A BIPARTITIONAL FUNCTION ARISING IN HALL'S ALGEBRA

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1. Introduction. Hall's algebra **(3)** is an algebra over the field $V(p)$ of rational functions in the indeterminate p with coefficients in the field V of complex numbers. The basis of the algebra consists of elements G_λ which are in one–one correspondence with the set of all partitions (λ) and whose multiplication "constants" are the "Hall polynomials" $g^\lambda_{\alpha\beta}(p)$, i.e.

$$G_\alpha G_\beta = \sum_\lambda g^\lambda_{\alpha\beta}(p) G_\lambda,$$

where (α), (β), and (λ) are partitions of m, n, and $m + n$ respectively. $g^\lambda_{\alpha\beta}(p)$ denotes the number of subgroups F of type (β) in an Abelian p-group E of type (λ) which have a quotient group E/F of type (α). Hall has proved the following important result concerning the $g^\lambda_{\alpha\beta}(p)$, a result which indicates an interesting connection with the Schur functions $\{\lambda\}$, $\{\alpha\}$, and $\{\beta\}$. (Hall's result is, in fact, stated in more general terms.)

THEOREM 1. *If $e^\lambda_{\alpha\beta}$ is the coefficient of $\{\lambda\}$ in the product $\{\alpha\}\{\beta\}$, then $g^\lambda_{\alpha\beta}(p) = 0$ for all primes p if $e^\lambda_{\alpha\beta} = 0$. Otherwise $g^\lambda_{\alpha\beta}(p)$ is a polynomial in p of degree $n_\lambda - n_\alpha - n_\beta$, n_μ denoting*

$$\sum_i \frac{\mu_i'(\mu_i' - 1)}{2}$$

(μ' being the partition conjugate to μ) and the coefficient of the highest power of p is precisely $e^\lambda_{\alpha\beta}$.

The determination of the polynomial $g^\lambda_{\alpha\beta}(p)$ for any given partitions (λ), (α), and (β) is, as yet, an unsolved problem. The author has solved the problem in a particular case **(1)**.

Bipartitional functions arise in Hall's algebra, just as they do in the classical algebra of symmetric functions of which Hall's algebra is a generalization. The bipartitional functions with which we are concerned can best be defined as the bipartitional function $R_{\lambda\mu}(p)$ satisfying

$$\{\lambda\} = \sum_\mu R_{\lambda\mu}(p) G_\mu,$$

(λ) and (μ) being partitions of an integer n, the summation being taken over all partitions (μ) of n. $R_{\lambda\mu}(p)$ is a generalization of the known bipartitional

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function connecting the Schur function $\{\lambda\}$ and the monomial symmetric function M_μ, given when $p = 1$, i.e. we can write

$$\{\lambda\} = \sum_\mu R_{\lambda\mu}(1) M_\mu.$$

The $R_{\lambda\mu}(p)$ are polynomials in p derived from the Hall polynomials and consist entirely of sums of powers of p, although this is difficult to prove in general, and it would be of great interest to be able to find explicit expressions for them in general. This is slightly different from the problem stated by Green (**3**, § 3); his $\Phi_{\lambda\mu}(t)$ are, in effect, the elements of the matrix inverse to the one formed by the $R_{\lambda\mu}(p)$ for all pairs of partitions (λ), (μ) of n.

In § 2 we discuss how the polynomials $R_{\lambda\mu}(p)$ can be calculated and prove some general results concerning them. In § 3 we prove a result which gives the polynomial $R_{\lambda\mu}(p)$ when (λ) is a hook partition but whose main interest lies in the fact that it leads to a combinatorial rule for evaluating the polynomials $R_{\lambda\mu}(p)$ in other cases.

2. Following Hall (**4**), we make the following definitions:

$$A_r^* = G_{1^r}, \qquad H_r^* = \sum_\lambda G_\lambda, \quad \text{summed over all partitions } (\lambda) \text{ of } r,$$

$$A_\lambda^* = A_{\lambda_1}^* A_{\lambda_2}^* \ldots, \quad H_\lambda^* = H_{\lambda_1}^* H_{\lambda_2}^* \ldots, \quad \text{when } (\lambda) = (\lambda_1, \lambda_2, \ldots).$$

Hall has identified the elements of his algebra and the elements of the algebra of symmetric functions by means of the relationship $A_r = p^{r(r-1)/2} A_r^*$, A_r being the elementary symmetric function. From this it is easy to show that $A_\lambda = p^{n_{\lambda'}} A_\lambda^*$, where (λ') is the partition conjugate to (λ) and n_μ has been defined in Theorem 1. Having chosen the above identification of the two algebras, it is necessary to relate H_r and H_r^* because of the form of the identity connecting the A^* and the H^*. We thus also have $H_\lambda = H_\lambda^*$.

We can now think of A_λ as a product of terms such as G_{1^r}, i.e. as a Hall polynomial, so that we can write

$$A_\lambda = \sum_\mu P_{\lambda\mu}(p) G_\mu,$$

where the polynomial $P_{\lambda\mu}(p)$ is derived from the Hall polynomials and $P_{\lambda\mu}(1)$ is the known bipartitional function relating the elementary symmetric function A_λ and the monomials M_μ. The bipartitional functions $P_{\lambda\mu}(p)$ are used in the calculation of the $R_{\lambda\mu}(p)$ as follows.

We can expand a Schur function $\{\lambda\}$ as a determinant of elementary symmetric functions in the form (**7**)

$$\{\lambda\} = \sum_\alpha \xi_{\lambda\alpha} A_\alpha,$$

where (λ) and (α) are partitions of an integer n. We thus have

$$\{\lambda\} = \sum_\alpha \left[\xi_{\lambda\alpha} \sum_\mu P_{\alpha\mu}(p) G_\mu\right],$$

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where (μ) is a partition of n and the summations are taken over all partitions (α) and (μ) of n. By changing the orders of summation, we obtain

$$\{\lambda\} = \sum_{\mu} \Big[\sum_{\mu} \xi_{\lambda\alpha} P_{\alpha\mu}(p)\Big] G_{\mu}$$

so that

$$R_{\lambda\mu}(p) = \sum_{\alpha} \xi_{\lambda\alpha} P_{\alpha\mu}(p),$$

where the sum extends over all partitions (α) of n. For $n = 3$, we have

$$\begin{aligned} A_3 &= p^3 G_{1^3}, \\ A_{21} &= p G_{1^2} G_1 = p[g^{1^3}_{1^2,1}(p) G_{1^3} + g^{21}_{1^2,1}(p) G_{21} + g^{3}_{1^2,1}(p) G_3] \\ &= p[(p^2 + p + 1) G_{1^3} + G_{21}], \end{aligned}$$

using the result for the polynomial $g^{\lambda}_{\alpha,1^m}(p)$ (cf. **1**),

$$\begin{aligned} A_{1^3} = G_1 A_{1^2} &= G_1[(p+1) G_{1^2} + G_2], \quad \text{assuming the result for } A_{1^2} \\ &= (p+1) G_1 G_{1^2} + G_1 G_2 \\ &= (p+1)(p^2+p+1) G_{1^3} + (2p+1) G_{21} + G_3. \end{aligned}$$

Now

$$\begin{aligned} \begin{bmatrix} \{1^3\} \\ \{21\} \\ \{3\} \end{bmatrix} &= \begin{bmatrix} 1 & . & . \\ -1 & 1 & . \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} A_3 \\ A_{21} \\ A_{1^3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & . & . \\ -1 & 1 & . \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} p^3 & . & . \\ p(p^2+p+1) & p & . \\ (p+1)(p^2+p+1) & (2p+1) & 1 \end{bmatrix} \begin{bmatrix} G_{1^3} \\ G_{21} \\ G_3 \end{bmatrix} \\ &= \begin{bmatrix} p^3 & . & . \\ p^2+p & p & . \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} G_{1^3} \\ G_{21} \\ G_3 \end{bmatrix}, \end{aligned}$$

i.e. in tabular form, we have the polynomials $R_{\lambda\mu}(p)$ for $n = 3$ as

	(1^3)	(21)	(3)
(1^3)	p^3	.	.
(21)	$p^2 + p$	p	.
(3)	1	1	1

It is quite obvious that this method of calculating the polynomials $R_{\lambda\mu}(p)$ is not very practicable for large values of n, so it would be advantageous to have some other method of finding them. Before discussing some such method, we shall first prove some general results about the polynomials $R_{\lambda\mu}(p)$.

In the proof of the theorem which follows, we shall require certain definitions regarding the ordering of partitions. Let $(\alpha) = (\alpha_1, \ldots, \alpha_r)$ and $(\beta) = (\beta_1, \ldots, \beta_s)$ be partitions of an integer n. We shall assume that $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ and $\beta_1 \geqslant \beta_2 \geqslant \ldots$. Then in the "natural ordering" we say that (α) precedes (β) (or $(\alpha) > (\beta)$) if the first of the parts α_i of (α) which is different from β_i is greater than β_i. Thus the partitions of 6 would have a natural ordering

$$6, 51, 42, 41^2, 3^2, 321, 31^3, 2^3, 2^21^2, 21^4, 1^6.$$

The partitions conjugate to these are respectively

$$1^6, 21^4, 2^21^2, 31^3, 2^3, 321, 41^2, 3^2, 42, 51, 6,$$

which are not in reversed natural order. This second order will be called the "conjugate ordering." For $n < 6$, the reversed natural order and the conjugate order are the same.

THEOREM 2. (i) $R_{\lambda\mu}(p) = 0$ *if* $(\mu) > (\lambda)$ *in the natural order,*
(ii) $R_{\lambda\lambda}(p) = p^{n_\lambda}$,
(iii) *The lowest power of* p *in* $R_{\lambda\mu}(p)$ *is* p^{n_λ}.

Proof. (i) We have

$$R_{\lambda\mu}(p) = \sum_{\alpha} \xi_{\lambda\alpha} P_{\alpha\mu}(p).$$

By considering chains of subgroups whose quotient groups are elementary abelian groups of orders $p^{\lambda_1}, p^{\lambda_2}, \ldots,$ where $(\lambda) = (\lambda_1, \lambda_2, \ldots)$, it can be shown **(3)** that $P_{\lambda\mu}(p) = 0$ if $(\mu') > (\lambda)$ in the conjugate ordering. Kostka **(6)** has shown also that $\xi_{\lambda\mu} = 0$ if $(\mu) > (\lambda')$ in the conjugate ordering.

Thus $R_{\lambda\mu}(p) = 0$ for all partitions (α) satisfying both $(\alpha) > (\lambda')$ and $(\mu') > (\alpha)$ in the conjugate ordering. $R_{\lambda\mu}(p) = 0$, therefore, when $(\mu') > (\lambda')$ in the conjugate ordering. If $(\mu') > (\lambda')$ in the conjugate ordering, then $(\mu) > (\lambda)$ in the natural ordering, which is the required result.

(ii) If $(\mu) = (\lambda)$, then

$$R_{\lambda\lambda}(p) = \sum_{\alpha} \xi_{\lambda\alpha} P_{\alpha\lambda}(p),$$

which will be non-zero only when $(\alpha) = (\lambda')$ because if $(\alpha) > (\lambda')$ in the conjugate ordering then $\xi_{\lambda\alpha} = 0$, and if $(\lambda') > (\alpha)$ in the conjugate ordering then $P_{\alpha\lambda}(p) = 0$.

When $(\alpha) = (\lambda')$, $\xi_{\lambda\alpha} = 1$ and $R_{\lambda\lambda}(p) = P_{\lambda'\lambda}(p)$. Now, from the definition of $P_{\lambda\mu}(p)$, we see that the highest power of p in

$$p^{n_{\lambda'}} G_{1\lambda_1} G_{1\lambda_2} \ldots$$

is

$$p^{n_{\lambda'} + n_\mu - n_{\lambda'}} = p^{n_\mu},$$

using Theorem 1, and the lowest power of p in $P_{\lambda\mu}(p)$ is $p^{n_{\lambda'}}$. Thus the only power of p in $P_{\lambda'\lambda}(p)$ is p^{n_λ} and, since it is clear that $P_{\lambda'\lambda}(1) = 1$, we have $P_{\lambda'\lambda}(p) = p^{n_\lambda}$ and the result follows.

(iii) It is clear that, if $(\lambda) > (\mu)$ in the natural ordering, then $n_\lambda \leqslant n_\mu$. Now, $\xi_{\lambda\alpha} = 0$ if $(\alpha) > (\lambda')$ in the conjugate ordering. If $(\lambda') \geqslant (\alpha)$ in the conjugate ordering, then $(\lambda) \geqslant (\alpha')$ in the natural ordering. The lowest power of p in $P_{\alpha\mu}(p)$ is $p^{n_{\alpha'}}$ so that the lowest power of p in

$$\sum_\alpha \xi_{\lambda\alpha} P_{\alpha\mu}(p),$$

which must be given when $(\alpha') = (\lambda)$, is p^{n_λ}, which is the required result.

3. This is as far as we can go in general at the moment but the next theorem does take us a little nearer to the solution of the problem of finding the polynomials $R_{\lambda\mu}(p)$ in that it gives a formula for $R_{\lambda\mu}(p)$ when (λ) is a hook partition. However, its interest in this context lies more in the fact that it leads one towards a combinatorial rule for evaluating the polynomials in other cases.

THEOREM 3. *If λ is a hook partition with r parts and (μ) is any partition with s parts, (λ) and (μ) being partitions of an integer n, then*

$$R_{\lambda\mu}(p) = p^{r(r-1)/2}\frac{(p^{s-1}-1)(p^{s-2}-1)\dots(p^{s-r+1}-1)}{(p^{r-1}-1)(p^{r-2}-1)\dots(p-1)}.$$

Proof. For convenience, we denote

$$\frac{(p^\alpha-1)\dots(p^{\alpha-\beta+1}-1)}{(p^\beta-1)\dots(p-1)}$$

by $[\alpha, \beta]$, putting $[\alpha, 0] = 1$, and $p^{x(x-1)/2}$ by $C(x)$.

By definition,

$$\{\lambda\} = \sum_\mu R_{\lambda\mu}(p) G_\mu.$$

If (λ) is of the form $(k, 1^{r-1})$, it is known that

(1) $$\begin{aligned}\{\lambda\} &= A_r H_{k-1} - \{k-1, 1^r\}\\ &= A_r H_{k-1} - A_{r+1}H_{k-2} + A_{r+2}H_{k-3} - \dots\\ &= \sum_{i=1}^{s-r+1} (-1)^{i-1} C(r+i-1) G_{1^{r+i-1}} H_{k-i}.\end{aligned}$$

We shall now prove that

$$G_{1^r} H_{n-r} = \sum_\phi [t, r] G_\phi,$$

where (ϕ) is any partition of n and has t parts.

$$G_{1^r} H_{n-r} = G_{1^r} H^*_{n-r} = G_{1^r} \sum_\nu G_\nu,$$

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where (ν) is a partition of $n - r$. Let the different partitions of $n - r$ be denoted by ν_i, $i = 1, 2, \ldots, m$. Then

$$G_{1^r} \sum_{\nu} G_{\nu} = \sum_{\phi} \left[\sum_{i=1}^{m} g^{\phi}_{1^r, \nu_i}(p) \right] G_{\phi},$$

where (ϕ) is a partition of n and $g^{\phi}_{\lambda\mu}(p)$ is a Hall polynomial. Each subgroup of type (1^r) in an Abelian p-group of type (ϕ) will have a quotient group of type one of the (ν_i) so that every subgroup of type (1^r) is included in the sum

$$\sum_{i=1}^{m} g^{\phi}_{1^r, \nu_i}(p).$$

This sum is therefore equal to the number of subgroups of type (1^r) in an Abelian p-group of type (ϕ), which, from the work of Yeh, Delsarte, and Kinosita **(12, 2, 5)**, is equal to $[t, r]$ if the partition (ϕ) has t parts and $t \geqslant r$. Thus

$$G_{1^r} H_{n-r} = \sum_{\phi} [t, r] G_{\phi}.$$

Using the expression (1) above, we see that the coefficient of G_μ in $\{\lambda\}$ is

$$\begin{aligned} R_{\lambda\mu}(p) &= \sum_{i=1}^{s-r+1} (-1)^{i-1} C(r + i - 1)[s, r + i - 1] \\ &= \sum_{i=1}^{s-r+1} (-1)^{i-1} C(r + i - 1)[s, s - r - i + 1] \\ &= (-1)^{s-r} \sum_{i=1}^{s-r+1} (-1)^{i-1} C(s - i + 1)[s, i - 1], \end{aligned}$$

the last step following by reversing the order of the terms in the summation.

Now assume that

$$\sum_{i=1}^{u} (-1)^{i-1} C(s - i + 1)[s, i - 1] = (-1)^{u-1} C(s - u + 1)[s - 1, u - 1].$$

Then

$$\begin{aligned} \sum_{i=1}^{u+1} (-1)^{i-1} C(s - i + 1)[s, i - 1] &= (-1)^{u-1} C(s - u + 1)[s - 1, u - 1] \\ &\qquad + (-1)^{u} C(s - u)[s, u] \\ &= (-1)^{u} C(s - u)[s - 1, u - 1] \cdot \\ &\qquad \left\{ \frac{p^s - 1}{p^u - 1} - p^{s-u} \right\} \\ &= (-1)^{u} C(s - u)[s - 1, u - 1] \cdot \frac{p^{s-u} - 1}{p^u - 1} \\ &= (-1)^{u} C(s - u)[s - 1, u]. \end{aligned}$$

The result is true for $u = 1, 2$ and so, by induction, we have

$$\sum_{i=1}^{s-r+1} (-1)^{i-1} C(s-i+1)[s, i-1] = (-1)^{s-r} C(r)[s-1, s-r]$$
$$= (-1)^{s-r} C(r)[s-1, r-1],$$

so that $R_{\lambda\mu}(p) = C(r)[s-1, r-1]$, which is the required result.

In the Introduction, it was stated that the polynomials $R_{\lambda\mu}(p)$ are *sums* of positive integral powers of p although the result was difficult to prove in general. When (λ) is a hook partition, the proof of this result follows from Theorem 3. Sylvester **(11)** proved initially and, later, proofs were given by MacMahon **(9)**, and Riordan **(10)** that $C(r)[s-1, r-1]$ is the generating function for partitions into exactly $r-1$ unequal parts, none greater than $s-1$; i.e. the coefficient of p^n in $C(r)[s-1, r-1]$ is the number of partitions of weight n satisfying the given conditions so that this coefficient will necessarily be a non-negative integer. For example, if $r = 4$, $s = 6$, the partitions into 3 unequal parts none greater than 5 are

$$321, 421, 431, 432, 521, 531, 532, 541, 542, 543$$

which are partitions of $6, 7, 8, 9, 8, 9, 10, 10, 11, 12$, so that, if (λ) is any hook partition of the form $(k1^3)$ and (μ) is a partition of $k+3$ with 6 parts, then

$$R_{\lambda\mu}(p) = p^6 + p^7 + 2p^8 + 2p^9 + 2p^{10} + p^{11} + p^{12}.$$

This presents us with a different way of considering the polynomials $R_{\lambda\mu}(p)$. In fact, it appears that $R_{\lambda\mu}(p)$ might be the generating function for a class of partitions, defined by the partition (λ), subject to a "condition," defined by the partition (μ), and it is probable that this condition is one connected with the numerical values of the parts in the partitions enumerated. What class of partitions can we associate with the partition (λ)? When (λ) is a hook partition $(k1^r)$, we have seen that the class of partitions associated with (λ) contains all partitions with exactly r unequal parts. What if (λ) is not a hook partition? We can answer this to a certain extent by introducing MacMahon's concept of the "lesser index" of a lattice permutation **(9)** defined as follows.

An arrangement of the assemblage $a^i b^j c^k \ldots$ is called a lattice permutation if a line can be drawn between any two letters of the arrangement so that the letters to the left of the line are an arrangement of $a^\alpha b^\beta c^\gamma \ldots$, where $\alpha \geqslant \beta \geqslant \gamma \geqslant \ldots$. If, in this arrangement, the r_1th, r_2th, r_3th, ... letters immediately precede letters which are later in alphabetical order, the "lesser index" of the lattice permutation is defined as $r_1 + r_2 + r_3 + \ldots = r$, so that we can consider $(r_1 r_2 r_3 \ldots)$ as a partition of r if $r_1 > r_2 > r_3 > \ldots$, e.g. for the lattice permutation $a\underline{a}b\underline{ab}c$ of the assemblage a^3b^2c, the lesser index is $2 + 4 + 5 = 11$.

If we enumerate the partitions of the lesser indices of the lattice permutations associated with a hook partition (λ), say $(\lambda) = (k1^r)$, we find, in fact, that we have numerated the partitions with exactly r unequal parts none greater than $k + r - 1$, e.g. take $(\lambda) = (31^3)$; we require the partitions of the lesser indices of the lattice permutations associated with the assemblage a^3bcd. These are:

$aa\underline{a}\underline{b}\underline{c}d$ (543), $a\underline{a}b\underline{a}\underline{c}d$ (542), $\underline{a}ba\underline{a}\underline{c}d$ (541), $a\underline{a}\underline{b}c\underline{a}d$ (532), $\underline{a}b\underline{a}c\underline{a}d$ (531), $\underline{a}\underline{b}ca\underline{a}d$ (521), $a\underline{a}\underline{b}\underline{c}da$ (432), $\underline{a}b\underline{a}\underline{c}da$ (431), $\underline{a}\underline{b}c\underline{a}da$ (421), $\underline{a}\underline{b}\underline{c}daa$ (321).

This, therefore, gives us a rule for finding the indices of p in the polynomials $R_{\lambda\mu}(p)$ when (λ) is a hook partition. We can adapt the rule for the cases in which (λ) is not a hook partition and (μ) is a hook partition. As an example, consider $(\lambda) = (42)$. The lattice permutations and the corresponding lesser index partitions are:

$aaa\underline{a}bb$ (4), $aa\underline{a}b\underline{a}b$ (53), $aa\underline{a}bba$ (3), $a\underline{a}ba\underline{a}b$ (52), $a\underline{a}b\underline{a}ba$ (42), $a\underline{a}bbaa$ (2), $\underline{a}baa\underline{a}b$ (51), $\underline{a}ba\underline{a}ba$ (41), $\underline{a}b\underline{a}baa$ (31).

For $(\mu) = (1^6)$, we consider partitions with no part greater than 5, i.e.

$$R_{42,1^6}(p) = p^2 + p^3 + 2p^4 + p^5 + 2p^6 + p^7 + p^8.$$

For $(\mu) = (21^4)$, we consider partitions with no part greater than 4, i.e.

$$R_{42,21^4}(p) = p^2 + p^3 + 2p^4 + p^5 + p^6.$$

Similarly

$$R_{42,31^3}(p) = p^2 + p^3 + p^4,$$

$$R_{42,41^2}(p) = p^2,$$

$$R_{42,51}(p) = 0,$$

$$R_{42,6}(p) = 0.$$

We thus have the following rule for evaluating the indices of p in the polynomial $R_{\lambda\mu}(p)$:

Enumerate the partitions of the lesser indices of the lattice permutations associated with the partition (λ).

(a) When (λ) is a hook partition, consider only those partitions with no part greater than $s - 1$, the partition (μ) having s parts. Every such partition of an integer gives one term p^n in $R_{\lambda\mu}(p)$.

(b) When (λ) is not a hook partition, consider only those partitions with no part greater than $s - 1$, (μ) being a hook partition with s parts. Every such partition of an integer n gives one term p^n in $R_{\lambda\mu}(p)$.

There appear to be inherent difficulties in producing a rule which is successful for general partitions (λ) and (μ). At the moment, the best one can hope to do is to obtain a rule which is correct in the greatest possible number of cases; the above rule appears to do this.

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