \mathbb{R}^d$. We have used two classical properties of the weak-$*$ convergence:

(i) a uniformly bounded sequence of functions has a subsequence which has a weak-$*$ limit;

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

(ii) the convergence

$$\int_{t_0}^{t_1} f_n g_n \,\mathrm{d}t \to \int f g \,\mathrm{d}t$$

holds if $f_n \rightharpoonup f$ weakly-$*$ in L^∞ and if $g_n \to g$ strongly in L^1.

□

Corollary 1.7. *If there exists a locally Lipschitz map* $\chi\colon \Omega \to \mathbb{R}^{d*}$ *on some open subset* Ω *of* $]t_0, t_1[$ *such that* $(t, q, \chi(t, q)) \subset G$ *for all* $(t, q) \in \Omega$*, then the function*

$$u(t, q) := w(t, q, \chi(t, q))$$

is C^1 *and it solves (HJ) on* Ω*. Moreover, we have* $\partial_q u = \chi$*.*

Proof. For each C^1 curve $(T(s), Q(s))$ in Ω, the curve

$$Y(s) = (T(s), Q(s), \chi(T(s), Q(s)))$$

is Lipschitz; hence, by proposition 1.6, we have

$$\begin{aligned}\partial_s u(T(s), Q(s)) &= \partial_s w(T(s), Q(s), \chi(T(s), Q(s))) \\ &= \chi(T(s), Q(s)) \cdot \partial_s Q(s) - H(T(s), Q(s), \chi(T(s), Q(s)))\partial_s T(s)\end{aligned}$$

almost everywhere. Since the right-hand side in this expression is continuous, we conclude that the Lipschitz functions $u(T(s), Q(s))$ is actually differentiable at each point, the equality above being satisfied everywhere. Since this holds for each C^1 curve $(T(s), Q(s))$, the function u has to be differentiable, with $\partial_q u(t, q) = \chi(t, q)$ and $\partial_t u(t, q) + H(t, q, \chi(t, q)) = 0$. □

We have reduced the existence problem to the study of the geometric solution G. We need an additional hypothesis to obtain a local existence result. We shall use the following one, which it is stronger than would really be necessary, but will allow us to rest on simple estimates in this course.

Hypothesis 1.8. *There exists a constant* M *such that*

$$\|\mathrm{d}^2 H(t, q, p)\| \leqslant M$$

for each (t, q, p)*.*

This hypothesis implies that the Hamiltonian vector field is Lipschitz, and hence that the Hamiltonian flow is complete. The hypothesis can be exploited further to estimate the differential

$$\mathrm{d}\varphi_0^t = \begin{bmatrix} \partial_q Q_0^t(x) & \partial_p Q_0^t(x) \\ \partial_q P_0^t(x) & \partial_p P_0^t(x) \end{bmatrix}$$

using the variational equation

$$\begin{bmatrix} \partial_q \dot{Q}_0^t(x) & \partial_p \dot{Q}_0^t(x) \\ \partial_q \dot{P}_0^t(x) & \partial_p \dot{P}_0^t(x) \end{bmatrix} = \begin{bmatrix} \partial_{qp} H(t, x) & \partial_{pp} H(t, x) \\ -\partial_{qq} H(t, x) & -\partial_{pq} H(t, x) \end{bmatrix} \begin{bmatrix} \partial_q Q_0^t(x) & \partial_p Q_0^t(x) \\ \partial_q P_0^t(x) & \partial_p P_0^t(x) \end{bmatrix}.$$

We obtain the following estimate:

$$\|\mathrm{d}\varphi_\tau^t - I\| \leqslant \mathrm{e}^{M|t-\tau|} - 1,$$

which implies, for $|t-\tau| \leqslant 1/M$, that

$$\|\mathrm{d}\varphi_\tau^t - I\| \leqslant 2M|t-\tau|, \tag{M}$$

or componentwise (taking $\tau = 0$, and assuming that $|t| \leqslant M$):

$$\|\partial_q Q_0^t - I\| \leqslant 2M|t|, \quad \|\partial_p P_0^t - I\| \leqslant 2M|t|, \quad \|\partial_q P_0^t\| \leqslant 2M|t|, \quad \|\partial_p Q_0^t\| \leqslant 2M|t|.$$

We can now prove the following.

THEOREM 1.9. *Let $H\colon \mathbb{R} \times \mathbb{R}^d \times (\mathbb{R}^d)^* $ be a C^2 Hamiltonian satisfying hypothesis 1.8. Let u_0 be a $C^{1,1}$ initial condition. There exist a time $T > 0$ and a $C^{1,1}_{\mathrm{loc}}$ solution $u(t,q)\colon]-T,T[$ of (HJ) such that $u(0,q) = u_0(q)$. Moreover, we can take*

$$T = (4M(1 + \mathrm{Lip}(\mathrm{d}u_0)))^{-1},$$

and we have

$$\mathrm{Lip}(\mathrm{d}u_t) \leqslant \mathrm{Lip}(\mathrm{d}u_0) + 4|t|M(1 + \mathrm{Lip}(\mathrm{d}u_0))^2,$$

when $|t| \leqslant T$. If the initial condition u_0 is C^2, then so is the solution $u(t,q)$.

Proof. Let (Γ_0, w_0) be the geometric initial condition associated to u_0, and let (G, w) be the geometric solution emanating from (Γ_0, w_0). We first prove that the restriction of G to $]-T,T[$ is a graph. It is enough to prove that the map

$$F(t,q) := (t, Q_0^t(q, \mathrm{d}u_0(q)))$$

is a bi-Lipschitz homeomorphism of $]-T,T[$. By (M), we have

$$\mathrm{Lip}(F - \mathrm{Id}) \leqslant 2|t|M(1 + \mathrm{Lip}(\mathrm{d}u_0)) < 1,$$

provided $|t| < (2M(1 + \mathrm{Lip}(\mathrm{d}u_0)))^{-1}$. We conclude using the classical proposition A.1 that F is a bi-Lipschitz homeomorphism of $]-T,T[$. Moreover, if u_0 is C^2, then F is a C^1 diffeomorphism. Since F is a homeomorphism preserving t, we can denote by $(t, Z(t,q))$ its inverse. By proposition A.1, we have

$$\mathrm{Lip}(Z) \leqslant \frac{1}{1 - 2|t|M(1 + \mathrm{Lip}(\mathrm{d}u_0))},$$

and, under the assumption that $|t| \leqslant T$ (as defined in the statement), we obtain

$$\mathrm{Lip}(Z) \leqslant 1 + 4M|t|(1 + \mathrm{Lip}(\mathrm{d}u_0)) \leqslant 2.$$

We have just used here that $(1-a)^{-1} \leqslant 1 + 2a$ for $a \in [0, \tfrac{1}{2}]$. We set

$$\chi(t,q) = P_0^t(Z(t,q), \mathrm{d}u_0(Z(t,q)))$$

in such a way that G is the graph of χ on $]-T,T[$. Observing that χ is Lipschitz, we conclude from corollary 1.7 that the function $u(t,q) := w(t,q,\chi(t,q))$ solves (HJ).

Moreover, we have $u(0,q)=u_0(q)$. Corollary 1.7 also implies that $\mathrm{d}u_t=\chi_t$; hence, in view of (M), we have

$$\begin{aligned}\mathrm{Lip}(\mathrm{d}u_t) &= \mathrm{Lip}(\chi_t) \leqslant 2M|t|\,\mathrm{Lip}(Z_t) + (1+2M|t|)\,\mathrm{Lip}(\mathrm{d}u_0)\,\mathrm{Lip}(Z_t)\\ &\leqslant 4M|t| + \mathrm{Lip}(\mathrm{d}u_0) + \mathrm{Lip}(\mathrm{d}u_0)(4M|t|(1+\mathrm{Lip}(\mathrm{d}u_0))) + 4M|t|\,\mathrm{Lip}(\mathrm{d}u_0)\\ &\leqslant \mathrm{Lip}(\mathrm{d}u_0) + 4M|t|(1+\mathrm{Lip}(\mathrm{d}u_0))(1+\mathrm{Lip}(\mathrm{d}u_0)).\end{aligned}$$

□

1.1. Exercise

Take $d=1$, $H(t,q,p)=\frac{1}{2}p^2$ and $u_0(q)=-q^2$, and prove that the C^2 solution cannot be extended beyond $t=\frac{1}{2}$.

2. Convexity, the twist property, and the generating function

We make an additional assumption on H. Once again, we make the assumption in a stronger form than would be necessary; this allows us to obtain simpler statements.

HYPOTHESIS 2.1. *There exists $m>0$ such that*

$$\partial^2_{pp}H \geqslant m\,\mathrm{Id}$$

for each (t,q,p), in the sense of quadratic forms.

Let us first study the consequences of this hypothesis on the structure of the flow.

PROPOSITION 2.2. *There exists $\sigma>0$ such that the map $p\mapsto Q_0^t(q,p)$ is $(\frac{1}{2}mt)$-monotone when $t\in\,]0,\sigma]$, in the sense that the inequality*

$$(Q_0^t(q,p')-Q_0^t(q,p))\cdot(p'-p) \geqslant \tfrac{1}{2}mt|p'-p|^2$$

holds for each $q\in\mathbb{R}^d$ and each $t\in[0,\sigma]$. As a consequence, it is a C^1 diffeomorphism onto $\mathbb{R}^d$.

We say that the flow has the *twist property.*

Proof. Fix a point q and denote by F^t the map $p\mapsto Q_0^t(q,p)$. We have $\mathrm{d}F^t(p)=\partial_p Q_0^t(q,p)$. In order to estimate this linear map, we recall the variational equation

$$\partial_p\dot{Q}_0^t(x)=\partial^2_{qp}H(t,\varphi_0^t(x))\partial_pQ_0^t(x)+\partial^2_{pp}H(t,\varphi_0^t(x))\partial_pP_0^t(x).$$

We deduce that

$$\partial_p\dot{Q}_0^t(x)-\partial^2_{pp}H(t,\varphi_0^t(x))=\partial^2_{qp}H(t,\varphi_0^t(x))\partial_pQ_0^t(x)+\partial^2_{pp}H(t,\varphi_0^t(x))(\partial_pP_0^t(x)-\mathrm{Id})$$

and then that

$$\|\partial_p\dot{Q}_0^t(x)-\partial^2_{pp}H(t,\varphi_0^t(x))\|\leqslant 2M^2t.$$

As a consequence, for $t\leqslant\sigma=m/(4M^2)$, we have

$$\partial_p\dot{Q}_0^t\geqslant(m-2M^2t)I\geqslant(\tfrac{1}{2}m)\,\mathrm{Id}$$

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

in the sense of quadratic forms (note that the matrix $\partial_p \dot{Q}_0^t$ is not necessarily symmetric). Since

$$\partial_p Q_0^t(x) = \int_0^t \partial_p \dot{Q}_0^s(x)\,\mathrm{d}s,$$

we conclude that

$$dF^t(p) = \partial_p Q_0^t(q,p) \geqslant (\tfrac{1}{2}m)\,\mathrm{Id},$$

which means that $(\mathrm{d}F^t(p)z, z) \geqslant (\frac{1}{2}m)|z|^2$ for each $z \in \mathbb{R}^{d*}$. This estimate can be integrated, and implies the monotony of the map F^t:

$$\begin{aligned}
(Q^t(q,p') - Q^t(q,p)) \cdot (p'-p) \\
&= \bigg(\int_0^1 \partial_p Q^t(q, p + s(p'-p)) \cdot (p'-p)\,\mathrm{d}s\bigg) \cdot (p'-p) \\
&= \int_0^1 (\partial_p Q^t(q, p+s(p'-p)) \cdot (p'-p))\,\mathrm{d}s \\
&\geqslant \int_0^1 (\tfrac{1}{2}m)t(p'-p)\cdot(p'-p)\,\mathrm{d}s \\
&\geqslant (\tfrac{1}{2}m)t(p'-p)\cdot(p'-p).
\end{aligned}$$

It is then a classical result that the map F^t is a C^1 diffeomorphism; see proposition A.2. □

Corollary 2.3. *The map $(t,q,p) \mapsto (t,q,Q_0^t(q,p))$ is a C^1 diffeomorphism from $]0,\sigma[$ onto its image $]0,\sigma[$.*

We denote by $\rho_0(t,q_0,q_1)$ the unique momentum p such that

$$Q_0^t(q_0, \rho_0(t,q_0,q_1)) = q_1.$$

In other words, $\rho_0(t,q_0,q_1)$ is the initial momentum $p(0)$ of the unique orbit

$$(q(s),p(s))\colon [0,t] \to \mathbb{R}^d \times \mathbb{R}^{d*}$$

of (HS) that satisfies $q(0) = q_0$ and $q(t) = q_1$. By the corollary 2.3, the map ρ_0 is C^1. Similarly, we denote by $\rho_1(t,q_0,q_1)$ the unique momentum p such that $Q_t^0(q_1, \rho_1(t,q_0,q_1)) = q_0$. We can equivalently define ρ_1 by

$$\rho_1(t,q_0,q_1) = P_0^t(q_0, \rho_0(t,q_0,q_1)).$$

Considering the geometric initial condition $(\Gamma_0 = \{q_0\} \times \mathbb{R}^{d*}, w_0 = 0)$, and the associated geometric solution (G,w), we see that

$$G = \{(t,q,\rho_1(t,q_0,q)),\ (t,q) \in\,]0,\sigma[\, \times \mathbb{R}^d\}.$$

We conclude from corollary 1.7 that there exists a genuine solution of (HJ) emanating from the geometric initial condition $(\{q_0\} \times \mathbb{R}^{d*}, 0)$. We denote this solution by $S^t(q_0,q)$. We have

$$S^t(q_0,q) = w(t,p,\rho_1(t,q_0,q))$$

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

and

$$\partial_q S^t(q_0, q) = \rho_1(t, q_0, q).$$

In view of the definition of geometric solutions, the function S can be written more explicitly:

$$S^t(q_0, q_1) = \int_0^t P_0^s(q_0, \rho_0(t, q_0, q_1))\dot{Q}_0^s(q_0, \rho_0(t, q_0, q_1)) - H(s, \varphi_0^s(q_0, \rho_0(t, q_0, q_1)))\,\mathrm{d}s.$$

In words, $S^t(q_0, q_1)$ is the action of the unique trajectory $(q(s), p(s))\colon [0, t] \to \mathbb{R}^d \times \mathbb{R}^{d*}$ of (HS) that satisfies $q(0) = q_0$ and $q(t) = q_1$.

We have defined the function $S^t(q_0, q_1)$ as the action of the unique orbit joining q_0 and q_1 between time 0 and time t. We can similarly define the function $S^t_\tau(q_0, q_1)$ as the action of the unique orbit joining q_0 to q_1 between time τ and time t, all this being well defined, provided $0 < t - \tau < \sigma$. It is possible to prove as above that the function $(s, q) \mapsto S^t_s(q, q_1)$ solves the Hamilton–Jacobi equation

$$\partial_s u + H(t, q, -\partial_q u) = 0,$$

on $s < t$, and that

$$\partial_q S^t(q, q_1) = \partial_q S^t_0(q, q_1) = -\rho_0(t, q, q_1).$$

Convention. We shall from now on denote by $\partial_0 S^t$ the partial differential with respect to the first variable (which in our notation is often q_0), and by $\partial_1 S^t$ the partial differential with respect to the second variable (which in our notation is often q_1).

The relations $\partial_0 S = -\rho_0$, $\partial_1 S = \rho_1$, $\partial_t S = -H(t, q_1, \rho_1) = -H(0, q_0, \rho_0)$ that we have proved imply that the function S is C^2. Moreover, since $\varphi_0^t(q_0, \rho_0(t, q_0, q_1)) = (q_1, \rho_1(t, q_0, q_1))$, we have

$$\varphi_0^t(q_0, -\partial_0 S(q_0, q_1)) = (q_1, \partial_1 S^t(q_0, q_1)).$$

We say that S^t is a *generating function* of the flow map φ_0^t. See [18, ch. 9] for more material on generating functions. It is useful to estimate the second differentials of S.

Lemma 2.4. *The function S is C^2 on $]0, \sigma[$, and the estimates*

$$\begin{aligned} \partial^2_{00} S^t &\geqslant \frac{c}{t}\,\mathrm{Id}, \\ \partial^2_{11} S^t &\geqslant \frac{c}{t}\,\mathrm{Id}, \\ \|\partial^2_{00} S^t\| + \|\partial^2_{01} S^t\| + \|\partial^2_{01} S^t\| &\leqslant \frac{C}{t} \end{aligned}$$

hold, with constants c and C which depend only on m and M.

Proof. Let us first observe that

$$\partial^2_{11} S^t(q_0, q_1) = (\partial_p P_0^t(q_0, \rho_0(t, q_0, q_1)))(\partial_p Q_0^t(q_0, \rho_0(t, q_0, q_1)))^{-1},$$

and recall the estimates

$$\|\partial_p P_0^t - \mathrm{Id}\,\| \leqslant 2Mt, \qquad \|\partial_p Q_0^t\| \leqslant 2Mt, \qquad \partial_p Q_0^t \geqslant (\tfrac{1}{2}mt)\,\mathrm{Id}\,.$$

We conclude that (see lemma A.3)

$$(\partial_p Q_0^t)^{-1} \geqslant \frac{m}{8M^2 t}\,\mathrm{Id}, \qquad \|(\partial_p Q_0^t)^{-1}\| \leqslant \frac{2}{mt}.$$

Finally, we obtain that

$$\partial_{11}^2 S(q_0, q_1) \geqslant \left(\frac{m}{8M^2 t} - \frac{4M}{m}\right)\mathrm{Id} \geqslant \frac{m}{16M^2 t}\,\mathrm{Id}$$

provided $t \leqslant m^2/(64M^3)$. The other estimates can be proved similarly, using the expressions

$$\partial_{00}^2 S^t(q_0, q_1) = -(\partial_p P_t^0(q_1, \rho_1(t, q_0, q_1)))(\partial_p Q_t^0(q_1, \rho_1(t, q_0, q_1)))^{-1},$$
$$\partial_{10}^2 S^t(q_0, q_1) = (\partial_p Q_0^t(q_0, \rho_0(t, q_0, p_0)))^{-1}.$$

□

Proposition 2.5. *Given times t_1 and t_2 such that $0 < t_1 < t_2 < \sigma$, we have the triangle inequality*

$$S_0^{t_2}(q_0, q_2) \leqslant S_0^{t_1}(q_0, q_1) + S_{t_1}^{t_2}(q_1, q_2)$$

for each q_0, q_1, q_2. Moreover, $S_0^{t_2}(q_0, q_2) = \min_q(S_0^{t_1}(q_0, q) + S_{t_1}^{t_2}(q, q_2))$.

Proof. Let us consider the map

$$q \mapsto f(q) = S_0^{t_1}(q_0, q) + S_{t_1}^{t_2}(q, q_2).$$

We have $\mathrm{d}^2 f \geqslant 2c$; hence, the map f is convex. Now let us denote by

$$(q(s), p(s))\colon [0, t_2] \to \mathbb{R}^d \times \mathbb{R}^{d*}$$

the unique orbit that satisfies $q(0) = q_0$ and $q(t_2) = q_2$. We can compute

$$\mathrm{d}f(q(t_1)) = \partial_1 S_0^{t_1}(q_0, q(t_1)) + \partial_0 S_{t_1}^{t_2}(q(t_1), q_2) = p(t_1) - p(t_1) = 0.$$

The point $q(t_1)$ is thus a critical point of the convex function f; hence, it is a minimum of this function. We conclude that

$$S_0^{t_1}(q_0, q) + S_{t_1}^{t_2}(q, q_2) \geqslant S_0^{t_1}(q_0, q(t_1)) + S_{t_1}^{t_2}(q(t_1), q_2) = S_0^{t_2}(q_0, q_2)$$

for all q. □

Under the convexity hypothesis (hypothesis 2.1), theorem 1.2 can be extended to C^1 solutions, as follows.

Theorem 2.6. *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ be an open set, and let $u(t, q)\colon \Omega \to \mathbb{R}$ be a C^1 solution of the Hamilton–Jacobi equation (HJ). Let $q(t)\colon [t_0, t_1] \to \mathbb{R}^d$ be a C^1 curve such that $(t, q(t)) \in \Omega$ and*

$$\dot{q}(t) = \partial_p H(q(t), \partial_q u(t, q(t)))$$

for each $t \in [t_0, t_1]$. Then, on setting $p(t) = \partial_q u(t, q(t))$, the curve $(q(t), p(t))$ solves (HS).

Proof. As in the proof of theorem 1.2, we consider a variation $q(t,s)=q(t)+s\theta(t)$ of $q(t)$, where θ is smooth and vanishes on the endpoints. We choose the vertical variation $p(t,s)$ in such a way that the equation

$$\dot q(t,s)=\partial_p H(t,q(t,s),p(t,s))$$

holds. The map $p(t,s)$ defined by this relation is differentiable in s, because q and $\dot q$ are differentiable in s and because the matrix $\partial^2_{pp}H$ is invertible. It is also useful to consider the other vertical variation:

$$P(t,s):=\partial_q u(t,q(t,s)).$$

Our hypothesis is that $\dot q(t)=\partial_p H(t,q(t),p(t))$, which is the first part of (HS). We start as in the proof of theorem 1.2 with the following equality:

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\biggl(\int_{t_0}^{t_1} p(t,s)\cdot\dot q(t,s)-H(t,q(t,s),p(t,s))\,\mathrm{d}t\biggr)=0. \tag{2.1}$$

We deduce this equality from the observation that $s=0$ is a local minimum of the function

$$s\mapsto F(s):=\int_{t_0}^{t_1} p(t,s)\cdot\dot q(t,s)-H(t,q(t,s),p(t,s))\,\mathrm{d}t.$$

This claim follows from the equality

$$F(0)=u(t_1,q(t_1))-u(t_0,q(t_0))=\int_{t_0}^{t_1} P(t,s)\cdot\dot q(t,s)-H(t,q(t,s),P(t,s))\,\mathrm{d}s,$$

which holds for all s, and from the inequality

$$F(s)\geqslant\int_{t_0}^{t_1} P(t,s)\cdot\dot q(t,s)-H(t,q(t,s),P(t,s))\,\mathrm{d}s$$

that results, in view of the convexity of H, from the computation

$$\begin{aligned}
&H(t,q(t,s),P(t,s))\\
&\qquad\geqslant (P(t,s)-p(t,s))\cdot\partial_p H(t,q(t,s),p(t,s))+H(t,q(t,s),p(t,s))\\
&\qquad\geqslant (P(t,s)-p(t,s))\cdot\dot q(t,s)+H(t,q(t,s),p(t,s)).
\end{aligned}$$

We have proved (2.1). As in the proof of theorem 1.2, we develop the left-hand side and, after a simplification, we get

$$\int_{t_0}^{t_1} p(t)\cdot\dot\theta(t)-\partial_q H(t,q(t),p(t))\cdot\theta(t)\,\mathrm{d}t=0.$$

In other words, we have proved that $\dot p(t)=\partial_q H(t,q(t),p(t))$ in the sense of distributions. Since the right-hand side is continuous, p is C^1 and the equality holds in the genuine sense. □

As in the C^2 case, we have the following corollary (see [12]).

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

Corollary 2.7. *Let $u(t,q)\colon]t_0,t_1[$ be a C^1 solution of (HJ). Then, for each s and t in $]t_0,t_1[$ we have*

$$\Gamma_t = \varphi_s^t(\Gamma_s),$$

where Γ_t is defined by

$$\Gamma_t := \{(q, \mathrm{d}u_t(q))\colon q \in \mathbb{R}^d\}.$$

Proof. This corollary follows from theorem 2.6 in the same way as corollary 1.4 follows from theorem 1.2. The only difference here is that the map

$$F(t,q) := \partial_p H(t,q,\partial_q u(t,q))$$

is only continuous. By the Cauchy–Peano theorem, this is sufficient to imply the existence of solutions to the associated differential equation, which is what we need to develop the argument. □

Another property of the functions S will be useful. Assume that we are considering a family $H_\mu, \mu \in I$ of Hamiltonians, where $I \subset \mathbb{R}$ is an interval, such that the whole function $H(\mu,t,q,p)$ is C^2 and such that each of the Hamiltonians H_μ satisfy our hypotheses 1.8 and 2.1, with uniform constants m and M. Then, for each value of μ, we have the function $S^t(\mu;q_0,q_1)$, which is defined for $t \in]0,\sigma]$, the bound $\sigma > 0$ being independent of μ. Since everything we have done so far has been based on the local inversion theorem, the function $S^t(\mu;q_0,q_1)$ is C^1 in μ, or, more precisely, the function $(\mu,t,q_0,q_1) \mapsto S^t(\mu;q_0,q_1)$ is C^1. Moreover, a computation similar to the proof of proposition 1.1 yields

$$\partial_\mu S^t(\mu;q_0,q_1) = -\int_0^t \partial_\mu H_\mu(s,q(\mu,s),p(\mu,s))\,\mathrm{d}s,$$

where $s \mapsto (q(\mu,s),p(\mu,s))$ is the only H_μ-trajectory satisfying $q(\mu,0) = q_0$ and $q(\mu,t) = q_1$. We can exploit this remark when H_μ is the linear interpolation $H_\mu = H_0 + \mu(H_1 - H_0)$ between two Hamiltonians H_0 and H_1, and conclude the important monotony property:

$$H_0 \leqslant H_1 \quad \Longrightarrow \quad S^t(0;q,q') \geqslant S^t(1;q,q'). \qquad \text{(monotone)}$$

2.1. Exercise

If $H(t,q,p) = h(p)$ is a function of p, then

$$S^t(q_0,q_1) = th^*\left(\frac{q_1 - q_0}{t}\right),$$

where h^* is the Legendre transform of h. As an example, when $H(t,q,p) = \frac{1}{2}a|p|^2$, we have

$$S^t(q_0,q_1) = \frac{1}{2ta}|q_1 - q_0|^2.$$

3. Extension of the generating function: the minimal action

A classical problem consists in finding an orbit $(q(t), p(t))$ of the Hamiltonian system such that $q(t_0) = q_0$ and $q(t_1) = q_1$, for given $[t_0, t_1] \subset \mathbb{R}$, $q_0, q_1 \in \mathbb{R}^d$. We have seen, under hypotheses 1.8 and 2.1, that this problem has a unique solution, provided $t_0 < t_1 < t_0 + \sigma$, where σ is a constant depending only on m and M. The situation is more subtle for larger values of $t_1 - t_0$. In order to study it, it is useful to consider the function

$$\mathfrak{S}\colon (\theta_1, \dots, \theta_{n-1}) \mapsto S_0^{t/n}(q_0, \theta_1) + S_{t/n}^{2t/n}(\theta_1, \theta_2) + \dots + S_{(n-1)t/n}^{t}(\theta_{n-1}, q_1),$$

where we have taken $t_0 = 0$ and $t_1 = t$ to simplify notation, and where n is an integer such that $t/n \leqslant \sigma$. The critical points of $\mathfrak{S}$ are in one-to-one correspondence with the solutions of our problem.

Lemma 3.1. *The point $(\theta_1, \dots, \theta_{n-1})$ is a critical point of $\mathfrak{S}$ if and only if there exists an orbit $(q(s), p(s))\colon [0, t] \to \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $q(0) = q_0$, $q(t) = q_1$, and $q(it/n) = \theta_i$ for $i = 1, \dots, n-1$. This orbit is then unique, and its action is $\mathfrak{S}(\theta_1, \dots, \theta_{n-1})$.*

Proof. Let $(q(s), p(s))$ be the piecewise orbit defined on $[it/n, (i+1)t/n]$ by the constraints $q(it/n) = \theta_i$ and $q((i+1)t/n) = \theta_{i+1}$. The action of this piecewise orbit is $\mathfrak{S}(\theta_1, \dots, \theta_{n-1})$. The statement follows from the simple computation

$$\partial_{\theta_i}\mathfrak{S} = \partial_1 S^{t/n}(\theta_{i-1}, \theta_i) + \partial_0 S^{t/n}(\theta_i, \theta_{i+1}) = p^-(it/n) - p^+(it/n).$$

□

Using this finite-dimensional variational functional is usually called the method of broken geodesics (see [9]). The function $\mathfrak{S}$ can be minimized under additional assumptions, for example, as follows.

Hypothesis 3.2.

$$\tfrac{1}{2}m|p|^2 - M \leqslant H(t, q, p) \leqslant \tfrac{1}{2}M|p|^2 + M.$$

By exploiting the monotony property (monotone), this hypothesis implies that

$$\frac{1}{2tM}|q_1 - q_0|^2 - Mt \leqslant S^t(q_0, q_1) \leqslant \frac{1}{2tm}|q_1 - q_0|^2 + Mt,$$

and then that

$$\mathfrak{S}(\theta_1, \dots, \theta_{n-1}) \geqslant \frac{n}{2tM}(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + \dots + |q_1 - \theta_{n-1}|^2) - Mt.$$

As a consequence, the function $\mathfrak{S}$ is coercive and C^2; hence, it has a minimum. Note that, although $\mathfrak{S}$ is convex separately in each of its variables, it is not jointly convex. It can have critical points which are not minima, and it can have several different minima. We denote by A^t the value function

$$\begin{aligned} A^t(q_0, q_1) &= \min \mathfrak{S} \\ &= \min_{\theta_1, \theta_2, \dots, \theta_{n-1}} (S_0^{t/n}(q_0, \theta_1) + S_{t/n}^{2t/n}(\theta_1, \theta_2) + S_{(n-1)t/n}^{t}(\theta_{n-1}, q_1)), \end{aligned} \tag{A}$$

where n is any integer such that $t/n < \sigma$. The functions $A^t_\tau(q_0, q_1)$ are defined similarly for each $t \geqslant \tau$. This notation is legitimate in view of the following.

LEMMA 3.3. *The value of A^t does not depend on n provided $t/n < \sigma$. Moreover, we have*

$$\frac{1}{2Mt}|q_1 - q_0|^2 - Mt \leqslant A^t(q_0, q_1) \leqslant \frac{1}{2tm}|q_1 - q_0|^2 + Mt.$$

This statement implies that $A^t = S^t$ when $t < \sigma$; hence, A^t can be seen as an extension of S^t beyond $t = \sigma$.

Proof. Since we have not yet proved the independence of n, we temporarily denote by $A^t(q_0, q_1; n)$ the value of the minimum. We have

$$\begin{aligned} A^t(q_0, q_1; n) &\geqslant \min_{\theta_1, \theta_2, \dots, \theta_{n-1}} \left(\frac{n}{2Mt}(|\theta_1 - q_0|^2 + \cdots + |q_1 - \theta_{n-1}|^2) - Mt \right) \\ &= \frac{1}{2Mt}|q_1 - q_0|^2 - Mt. \end{aligned}$$

If $t < \sigma$, then the equality $S^t(q_0, q_1) = A^t(q_0, q_1; n)$ can be proved by recurrence for each n using proposition 2.5. For general t, let us prove that $A^t(n)$ is independent of n. We take two integers n and m such that $t/n < \sigma, t/m < \sigma$ and want to prove that $A^t(n) = A^t(m)$. We shall prove that $A^t(n) = A^t(nm) = A^t(m)$. Since $t/m < \sigma$, we have

$$A^{\tau + t/m}_\tau(q_0, q_1; n) = S^{\tau + t/m}_\tau(q_0, q_1)$$

for each τ and n; hence,

$$\begin{aligned} &A^t(q_0, q_1; nm) \\ &= \min_{\theta_1, \theta_2, \dots, \theta_{nm-1}} [S_0^{t/nm}(q_0, \theta_1) + S^{2t/mn}_{t/nm}(\theta_1, \theta_2) + \cdots + S^{t/m}_{(n-1)t/nm}(\theta_{n-1}, \theta_n) \\ &\qquad + S^{(n+1)t/nm}_{t/m}(\theta_n, \theta_{n+1}) + \cdots + S^{2t/m}_{(2n-1)t/nm}(\theta_{2n-1}, \theta_{2n}) + \cdots \\ &\qquad + S^{(m-1)t/m + t/nm}_{(m-1)t/m}(\theta_{(m-1)n}, \theta_{(m-1)n+1}) + \cdots \\ &\qquad + S^t_{(1-1/nm)t}(\theta_{mn-1}, q_1)] \\ &= \min_{\theta_{2n}, \theta_{3n}, \dots, \theta_{(m-1)n}} [S_0^{t/m}(q_0, \theta_n) + S^{2t/m}_{t/m}(\theta_n, \theta_{2n}) + \cdots + S^t_{(m-1)t/m}(\theta_{(m-1)n}, q_1)] \\ &= A^t(q_0, q_1; m). \end{aligned}$$

We have proved that $A^t(nm) = A^t(m)$; by symmetry we also have $A^t(nm) = A^t(n)$. Hence, $A^t(n) = A^t(m)$. Finally, we have

$$\mathfrak{S}(\theta_1, \dots, \theta_{n-1}) \leqslant \frac{n}{2mt}(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + |q_1 - \theta_{n-1}|^2) + Mt;$$

hence,

$$\begin{aligned} A^t(q_0, q_1) &\leqslant \min_{\theta_1, \theta_2, \dots, \theta_{n-1}} \frac{n}{2mt}(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + |q_1 - \theta_{n-1}|^2) + Mt \\ &= \frac{1}{2mt}|q_1 - q_0|^2 + Mt. \end{aligned}$$

□

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

The following property concerning A follows easily from the definition:

$$A_{t_0}^{t_2}(q_0, q_2) = \min_{q_1}(A_{t_0}^{t_1}(q_0, q_1) + A_{t_1}^{t_2}(q_1, q_2)), \tag{T}$$

when $0 \leqslant t_0 \leqslant t_1 \leqslant t_2$. The following consequence of hypothesis 3.2 will also be useful.

Lemma 3.4.

$$p \cdot \partial_p H(t, q, p) - H(t, q, p) \geqslant \frac{m}{M} H(t, q, p) - (m + M).$$

Proof. We deduce from hypothesis 2.1 that

$$H(t, q, 0) \geqslant H(t, q, p) - p \cdot \partial_p H(t, q, p) + \tfrac{1}{2} m|p|^2.$$

We deduce that

$$p \cdot \partial_p H(t, q, p) - H(t, q, p) \geqslant \tfrac{1}{2} m|p|^2 - H(t, q, 0) \geqslant \frac{m}{M}(H(t, q, p) - M) - M.$$

□

The minimal action $A^t(q_0, q_1)$ is not necessarily C^1; we need some definitions before we can study its regularity. The linear form l is called a K-superdifferential of the function u at point q if the inequality

$$u(\theta) \leqslant u(q) + l(\theta - q) + K|\theta - q|^2$$

holds in a neighbourhood of q. The linear form l is a proximal superdifferential of u at point q if it is a K-superdifferential for some K. The form l is a proximal superdifferential of u at q if and only if there exists a C^2 function v such that $\mathrm{d}v(q) = l$ and such that the difference $v - u$ has a minimum at q. More generally, we shall say that l is a superdifferential of u at q if there exists a C^1 function v such that $\mathrm{d}v(q) = l$ and such that the difference $v - u$ has a minimum at q. A superdifferential is not necessarily a proximal superdifferential.

A function $u\colon \mathbb{R}^d \to \mathbb{R}$ is called K-semi-concave if it admits a K-superdifferential at each point. It is equivalent to requiring that the function $\theta \mapsto u(\theta) - K|\theta|^2$ is concave. A function is called semi-concave if it is K-semi-concave for some K. If u is a K-semi-concave function, and if l is a superdifferential at u, then the inequality

$$u(\theta) \leqslant u(q) + l(\theta - q) + K|\theta - q|^2$$

holds for each θ. In particular, l is a K-superdifferential.

Lemma 3.5. *The function A^t is $C(1 + 1/t)$-semi-concave, with some constant C that depends only on m and M.*

Proof. Let us first assume that $t \in]0, \sigma[$. In this case, $A_0^t = S_0^t$, this function is C^2 and its second derivative was estimated in lemma 2.4. Let us now assume that $t \geqslant \sigma$. Then, there exists $n \in \mathbb{N}$ such that $t/n \in [\frac{1}{3}\sigma, \frac{1}{2}\sigma[$. We have

$$A_0^t(q, q') = \min_{\theta, \theta'}(S_0^{t/n}(q, \theta) + A_{t/n}^{t-t/n}(\theta, \theta') + S_{t-t/n}^t(\theta', q')).$$

Considering a minimizing pair (θ_0, θ_1) in the expression above at (q_0, q_1), we see that the C^2 function

$$(q, q') \mapsto S_0^{t/n}(q, \theta_0) + A_{t/n}^{t-t/n}(\theta_0, \theta_1) + S_{t-t/n}^{t}(\theta_1, q')$$

is touching from above the function A_0^t at point (q_0, q_1). In view of lemma 2.4, this provides a uniform (for $t \geqslant \sigma$) semi-concavity constant for A_0^t. □

4. The Lax–Oleinik operators

Given $t_0 < t_1$, we define the Lax–Oleinik operators $\boldsymbol{T}_{t_0}^{t_1}$ and $\check{\boldsymbol{T}}_{t_1}^{t_0}$, which, to each function $u\colon \mathbb{R}^d \to \mathbb{R}$, associate the functions

$$\boldsymbol{T}_{t_0}^{t_1} u(q) := \inf_{\theta \in \mathbb{R}^d} (u(\theta) + A_{t_0}^{t_1}(\theta, q)), \qquad \check{\boldsymbol{T}}_{t_1}^{t_0} u(q) := \sup_{\theta \in \mathbb{R}^d} (u(\theta) - A_{t_0}^{t_1}(q, \theta)).$$

We have the Markov (or semi-group) property:

$$\boldsymbol{T}_{t_1}^{t_2} \circ \boldsymbol{T}_{t_0}^{t_1} = \boldsymbol{T}_{t_0}^{t_2}, \qquad \check{\boldsymbol{T}}_{t_1}^{t_0} \circ \check{\boldsymbol{T}}_{t_2}^{t_1} = \check{\boldsymbol{T}}_{t_2}^{t_0}$$

for $t_0 < t_1 < t_2$. Note, however, that $\boldsymbol{T}_{t_0}^{t_1} \circ \check{\boldsymbol{T}}_{t_1}^{t_0}$ and $\check{\boldsymbol{T}}_{t_1}^{t_0} \circ \boldsymbol{T}_{t_0}^{t_1}$ are not the identity. Concerning these operators, we only have the inequalities

$$\check{\boldsymbol{T}}_{t_1}^{t_0} \circ \boldsymbol{T}_{t_0}^{t_1}(u) \leqslant u, \qquad \boldsymbol{T}_{t_0}^{t_1} \circ \check{\boldsymbol{T}}_{t_1}^{t_0}(u) \geqslant u,$$

the easy proof of which is left to the reader. Each property concerning the Lax–Oleinik operator $\boldsymbol{T}$ has a counterpart for the dual operator $\check{\boldsymbol{T}}$, which we shall not always bother to state but never hesitate to use. The family of operators $\boldsymbol{T}_{t_0}^{t_1}$ is characterized by the fact that $\boldsymbol{T}_{t_0}^{t_1} u(q) = \inf_\theta (u(\theta) + S_{t_0}^{t_1}(q_0, q_1))$ when $t_0 \leqslant t_1 \leqslant t_0 + \sigma$ and by the Markov property. The Lax–Oleinik operators solve (HJ) in various important ways that will be detailed in this section. It is useful first to settle some regularity issues.

LEMMA 4.1. *There exists a constant C, depending only on m and M, such that for each $t \in]0, \sigma]$, the function $\boldsymbol{T}^t u$ is (C/t)-semi-concave provided it is finite at each point.*

Proof. The function $\boldsymbol{T}^t u$ is the infimum of the functions $f = u(\theta) + S^t(\theta, \cdot)$, which are C^2 with the uniform bound $\|\mathrm{d}^2 f\| \leqslant C/t$. It is then an easy exercise to conclude that the function $\boldsymbol{T}^t u$ is C/t-semi-concave; see lemma A.5. □

Given an arbitrary function u_0, the infimum in the definition of $\boldsymbol{T}_0^t u_0$ is not necessarily finite, and, even if it is finite, it is not necessarily a minimum. It is clear from proposition 3.3 that the infimum is a finite minimum under the assumption that u_0 is continuous and *Lipschitz in the large*, which means that there exists a constant k such that

$$u_0(q') - u_0(q) \leqslant k(1 + |q' - q|)$$

for each q and q'.

LEMMA 4.2. *If u_0 is Lipschitz in the large, then so are the functions $\boldsymbol{T}_0^t u_0$ for all $t \geqslant 0$. The function $(t, q) \mapsto u(t, q) = \boldsymbol{T}_0^t u_0(q)$ is locally semi-concave, and hence locally Lipschitz on $]0, \infty) \times \mathbb{R}^d$. The function u solves (HJ) at all its points of differentiability (hence almost everywhere).*

Proof. Since u_0 is Lipschitz in the large, the function $\boldsymbol{T}_0^t u_0 - u_0$ is bounded for each $t > 0$, as follows from the inequalities

$$\inf_\theta (u_0(q) - k - k|\theta - q| + S^t(\theta, q)) \leqslant \boldsymbol{T}_0^t u_0 \leqslant u_0(q) + S^t(q, q),$$

which imply (setting $\Delta = \theta - q$) that

$$\inf_{\Delta \in \mathbb{R}^d} \left(-k - k|\Delta| + \frac{1}{2tM}|\Delta|^2 - tM \right) \leqslant \boldsymbol{T}_0^t u_0(q) - u_0(q) \leqslant Mt.$$

We conclude that the function $\boldsymbol{T}_0^t u_0 = (\boldsymbol{T}_0^t u_0 - u_0) + u_0$ is Lipschitz in the large. In the computations above, we also see that the infimum can be taken on $|\Delta| \leqslant K$, where K is a constant independent from q.

Let us now prove that the function $u(t, q) := \boldsymbol{T}_0^t u_0(q)$ is locally Lipschitz on $t > 0$. In view of the Markov property, it is enough to prove that the function u is Lipschitz on $]\tau, \frac{1}{2}\sigma[$ for each closed ball $B \subset \mathbb{R}^d$ and each time $\tau \in]0, \frac{1}{2}\sigma[$. Since $u(q)$ is Lipschitz in the large, there exists a radius $R > 0$ such that

$$u(t, q) = \inf_{|\theta| \leqslant R} u(\theta) + S^t(\theta, q)$$

for $(t, q) \in]\tau, \frac{1}{2}\sigma[$. Since S is C^2, the functions $(t, q) \mapsto u(\theta) + S^t(\theta, q), |\theta| \leqslant R$ have uniform C^2 bounds on $]\tau, \frac{1}{2}\sigma[$. Their infimum $u(t, q)$ is then semi-concave, and hence Lipschitz on that set; see lemma A.5.

Finally, let (t, q) be a point of differentiability of u, and let $\tau \in]\max(0, t-\sigma), t[$ be given. Since u_τ is Lipschitz in the large and locally Lipschitz, there exists θ such that $\boldsymbol{T}_\tau^t u_\tau(q) = u_\tau(\theta) + S_\tau^t(\theta, q)$. For a different point (s, y), we have $\boldsymbol{T}_\tau^s u_\tau(y) \leqslant u_\tau(\theta) + S_\tau^t(\theta, y)$; hence, the function $(s, y) \mapsto u(s, y) - S_\tau^s(\theta, y)$ has a maximum at (t, q), which implies that the functions $u(s, y)$ and $S_\tau^s(\theta, y)$, each of which is differentiable at (t, q), have the same differential at (t, q). Since the functions $(s, y) \mapsto S_\tau^s(\theta, y)$ solve (HJ), the function u also solves (HJ) at (t, q). □

Let us now establish the relation of our operators with regular solutions.

PROPOSITION 4.3. *Let $u(t, q)\colon]t_0, t_1[$ be a C^1 solution of HJ. Then $\boldsymbol{T}_\tau^t u_\tau = u_t$ and $\check{\boldsymbol{T}}_t^\tau u_t = u_\tau$ for each $\tau \leqslant t$ in $]t_0, t_1[$. The function u is locally $C^{1,1}$.*

This property is one of the main motivations to introduce the Lax–Oleinik operators. The observation that C^1 solutions are actually locally $C^{1,1}$ comes from Fathi's paper [12], itself inspired by anterior works of Herman. Another consequence of this Theorem is that uniqueness extends to C^1 solutions under the convexity assumption.

Proof. In view of the Markov property, it is enough to prove the result for $0 < t - \tau < \sigma$. Given q and θ in $\mathbb{T}^d$, we consider the unique orbit $(q(s), p(s))$ such that $q(\tau) = \theta$ and $q(t) = q$. By the convexity of H, we have

$$H(q(s), \partial_q u(s, q(s))) \geqslant H(q(s), p(s)) + (\partial_q u(s, q(s)) - p(s)) \cdot \partial_p H(s, q(s), p(s)).$$

Noting that $\dot{q}(s) = \partial_p H(s, q(s), p(s))$ and integrating gives

$$\begin{aligned} S_\tau^t(\theta, q) &= \int_\tau^t p(s) \cdot \dot{q}(s) - H(s, q(s), p(s))\,\mathrm{d}s \\ &\geqslant \int_\tau^t \partial_q u(s, q(s)) \cdot \dot{q}(s) - H(s, q(s), \partial_q u(s, q(s)))\,\mathrm{d}s \\ &= u(t, q) - u(\tau, \theta), \end{aligned}$$

with equality if $p(s) = \partial_q u(s, q(s))$ for each s. We conclude that

$$\boldsymbol{T}_\tau^t u_\tau(q) \geqslant u_t(q),$$

with equality if there exists an orbit $(q(s), p(s))\colon [\tau, t] \to \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $p(s) = \partial_q u(s, q(s))$ and $q(t) = q$. By corollary 2.7, the orbit of the point $(q, \partial_q u(t, q))$ satisfies this property; hence, the equality holds.

To prove the regularity of u we consider a subinterval $[\tilde{t}_0, \tilde{t}_1] \subset]t_0, t_1[$, and prove that u is locally $C^{1,1}$ on $]\tilde{t}_0, \tilde{t}_1[$. We have

$$u(t, q) = \boldsymbol{T}_{\tilde{t}_0}^t u_{\tilde{t}_0}(q) = \check{\boldsymbol{T}}_t^{\tilde{t}_1} u_{\tilde{t}_1}(q)$$

for each $t \in]\tilde{t}_0, \tilde{t}_1[$. If the functions u_t were Lipschitz in the large, we could apply lemma 4.2 and deduce that u is both locally semi-concave and locally semi-convex, and hence locally $C^{1,1}$, on $]\tilde{t}_0, \tilde{t}_1[$. Here we do not make any growth assumption, so we need a slightly different argument to prove the semi-concavity of u (and, similarly, its semi-convexity). We have seen that the infimum in the definition $\boldsymbol{T}_{\tilde{t}_0}^t u_{\tilde{t}_0}(q)$ is a minimum, which is attained at the point $\theta = Q_t^{\tilde{t}_0}(q, \partial_q u(t, q))$. This gives us an *a priori* bound on θ, and we can continue the proof as in lemma 4.2. □

Let us sum up some properties of the Lax–Oleinik operators $\boldsymbol{T}_\tau^t$ associated to a Hamiltonian satisfying hypotheses 1.8, 2.1 and 3.2.

PROPERTY 4.4.

Markov property: $\boldsymbol{T}_s^t \circ \boldsymbol{T}_\tau^s = \boldsymbol{T}_\tau^t$ when $\tau \leqslant s \leqslant t$.

Monotony: $u \geqslant v \Rightarrow \boldsymbol{T}_\tau^t u \geqslant \boldsymbol{T}_\tau^t v$ for each $t \geqslant \tau$.

Compatibility with (HJ): if $u(t, q)\colon]t_0, t_1[$ is a C^2 solution of (HJ), then $\boldsymbol{T}_\tau^t u_\tau = u_t$ when $t_0 < \tau < t < t_1$.

Boundedness: if u_τ is Lipschitz in the large, then the functions $\boldsymbol{T}_\tau^t u_\tau$, $t \in [\tau, T]$, are uniformly Lipschitz in the large for each $T \geqslant \tau$.

Regularity: if u_τ is Lipschitz in the large, the function $(t, q) \mapsto \boldsymbol{T}_\tau^t u_\tau(q)$ is locally Lipschitz on $]\tau, \infty) \times \mathbb{R}^d$.

Translation invariance: $\boldsymbol{T}_\tau^t(c + u) = c + \boldsymbol{T}_\tau^t u$ for each constant $c \in \mathbb{R}$.

The Lax–Oleinik operators solve the Cauchy problem for (HJ) in the viscosity sense. Actually, this follows from property 4.4.

Proposition 4.5. *Let H be a Hamiltonian satisfying hypothesis 1.8. Assume that there exists a family $\boldsymbol{T}_\tau^t$, $0 \leqslant \tau \leqslant t$ of operators satisfying the Markov property, the monotony, the compatibility with (HJ) and the boundedness as expressed in property 4.4. Then if u_0 is an initial condition which is Lipschitz in the large, the function*

$$(t,q) \mapsto u(t,q) = \boldsymbol{T}_0^t u_0(q)$$

is a viscosity solution of (HJ) on $]0,\infty) \times \mathbb{R}^d$.

Note that we did not make any convexity assumption. This kind of axiomatic characterization of viscosity solutions is reminiscent of [1] (see also [8]). It may also help to understand the links between viscosity solutions and variational solutions in the non-convex setting. Such links were suggested by Claude Viterbo and Marc Chaperon, and established in [22].

Proof of proposition 4.5. Let us prove that u is a viscosity subsolution, a similar proof yields that it is also a supersolution. We consider a point $(T,Q) \in]0,\infty) \times \mathbb{R}^d$ and a superdifferential (h,p) of the function u at (T,Q). To prove that $h + H(T,Q,p) \leqslant 0$, we assume, by contradiction, that

$$h + H(T,Q,p) > 0.$$

As is usual for viscosity solutions we shall use a test function ϕ. We shall assume that $\phi\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is smooth and satisfies the following properties:

(i) $\phi(T,Q) = u(T,Q)$, $\partial_t \phi(T,Q) = h$, $\partial_q \phi(T,Q) = p$;

(ii) $\phi \geqslant u$ on $[-\frac{1}{2}T, 2T] \times \mathbb{R}^d$;

(iii) there exists a constant $C > 0$ such that $\phi(t,q) = C\sqrt{1+|q|^2}$ when $|q| + |t| \geqslant C$.

Note that $\mathrm{d}^2\phi$ is bounded. Such a test function exists because the functions u_t, $t \in [\frac{1}{2}T, 2T]$, are uniformly Lipschitz in the large, as follows from the boundedness property assumed on the operators.

Claim 4.6. *There exist $S > 0$ and a C^2 function $w(\tau,t,q)$ defined on the open set*

$$\{(\tau,t,q) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \colon \tau - S < t < \tau + S\} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$$

such that, for each fixed τ, the function $w_\tau\colon (t,q) \mapsto w(\tau,t,q)$ is the solution of the Cauchy problem

$$\left.\begin{aligned} \partial_t w_\tau + H(t,q,\partial_q w_\tau) &= 0,\\ w_\tau(\tau,q) &= \phi(\tau,q). \end{aligned}\right\} \tag{4.1}$$

The existence of a solution w_τ to this problem follows from theorem 1.9. However, to see that w is C^2 in all its variables, we find it more convenient to consider the Cauchy problem

$$\left.\begin{aligned} \partial_s u + (\partial_z u + H(z,q,\partial_q u(s,z,q))) &= 0,\\ u(0,z,q) &= \phi(z,q). \end{aligned}\right\} \tag{4.2}$$

By theorem 1.9, applied to the Hamiltonian

$$\hat{H}(s,z,q,\xi,p)\colon \mathbb{R}\times(\mathbb{R}\times\mathbb{R}^d)\times(\mathbb{R}\times\mathbb{R}^d)^*\to\mathbb{R},$$
$$(s,z,q,\xi,p)\mapsto \xi + H(z,q,p),$$

there exists $S>0$ *and a* C^2 *solution* $u(s,z,q)\colon\,]-S,S[$ *of this Cauchy problem. Setting*

$$w(\tau,t,q) := u(t-\tau,t,q),$$

we verify that

$$\begin{aligned}\partial_t w(t,q) &+ H(t,q,\partial_q w(t,q))\\ &= \partial_s u(t-\tau,t,q)+\partial_z u(t-\tau,t,q)+H(t,q,\partial_q u(t-\tau,t,q))\\ &= 0\end{aligned}$$

and that $w(\tau,\tau,q)=u(0,\tau,q)=\phi(\tau,q)$.

Claim 4.7. *There exists* $\tau\in\,]T-S,T[$ *such that* $w(\tau,T,Q)<\phi(T,Q)$.
Since $w(T,T,q)=\phi(T,q)$, *we have*

$$\partial_t w(T,T,Q) = -H(T,Q,\partial_q w(T,Q)) = -H(T,Q,\partial_q\phi(T,Q)) < \partial_t\phi(T,Q).$$

As a consequence, there exists $\delta>0$ *such that*

$$\partial_t w(\tau,t,Q)-\partial_t\phi(t,Q)<0$$

for $\tau,t\in\,]T-\delta,T[$. *Since* $w(\tau,\tau,Q)=\phi(\tau,Q)$, *we deduce by integration that*

$$w(\tau,T,Q)-\phi(T,Q)=\int_\tau^T \partial_t w(\tau,t,Q)-\partial_t\phi(t,Q)\,\mathrm{d}t<0$$

provided $\tau\in\,]T-\delta,T[$, *which proves our claim.*

Since we are considering monotone operators compatible with (HJ) we have

$$w(\tau,T,Q)=\boldsymbol{T}_\tau^T w_\tau(Q)=\boldsymbol{T}_\tau^T\phi_\tau(Q)\geqslant \boldsymbol{T}_\tau^T u_\tau(Q)=u(T,Q);$$

hence $\phi(T,Q)>u(T,Q)$, which is a contradiction. □

This aside through viscosity solutions being complete, let us turn our attention to more geometric aspects of the Lax–Oleinik operators. We denote by Γ_u the graph of the differential of u on its domain of definition,

$$\Gamma_u := \{(q,\mathrm{d}u(q))\colon q\in\mathbb{R}^d,\ \mathrm{d}u(q)\text{ exists}\}.$$

Proposition 4.8. *Let* u *be a semi-concave and Lipschitz function. The set*

$$\varphi_t^0(\bar{\Gamma}_{\boldsymbol{T}_0^t u})$$

is contained in Γ_u *for each* $t>0$, *and it is a Lipschitz graph.*

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

Proof. In view of the Markov property, it is enough to prove the result for $t \in]0, \sigma]$. Let (q, p) be a point of $\Gamma_{\boldsymbol{T}_0^t u}$, which means that the function $\boldsymbol{T}_0^t u$ is differentiable at q and that $d(\boldsymbol{T}_0^t u)(q) = p$. Let Θ be a minimizing point in the expression $\boldsymbol{T}_0^t u(q) = \min_\theta u(\theta) + S_0^t(\theta, q)$. Since each of the functions u and $S_0^t(\cdot, q)$ are semi-concave, this implies that they are both differentiable at Θ, and that $\mathrm{d}u(\Theta) + \partial_0 S_0^t(\Theta, q) = 0$. Moreover, this implies that the function $u(\Theta) + S_0^t(\Theta, \cdot)$ touches the function $\boldsymbol{T}_0^t u$ from above at point q, and hence that $S_0^t(\Theta, \cdot)$ is differentiable at q, with a differential equal to p. We then have

$$\varphi_t^0(q, p) = \varphi_t^0(q, \partial_1 S_0^t(\Theta, q)) = (\Theta, -\partial_0 S_0^t(\Theta, q)) = (\Theta, \mathrm{d}u(\Theta)) \subset \Gamma_u.$$

We have proved that $\varphi_t^0(\Gamma_{\boldsymbol{T}_0^t u}) \subset \Gamma_u$. Moreover, we have $Q_t^0(\Gamma_{\boldsymbol{T}_0^t u}) \subset \mathcal{I}$, where $\mathcal{I} \subset \mathbb{R}^d$ is the set of points θ that are minimizing in the definition of $\boldsymbol{T}_0^t u(q)$ for some point q.

Claim 4.9. *The function u is $C^{1,1}$ on $\mathcal{I}$. This means that u is differentiable at each point of $\mathcal{I}$, and that the map $\theta \mapsto \mathrm{d}u(\theta)$ is Lipschitz on $\mathcal{I}$. In other words, the projection of Γ_u to $\mathbb{R}^d$ contains $\mathcal{I}$, and the set*

$$\Gamma_u|_{\mathcal{I}} := \{(\theta, \mathrm{d}u(\theta)),\ \theta \in \mathcal{I}\}$$

is a Lipschitz graph.

To prove the claim, we first prove that u has C-superdifferentials and C-subdifferentials at each point of $\mathcal{I}$, where C is a common semi-concavity constant of all the functions $-S_0^t(\cdot, q)$ and of the function u. The existence of a C-superdifferential follows from the C-semi-concavity of u. To prove the existence of a C-subdifferential at a point $\Theta \in \mathcal{I}$, we consider a point q such that $u(\Theta) + S_0^t(\Theta, q) = T_0^t u(q)$. Such a point exists by definition of $\mathcal{I}$. This implies that the function $\theta \mapsto u(\theta) + S_0^t(\theta, q)$ has a minimum at $\theta = \Theta$; hence, each C-subdifferential of $-S_0^t(\cdot, q)$ is a C-subdifferential of u. The claim then follows from a result of Fathi (see proposition A.4).

Let now (q, p) be a point in the closure $\bar{\Gamma}_{\boldsymbol{T}_0^t u}$ of $\Gamma_{\boldsymbol{T}_0^t u}$. There exists a sequence (q_n, p_n) of points of $\Gamma_{\boldsymbol{T}_0^t u}$ that converges to (q, p). By definition, the function $\boldsymbol{T}_0^t u$ is differentiable at q_n, and $p_n = d(\boldsymbol{T}_0^t u)(q_n)$. Let $\Theta_n = Q_t^0(q_n, p_n)$ be the sequence of points such that

$$\boldsymbol{T}_0^t u(q_n) = u(\Theta_n) + S_0^t(\Theta_n, q_n).$$

The sequence Θ_n is converging to $\Theta = Q_t^0(q, p)$, and, at the limit, we see that

$$\boldsymbol{T}_0^t u(q) = u(\Theta) + S_0^t(\Theta, q).$$

We conclude that $\Theta \in \mathcal{I}$. Since we have already proved the Lipschitz regularity of $\mathrm{d}u$ on $\mathcal{I}$, we deduce that

$$\varphi_t^0(q, p) = \lim(\varphi_t^0(q_n, p_n)) = \lim(\Theta_n, \mathrm{d}u(\Theta_n)) = (\Theta, \mathrm{d}u(\Theta)) \in \Gamma_u|_{\mathcal{I}} \subset \Gamma_u.$$

□

The action of the Lax–Oleinik operators on semi-convex functions also has a remarkable property; see [4]. It is useful to denote by L_u the set of point (Q, P) such that P is a subdifferential of u at Q. Note that $\Gamma_u \subset L_u$.

Proposition 4.10. *If u is K-semi-convex, then for each $\delta \in\,]0,1[$ there exists $T>0$ such that $\boldsymbol{T}_0^t u$ is $(K+\delta)$-semi-convex, and hence $C^{1,1}$, for each $t \in\,]0,T]$. One can take*

$$T = \frac{\delta}{2M(3+2K)^2}.$$

Proof. Since u is K-semi-convex, for each $(Q,P) \in L_u$, we have

$$u(q) \geqslant u(Q) + P(q-Q) - K|q-Q|^2.$$

We denote by $l_{Q,P}(q)$ the function on the right in this inequality, so that

$$u = \max_{(Q,P)\in L_u} l_{Q,P}.$$

Taking T as in the statement, it follows from theorem 1.9 that the functions $\boldsymbol{T}_0^t(l_{Q,P})$, $t \in [-T,T]$, are C^2 with a second derivative bounded by $2K + 4tM(1+2K)^2 \leqslant 2K + 2\delta$. We claim that

$$\boldsymbol{T}_0^t u = \max_{(Q,P)\in L} \boldsymbol{T}_0^t(l_{Q,P}),$$

for $t \in [0,T] \cap [0,\sigma]$, which implies that $\boldsymbol{T}_0^t u$ is $(K + 4tM(1+2K)^2)$-semi-convex. We prove the claim in two steps. First, the inequality

$$\boldsymbol{T}_0^t u \geqslant \max_{(Q,P)\in L} \boldsymbol{T}_0^t(l_{Q,P})$$

follows immediately from the fact that $u \geqslant l_{Q,P}$ for each $(Q,P) \in L$ in view of the monotony of $\boldsymbol{T}_0^t$ (see property 4.4). Let us fix a point (t,q) and prove the converse inequality at this point. Since

$$u(\theta) + S_0^t(\theta,q) \geqslant u(q) + P(\theta - q) - K(\theta-q)^2 + \frac{1}{2tM}|\theta-q|^2 - tM$$

and since $K \leqslant 1/2tM$, there exists a point θ such that $\boldsymbol{T}_0^t u(q) = u(\theta) + S_0^t(\theta,q)$. Assuming that $t \leqslant \sigma$, this implies that the point $(\theta,\zeta) = (\theta, -\partial_0 S_0^t(\theta,q))$ belongs to L_u, and that $q = Q_0^t(\theta,\zeta)$. Then, we have

$$\boldsymbol{T}_0^t(l_{\theta,\zeta})(q) = l_{\theta,\zeta}(\theta) + S_0^t(\theta,q) = u(\theta) + S_0^t(\theta,q) = \boldsymbol{T}_0^t u(q);$$

hence

$$\boldsymbol{T}_0^t u(q) \leqslant \max_{(Q,P)\in L} \boldsymbol{T}_0^t(l_{Q,P})(q),$$

provided $t \leqslant \sigma$. We conclude that $\boldsymbol{T}_0^t u$ is semi-concave with constant $K + 2tM(1+2K)^2$ for $t \in [0,\sigma] \cap [0,T]$. We can then apply this result to $\boldsymbol{T}_0^\sigma u$, and, since $K + tM(1+2K)^2 \leqslant K+1$, we conclude that the function $\boldsymbol{T}_\sigma^t \boldsymbol{T}_0^\sigma u$ is semi-concave with constant

$$K + 2\sigma M(1+2K)^2 + 2tM(3+2K)^2 \leqslant K + 2(\sigma+t)M(3+2K)^2 \leqslant K+1$$

for $t \in [0,\sigma] \cap [0, T-\sigma]$. In other words, the functions $\boldsymbol{T}_0^t u$ are semi-concave with constant $K + 2tM(3+2K)^2$ for $t \in [0,2\sigma] \cap [0,T]$. We can apply this argument as many times as necessary and obtain that the functions $\boldsymbol{T}_0^t u$ are semi-concave with constant $K + 2tM(3+2K)^2$ for each $t \in [0,T]$. □

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

The following was first stated explicitly by Arnaud in [2].

Addendum 4.11. Under the hypotheses of proposition 4.10, we have

$$L_u = \varphi_t^0(\Gamma_{\boldsymbol{T}_0^t u})$$

for each $t \in \,]0,T[$. Moreover, for each q, we have $\boldsymbol{T}_0^t u(q) = u(\theta) + S_0^t(\theta, q)$, with $\theta = Q_t^0(q, \mathrm{d}(\boldsymbol{T}_0^t u)(q))$.

Proof. For each $q \in \mathbb{R}^d$, we have seen that there exists $(\theta,\zeta) \in L_u$ such that $\boldsymbol{T}_0^t u(q) = u(\theta) + S_0^t(\theta, q)$ and $\zeta = -\partial_0 S_0^t(\theta, q)$. Since we know that $\boldsymbol{T}_0^t u$ is C^1, the first of these equalities implies that $d(\boldsymbol{T}_0^t u)(q) = \partial_1 S_0^t(\theta, q)$, while the second implies that $\varphi_0^t(\theta,\zeta) = (q, \partial_1 S_0^t(\theta, q))$. We conclude that $\varphi_t^0(\Gamma_{\boldsymbol{T}_0^t u}) \subset L_u$. Moreover, $\theta = Q_t^0(q, \mathrm{d}(\boldsymbol{T}_0^t u)(q))$.

Conversely, let us consider a point $(\theta,\zeta) \in L$, and denote by l the associated function $l_{\theta,\zeta}$. By proposition 4.3, the function $(t, q) \mapsto \boldsymbol{T}_0^t l(q)$ is the restriction to $]0,T[$ of the C^2 solution of (HJ) emanating from l. As a consequence, we have

$$\boldsymbol{T}_0^t l(Q_0^t(\theta,\zeta)) = l(\theta) + S_0^t(\theta, Q_0^t(\theta,\zeta)) = u(\theta) + S_0^t(\theta, Q_0^t(\theta,\zeta)) \geqslant \boldsymbol{T}_0^t u(Q_0^t(\theta,\zeta)).$$

Since we know from the monotony property that $\boldsymbol{T}_0^t l \leqslant \boldsymbol{T}_0^t u$, we conclude that this last inequality is actually an equality. Setting $q_1 = Q_0^t(\theta,\zeta)$, this implies that

$$(\theta,\zeta) = (\theta, -\partial_0 S_0^t(\theta, q_1)) = \varphi_t^0(q_1, \partial_1 S_0^t(\theta, q_1)) = \varphi_t^0(q_1, \mathrm{d}\boldsymbol{T}_0^t u(q_1)) \subset \varphi_t^0(\Gamma_{\boldsymbol{T}_0^t u}).$$

We conclude that $L_u \subset \varphi_t^0(\Gamma_{\boldsymbol{T}_0^t u})$. □

Addendum 4.12. Under the hypotheses of proposition 4.10, we have $\check{\boldsymbol{T}}_t^0 \circ \boldsymbol{T}_0^t u = u$ for each $t \in \,]0,T[$.

Proof. Let us define the map $F\colon q \mapsto Q_t^0(q, \mathrm{d}(\boldsymbol{T}_0^t u(q)))$. By addendum 4.11, the image of F is equal to the projection of L_u on $\mathbb{R}^d$; hence, the map F is onto. Given a point $\theta \in \mathbb{R}^d$, we consider a preimage q of θ by F, and write

$$\check{\boldsymbol{T}}_t^0 \circ \boldsymbol{T}_0^t u(\theta) \geqslant \boldsymbol{T}_0^t u(q) - S_0^t(\theta, q) = u(\theta),$$

where the last equality comes from addendum 4.11. We conclude that $\check{\boldsymbol{T}}_t^0 \circ \boldsymbol{T}_0^t u \geqslant u$; hence, $\check{\boldsymbol{T}}_t^0 \circ \boldsymbol{T}_0^t u = u$. □

The following extrapolates the ideas in [7]. For $t_0 \in \mathbb{R}$ and $\delta, t > 0$, let us define the operators

$$\boldsymbol{R}^t := \check{\boldsymbol{T}}_{t_0+\delta t}^{t_0} \circ \boldsymbol{T}_{t_0-t}^{t_0+\delta t} \circ \check{\boldsymbol{T}}_{t_0}^{t_0-t}, \qquad \check{\boldsymbol{R}}^t := \boldsymbol{T}_{t_0-\delta t}^{t_0} \circ \check{\boldsymbol{T}}_{t_0+t}^{t_0-\delta t} \circ \boldsymbol{T}_{t_0}^{t_0+t}.$$

Theorem 4.13. *There exists $\delta \in \,]0,1[$, which depends only on m and M such that the operators $\boldsymbol{R}^t, \check{\boldsymbol{R}}^t$ have the following properties:*

(i) *for each $t_0 \in \mathbb{R}$ and $t \in \,]0,1[$, the finite-valued functions in the images of $\boldsymbol{R}^t$ and $\check{\boldsymbol{R}}^t$ are uniformly $C^{1,1}$;*

(ii) *for each semi-concave function u, there exists $T > 0$ such that $\boldsymbol{R}^t u \leqslant u$ and $\check{\boldsymbol{R}}^t u \leqslant u$ for each $t_0 \in \mathbb{R}$ and $t \in \,]0,T[$;*

(iii) *for each semi-convex function u, there exists $T > 0$ such that $\boldsymbol{R}^t u \geqslant u$ and $\check{\boldsymbol{R}}^t u \geqslant u$ for each $t_0 \in \mathbb{R}$ and $t \in]0, T[$;*

(iv) *for each $C^{1,1}$ function u, there exists $T > 0$ such that $\boldsymbol{R}^t u = u$ and $\check{\boldsymbol{R}}^t u = u$ for each $t_0 \in \mathbb{R}$ and $t \in]0, T[$.*

Proof. The finite-valued functions in the image of $\boldsymbol{T}_{t_0-t}^{t_0+\delta t}$ are C/t-semi-concave, by lemma 4.1 (we assume that $t \in]0, 1[$). Then, by proposition 4.10, the finite-valued functions in the image of $\check{\boldsymbol{T}}_{t_0+\delta t}^{t_0} \circ \boldsymbol{T}_{t_0-t}^{t_0+\delta t}$ are $(2C/t)$-semi-concave, provided

$$\delta t \leqslant \frac{C}{tM(3 + 2C/t)^2} = \frac{Ct}{M(3t + 2C)^2},$$

which holds if $\delta \leqslant C/(M(3 + 2C))$. For such a δ, the finite-valued functions in the image of $\boldsymbol{R}^t$ are uniformly semi-concave. They are also uniformly semi-convex, and hence uniformly $C^{1,1}$. The proof is similar for $\check{\boldsymbol{R}}$. Let us now write

$$\boldsymbol{R}^t := (\check{\boldsymbol{T}}_{t_0+\delta t}^{t_0} \circ \boldsymbol{T}_{t_0}^{t_0+\delta t}) \circ (\boldsymbol{T}_{t_0-t}^{t_0} \circ \check{\boldsymbol{T}}_{t_0}^{t_0-t}),$$

which implies, using the monotony, that

$$\boldsymbol{R}^t u \geqslant \check{\boldsymbol{T}}_{t_0+\delta t}^{t_0} \circ \boldsymbol{T}_{t_0}^{t_0+\delta t} u \quad \text{and} \quad \boldsymbol{R}^t u \leqslant \boldsymbol{T}_{t_0-t}^{t_0} \circ \check{\boldsymbol{T}}_{t_0}^{t_0-t} u.$$

By addendum 4.12 we conclude that $\boldsymbol{R}^t u \geqslant u$ for small t when u is semi-convex. All the statements of (ii) and (iii) follow by similar considerations. Statement (iv) follows from (ii) and (iii). □

5. Subsolutions of the stationary Hamilton–Jacobi equation

We assume from now on that the Hamiltonian does not explicitly depend on time. Then, in addition to (HJ), we can consider the stationary Hamilton–Jacobi equation

$$H(q, \mathrm{d}u(q)) = a, \tag{HJa}$$

for each real parameter a. This stationary equation is the main character of Fathi's companion lecture (see also [16]). Formally, a function $u(q)$ solves (HJa) if and only if the function $(t, q) \mapsto u(q) - at$ solves (HJ). It is not hard to check that this also holds in the sense of viscosity solutions: the function $u(q)$ is a viscosity solution of (HJa) if and only if the function $(t, q) \mapsto u(q) - at$ is a viscosity solution of (HJ). Let us give a summary for later reference.

Hypothesis 5.1. *We say that H is autonomous if it does not depend on the time variable.*

In this autonomous context, we have $\boldsymbol{T}_\tau^{\tau+t} = \boldsymbol{T}_0^t$. We shall denote this operator by $\boldsymbol{T}^t$. The Markov property becomes the equality $\boldsymbol{T}^t \circ \boldsymbol{T}^s = \boldsymbol{T}^{t+s}$. In other words, the Lax–Oleinik operators form a semi-group, the famous Lax–Oleinik semi-group. Another important specificity of the autonomous context is that the Hamiltonian H is constant along Hamiltonian orbits, as can be checked by an easy computation.

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

Proposition 5.2. *Given a Hamiltonian H satisfying hypotheses 1.8, 2.1, 3.2 and 5.1, the following properties are equivalent for a function u:*

(i) *the function u is Lipschitz and it solves the inequality $H(q, \mathrm{d}u(q)) \leqslant a$ almost everywhere;*

(ii) *the inequality $u(q_1) - u(q_0) \leqslant A^t(q_0, q_1) + at$ holds for each $q_0 \in \mathbb{R}^d, q_1 \in \mathbb{R}^d, t > 0$;*

(iii) *the inequality $u \leqslant \boldsymbol{T}^t u + ta$ holds for each $t \geqslant 0$;*

(iv) *the function u is a viscosity subsolution of the Hamilton–Jacobi equation $H(q, \mathrm{d}u(q)) = a$;*

(v) *the function u is Lipschitz and the inequality $H(q, \mathrm{d}u(q)) \leqslant a$ holds at each point of differentiability q of u (by Rademacher's theorem, the set of points of differentiability has full measure).*

The function u is called a subsolution at level a, or a subsolution of (HJa), if it satisfies these properties.

Proof. It is tautological that (v) $\Longrightarrow$ (i) and easy that (ii) $\Longleftrightarrow$ (iii). Let us prove that (i) $\Longrightarrow$ (ii), following Fathi. If (i) holds, then there exists a set $M \subset \mathbb{R}^d$ of full measure composed of points of differentiability q of u such that $H(q, \mathrm{d}u(q)) \leqslant a$. We first assume that $t < \sigma$ and prove (ii) (recall that $A^t = S^t$). Let us consider the map

$$(q_0, q_1, \tau) \mapsto (q(\tau), q_1, \tau),$$

where $q(\tau)$ is the value at time τ of the unique orbit $(q(s), p(s))$ that satisfies $q(0) = q_0$ and $q(t) = q_1$. This map is a diffeomorphism of $\mathbb{R}^d \times \mathbb{R}^d \times]0, t[$, the inverse diffeomorphism being

$$(\theta, q_1, \tau) \mapsto (q(0), q_1, \tau),$$

where $(q(s), p(s))$ is the unique orbit such that $q(\tau) = \theta$ and $q(t) = q_1$. As a consequence, for almost every pair (q_0, q_1), the function u is differentiable at the point $q(s)$ for almost every $s \in]0, t[$. If (q_0, q_1) is such a pair, we have, using the convexity of H in p,

$$\begin{aligned} u(q_1) - u(q_0) &= u(q(t)) - u(q(0)) \\ &= \int_0^t du_{q(s)} \cdot \dot{q}(s) \, \mathrm{d}s \\ &= \int_0^t du_{q(s)} \cdot \partial_p H(q(s), p(s)) \, \mathrm{d}s \\ &\leqslant \int_0^t H(q(s), \mathrm{d}u_{q(s)}) + \partial_p H(q(s), p(s)) \cdot p(s) - H(q(s), p(s)) \, \mathrm{d}s \\ &\leqslant at + S^t(q(0), q(t)) \\ &= at + A^t(q_0, q_1). \end{aligned}$$

We have proved the desired inequality for almost every pair (q_0, q_1), and hence on a dense subset of pairs. Since both sides of the inequality are continuous, we deduce that the inequality holds for all pairs (q_0, q_1), provided $t < \sigma$. In order to deduce the inequality when $t \geqslant \sigma$, we write, for n large enough,

$$\begin{aligned} A^t(q_0, q_1) + at &= \min_{\theta_1,\dots,\theta_{n-1}} (S^{t/n}(q_0, \theta_1) + at/n + \cdots + S^{t/n}(q_{n-1}, q_1) + at/n) \\ &\geqslant \min_{\theta_1,\dots,\theta_{n-1}} (u(\theta_1) - u(q_0) + \cdots + u(q_1) - u(\theta_{n-1})) \\ &= u(q_1) - u(q_0). \end{aligned}$$

Let us now prove that $3 \Rightarrow 4$. Let u be a function satisfying (iii). This function then satisfies (ii); hence, it is Lipschitz. We consider a C^2 function $v(q)$ that touches u from above at some point θ, which means that $v - u$ has a global minimum at θ. Since the function u is Lipschitz, we can modify v at infinity and assume that it has bounded second differential. Then, there exists a C^2 solution $V(t, q)$ of (HJ) defined on $]-T, T[$ with $T > 0$, and such that $V(0, q) = v(q)$. For $t \geqslant 0$, we have $V_t = \boldsymbol{T}^t v$, by proposition 4.3. Since $v \geqslant u$, we obtain that

$$V(t, q) = \boldsymbol{T}^t v(q) \geqslant \boldsymbol{T}^t u(q) \geqslant u(q) - at$$

for $t \in\,]0, T[$; hence, $\partial_t V(0, \theta) \geqslant -a$ (recall that θ is the point of contact between u and v). Since we know that V solves (HJ), we conclude that

$$H(\theta, \partial_q V(0, \theta)) = H(\theta, \mathrm{d}v(\theta)) \leqslant a.$$

The proof that (iv) $\implies$ (v) is classical and can be found in [13], but we recall it here for completeness. If q is a point of differentiability of u, then $\mathrm{d}u(q)$ is a superdifferential (but not necessarily a proximal superdifferential) of u at q; hence, $H(q, \mathrm{d}u(q)) \leqslant a$. We shall now prove that the function u is locally Lipschitz. The estimate $H(q, \mathrm{d}u(q)) \leqslant a$, which holds at each point of differentiability of u, then implies that it is globally Lipschitz in view of hypothesis 3.2.

Let $B(Q, 1)$ be a closed ball, of radius 1. Let us set

$$r = \max_{\theta \in B(Q,2), q \in B(Q,1)} (u(\theta) - u(q)).$$

Let k be a positive number greater than r and such that $|p| \geqslant k \Rightarrow H(q, p) > a$ for each q. Such a k exists by hypothesis 3.2. Given q in $B(Q, 1)$, the function

$$\theta \mapsto k|\theta - q| - u(\theta)$$

then has a local minimum in the interior of the ball $B(Q, 2)$. If this minimum is reached at a point $q_1 \neq q$, then the function $v(\theta) := k|\theta - q|$ is smooth at q_1 and, since u is a viscosity subsolution, we have $H(q_1, \mathrm{d}v(q_1)) \leqslant a$, which contradicts the fact that $|\mathrm{d}v(q_1)| = k$. Hence, the minimum must be reached at q, which implies that $k|\theta - q| - u(\theta) \geqslant -u(q)$ or, equivalently, that

$$u(\theta) - u(q) \leqslant k|\theta - q|$$

for each $\theta \in B(Q, 2)$ and all $q \in B(Q, 1)$. We conclude that u is k-Lipschitz on $B(Q, 1)$. □

Corollary 5.3. *If u is a subsolution of (HJa), then, for each $t \geqslant 0$, $\boldsymbol{T}^t u$ is a subsolution of (HJa), and so is $\check{\boldsymbol{T}}^t u$.*

Proof. The function u is a subsolution if and only if $\boldsymbol{T}^s u + as \geqslant u$ for each $t \geqslant 0$. Applying $\boldsymbol{T}^t$, we obtain $\boldsymbol{T}^t\boldsymbol{T}^s u + as = \boldsymbol{T}^s\boldsymbol{T}^t u + as \geqslant \boldsymbol{T}^t u$. Since this inequality holds for each $s \geqslant 0$, we conclude that $\boldsymbol{T}^t u$ is a subsolution. □

Corollary 5.4. *If the function u is Lipschitz, and if the Hamiltonian is autonomous, then the functions $\boldsymbol{T}^t u, t \geqslant 0$ are equi-Lipschitz.*

Proof. If the function u is k-Lipschitz, then $\mathrm{d}u(q) \leqslant k$ almost everywhere; hence, u is a subsolution to (HJa) for some a (one can take $a = \sup_{|p|\leqslant k} H(q,p)$). As a consequence, the functions $\boldsymbol{T}^t u$, $t \geqslant 0$ are all subsolutions to (HJa); hence, they are K-Lipschitz, with $K = \sup\{|p|, H(q,p) \leqslant a\}$. □

6. Weak KAM solutions and invariant sets

We derive here the first dynamical consequences from the theory.

Definition 6.1. The function u is called a weak KAM solution at level a if $\boldsymbol{T}^t u + ta = u$ for each $t \geqslant 0$. Weak KAM solutions at level a are viscosity solutions of (HJa). We say that the function u is a weak KAM solution if it is a weak KAM solution at some level a.

If u is a weak KAM solution, then it is semi-concave (with a semi-concavity constant that depends only on M and m). By theorem 4.8, for $t > 0$, we have the inclusion

$$\varphi^{-t}(\bar{\Gamma}_u) \subset \Gamma_u,$$

and this set is a Lipschitz graph. The set

$$\mathcal{I}^*(u) := \bigcap_{n\in\mathbb{N}} \varphi^{-n}(\bar{\Gamma}_u)$$

is a closed invariant set contained in a Lipschitz graph. It would be a very nice result to have obtained a distinguished closed invariant subset of our Hamiltonian system contained in a Lipschitz graph. Unfortunately, at this point, we cannot prove (because it is not necessarily true) that the set $\mathcal{I}^*(u)$ is not empty. In order to obtain interesting dynamical consequences from this theory, we need an additional assumption.

Hypothesis 6.2. *We say that the Hamiltonian H is periodic if $H(q+w,p) = H(q,p)$ for each $w \in \mathbb{Z}^d$, $q \in \mathbb{R}^d$ and $p \in \mathbb{R}^{d*}$.*

Under this hypothesis, we should see the Hamiltonian system as defined on the phase space $\mathbb{T}^d \times \mathbb{R}^{d*}$, with $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Indeed, the flow φ^t commutes with the translations $(q,p) \mapsto (q+w,p)$, $w \in \mathbb{Z}^d$. The compactness of this new configuration space has remarkable consequences, summed up in the following Theorem. We assume in the rest of this section that the Hamiltonian H satisfies hypotheses 1.8, 2.1, 3.2, 5.1 and 6.2.

Theorem 6.3. *If the Hamiltonian is autonomous and periodic, then there exists a periodic weak KAM solution. The corresponding set $\mathcal{I}^*(u)$ is a non-empty closed invariant set that is contained in a Lipschitz graph and is invariant under the translations $(q,p) \mapsto (q+w,p)$, $w \in \mathbb{Z}^d$.*

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

This latter property on the invariance under translations means that $\mathcal{I}^*(u)$ naturally gives rise to an invariant space on the quotient phase space $\mathbb{T}^d \times \mathbb{R}^{d*}$.

Proof. Let us first prove the second part of the theorem. If u is a periodic weak KAM solution, then the set $\bar{\Gamma}_u$ is contained in $\{|p| \leqslant C\}$ for some constant C, and it is invariant under the integral translations; hence, it descends to a compact subset of $\mathbb{T}^d \times \mathbb{R}^{d*}$, which we still denote by $\bar{\Gamma}_u$. Then the sets $\varphi^{-n}(\bar{\Gamma}_u)$ form a decreasing sequence of non-empty compact sets; hence, their intersection is a non-empty compact set.

Let us now prove that there exists a periodic weak KAM solution. We follow the proof of [6], which is slightly different from the original proof of Fathi. Observe first that the functions $A^t(q_0, q_1)$ are periodic in the sense that $A^t(q_0 + w, q_1 + w) = A^t(q_0, q_1)$ for each $w \in \mathbb{Z}^d$. This implies that $\boldsymbol{T}^t u$ is periodic when u is periodic. Considering the Cauchy problem for (HJ) with initial condition equal to zero, we define $v(t, q) := \boldsymbol{T}^t 0(q)$. The quantities $a^+(t) = \max_q v_t(q)$ and $a^-(t) = \min_q v_t(q)$ will be useful. Since the functions $v_t, t \geqslant 0$ are equi-Lipschitz, there exists a constant K such that $a^+(t) - a^-(t) \leqslant K$ for all $t \geqslant 0$. We have

$$\begin{aligned} a^+(t+s) &= \max \boldsymbol{T}^{t+s} 0 \\ &= \max \boldsymbol{T}^t(\boldsymbol{T}^s 0) \\ &\leqslant \boldsymbol{T}^t(a^+(s)) \\ &= a^+(s) + \boldsymbol{T}^t(0) \\ &\leqslant a^+(s) + a^+(t), \end{aligned}$$

and, similarly,

$$a^-(t+s) \geqslant a^-(t) + a^-(s).$$

By standard results on subadditive functions, we conclude that $a^+(t)/t$ and $a^-(t)/t$ converge, respectively, to $\inf_{t \geqslant 0} a^+(t)/t$ and $\sup_{t \geqslant 0} a^-(t)/t$. Since $a^+ - a^-$ is bounded, these two limits have the same value: let us call it $-a$. We have

$$K - ta \geqslant a^-(t) + K \geqslant a^+(t) \geqslant -ta \geqslant a^-(t) \geqslant a^+(t) - K \geqslant K - at$$

for all $t \geqslant 0$; hence,

$$K \geqslant v(t, q) + ta \geqslant -K.$$

We can now define

$$u(q) := \liminf_{t \to \infty} (v(t, q) + ta).$$

We claim that u is a weak KAM solution at level a. Since the functions $v_t + ta$ are equi-Lipschitz and equi-bounded, the function u is well defined and Lipschitz. We have to prove that $\boldsymbol{T}^t u + ta = u$ for all $t \geqslant 0$.

We have

$$v(t+s, q_1) + (t+s)a \leqslant v(s, q_0) + sa + A^t(q_0, q_1) + ta$$

for each q_0, q_1 and $t \geqslant 0$, $s \geqslant 0$. Taking the infimum limit in s yields

$$u(q_1) \leqslant u(q_0) + A^t(q_0, q_1) + ta.$$

We have proved that u is a subsolution to (HJa).

https://doi.org/10.1017/S0308210511000059 Published online by Cambridge University Press

Conversely, we have to prove that $\boldsymbol{T}^t u + ta \geqslant u$. Let us pick a point q and consider a sequence t_n such that $v(t_n, q) + t_n a \to u(q)$. Fixing $t > 0$, we consider a sequence q_n in $\mathbb{R}^d$ such that

$$v(t_n, q) + t_n a = v(t_n - t, q_n) + (t_n - t)a + A^t(q_n, q) + ta.$$

This equality implies that the sequence q_n is bounded, and we assume by taking a subsequence that it has a limit q'. We can also assume that the sequence $v(t_n - t, q') + (t_n - t)a$ has a limit, which we denote by l. Note that $l \geqslant u(q')$. Since the functions v_t are equi-Lipschitz, we have $v(t_n - t, q_n) + (t_n - t)a \to l$; hence, taking the limit in the equality above,

$$u(q) = l + A^t(q', q) + at \geqslant u(q') + A^t(q', q) + at \geqslant \boldsymbol{T}^t u(q) + at.$$

We have proved that u is a periodic weak KAM solution at level a. □

The periodic weak KAM solutions at level a are the periodic viscosity solutions of (HJa), as is proved in Fathi's companion paper. The existence of periodic viscosity solutions was first obtained by Lions *et al.* in a famous unpublished preprint (see [17]). The most important aspect of Fathi's weak KAM theorem that we just exposed is that these viscosity solutions have a dynamical relevance and give rise to invariant sets.

Let us comment a bit further in that direction, and explain the name 'weak KAM'. Consider a periodic Lipschitz function u, and the associated set Γ_u, which we consider here as a subspace of $\mathbb{T}^d \times \mathbb{R}^{d*}$.

Assume first that u is C^2, so that Γ_u is a C^1 graph. This graph is invariant if and only if there exists a such that u solves (HJa). This follows from § 1: if u solves (HJa), then the function $U(t, q) = u(q) - at$ solves (HJ); hence,

$$\varphi^t(\Gamma_u) = \Gamma_{U_t} = \Gamma_u.$$

Conversely, if Γ_u is invariant, then $\Gamma_{\boldsymbol{T}^t u} = \varphi^t(\Gamma_u) = \Gamma_u$, by corollary 1.4; hence, $\boldsymbol{T}^t u$ is equal to u up to an additive constant $a(t)$. Since $\boldsymbol{T}^t$ is a semi-group, it is easy to deduce that $a(t) = at$ for some $a \in \mathbb{R}$. As a consequence, u is a C^2 weak KAM solution; hence, a classical solution of (HJa).

The classical KAM theorem gives the existence, in certain very specific settings, of some invariant C^1 graphs of the form Γ_u. From what we just explained, the theorem can be interpreted as giving the existence of C^2 solutions of (HJa), although this point of view is not the right one to obtain its proof. It is natural to expect that the Hamilton–Jacobi equation could be used to produce invariant sets in more general situations. Since we do not know any direct method to prove the existence of C^2 solutions of (HJa), we should deal with some kind of weak solutions. However, if u is just a Lipschitz solution almost everywhere, we cannot say much about the dynamical properties of Γ_u. It is remarkable that the inclusion $\varphi^t(\Gamma_u) \supset \bar{\Gamma}_u$ holds for viscosity solutions (or, equivalently, weak KAM solutions) in the convex case. This is the starting point of Fathi's construction of the invariant set $\mathcal{I}^*(u)$ that we have presented in this section.

7. Regular subsolutions and the Aubry set

We abandon for a moment hypothesis 6.2, and consider a Hamiltonian satisfying hypotheses 1.8, 2.1, 3.2 and 5.1. We describe a new construction of invariant sets based on the study of regular subsolutions, and define the Aubry set. We mostly follow [4] in this section. The following result is the basis of our constructions (see [2, 4, 15]).

THEOREM 7.1. *If (HJa) admits a subsolution, then it admits a $C^{1,1}$ subsolution. Moreover, the set of $C^{1,1}$ subsolutions is dense in the set of all subsolutions for the uniform topology.*

Proof. Let u be a subsolution at level a. We use the operator $\boldsymbol{R}^t = \breve{\boldsymbol{T}}^{\delta t} \circ \boldsymbol{T}^{(\delta+1)t} \circ \breve{\boldsymbol{T}}^t$ of theorem 4.13 to regularize u. Since the operators $\boldsymbol{T}^t$ and $\breve{\boldsymbol{T}}^t$ preserve subsolutions, so does $\boldsymbol{R}^t$. We claim that

$$u - (C+a)(1+\delta)t \leqslant \boldsymbol{R}^t u \leqslant u + (C+a)(1+\delta)t,$$

with a constant C that depends only on m and M. This implies that the function $\boldsymbol{R}^t u$ is finite valued. If the parameter δ has been chosen small enough, then, by theorem 4.13, the functions $\boldsymbol{R}^t$ are $C^{1,1}$ subsolutions, which converge uniformly to u as $t \to 0$. The bound on $\boldsymbol{R}^t u$ claimed above follows from the following ones in view of property 4.4:

$$v - sa \leqslant \boldsymbol{T}^s v \leqslant v + Cs, \qquad v - Cs \leqslant \breve{\boldsymbol{T}}^s v \leqslant v + sa,$$

which hold for each $s \geqslant 0$ and each subsolution v at level a. The first one can be seen by writing

$$u(q) - as \leqslant \boldsymbol{T}^s u(q) \leqslant u(q) + A^s(q,q) \leqslant u(q) + Cs.$$

This ends the proof of theorem 7.1. Observe that we could have used the simpler operator $\breve{\boldsymbol{T}}^{\delta t} \circ \boldsymbol{T}^t$, as was done in [4], but the operator $\boldsymbol{R}^t$ deserves attention for some nicer properties. □

DEFINITION 7.2. The critical value of H is the real number α (or $\alpha(H)$) defined as the infimum of all real numbers a such that (HJa) has a subsolution. The subsolutions of (HJα) are called critical subsolutions.

LEMMA 7.3. *We have the estimate $-M \leqslant \alpha \leqslant M$.*

Proof. The function $u = 0$ is a subsolution at level M; hence, $\alpha \leqslant M$. Conversely, since $H \geqslant -M$, there exists no subsolution at level a when $a < -M$. □

PROPOSITION 7.4. *There exists a $C^{1,1}$ subsolution of (HJα).*

Proof. Let a_n be a sequence decreasing to α. Since $a_n > \alpha$, the Hamilton–Jacobi equation at level a_n has a subsolution u_n. The sequence u_n is equi-Lipschitz, and we can assume by adding constants that it is also equi-bounded. Taking a subsequence, we can also assume that it converges locally uniformly to a limit u. Taking the limit $n \to \infty$ in the inequalities $u_n(q_1) - u_n(q_0) \leqslant A^t(q_0, q_1) + ta_n$ gives $u(q_1) - u(q_0) \leqslant A^t(q_0, q_1) + ta_n$. This holds for all q_0, q_1 and $t > 0$; hence, u is a subsolution at level

α, or, in other words, a critical subsolution. Since there exists a critical subsolution, theorem 7.1 implies that there exists a $C^{1,1}$ critical subsolution. □

Definition 7.5. The projected Aubry set is the set $\mathcal{A} \subset \mathbb{R}^d$ of points q such that the equality $H(q, \mathrm{d}u(q)) = \alpha$ holds for all C^1 critical subsolutions u.

We point out that $\mathcal{A}$ might be empty without additional hypotheses.

Lemma 7.6. *If $q \in \mathcal{A}$, then all C^1 critical subsolutions u have the same differential at q. In other words, the restriction $\Gamma_{u|\mathcal{A}}$ does not depend on the C^1 critical subsolution u.*

Proof. Let u and v be two critical subsolutions, and q a point in $\mathcal{A}$. We have to prove that $\mathrm{d}u(q) = \mathrm{d}v(q)$. Assume, by contradiction, that this equality does not hold and consider the subsolution $w = \frac{1}{2}(u + v)$. Since $H(q, \mathrm{d}u(q)) = H(q, \mathrm{d}v(q)) = \alpha$, the strict convexity of $H(q, \cdot)$ implies that $H(q, \mathrm{d}w(q)) < \alpha$, which contradicts the definition of $\mathcal{A}$. □

Lemma 7.7. *There exists a $C^{1,1}$ subsolution u_0 that satisfies the strict inequality $H(q, \mathrm{d}u_0(q)) < \alpha$ for all q in the complement of $\mathcal{A}$.*

Proof. The set of C^1 functions is separable for the topology of uniform C^1 convergence on compact sets. This topology can be defined, for example, by the distance

$$d(u, v) = \sum_n \frac{\sup_{|q| \leqslant n} \arctan(|u(q)| + |\mathrm{d}u(q)|)}{2^n}.$$

Since a subset of a separable space is separable, there exists a sequence u_n of C^1 critical subsolutions which is dense for this topology in the set of all C^1 critical subsolutions. Let us set

$$a_n = \frac{a_0}{2^n \sup_{k \leqslant n, |q| \leqslant n}(1 + |u_k(q)| + |\mathrm{d}u_k(q)|)}$$

and choose a_0 such that $\sum_{n \geqslant 1} a_n = 1$. The sum $\sum_{n \geqslant 1} a_n u_n$ converges uniformly with its differentials on each compact sets to a C^1 limit v_0. The function v_0 is a critical subsolution, and we claim that $H(q, \mathrm{d}v_0(q)) = \alpha$ if and only if q belongs to $\mathcal{A}$. Indeed, this equality holds only if all the inequalities $H(q, \mathrm{d}u_n(q)) \leqslant \alpha$ are equalities, which, in view of the density of the sequence u_n, implies that $H(q, \mathrm{d}u(q)) = \alpha$ for all C^1 subsolutions u. By definition, this implies that q belongs to $\mathcal{A}$. We have constructed a C^1 subsolution v_0 such that

$$H(q, \mathrm{d}v_0(q)) < \alpha$$

outside of $\mathcal{A}$. We have to prove the existence of a $C^{1,1}$ critical subsolution with the same property. We consider a smooth function $V(q)$, bounded in C^2, which is positive outside of $\mathcal{A}$ and such that

$$0 \leqslant V(q) \leqslant \alpha - H(q, \mathrm{d}v_0(q))$$

for all $q \in \mathbb{R}^n$. The modified Hamiltonian $\tilde{H}(q, p) = H(q, p) + V(q)$ satisfies all our hypotheses. Since $\tilde{H} \geqslant H$, the corresponding critical value $\tilde{\alpha}$ satisfies $\tilde{\alpha} \geqslant \alpha$. Since

v_0 is a subsolution of the inequality

$$\tilde{H}(q, \mathrm{d}v_0(q)) \leqslant \alpha,$$

we can apply theorem 7.1 to $\tilde{H}$ at level α, and obtain the existence of a $C^{1,1}$ subsolution u_0 to the same inequality. The inequality

$$H(q, \mathrm{d}u_0(q)) \leqslant \alpha - V(q)$$

implies that u_0 is a critical subsolution for H that is strict on the set $\{V > 0\}$, which, from our construction of V, is the complement of $\mathcal{A}$. □

Definition 7.8. The Aubry set $\mathcal{A}^*$ is defined as

$$\mathcal{A}^* = \bigcap_u \Gamma_{u|\mathcal{A}} = \bigcap_u \Gamma_u,$$

where the intersections are taken on the set of C^1 critical subsolutions.

In view of lemma 7.6 we have $\mathcal{A}^* = \Gamma_{u|\mathcal{A}}$ for each C^1 subsolution u; hence, $\pi(\mathcal{A}^*) = \mathcal{A}$, where $\pi\colon \mathbb{R}^d \times \mathbb{R}^{d*} \to \mathbb{R}^d$ is the projection on the first factor. To check the second inequality, it is sufficient to prove that $\bigcap_u \Gamma_u \subset \mathcal{A}^*$. Let u_0 be a C^1 critical subsolution such that $H(q, \mathrm{d}u_0(q)) < \alpha$ outside of $\mathcal{A}$. Given a point (q_0, p_0) in $\Gamma_{u_0} - \mathcal{A}^*$, we can slightly perturb the critical subsolution u_0 around q_0 to a critical subsolution u_1 such that $\mathrm{d}u_1(q_0) \neq \mathrm{d}u_0(q_0)$ (we use the strict inequality $H(q, \mathrm{d}u_0(q)) < \alpha$). The point (q_0, p_0) does not belong to Γ_{u_1}; hence, it does not belong to $\bigcap_u \Gamma_u$, which ends our proof.

The set $\mathcal{A}^*$ is contained in the Lipschitz graph Γ_{u_0} for each $C^{1,1}$ subsolution u_0. As in § 6, we have obtained an invariant set contained in a Lipschitz graph, but which may be empty in general.

Proposition 7.9. *The Aubry set is a closed invariant set.*

Proof. Let u_0 be a $C^{1,1}$ critical solution such that $H(q, \mathrm{d}u_0(q)) < \alpha$ outside of $\mathcal{A}$. By proposition 4.10, there exists $T > 0$ such that $\boldsymbol{T}^t u_0$ is still $C^{1,1}$ for $t \in [-T, T]$. Given $(q, p) \in \mathcal{A}^*$, we conclude that, for $t \in [0, T]$, we have $p = d(\boldsymbol{T}^t u_0)(q)$. Setting $\theta = Q^{-t}(q, p)$, the addendum to proposition 4.10 implies that $\boldsymbol{T}^t u_0(q) = u_0(\theta) + S^t(\theta, q)$, and that

$$\varphi^t(\theta, \mathrm{d}u_0(\theta)) = (q, p).$$

Since the flow preserves the Hamiltonian, we get that $H(\theta, \mathrm{d}u_0(\theta)) = \alpha$; hence, the point θ belongs to $\mathcal{A}$, and then

$$\varphi^{-t}(q, p) = (\theta, \mathrm{d}u_0(\theta)) \in \mathcal{A}^*.$$

We have proved that $\varphi^{-t}(\mathcal{A}^*) \subset \mathcal{A}^*$ for $t \in [0, T]$. We can prove in a similar way, using the $C^{1,1}$ subsolution $\check{\boldsymbol{T}}^t u_0$ instead of $\boldsymbol{T}^t u_0$, that $\varphi^t(\mathcal{A}^*) \subset \mathcal{A}^*$ for $t \in [0, T]$, and hence that

$$\varphi^t(\mathcal{A}^*) = \mathcal{A}^*$$

for each $t \in [-T, T]$, which clearly implies that this equality holds for all t. We have proved the invariance of $\mathcal{A}^*$. □

Proposition 7.10. *The equality*

$$\breve{\boldsymbol{T}}^t u(q) - t\alpha = u(q) = \boldsymbol{T}^t u(q) + t\alpha$$

holds for each critical subsolution u, each $t \geqslant 0$ and each $q \in \mathcal{A}$. The inclusion $\mathcal{A}^* \subset \Gamma_u$ holds for each critical subsolution; hence, the inclusion $\mathcal{A}^* \subset \mathcal{I}^*(u)$ holds for each weak KAM solution at level α.

Proof. Let $(q(s), p(s))$ be a trajectory contained in $\mathcal{A}^*$, and let $t \geqslant 0$ be given. For each C^1 critical subsolution u, we have $p(s) = du_{q(s)}$, and

$$\begin{aligned} u(q(t)) - u(q(0)) &= \int_0^t du_{q(s)} \dot{q}(s) \, \mathrm{d}s \\ &= t\alpha + \int_0^t du_{q(s)} \dot{q}(s) - H(q, \mathrm{d}u_{q(s)}) \, \mathrm{d}s \\ &\geqslant A^t(q(0), q(t)) + t\alpha. \end{aligned}$$

Since u is a critical subsolution, the second point in proposition 5.2 implies that the last inequality must be an equality; hence,

$$u(q(t)) - u(q(s)) = A^{t-s}(q(s), q(t)) + (t-s)\alpha$$

for each $t \geqslant s$. In the terminology of Fathi, we have proved that the curve $q(s)$ is calibrated by the subsolution u. We can now write

$$u(q(t)) \leqslant \boldsymbol{T}^t u(q(t)) + t\alpha \leqslant u(q(0)) + A^t(q(0), q(t)) + t\alpha = u(q(t)).$$

This implies that $\boldsymbol{T}^t u + t\alpha = u$ on $\mathcal{A}$, and, similarly, $\breve{\boldsymbol{T}}^t u - t\alpha = u$ on $\mathcal{A}$. Let us now fix $t \in \,]0, \sigma[$. Given an orbit $(q(s), p(s))$ in $\mathcal{A}^*$, we have

$$u(q(0)) \leqslant u(\theta) + S^t(\theta, q(0)) + t\alpha$$

for each subsolution u and each θ, with equality at $\theta = q(-t)$. This implies that $\partial_1 S(q(-t), q(0))$ is a superdifferential of u at $q(0)$. This holds, in particular, for C^1 subsolutions, which satisfy $\mathrm{d}u(q(0)) = p(0)$; hence, $\partial_1 S(q(-t), q(0)) = p(0)$. We have proved that $p(0)$ is a superdifferential of u at $q(0)$. Similarly, using the inequality

$$u(q(0)) \geqslant u(\theta) - S^t(q(0), \theta) - t\alpha,$$

with equality at $\theta = q(t)$, we conclude that $p(0)$ is a subdifferential of u at $q(0)$. This implies that u is differentiable at $q(0)$, and that $\mathrm{d}u(q(0)) = p(0)$. As a consequence, $\mathcal{A}^* \subset \Gamma_u$ for each subsolution u. □

In the course of the above proof, we have established the following lemma, which will be needed later.

Lemma 7.11. *Let u be a subsolution at level a, and let $(q(s), p(s))$ be a Hamiltonian trajectory contained in $\Gamma_u \cap \{H = a\}$ (note that this set is not necessarily invariant in general). Then, the equality $\breve{\boldsymbol{T}}^t u(q(s)) - ta = u(q(s)) = \boldsymbol{T}^t u(q(s)) + ta$ holds for each $t \geqslant 0$ and each $s \in \mathbb{R}$.*

8. The Mañé potential

In this section, we work with a Hamiltonian satisfying hypotheses 1.8, 2.1, 3.2 and 5.1. The Mañé potential at level a is the function

$$\Phi^a(q_0, q_1) := \inf_{t>0}(A^t(q_0, q_1) + at).$$

This function was first introduced by Ricardo Mañé (see [19]). We leave as an easy exercise for the reader the proof of the triangle inequality

$$\Phi^a(q_0, q_1) \leqslant \Phi^a(q_0, \theta) + \Phi^a(\theta, q_1).$$

In view of proposition 5.2, each subsolution u at level a satisfies

$$u(q_1) - u(q_0) \leqslant \Phi^a(q_0, q_1)$$

for each q_0 and q_1. We conclude that Φ^a is finite if there exists a subsolution at level a, which holds if and only if $a \geqslant \alpha$. Conversely, If the function Φ^a is finite, then we see from the triangle inequality that the function $q \mapsto \Phi^a(q_0, q)$ is a subsolution at level a, which implies that $a \geqslant \alpha$. The estimates of lemma 3.3 imply that

$$\Phi^a(q_0, q_1) \leqslant 2\sqrt{2m(M + a)}|q_1 - q_0|$$

provided $a \geqslant \alpha$ (note that $\alpha \geqslant -M$). We have proved that the Mañé potential is the function called the viscosity semi-distance in [13].

Proposition 8.1. *If $a \geqslant \alpha$, then the function $q \mapsto \Phi^a(q_0, q)$ is the maximum of all subsolutions u at level a that vanish at q_0. If $a < \alpha$, then there is no such subsolution and Φ^a is identically equal to $-\infty$.*

This statement also implies that the Mañé Potential at level a depends only on the energy level $\{H = a\}$. More precisely, let G be another Hamiltonian satisfying our hypotheses and such that $H = a \Leftrightarrow G = a$. Then, the sets $\{H \leqslant a\}$ and $\{G \leqslant a\}$ are equal, which implies, in view of the first characterization of subsolutions in proposition 5.2, that G and H have the same subsolutions at level a. As a consequence, they have the same Mañé potential at level a. This is also reflected in the following proposition by the fact that the involved orbits are contained in the set $\{H = a\}$.

Proposition 8.2. *Given $q_0 \neq q_1$, there exist $\tau \in]0, \infty]$ and an orbit*

$$(q(s), p(s)) \colon (-\tau, 0] \to \mathbb{R}^d \times \mathbb{R}^{d*}$$

such that $q(0) = q_1$, $A_s^0(q_0, q(s)) - as = \Phi^a(q(s), q_1)$,

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) = \Phi^a(q_0, q_1)$$

and $H(q(s), p(s)) = a$ for each $s \in (-\tau, 0]$. If, moreover, τ is finite, then $q(-\tau) = q_0$.

Proof. If $q_0 \neq q_1$, then either the functions $t \mapsto A^t(q_0, q_1) + at$ reach their minimum at some finite time $\tau > 0$, or they have a minimizing sequence $\tau_n \to \infty$. This follows from lemma 3.3.

In the first case, there exists an orbit $(q(t), p(t))\colon [-\tau, 0] \to \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $q(-\tau) = q_0$, $q(0) = q_1$, and

$$\int_{-\tau}^{0} p \cdot \dot{q} - H(q,p)\,\mathrm{d}t = A^{\tau}(q_0, q_1) = \Phi^a(q_0, q_1) - \tau a.$$

We obtain, for each $s \in [-\tau, 0]$, that

$$\begin{aligned}
\Phi^a(q_0, q_1) - a\tau &= \int_{-\tau}^{0} p \cdot \dot{q} - H(q,p)\,\mathrm{d}t \\
&= \int_{-\tau}^{s} p \cdot \dot{q} - H(q,p)\,\mathrm{d}t + \int_{s}^{0} p \cdot \dot{q} - H(q,p)\,\mathrm{d}t \\
&\geqslant A^{s+\tau}(q_0, q(s)) + A^{-s}(q(s), q_1) \\
&\geqslant \Phi^a(q_0, q(s)) - a(s+\tau) + \Phi^a(q(s), q_1) + as \\
&\geqslant \Phi^a(q_0, q_1) - a\tau.
\end{aligned}$$

We conclude that all these inequalities are equalities; hence,

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) = \Phi^a(q_0, q_1).$$

We also deduce from the above chain of inequalities that $A^{-s}(q(s), q_1) - as = \Phi^a(q(s), q_1)$, which implies that the function $t \mapsto A^t(q(s), q_1) + at$ is minimal for $t = -s$. Taking $s \in \left]-\sigma, 0\right[$, we can differentiate with respect to t at $t = -s$ and get

$$\partial_{t|t=-s} S^t(q(s), q_1) + a = 0.$$

Recalling the equality

$$\partial_t S^{-s}(q(s), q_1) + H(q_1, p(0)) = 0$$

(because $p(0) = \rho_1(-s, q(s), q_1)$ in the notation of § 2), we conclude that

$$H(q_1, p(0)) = a$$

and, since the Hamiltonian is constant on Hamiltonian orbits, $H(q(t), p(t)) = a$ for each t.

In the second case, there exists a sequence of orbits $(q_n(t), p_n(t))$ on $[-\tau_n, 0]$ such that

$$\int_{-\tau_n}^{0} p_n \cdot \dot{q}_n - H(q_n, p_n)\,\mathrm{d}t + a\tau_n = A^{\tau_n}(q_0, q_1) + a\tau_n \leqslant \Phi^a(q_0, q_1) + \delta_n,$$

where $\delta_n \to 0$. Let us denote $h_n := H(q_n(s), p_n(s))$; this does not depend on s. By lemma 3.4 and the above inequality, we have

$$\frac{m}{M}\tau_n h_n - (m+M)\tau_n \leqslant \int_{-\tau_n}^{0} p_n \cdot \partial_p H(q_n, p_n) - H(q_n, p_n)\,\mathrm{d}t \leqslant \Phi^a(q_0, q_1) + \delta_n;$$

hence, the sequence h_n is bounded. As a consequence, the curves $p_n(s)$ are uniformly bounded; hence, so is $\dot{q}_n(s) = \partial_p H(q_n(s), p_n(s))$. On each compact interval of time $[s, 0]$, the curves $x_n(t) = (q_n(t), p_n(t))$ are thus uniformly bounded, and hence uniformly Lipschitz. Up to taking a subsequence, we can thus assume that the curves $x_n(t)$ converge, uniformly on compact time intervals, to a Hamiltonian orbit $x(t) = (q(t), p(t))\colon (-\infty, 0] \to \mathbb{R}^d \times \mathbb{R}^{d*}$. Passing to the limit in the inequality

$$\Phi^a(q_0, q_n(s)) + \Phi^a(q_n(s), q_1) \leqslant \Phi^a(q_0, q_1) + \delta_n,$$

which holds for each $s \in [-\tau_n, 0]$, yields

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) \leqslant \Phi^a(q_0, q_1),$$

which must actually be an equality. We prove as in the first case that $H(q_1, p(0)) = a$, thus $H(q(s), p(s)) \equiv a$. □

The projected Aubry set $\mathcal{A}$ can be characterized in terms of the Mañé potential (see also [13]).

PROPOSITION 8.3. *The following statements are equivalent for a point q_0 and a real number a, where we denote by u the function $\Phi^a(q_0, \cdot)$:*

(i) $q_0 \in \mathcal{A}$ *and* $a = \alpha$*;*

(ii) $\boldsymbol{T}^t u(q_0) + ta = u(q_0) = 0$ *for each* $t \geqslant 0$*;*

(iii) *the function u is a weak KAM solution at level a;*

(iv) *u is differentiable at q_0.*

Proof. (i) $\implies$ (ii). This follows from proposition 7.10, since u is a subsolution at level $a = \alpha$.

(ii) $\implies$ (iii). Let us fix $t > 0$ and q_1. We have to prove that there exists θ such that $u(q_1) \geqslant u(\theta) + A^t(\theta, q_1) + ta$ (this inequality is then an equality). If $q_1 = q_0$, the existence of this point follows from the equality $\boldsymbol{T}^t u(q_0) + ta = u(q_0)$.

If $q_1 \neq q_0$, we can apply proposition 8.2 to this pair of points. With the notation of proposition 8.2, if $\tau \geqslant t$, then the point $\theta = q(-t)$ fulfils our demand. If $\tau < t$, then we set $s = t - \tau$. We have $q(-\tau) = q_0$ and $A^\tau(q_0, q_1) + a\tau = u(q_1)$. Since $\boldsymbol{T}^s u(q_0) + sa = u(q_0)$, there exists θ such that $u(\theta) + A^s(\theta, q_0) + sa = u(q_0) = 0$. The infimum in the definition of $\boldsymbol{T}^s u(q_0)$ exists because u is Lipschitz. We conclude that

$$u(\theta) + A^t(\theta, q_1) + at \leqslant u(\theta) + A^s(\theta, q_0) + sa + A^\tau(q_0, q_1) + a\tau = u(q_1).$$

(iii) $\implies$ (iv). If u is a weak KAM solution, then it has a proximal superdifferential at each point. Conversely, if v is a C^1 subsolution, then $u - v$ has a minimum at q_0; hence, $\mathrm{d}v(q_0)$ is a subdifferential of u at q_0. The function u both has a superdifferential and a subdifferential at q_0; hence, it is differentiable at q_0.

(iv) $\implies$ (i). If $a > \alpha$, or if q_0 does not belong to $\mathcal{A}$, then there exists a C^1 subsolution v at level a that is strict near q_0. We can then slightly perturb the function v near q_0 and build a subsolution w such that $\mathrm{d}w(q_0) \neq \mathrm{d}v(q_0)$. In view of the characterization of u as the largest subsolution vanishing at q_0, we conclude that $\mathrm{d}v(q_0)$ and $\mathrm{d}w(q_0)$ are subdifferentials of u at q_0; hence, u is not differentiable at this point. □

The Mañé potential also allows us to build weak KAM solutions in the non-periodic case by the Busemann method (see [10,13]). Let q_n be a sequence of points of $\mathbb{R}^d$ such that $|q_n| \geqslant n$. We consider the sequence of functions

$$u_n(q) = \Phi^a(q_n, q) - \Phi^a(q_n, q_0).$$

By construction, $u_n(q_0) = 0$, and it follows from the triangle inequality that the functions u_n are equi-Lipschitz. We can then assume, without loss of generality, that the functions u_n converge, uniformly on compact sets, to a Lipschitz limit $u(q)$.

Proposition 8.4. *The limit function $u(q)$ is a weak KAM solution at level a.*

Proof. The functions u_n are all subsolutions at level a, which means that $u_n(q_1) - u_n(q_0) \leqslant A^t(q_0, q_1) + ta$ for each $t \geqslant 0$, q_0, q_1. At the limit $n \to \infty$, we obtain that $\boldsymbol{T}^t u + ta \geqslant u$ for each $t \geqslant 0$.

We have to prove that $\boldsymbol{T}^t u + ta \leqslant u$ for all $t \geqslant 0$. Let us fix a point q and a time $t \geqslant 0$, and consider a sequence t_n such that

$$A^{t_n}(q_n, q) + at_n \leqslant \Phi^a(q_n, q) + 1/n.$$

This inequality implies that

$$\frac{1}{2Mt_n}|q_n - q|^2 \leqslant 1 + (M - a)t_n + 2\sqrt{2m(M + a)}|q_n - q|$$

and, since $|q_n - q| \to \infty$, that $t_n \to \infty$. When n is large enough, we have $t_n \geqslant t$ and there exists $\theta_n \in \mathbb{R}^d$ such that $A^{t_n}(q_n, q) = A^{t_n - t}(q_n, \theta_n) + A^t(\theta_n, q)$. This implies that

$$\begin{aligned}\Phi^a(q_n, q) &\geqslant A^{t_n}(q_n, q) + at_n - 1/n \\ &\geqslant A^{t_n - t}(q_n, \theta_n) + a(t_n - t) + A^t(\theta_n, q) + at - 1/n \\ &\geqslant \Phi^a(q_n, \theta_n) + A^t(\theta_n, q) + at - 1/n.\end{aligned}$$

This inequality implies that

$$u_n(q) \geqslant u_n(\theta_n) + A^t(\theta_n, q) + at - 1/n.$$

Since the functions u_n are equi-Lipschitz, this implies that the sequence θ_n is bounded, by lemma 3.3. By taking a subsequence, we assume that θ_n has a limit θ, and, at the limit, we obtain

$$u(q) \geqslant u(\theta) + A^t(\theta, q) + at,$$

which implies that $u(q) \geqslant \boldsymbol{T}^t u(q) + ta$. □

9. A return to the periodic case

A more precise link can be established between the contents of §§ 6 and 7 under the assumption that H is periodic (see hypothesis 6.2). It is useful first to expose a variant of § 7 adapted to the periodic case. We leave as exercises the proofs that are direct adaptations of those given above. From now on, we assume hypotheses 1.8, 2.1, 3.2, 5.1 and 6.2 hold.

THEOREM 9.1. *If (HJa) admits a periodic subsolution, then it admits a periodic $C^{1,1}$ subsolution. Moreover, the set of periodic $C^{1,1}$ subsolutions is dense in the set of all periodic subsolutions for the uniform topology.*

DEFINITION 9.2. The periodic critical value of H is the real number $\alpha(0)$, defined as the infimum of all real numbers a such that (HJa) has a periodic subsolution. The periodic subsolutions at level $\alpha(0)$ are called critical periodic subsolutions.

DEFINITION 9.3. The projected periodic Aubry set is the set $\mathcal{A}(0) \subset \mathbb{T}^d$ of points q such that the equality $H(q, \mathrm{d}u(q)) = \alpha(0)$ holds for all C^1 periodic critical subsolutions u.

LEMMA 9.4. *If $q \in \mathcal{A}(0)$, then all C^1 critical periodic subsolutions u have the same differential at q. In other words, the restriction $\Gamma_{u|\mathcal{A}}$ does not depend on the C^1 critical periodic subsolution u.*

PROPOSITION 9.5. *There exists a $C^{1,1}$ periodic critical subsolution u_0 such that $H(q, \mathrm{d}u_0(q)) < \alpha(0)$ outside of $\mathcal{A}(0)$.*

Without surprise, we define the periodic Aubry set $\mathcal{A}^*(0)$ as

$$\mathcal{A}^*(0) := \Gamma_{u_0|\mathcal{A}},$$

with u_0 given by the proposition (there is not a single u_0, but the Aubry set is well defined).

PROPOSITION 9.6. *The set $\mathcal{A}^*(0) \subset \mathbb{T}^d \times \mathbb{R}^{d*}$ is compact, non-empty and invariant.*

Proof. Let us prove that $\mathcal{A}(0)$; hence, $\mathcal{A}^*(0)$ is not empty. Assuming by contradiction that it is empty, then the equality $H(q, \mathrm{d}u_0(q)) < \alpha(0)$ would hold for all $q \in \mathbb{R}^d$. Since the function $q \mapsto H(q, \mathrm{d}u_0(q))$ is periodic, we could conclude that $\sup_q H(q, \mathrm{d}u_0(q)) < \alpha(0)$, which is contradicts the definition of $\alpha(0)$. □

We are now in a position to specify the connection with the invariant sets introduced in § 6.

PROPOSITION 9.7. *In the periodic case, we have the equality*

$$\mathcal{A}^*(0) = \bigcap_u \mathcal{I}^*(u),$$

where the intersection is taken on all periodic weak KAM solutions.

Proof. The inclusion $\mathcal{A}^*(0) \subset \bigcap_u \mathcal{I}^*(u)$ is proved as in § 7. Our goal is to prove the other inclusion. Let u_0 be a $C^{1,1}$ periodic subsolution that is strict outside of $\mathcal{A}(0)$. The map $t \mapsto \boldsymbol{T}^t u_0 + t\alpha(0)$ is non-decreasing. In addition, the functions $\boldsymbol{T}^t u_0 + t\alpha(0)$ are equi-Lipschitz, and they coincide with u_0 on $\mathcal{A}$; hence, they are equi-bounded. As a consequence, $\boldsymbol{T}^t u_0 + t\alpha \to u_\infty$ uniformly as $t \to \infty$.

CLAIM 9.8. *The limit u_∞ is a periodic weak KAM solution such that $u_0 < u_\infty$ outside of $\mathcal{A}(0)$.*

In order to prove that u_∞ is a weak KAM solution, it is enough to notice that the function $\boldsymbol{T}^{t+s}u_0+(t+s)\alpha(0)$ converges both to u_∞ and to $\boldsymbol{T}^s u_\infty + s\alpha(0)$ when $t\to\infty$. This implies, as desired, that $\boldsymbol{T}^s u_\infty + s\alpha(0) = u_\infty$ for each $s\geqslant 0$.

We know that $u_\infty \geqslant u_0$, with equality on $\mathcal{A}(0)$. Conversely, let us consider a point q such that $u_\infty(q)=u_0(q)$. The point q is minimizing the difference $u_\infty - u_0$. Since u_∞ is semi-concave and u_0 is C^1, the function u_∞ must be differentiable at q with $du_\infty(q)=du_0(q)$. Since u_∞ solves the Hamilton–Jacobi equation at its points of differentiability, we conclude that $H(q,\mathrm{d}u_0(q))=H(q,\mathrm{d}u_\infty(q))=\alpha(0)$; hence, $q\in\mathcal{A}(0)$. We have proved the claim.

Let us now establish that $\mathcal{I}(u_\infty)=\mathcal{A}(0)$, which implies the proposition. By Lemma 7.11, we have $\check{\boldsymbol{T}}^t u_\infty - t\alpha = u_\infty$ on $\mathcal{I}(u_\infty)$ for each $t\geqslant 0$. Setting $\epsilon(t)=\sup(u_\infty - \boldsymbol{T}^t u_0 - t\alpha(0))$, we have

$$u_\infty \geqslant u_0 \geqslant \check{\boldsymbol{T}}^t\circ \boldsymbol{T}^t u_0 \geqslant \check{\boldsymbol{T}}^t(u_\infty-\epsilon(t)-t\alpha(0)) \geqslant \check{\boldsymbol{T}}^t u_\infty - \epsilon(t) - t\alpha(0) = u_\infty - \epsilon(t)$$

on $\mathcal{I}(u_\infty)$. Since this holds for all $t\geqslant 0$, and since $\lim_{t\to\infty}\epsilon(t)=0$, we conclude that $u_0=u_\infty$ on $\mathcal{I}(u_\infty)$. On the other hand, we have seen that $u_0<u_\infty$ outside of $\mathcal{A}(0)$; hence, $\mathcal{I}(u_\infty)\subset\mathcal{A}(0)$. □

We finish with a simple remark, which is specific to the periodic case.

Proposition 9.9. *All periodic weak KAM solutions have level* $\alpha(0)$.

Proof. Let u_0 be a critical periodic subsolution, and let u be a periodic weak KAM solution at level a. Since u is a periodic subsolution at level a, the definition of $\alpha(0)$ implies that $a\geqslant\alpha(0)$. On the other hand, there exists a constant C such that $u-C\leqslant u_0\leqslant u+C$, which implies

$$u=\boldsymbol{T}^t u + ta \geqslant \boldsymbol{T}^t u_0 - C + ta \geqslant u_0 + t(a-\alpha(0)) - C \geqslant u + t(a-\alpha(0)) - 2C.$$

We obtain that $t(a-\alpha(0))\leqslant 2C$ for each $t\geqslant 0$; hence, $a-\alpha(0)\leqslant 0$. □

10. The Lagrangian

In most expositions of weak KAM theory (see, for example, [3,5,11,20]), the Lagrangian plays an important role. In this section, we relate it to our main objects in order to facilitate the connection with the core of the literature, where what we state here as properties are usually taken as definitions. We define the Lagrangian as

$$\begin{aligned} L\colon \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d &\to \mathbb{R}, \\ (t,q,v) &\mapsto \sup_{p\in(\mathbb{R}^d)^*}(p\cdot v - H(t,q,p)). \end{aligned}$$

By standard results on convex analysis (see, for example, [21]) we then have

$$H(t,q,p)=\sup_{v\in\mathbb{R}^d}(p\cdot v - L(t,q,v)).$$

We obviously have the Legendre inequality

$$H(t,q,p)+L(t,q,v)\geqslant p\cdot v$$

for all t, q, p, v. This inequality is an equality if and only if

$$p = \partial_v L(t, q, v) \quad \text{or, equivalently,} \quad v = \partial_p H(t, q, p).$$

Let $q(t)\colon]t_0, t_1[$ be a curve, The *action* of q is the number

$$\int_{t_0}^{t_1} L(t, q(t), \dot{q}(t)) \, \mathrm{d}t.$$

We can also call it Lagrangian action if we want to distinguish from the previously defined Hamiltonian action. The Lagrangian and Hamiltonian actions are related as follows.

The Hamiltonian action of a curve $(q(t), p(t))$ is smaller than the Lagrangian action of its projection $q(t)$, with equality if and only if $p(t) \equiv \partial_v L(t, q(t), \dot{q}(t))$. In particular, the Hamiltonian action of an orbit is equal to the Lagrangian action of its projection.

Lemma 10.1. *Let q_0 and q_1 be two points of $\mathbb{R}^d$, and let t_0 and t_1 be two times, with $0 < t_1 - t_0 < \sigma$. If $(q(s), p(s))$ is the orbit satisfying $q(t_0) = q_0$, $q(t_1) = q_1$, we have*

$$S_{t_0}^{t_1}(q_0, q_1) = \int_{t_0}^{t_1} L(s, q(s), \dot{q}(s)) \, \mathrm{d}s = \min_{\theta(s)} \int_{t_0}^{t_1} L(s, \theta(s), \dot{\theta}(s)) \, \mathrm{d}s,$$

where the minimum is taken on the set of Lipschitz curves $\theta\colon [t_0, t_1] \to \mathbb{R}^d$ that satisfy $\theta(t_0) = q_0$ and $\theta(t_1) = q_1$.

Proof. Since $S_{t_0}^{t_1}(q_0, q_1)$ is the Hamiltonian action of the unique orbit $(q(t), p(t))$, it is also the Lagrangian action of the curve $q(t)$:

$$S_{t_0}^{t_1}(q_0, q_1) = \int_{t_0}^{t_1} L(s, q(s), \dot{q}(s)) \, \mathrm{d}s.$$

The function $u(t, q) := S_{t_0}^{t}(q_0, q)$ solves (HJ) on $]t_0, t_1[$. Let us now consider any Lipschitz curve $\theta(s)\colon [t_0, t_1] \to \mathbb{R}^d$ satisfying $\theta(t_0) = q_0$ and $\theta(t_1) = q_1$, and write

$$\begin{aligned}
\int_{t_0}^{t_1} L(s, \theta(s), \dot{\theta}(s)) \, \mathrm{d}s &\geqslant \int_{t_0}^{t_1} \partial_q u(s, \theta(s)) \cdot \dot{\theta}(s) - H(s, \theta(s), \partial_q u(s, \theta(s))) \, \mathrm{d}s \\
&= \int_{t_0}^{t_1} \partial_q u(s, \theta(s)) \cdot \dot{\theta}(s) - \partial_t u(s, \theta(s)) \, \mathrm{d}s \\
&= u(t_1, q_1) - u(t_0, q_0) \\
&= S_{t_0}^{t_1}(q_0, q_1).
\end{aligned}$$

□

The following proposition is usually taken as the definition of A.

Proposition 10.2. *Given two points q_0 and q_1 and two times $t_0 < t_1$, we have*

$$A_{t_0}^{t_1}(q_0, q_1) = \min_{\theta(s)} \int_{t_0}^{t_1} L(s, \theta(s), \dot{\theta}(s)) \, \mathrm{d}s,$$

where the minimum is taken on the set of Lipschitz curves $\theta\colon [t_0, t_1] \to \mathbb{R}^d$ *that satisfy* $\theta(t_0) = q_0$ *and* $\theta(t_1) = q_1$.

It is part of the statement that the minimum is achieved. This is usually called Tonelli's theorem. The statement can be extended to absolutely continuous curves instead of Lipschitz curves, but this setting is not useful for our discussion.

Proof. For n large enough, we have $(t_1 - t_0)/n < \sigma$; hence, setting

$$\tau_i = t_0 + i(t_1 - t_0)/n,$$

we obtain

$$\begin{aligned} &A_{t_0}^{t_1}(q_0, q_1) \\ &\quad = \min_{(\theta_1,\dots,\theta_{n-1})} (S_{t_0}^{\tau_1}(q_0, \theta_1) + S_{\tau_1}^{\tau_2}(\theta_1, \theta_2) + \cdots + S_{\tau_{n-1}}^{t_1}(\theta_{n-1}, q_1)) \\ &\quad = \min_{(\theta_1,\dots,\theta_{n-1})} \left(\min_{\theta(s)} \int_{t_0}^{\tau_1} L(s, \theta(s), \dot\theta(s))\,\mathrm{d}s + \cdots + \min_{\theta(s)} \int_{\tau_{n-1}}^{t_1} L(s, \theta(s), \dot\theta(s))\,\mathrm{d}s \right) \\ &\quad = \min_{\theta(s)} \int_{t_0}^{t_1} L(s, \theta(s), \dot\theta(s))\,\mathrm{d}s. \end{aligned}$$

□

Acknowledgements

The author thanks Lyonell Boulton and Sergei Kuksin for organizing the CANPDE session on weak KAM theory, and Albert Fathi for many useful comments and suggestions.

Appendix A. Some technical results

Proposition A.1. *A Lipschitz map* $F\colon \mathbb{R}^d \to \mathbb{R}^d$ *that satisfies* $\mathrm{Lip}(F - \mathrm{Id}) < 1$ *is a bi-Lipschitz homeomorphism of* $\mathbb{R}^d$. *Its inverse is Lipschitz, and* $\mathrm{Lip}(F^{-1}) \leqslant (1-k)^{-1}$. *If* F *is* C^1, *then so is* F^{-1}.

Proof. The equation $F(q) = \theta$ can be rewritten

$$\theta - (F(q) - q) = q.$$

The map on the left being contracting, we conclude that F is invertible. We now write

$$|x_1 - x_0| - |F(x_1) - F(x_0)| \leqslant |(F(x_1) - x_1) - (F(x_0) - x_0)| \leqslant k|x_1 - x_0|$$

and deduce that $|F(x_1) - F(x_2)| \geqslant (1-k)|x_1 - x_0|$. □

Proposition A.2. *Let* $F\colon \mathbb{R}^d \to \mathbb{R}^d$ *be a* C^1, c*-monotone map on* $\mathbb{R}^d$, *with* $c > 0$. *Then* F *is a diffeomorphism from* $\mathbb{R}^d$ *onto itself.*

Proof. Let us consider a point $\theta \in \mathbb{R}^d$, and the line $\theta(s) = F(0) + s(\theta - F(0))$. Since F is a local diffeomorphism around 0, the points $\theta(s)$ for small s have a unique preimage $p(s)$. Let S be the infimum of the positive real numbers s such that the equation $F(p) = \theta(s)$ does not have a solution in $\mathbb{R}^d$. The curve $p(s)$ is well defined, C^1 and Lipschitz on $[0, S[$; hence, if S is finite, it extends at S with $F(p(S)) = \theta(S)$. Since F is a local diffeomorphism at $p(S)$, the points near $\theta(S)$ have preimages, which contradicts the definition of S. Hence, S cannot be finite. □

Lemma A.3. *Let A be a $d \times d$ matrix, such that $A \geqslant a\,\mathrm{Id}$ in the sense of quadratic forms, and $\|A\| \leqslant b$. Then $A^{-1} \geqslant (a/b^2)I$ in the sense of quadratic forms.*

Proof. We have

$$(A^{-1}v, v) = (AA^{-1}v, A^{-1}v) \geqslant a|A^{-1}v|^2 \geqslant a(|v|/b)^2.$$

□

The following important result appears in Fathi's book (see [14]) on weak KAM theory (the proof is also his).

Proposition A.4. *Let $u\colon \mathbb{R}^d \to \mathbb{R}$ be a function and K be a positive number. Let $\mathcal{I} \in \mathbb{R}^d$ be the set of points where u has both a K-superdifferential and a K-subdifferential. Then, the function u is differentiable at each point of $\mathcal{I}$, and the function $q \mapsto \mathrm{d}u(q)$ is $6K$-Lipschitz on $\mathcal{I}$.*

Proof. For each $q \in \mathcal{I}$, there exists a unique $l(q) \in \mathbb{R}^{d*}$ such that

$$|u(q+\theta) - u(q) - l(q)\cdot\theta| \leqslant K\|\theta\|^2.$$

We conclude that $l(q)$ is the differential of u at q, and we have to prove that the map $q \mapsto l(q)$ is Lipschitz on $\mathcal{I}$. We have, for each q, θ and y in H:

$$\begin{aligned} l(q)\cdot(y+\theta) - K\|y+\theta\|^2 \leqslant u(q+y+\theta) - u(q) \leqslant l(q)\cdot(y+\theta) + K\|y+\theta\|^2,\\ l(q+y)\cdot(-y) - K\|y\|^2 \leqslant u(q) - u(q+y) \leqslant l(q+y)\cdot(-y) + K\|y\|^2,\\ l(q+y)\cdot(-\theta) - K\|\theta\|^2 \leqslant u(q+y) - u(q+y+\theta) \leqslant l(q+y)\cdot(-\theta) + K\|\theta\|^2. \end{aligned}$$

Taking the sum, we obtain

$$|(l(q+y) - l(q))\cdot(y+\theta)| \leqslant K\|y+\theta\|^2 + K\|y\|^2 + K\|z\|^2.$$

By a change of variables, we get

$$|(l(q+y) - l(q))\cdot\theta| \leqslant K\|\theta\|^2 + K\|y\|^2 + K\|\theta - y\|^2.$$

Taking $\|\theta\| = \|y\|$, we obtain

$$|(l(q+y) - l_(q))\cdot(\theta)| \leqslant 6K\|\theta\|\|y\|$$

for each θ such that $\|\theta\| = \|y\|$. We conclude that

$$\|l(q+y) - l(q)\| \leqslant 6K\|y\|.$$

□

Lemma A.5. *Let u be a finite-valued function which is the infimum of a family $\mathcal{F}$ of equi-semi-concave functions: $u = \inf_{f\in\mathcal{F}} f$. Then the function u is semi-concave.*

It is important in the statement to assume that u is really finite valued at each point.

Proof. Let us assume that the functions in $\mathcal{F}$ are k-semi-concave. Given a point $q_0 \in \mathbb{R}^d$ let $f_n(q) = a_n + p_n \cdot q + k/2\|q\|^2$ be a sequence of functions of $\mathcal{F}$ such that $f_n(q_0) \to u(q_0)$. We have $f_n(q) \leqslant f_n(q_0) + p_n \cdot (q - q_0) + k/2\|q - q_0\|^2$ for some sequence $p_n \in \mathbb{R}^*$. If the sequence p_n is bounded, then we can take the limit along a subsequence and get the inequality

$$u(q) \leqslant u(q_0) + p \cdot (q - q_0) + k/2\|q - q_0\|^2.$$

If this holds for each q_0, we conclude that u is k-semi-concave. Let us now prove that p_n is bounded. If this is not true, there would exist a point q such that $p_n \cdot (q - q_0)$ is not bounded from below. This would imply that

$$u(q) = \inf_{f \in \mathcal{F}} f(q) \leqslant \inf_n f_n(q) \leqslant \inf_n (f_n(q_0) + p_n \cdot (q - q_0) + k/2\|q - q_0\|^2) = -\infty,$$

which would contradict the finiteness of u at q. □

References

1 L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel. Axioms and fundamental equations of image processing. *Arch. Ration. Mech. Analysis* **123** (1993), 199–257.

2 M. C. Arnaud. Pseudographs and the Lax–Oleinik semi-group: a geometric and dynamical interpretation'. *Nonlinearity* **24** (2011), 71.

3 P. Bernard. Connecting orbits of time dependent Lagrangian systems. *Annales Inst. Fourier* **52** (2002), 1533–1568.

4 P. Bernard. Existence of $C^{1,1}$ critical subsolutions of the Hamilton–Jacobi equation on compact manifolds. *Annales Scient. Éc. Norm. Sup.* **40** (2007), 445–452.

5 P. Bernard. Symplectic aspects of Mather theory. *Duke Math. J.* **136** (2007), 401–420.

6 P. Bernard. The dynamics of pseudographs in convex Hamiltonian systems. *J. Am. Math. Soc.* **21** (2008), 615–669.

7 P. Bernard. Lasri–Lions regularisation and a lemma of Ilmanen. *Rend. Sem. Mat. Univ. Padova* **124** (2010), 221–229.

8 S. Biton. Nonlinear monotone semigroups and viscosity solutions. *Annales Inst. H. Poincaré Analyse Non Linéaire* **18** (2001), 383–402.

9 M. Chaperon. Une idée du type géodésiques brisées pour les systèmes hamiltoniens. *C. R. Acad. Sci. Paris I* **298** (1984), 293–296.

10 G. Contreras. Action potential and weak KAM solutions. *Calc. Var. PDEs* **13** (2001), 427–458.

11 G. Contreras, R. Iturriaga, G. P. Paternain and M. Paternain. Lagrangian graphs, minimizing measures and Mañé's critical values. *Geom. Funct. Analysis* **8** (1998), 788–809.

12 A. Fathi. Regularity of C^1 solutions of the Hamilton–Jacobi equation. *Annales Fac. Sci. Toulouse Math.* **12** (2003), 479–516.

13 A. Fathi. Weak KAM from a PDE point of view: viscosity solutions of the Hamilton–Jacobi equation and Aubry set. *Proc. R. Soc. Edinb.* A **142**, 1193–1236.

14 A. Fathi. *Weak KAM theorem in Lagrangian dynamics.* (In preparation.)

15 A. Fathi and A. Siconolfi. Existence of C^1 critical subsolutions of the Hamilton–Jacobi equation. *Invent. Math.* **155** (2004), 363–388.

16 A. Fathi and A. Siconolfi. PDE aspects of Aubry–Mather theory for quasiconvex Hamiltonians. *Calc. Var. PDEs* **22** (2005), 185–228.

17 P. L. Lions, G. Papanicolaou and S. R. S. Varadhan. Homogenization of Hamilton–Jacobi equations. Preprint (1988).

18 D. M. McDuff and D. Salamon. *Introduction to symplectic topology* (Oxford University Press, 1995).

19 R. Mané. Lagrangian flows: the dynamics of globally minimizing orbits. *Bull. Brazil. Math. Soc.* **28** (1997), 141–153.

20 J. N. Mather. Variational construction of connecting orbits. *Annales Inst. Fourier* **43** (1993), 1349–1386.

21 R. T. Rockafellar. *Convex analysis* (Princeton University Press, 1997).

22 Q. Wei. PhD thesis.

(*Issued 7 December 2012*)