Proceedings of the Edinburgh Mathematical Society (2015) **58**, 661–682
DOI:10.1017/S0013091514000248

STABILIZATION OF REGULAR SOLUTIONS FOR THE ZAKHAROV–KUZNETSOV EQUATION POSED ON BOUNDED RECTANGLES AND ON A STRIP

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(Received 14 February 2013)

Abstract Initial–boundary-value problems for the two-dimensional Zakharov–Kuznetsov equation posed on bounded rectangles and on a strip are considered. Spectral properties of a linearized operator and critical sizes of domains are studied. An exponential decay rate of regular solutions for the original nonlinear problems is proved.

Keywords: ZK equation; stabilization; exponential decay; critical domains

2010 *Mathematics subject classification:* Primary 35Q53; 35B35

1. Introduction

We are concerned with initial–boundary-value problems (IBVPs) posed on bounded rectangles and on a strip located in the right half-plane $\{(x,y) \in \mathbb{R}^2 : x > 0\}$ for the Zakharov–Kuznetsov (ZK) equation

$$u_t + (\alpha + u)u_x + u_{xxx} + u_{xyy} = 0, \tag{1.1}$$

where α is equal to 1 or to 0. Equation (1.1) is a two-dimensional analog of the well-known Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.2}$$

which has applications in plasma physics [**36**].

Equations (1.1) and (1.2) are typical examples of so-called dispersive equations, which have attracted considerable attention from pure and applied mathematicians over the past decades. The KdV equation is probably the more studied of the two in this context. The theory of the initial-value problem (IVP henceforth) for (1.2) is considerably advanced today [**1**, **4**, **7**, **15**–**17**, **32**, **35**].

Recently, due to needs in physics and numerics, publications on IBVPs in bounded and unbounded domains for dispersive equations have appeared [**2**, **3**, **5**, **6**, **9**, **20**, **21**, **28**, **37**]. In particular, it has been discovered that the KdV equation posed on a bounded interval possesses an implicit internal dissipation. This enabled the proof of an exponential decay rate of small solutions for (1.2) posed on bounded intervals without the addition of any artificial damping term [**20**]. Similar results were proved for a wide class of dispersive equations of any odd order with one space variable [**12**].

Equation (1.2) is a satisfactory approximation for real wave phenomena when the equation is posed on the whole real line ($x \in \mathbb{R}$). If cutting-off domains are taken into account, however, (1.2) is no longer expected to mirror reality. The correct equation in this case (see, for example, [**2**, **37**]) should be written as

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.3}$$

Indeed, if $x \in \mathbb{R}$, $t > 0$, the linear term u_x in (1.3) can easily be scaled out by a simple change of variables, but it cannot be safely ignored for problems posed on finite and semi-infinite intervals without changes in the original domain.

Once bounded domains are considered as spatial regions of wave propagation, the size of these domains appears to be restricted by certain critical conditions. An important result regarding these conditions is the explicit description of a spectrum-related countable critical set [**29**]

$$\mathcal{N} = \frac{2\pi}{\sqrt{3}}\sqrt{k^2 + kl + l^2}, \quad k, l \in \mathbb{N}.$$

While studying the controllability and stabilization of solutions for (1.3), the set $\mathcal{N}$ provides qualitative difficulties when the length of a spatial interval coincides with some of its elements. In fact, the function

$$u(x) = 1 - \cos x$$

is a stationary (non-decaying) solution for a linearized (1.3) posed on $(0, 2\pi)$, and $2\pi \in \mathcal{N}$.

It was shown in [**29**] that control of the linear KdV equation with the term u_x may fail for critical lengths. This means that there is no decay of solutions for a countable set of critical domains; hence, there is no decay in a quarter-plane, at least without the inclusion of some additional internal damping [**24**, **27**]. We recall, however, that if the term u_x is neglected, then (1.3) becomes (1.2) and it is possible to prove the exponential decay rate of small solutions of (1.2) posed on any bounded interval. More recent results on control and stabilizability for the KdV equation can be found in [**30**, **31**].

Quite recently the interest in dispersive equations began to extend to multi-dimensional models such as the Kadomtsev–Petviashvili and ZK equations. As far as the ZK equation is concerned, the results on both IVPs and IBVPs can be found in [**10**, **11**, **13**, **23**, **25**, **26**]. Our work is inspired by [**33**], where (1.1) posed on a strip bounded in the x variable was considered. In studying this paper, we found that the term u_{xyy} in (1.1) delivers additional dissipation that may ensure the decay of small solutions. For example, the term u_{xyy} provides the exponential decay of small solutions in a channel-type domain; namely, in a half-strip unbounded in the x-direction [**22**]. However, there are restrictions on the width of the channel for the case in which $\alpha = 1$ whereas no restrictions are needed if $\alpha = 0$.

In the present paper we put forward the hypothesis that there are critical restrictions upon the size of both bounded and unbounded domains. Indeed, the function

$$u(x,y)=\cos(\tfrac{1}{2}y)\left(1-\cos\left(\frac{x\sqrt{3}}{2}\right)\right)$$

solves the linearized (1.1) with $\alpha=1$, i.e. the equation

$$u_t+u_x+u_{xxx}+u_{xyy}=0$$

considered on the rectangle

$$(x,y)\in(0,4\pi/\sqrt{3})\times(-\pi,\pi),$$

and clearly it does not decay as $t\to\infty$.

Explicit conditions (like the set $\mathcal{N}$ for (1.3)) have been established in the present paper to describe the critical size of domains in which the decay of solutions fails, at least for linear models (see (6.13)).

The main goal of our work is to prove the existence and uniqueness of global-in-time regular solutions of (1.1) posed both on bounded rectangles and on a strip, and the exponential decay rate of these solutions for sufficiently small initial data.

The paper is organized as follows. The formulation of the problem and auxiliaries are contained in § 2. In § 3 a parabolic regularization is used to prove the existence theorem in rectangles. Uniqueness is proved in § 4. The existence of a unique regular solution on a strip is established in § 5. In § 6 we provide spectral arguments motivating the principal stabilization results to be obtained in § 7. Concerning the nonlinear ZK equation, the linear spectral arguments seem to be technically more difficult to apply for stabilizability than in the one-dimensional case. Because of this, weight estimates are used in § 7 to prove the decay of solutions as opposed to more modern unique continuation methods [**8**].

After this work was completed, the paper [**34**] related to [**33**] appeared. It deals with initial boundary-value problems in bounded domains for the ZK equation and emphasis is given on the function spaces by using the refined methods of modern analysis. The results there are different from ours but (from private discussions) one can say that all the results complement each other. Moreover, we partly answer the question addressed in Remark 3.7 from [**34**] (see the conclusion for more details).

2. Problem and preliminaries

Let L, B and T be finite positive numbers. Define

$$\mathcal{D}=\{(x,y)\in\mathbb{R}^2\colon x\in(0,L),\ y\in(-B,B)\},\quad \mathcal{Q}_T=\mathcal{D}\times(0,T).$$

For $\alpha=1$ or $\alpha=0$ we consider the IBVP

$$A_\alpha u\equiv u_t+(\alpha+u)u_x+u_{xxx}+u_{xyy}=0\quad \text{in } \mathcal{Q}_T, \tag{2.1}$$

$$u(x,-B,t)=u(x,B,t)=0,\quad x\in(0,L),\ t>0, \tag{2.2}$$

$$u(0,y,t)=u(L,y,t)=u_x(L,y,t)=0,\quad y\in(-B,B),\ t>0, \tag{2.3}$$

$$u(x,y,0)=u_0(x,y),\quad (x,y)\in\mathcal{D}, \tag{2.4}$$

where $u_0\colon\mathcal{D}\to\mathbb{R}$ is a given function.

Hereafter, subscripts u_x, u_{xy}, etc., indicate partial derivatives, as do ∂_x or ∂^2_{xy} when it is convenient. Operators ∇ and Δ are the gradient and Laplacian acting over $\mathcal{D}$. By $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the inner product and the norm in $L^2(\mathcal{D})$ and $\|\cdot\|_{H^k}$ stands for the norm in L^2-based Sobolev spaces.

We will need the following result [**18**].

Lemma 2.1. *Let $u \in H^1(\mathcal{D})$ and γ be the boundary of $\mathcal{D}$.*

If $u|_\gamma = 0$, then

$$\|u\|_{L^q(\mathcal{D})} \leqslant \beta \|\nabla u\|^\theta \|u\|^{1-\theta}, \tag{2.5}$$

where $q = 3$ or $q = 4$, $\theta = 2((1/2) - (1/q))$ and $\beta = 2^\theta$.

If $u|_\gamma \neq 0$, then

$$\|u\|_{L^q(\mathcal{D})} \leqslant C_{\mathcal{D}} \|u\|^\theta_{H^1(\mathcal{D})} \|u\|^{1-\theta}, \tag{2.6}$$

where $C_{\mathcal{D}}$ does not depend on the size of $\mathcal{D}$.

3. Existence theorem

In this section we state the existence result for a bounded domain.

Theorem 3.1. *Let $\alpha = 1$ and let u_0 be a given function such that $u_0|_\gamma = u_{0x}|_{x=L} = 0$ and*

$$I_0 \equiv \|u_0\|^2_{H^1_0(\mathcal{D})} + \|\partial^2_y u_0\|^2 + \|u_0 u_{0x} + \Delta u_{0x}\|^2 < \infty.$$

Then, for all finite positive B, L and T, there exists a unique regular solution to (2.1)–(2.4) such that

$$\begin{aligned} u &\in L^\infty(0,T;H^2(\mathcal{D})) \cap L^2(0,T;H^3(\mathcal{D})),\\ \Delta u_x &\in L^\infty(0,T;L^2(\mathcal{D})) \cap L^2(0,T;H^1(\mathcal{D})),\\ u_t &\in L^\infty(0,T;L^2(\mathcal{D})) \cap L^2(0,T;H^1(\mathcal{D})) \end{aligned}$$

and

$$\begin{aligned} &\|u\|^2_{H^2(\mathcal{D})}(t) + \|\Delta u_x\|^2(t) + \|u_t\|^2(t) + \|u_x(0,y,t)\|^2_{H^1_0(-B,B)}\\ &\quad + \int_0^T \{\|u\|^2_{H^3(\mathcal{D})}(t) + \|\Delta u_x\|^2_{H^1(\mathcal{D})}(t) + \|u_x(0,y,t)\|^2_{H^2(-B,B)}\}\,\mathrm{d}t \leqslant CI_0 \quad \forall t \in [0,T], \end{aligned} \tag{3.1}$$

where the constant C depends on L, $\|u_0\|$ and T but does not depend on $B > 0$.

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To prove this theorem we consider, for all real $\varepsilon > 0$, the following parabolic regularization of (2.1)–(2.4):

$$A^\varepsilon u_\varepsilon \equiv A_1 u_\varepsilon + \varepsilon(\partial_x^4 u_\varepsilon + \partial_y^4 u_\varepsilon)7 = 0 \quad \text{in } \mathcal{Q}_T, \tag{3.2}$$

$$u_\varepsilon(x,-B,t) = u_\varepsilon(x,B,t) = \partial_y^2 u_\varepsilon(x,-B,t) = \partial_y^2 u_\varepsilon(x,B,t) = 0, \quad x \in (0,L),\ t > 0, \tag{3.3}$$

$$u_\varepsilon(0,y,t) = u_\varepsilon(L,y,t) = \partial_x^2 u_\varepsilon(0,y,t) = \partial_x u_\varepsilon(L,y,t) = 0, \quad y \in (-B,B),\ t > 0, \tag{3.4}$$

$$u_\varepsilon(x,y,0) = u_0(x,y), \quad (x,y) \in \mathcal{D}. \tag{3.5}$$

For all $\varepsilon > 0$, (3.2)–(3.5) admits, at least for small $T > 0$, a unique regular solution in $\mathcal{Q}_T$ [**19**]. We assume here that u_0 is a sufficiently smooth function satisfying necessary compatibility conditions. Exact restrictions on u_0 will follow from *a priori* estimates that are uniform for $\varepsilon > 0$. These estimates justify passage to the limit $\varepsilon \to 0$, which proves the existence part of Theorem 3.1. Uniqueness will be studied later.

In the following subsections we obtain *a priori* estimates independent of $\varepsilon > 0$ and $B > 0$. The subscript ε will be omitted whenever it is unambiguous.

3.1. Estimate I

Multiply (3.2) by u_ε and integrate over $\mathcal{D}$ to obtain

$$\|u_\varepsilon\|^2(t) + 2\varepsilon \int_0^t (\|\partial_x^2 u_\varepsilon\|^2(\tau) + \|\partial_y^2 u_\varepsilon\|^2(\tau))\,\mathrm{d}\tau + \int_0^t \int_{-B}^B u_{\varepsilon x}^2(0,y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau = \|u_0\|^2, \quad t \in (0,T). \tag{3.6}$$

3.2. Estimate II

Write the inner product

$$2(A^\varepsilon u_\varepsilon, (1+x)u_\varepsilon)(t) = 0$$

as

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x),u^2)(t) + (1-2\varepsilon)\int_{-B}^B u_x^2(0,y,t)\,\mathrm{d}y + 3\|u_x\|^2(t) + \|u_y\|^2(t) + 2\varepsilon[((1+x),u_{xx}^2)(t) + ((1+x),u_{yy}^2)(t)] = \|u\|^2(t) + \frac{2}{3}\int_{\mathcal{D}} u^3\,\mathrm{d}x\,\mathrm{d}y.$$

Making use of (2.5), we compute

$$\begin{aligned}\frac{2}{3}\int_{\mathcal{D}} u^3 \mathrm{d}x\,\mathrm{d}y &\leqslant \tfrac{2}{3}\|u\|_{L^3(\mathcal{D})}^3(t) \\ &\leqslant \tfrac{2}{3}[2^{1/3}\|\nabla u\|^{1/3}(t)\|u\|^{2/3}(t)]^3 \\ &= \tfrac{4}{3}\|\nabla u\|(t)\|u\|^2(t) \\ &\leqslant \delta\|\nabla u\|^2(t) + \frac{4}{9\delta}\|u\|^4(t).\end{aligned}$$

Taking $\varepsilon \in (0, \frac{1}{4})$ and $\delta = \frac{1}{2}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x), u^2)(t) + \tfrac{1}{2}\|\nabla u\|^2(t) + \frac{1}{2}\int_{-B}^{B} u_x^2(0,y,t)\,\mathrm{d}y + \varepsilon(\|u_{xx}\|^2(t) + \|u_{yy}\|^2(t)) \leqslant \|u\|^2(t) + \tfrac{8}{9}\|u\|^4(t).$$

Integration over $(0, t)$ and (3.6) then imply that

$$((1+x), u_\varepsilon^2)(t) + \int_0^t \int_{-B}^{B} u_{\varepsilon x}^2(0,y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau + \int_0^t \|\nabla u_\varepsilon\|^2(\tau)\,\mathrm{d}\tau + \varepsilon\int_0^t [u_{\varepsilon xx}^2(\tau) + u_{\varepsilon yy}^2(\tau)]\,\mathrm{d}\tau \leqslant C((1+x), u_0^2), \quad (3.7)$$

where the constant C does not depend on $B, \varepsilon > 0$ but does depend on T and $\|u_0\|$.

3.3. Estimate III

Transforming the inner product

$$-2((1+x)\partial_y^2 u_\varepsilon, A^\varepsilon u_\varepsilon)(t) = 0$$

into the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x), u_y^2)(t) + 3\|u_{xy}\|^2(t) + (1-2\varepsilon)\int_{-B}^{B} u_{xy}^2(0,y,t)\,\mathrm{d}y + \|u_{yy}\|^2(t) + 2\varepsilon[((1+x), |\partial_y^2 u_x|^2)(t) + ((1+x), |\partial_y^3 u|^2)(t)] = \|u_y\|^2(t) - 2((1+x)uu_x, u_{yy})(t), \quad (3.8)$$

we estimate

$$\begin{aligned} I &\equiv 2((1+x)uu_x, \partial_y^2 u)(t) \\ &= -2((1+x)(uu_y)_x, u_y)(t) \\ &= (u, u_y^2)(t) - ((1+x)u_x, u_y^2)(t) \\ &\equiv I_1 + I_2. \end{aligned}$$

Since $u_{y|_{y=-B,B}} \neq 0$, we use (2.6) to estimate

$$\begin{aligned} I_1 &\leqslant \|u\|(t)\|u_y\|^2_{L^4(\mathcal{D})} \\ &\leqslant C_{\mathcal{D}}\|u\|(t)\|u_y\|(t)\|u_y\|_{H^1(\mathcal{D})}(t) \\ &\leqslant \delta\|u_y\|^2_{H^1(\mathcal{D})}(t) + \frac{C_{\mathcal{D}}^2}{4\delta}\|u\|^2(t)\|u_y\|^2(t), \quad \delta > 0, \end{aligned}$$

and

$$I_2 \leqslant (1+L)C_{\mathcal{D}}\|u_x\|(t)\|u_y\|(t)\|u_y\|_{H^1(\mathcal{D})}(t)$$
$$\leqslant \delta\|u_y\|^2_{H^1(\mathcal{D})}(t) + \frac{1}{4\delta}(1+L)^2 C^2_{\mathcal{D}}\|\nabla u\|^2(t)((1+x), u_y^2)(t).$$

Estimates of I_1, I_2 and (3.7) give

$$I \leqslant 2\delta\|\nabla u_y\|^2(t) + \frac{C(L)}{\delta}(1+\|\nabla u\|^2(t))((1+x), u_y^2)(t).$$

Setting $\varepsilon \in (0, \frac{1}{4})$ and $\delta = \frac{1}{4}$, (3.8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x), u_y^2)(t) + \tfrac{1}{2}\|\nabla u_y\|^2(t) + \frac{1}{2}\int_{-B}^{B} u_{xy}^2(0,y,t)\,\mathrm{d}y$$
$$+ 2\varepsilon(\|\partial_y^2 u_x\|^2(t) + \|\partial_y^3 u\|^2(t)) \leqslant C(L)(1+\|\nabla u\|^2(t))((1+x), u_y^2)(t). \quad (3.9)$$

Hence, by the Gronwall lemma and (3.7),

$$((1+x), u_y^2)(t) \leqslant CI_0$$

and, finally,

$$\|\partial_y u_\varepsilon\|^2(t) + \int_0^t \|\nabla(\partial_y u_\varepsilon)\|^2(\tau)\,\mathrm{d}\tau + \int_0^t \int_{-B}^{B} (\partial_{xy}^2 u_\varepsilon)^2(0,y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau$$
$$+ \varepsilon\int_0^t (\|\partial_y^2\partial_x u_\varepsilon\|^2(\tau) + \|\partial_y^3 u_\varepsilon\|^2(\tau))\,\mathrm{d}\tau \leqslant C(L)I_0, \quad (3.10)$$

where the constant $C(L)$ depends neither on $\varepsilon > 0$ nor on $B > 0$, but does depend on T and $\|u_0\|$.

To obtain the next estimate, we need the following simple result.

Proposition 3.2. *Let $u \in H^1(\mathcal{D})$ and $u_{xy} \in L^2(\mathcal{D})$. Then*

$$\sup_{(x,y)\in\mathcal{D}} u^2(x,y,t) \leqslant \|u\|^2_{H^1(\mathcal{D})}(t) + \|u_{xy}\|^2_{L^2(\mathcal{D})}(t).$$

Proof. For a fixed $x \in (0, L)$ and for any $y \in (-B, B)$, it holds that

$$u^2(x,y,t) = \int_{-B}^{y} \partial_s u^2(x,s,t)\,\mathrm{d}s \leqslant \int_{-B}^{B} u^2(x,y,t)\,\mathrm{d}y + \int_{-B}^{B} u_y^2(x,y,t)\,\mathrm{d}y \equiv \rho^2(x,t).$$

On the other hand,

$$\sup_{(x,y)\in\mathcal{D}} u^2 \leqslant \sup_{x\in(0,L)} \rho^2(x,t)$$
$$= \sup_{x\in(0,L)} \left|\int_0^x \partial_s\rho^2(s,t)\,\mathrm{d}s\right|$$
$$\leqslant \int_0^L \int_{-B}^{B} (u^2 + u_x^2 + u_y^2 + u_{xy}^2)\,\mathrm{d}x\,\mathrm{d}y.$$

The proof is complete. □

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3.4. Estimate IV

Write

$$2((1+x)\partial_y^4 u_\varepsilon, A^\varepsilon u_\varepsilon)(t) = 0$$

in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}((1+x), u_{yy}^2)(t) + (1-2\varepsilon)\int_{-B}^{B} u_{xyy}^2(0,y,t)\,\mathrm{d}y + 3\|\partial_y^2 u_x\|^2(t) \\ + \|\partial_y^3 u\|^2(t) + 2\varepsilon[((1+x), |\partial_y^2\partial_x^2 u|^2)(t) + ((1+x), |\partial_y^4 u|^2)(t)] \\ = \|\partial_y^2 u\|^2(t) - ((1+x)u_{yy}, (u^2)_{yyx})(t). \quad (3.11)\end{aligned}$$

Define

$$I = -((1+x)u_{yy}, (u^2)_{yyx})(t) = (u_{yy}, (u^2)_{yy})(t) + ((1+x)u_{xyy}, (u^2)_{yy})(t) \equiv I_1 + I_2,$$

where

$$I_1 = 2(u_{yy}, uu_{yy} + u_y^2)(t) = I_{11} + I_{12}.$$

By Proposition 3.2,

$$\begin{aligned}I_{11} &= 2(u, u_{yy}^2)(t) \\ &\leqslant 2 \sup_{(x,y)\in\mathcal{D}} |u(x,y,t)| \|u_{yy}\|^2(t) \\ &\leqslant (1 + \|u\|^2(t) + \|\nabla u\|^2(t) + \|\nabla u_y\|^2(t))((1+x), u_{yy}^2)(t)\end{aligned}$$

and

$$\begin{aligned}I_{12} &= 2(u_{yy}, u_y^2)(t) \\ &\leqslant 2\|u_{yy}\|(t)\|u_y\|^2_{L^4(\mathcal{D})}(t) \\ &\leqslant 2C_{\mathcal{D}}\|u_{yy}\|(t)\|u_y\|(t)\|u_y\|_{H^1(\mathcal{D})}(t) \\ &\leqslant C\|u_y\|^2_{H^1(\mathcal{D})}(t)\|u_y\|(t).\end{aligned}$$

Similarly,

$$\begin{aligned}I_2 &\leqslant 2(1+L)(u_{xyy}, uu_{yy} + u_y^2)(t) \\ &\leqslant 2(1+L)\|u_{xyy}\|(t)(\|uu_{yy}\|(t) + \|u_y^2\|(t)) \\ &\leqslant \delta\|u_{xyy}\|^2(t) + \frac{2(1+L)^2}{\delta}(\|uu_{yy}\|^2(t) + \|u_y^2\|^2(t)) \\ &\leqslant \delta\|u_{xyy}\|^2(t) + \frac{2(1+L)^2}{\delta}[\sup_{\mathcal{D}} |u(x,y,t)|^2((1+x), u_{yy}^2)(t) \\ &\qquad + 2C_{\mathcal{D}}\|u_y\|^2(t)\|u_y\|^2_{H^1(\mathcal{D})}(t)].\end{aligned}$$

Estimates of I_{11}, I_{12} and I_2 then imply that

$$\begin{aligned}I \leqslant \delta\|u_{xyy}\|^2(t) + \frac{C(L)}{\delta}[1 + \|u\|^2(t) + \|\nabla u\|^2(t) + \|u_{xy}\|^2(t)]((1+x), u_{yy}^2)(t) \\ + \frac{C(L)}{\delta}\|u_y\|^2(t)\|u_y\|^2_{H^1(\mathcal{D})}(t).\end{aligned}$$

Inserting I into (3.11) and taking $\delta > 0$ and $\varepsilon > 0$ sufficiently small, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}((1+x),u_{yy}^2)(t) &+ \frac{1}{2}\int_{-B}^{B} u_{xyy}^2(0,y,t)\,\mathrm{d}y \\ &+ \|\nabla u_{yy}\|^2(t) + \varepsilon(\|\partial_x^2\partial_y^2 u\|^2(t) + \|\partial_y^4 u\|^2(t)) \\ &\leqslant C(L)\|u_y\|^2(t)(\|u_y\|^2(t) + \|\nabla u_y\|^2(t)) \\ &\quad + C(L)[1 + \|u\|^2(t) + \|\nabla u\|^2(t) + \|\nabla u_y\|^2(t)]((1+x),u_{yy}^2)(t). \qquad (3.12)\end{aligned}$$

Making use of (3.10) and the Gronwall lemma, we infer that

$$\|u_{yy}\|^2(t) \leqslant ((1+x),u_{yy}^2)(t) \leqslant C(L)I_0.$$

Returning to (3.12), we conclude that

$$\begin{aligned}\|\partial_y^2 u_\varepsilon\|^2(t) &+ \int_0^t \|\nabla(\partial_y^2 u_\varepsilon)\|^2(\tau)\,\mathrm{d}\tau \\ &+ \int_0^t\int_{-B}^{B} (\partial_{xyy}^3 u_\varepsilon)^2(0,y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &+ \varepsilon\int_0^t (\|\partial_y^2\partial_x^2 u_\varepsilon\|^2(\tau) + \|\partial_y^4 u_\varepsilon\|^2(\tau))\,\mathrm{d}\tau \\ &\leqslant C(L)((1+x),(u_0^2 + u_{0y}^2 + u_{0yy}^2)) \leqslant C(L)I_0 \qquad (3.13)\end{aligned}$$

with $C(L)$ independent of $\varepsilon > 0$, $B > 0$.

3.5. Estimate V

Write the inner product

$$2((1+x)\partial_t u_\varepsilon, \partial_t(A^\varepsilon u_\varepsilon))(t) = 0$$

as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}((1+x),u_t^2)(t) &+ (1-2\varepsilon)\int_{-B}^{B} u_{xt}^2(0,y,t)\,\mathrm{d}y + 3\|u_{xt}\|^2(t) \\ &+ \|u_{yt}\|^2(t) + 2\varepsilon[((1+x),u_{xxt}^2)(t) + ((1+x),u_{yyt}^2)(t)] \\ &= \|u_t\|^2(t) + 2((1+x)uu_t,u_{xt})(t) + 2(u,u_t^2)(t). \qquad (3.14)\end{aligned}$$

We calculate

$$\begin{aligned}I_1 &= 2((1+x)uu_t,u_{xt})(t) \\ &\leqslant 2(1+L)^{1/2}\|u_{xt}\|(t)\sup_{\mathcal{D}}|u(x,y,t)|\|(1+x)^{1/2}u_t\|(t) \\ &\leqslant \delta\|u_{xt}\|^2(t) + \left(\frac{1+L}{\delta}\right)[\|u\|_{H^1(\mathcal{D})}^2(t) + \|u_{xy}\|^2(t)]((1+x),u_t^2)(t).\end{aligned}$$

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Analogously,

$$\begin{aligned} I_2 &= 2(u, u_t^2)(t) \\ &\leqslant 2[1 + \|u\|^2(t) + \|\nabla u\|^2(t) + \|u_{xy}\|^2(t)]((1+x), u_t^2)(t). \end{aligned}$$

Taking $\delta > 0$, $\varepsilon > 0$ sufficiently small, we transform (3.14) into the inequality

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}((1+x), u_t^2)(t) + \frac{1}{2}\int_{-B}^{B} u_{xt}^2(0, y, t)\,\mathrm{d}y \\ &\quad + \|\nabla u_t\|^2(t) + \varepsilon[\|\partial_x^2 u_t\|^2(t) + \|\partial_y^2 u_t\|^2(t)] \\ &\qquad \leqslant C(L)[1 + \|u\|^2(t) + \|\nabla u\|^2(t) + \|u_{xy}\|^2(t)]((1+x), u_t^2)(t). \end{aligned} \tag{3.15}$$

By the Gronwall lemma,

$$((1+x), u_t^2)(t) \leqslant C(L) I_0.$$

Therefore, (3.15) becomes

$$\begin{aligned} &((1+x), u_{\varepsilon t}^2)(t) + \int_0^t \int_{-B}^{B} (\partial_{x\tau}^2 u_\varepsilon)^2(0, y, t)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad + \int_0^t \|\nabla \partial_\tau u_\varepsilon\|^2(\tau)\,\mathrm{d}\tau + \varepsilon \int_0^t [\|\partial_x^2 \partial_\tau u_\varepsilon\|^2(\tau) + \|\partial_y^2 \partial_\tau u_\varepsilon\|^2(\tau)]\,\mathrm{d}\tau \leqslant C I_0, \end{aligned} \tag{3.16}$$

where the constant C depends on $L > 0$, but does not depend on $B, \varepsilon > 0$.

3.6. Estimate VI

From the inner product

$$2((1+x)A^\varepsilon u_\varepsilon, u_\varepsilon)(t) = 0$$

we get

$$\begin{aligned} &(1 - 2\varepsilon)\int_{-B}^{B} u_x^2(0, y, t)\,\mathrm{d}y + 3\|u_x\|^2(t) \\ &\quad + \|u_y\|^2(t) + 2\varepsilon((1+x), [u_{xx}^2 + u_{yy}^2])(t) \\ &\qquad = \|u\|^2(t) + \frac{2}{3}\int_{\mathcal{D}} u^3\,\mathrm{d}x\,\mathrm{d}y - 2((1+x)u_t, u)(t). \end{aligned} \tag{3.17}$$

Acting as in § 3.2, we find, for all $\delta > 0$,

$$I = \frac{2}{3}\int_{\mathcal{D}} u^3\,\mathrm{d}x\,\mathrm{d}y \leqslant \delta\|\nabla u\|^2(t) + \frac{C}{\delta}\|u\|^4(t),$$

whence, taking $\delta > 0$, $\varepsilon > 0$ sufficiently small and using (3.7) and (3.16), we reduce (3.17) to the form

$$\int_{-B}^{B} (\partial_x u_\varepsilon)^2(0, y, t)\,\mathrm{d}y + \|\nabla u_\varepsilon\|^2(t) \leqslant C I_0 \quad \forall t \in (0, T). \tag{3.18}$$

Now transform

$$-2((1+x)\partial_y^2 u_\varepsilon, A^\varepsilon u_\varepsilon)(t) = 0$$

into the equality

$$\begin{aligned}(1-2\varepsilon)\int_{-B}^{B} u_{xy}^2(0,y,t)\,\mathrm{d}t + 3\|u_{xy}\|^2(t) + \|\partial_y^2 u\|^2(t)\\ - 2((1+x)u_t, u_{yy})(t) + 2\varepsilon((1+x),[|\partial_y^2 u_x|^2 + |\partial_y^3 u|^2])(t)\\ = \|u_y\|^2(t) - ((1+x)\partial_{yx}^2(u^2), u_y)(t). \qquad (3.19)\end{aligned}$$

Repeating the computations of Estimate III and taking into account (3.18), we find that

$$I_1 = -((1+x)(u^2)_{yx}, u_y)(t) \leqslant \delta\|\nabla u_y\|^2(t) + \frac{C}{\delta}\|u\|^2(t)(\|\nabla u\|^4(t) + \|\nabla u\|^2(t)),$$

that is

$$I_1 \leqslant \delta\|\nabla u_y\|^2(t) + \frac{C}{\delta} I_0.$$

For $\delta, \varepsilon > 0$ sufficiently small, (3.19) reads

$$\|\nabla u_{\varepsilon y}\|^2(t) + \int_{-B}^{B}(\partial_{xy}^2 u_\varepsilon)^2(0,y,t)\,\mathrm{d}y + \varepsilon[\|\partial_x^2 u_\varepsilon\|^2 + \|\partial_y^2 u_\varepsilon\|^2](t) \leqslant C I_0.$$

The constant C depends on L and I_0 but does not depend on $B > 0$ or $\varepsilon > 0$.

We combine Estimates I–VI as follows:

$$\begin{aligned}\|\nabla u_\varepsilon\|^2(t) + \|\nabla u_{\varepsilon y}\|^2(t) + \|u_{\varepsilon t}\|^2(t) + \|u_{\varepsilon x}(0,y,t)\|^2_{H_0^1(-B,B)}\\ + \int_0^T\{\|\nabla u_{\varepsilon yy}\|^2(t) + \|u_{\varepsilon t}\|^2_{H^1(\mathcal{D})}(t) + \|u_{\varepsilon x}(0,y,t)\|^2_{H^2(-B,B)}\}\,\mathrm{d}t \leqslant C(L,T)I_0 \qquad (3.20)\end{aligned}$$

and

$$\varepsilon[\|u_{\varepsilon xx}\|^2(t) + \|u_{\varepsilon yy}\|^2(t)] \leqslant C(L,T)I_0, \qquad (3.21)$$

where the constant $C(L,T)$ depends on L and T but does not depend on B or ε.

3.7. Passage to the limit as $\varepsilon \to 0$

It follows from (3.21) that for all $\psi \in H_0^2(\mathcal{D})$

$$\lim_{\varepsilon\to 0}\varepsilon(\partial_x^2 u_\varepsilon, \psi_{xx})(t) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\varepsilon(\partial_y^2 u_\varepsilon, \psi_{yy})(t) = 0.$$

Since the constants in (3.20) and (3.21) do not depend on $\varepsilon > 0$ or $B > 0$, one may pass to the limit as $\varepsilon \to 0$ in

$$\begin{aligned}\int_0^T\int_{\mathcal{D}}[\partial_t u_\varepsilon + (1+u_\varepsilon)\partial_x u_\varepsilon + \Delta\partial_x u_\varepsilon]\psi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ + \varepsilon\int_0^T\int_{\mathcal{D}}[\partial_x^2 u_\varepsilon\psi_{xx} + \partial_y^2 u_{\varepsilon yy}\psi_{yy}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0 \qquad (3.22)\end{aligned}$$

https://doi.org/10.1017/S0013091514000248 Published online by Cambridge University Press

to obtain

$$\int_0^T \int_{\mathcal{D}} [u_t + (1+u)u_x + \Delta u_x]\psi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0. \tag{3.23}$$

Estimating Δu_x from (3.22) and making use of (3.20) and (3.21), we establish the following lemma.

Lemma 3.3. *Let all the conditions of Theorem 3.1 hold. Then there exists a weak solution $u(x,y,t)$ of (2.1)–(2.4) such that*

$$\begin{aligned} &\|\nabla u\|^2(t) + \|\nabla u_y\|^2(t) + \|u_t\|^2(t) + \|\Delta u_x\|^2(t) + \|u_x(0,y,t)\|^2_{H_0^1(-B,B)} \\ &\quad + \int_0^T \{\|\nabla u_{yy}\|^2(t) + \|\Delta u_x\|^2(t) + \|u_t\|^2_{H_0^1(\mathcal{D})}(t) + \|u_x(0,y,t)\|^2_{H^2(-B,B)}\}\,\mathrm{d}t \\ &\qquad \leqslant C(L,T)I_0 \quad \text{for almost every } t \in (0,T), \end{aligned} \tag{3.24}$$

where $C(L,T)$, as earlier, depends on L, T and $\|u_0\|$ but does not depend on $B > 0$.

In order to complete the proof of the existence part of Theorem 3.1, it suffices to show that

$$u \in L^2(0,T;H^3(\mathcal{D})), \qquad \Delta u_x \in L^2(0,T;H^1(\mathcal{D}))$$

and

$$u \in L^\infty(0,T;H^2(\mathcal{D})), \qquad \Delta u_x \in L^\infty(0,T;L^2(\mathcal{D})).$$

These inclusions will be proved in the following lemmas.

Lemma 3.4. *A weak solution from Lemma 3.3 satisfies*

$$\int_0^T \{\|u\|^2_{H^3(\mathcal{D})}(t) + \|\Delta u_x\|^2_{H^1(\mathcal{D})}(t)\}\,\mathrm{d}t \leqslant C I_0, \tag{3.25}$$

where C does not depend on $B > 0$.

Proof. Taking into account (3.24) and Proposition 3.2, we write (3.23) in the form

$$\begin{gathered} \Delta u_x = -u_t - (1+u)u_x \equiv f(x,y,t) \in L^\infty(0,T;L^2(D)), \\ u_x(0,y,t) \equiv \varphi(y,t) \in L^2(0,T;H^2(-B,B)) \cap L^\infty(0,T;H_0^1(-B,B)), \\ u_x(x,-B,t) = u_x(x,B,t) = u_x(L,y,t) = 0. \end{gathered}$$

Define $\Phi(x,y,t) = \varphi(y,t)(1 - x/L)$ in $\mathcal{Q}_T$. Obviously,

$$\Phi \in L^2(0,T;H^2(\mathcal{D})).$$

Then the function

$$v = u_x - \Phi(x,y,t)$$

solves in $\mathcal{D}$ the elliptic problem

$$\Delta v = f(x,y,t) - \Phi_{yy}(x,y,t), \quad v|_\gamma = 0,$$

which admits a unique solution $v \in L^2(0,T;H^2(\mathcal{D}))$ (see [**18**]). Consequently, $u_x \in L^2(0,T;H^2(\mathcal{D}))$. Therefore, (3.24) implies (3.25). It remains to show that the constant in (3.25) does not depend on $B>0$. To prove this, consider the equality

$$\int_0^T (\Delta v - v, \Delta v - v)(t)\,\mathrm{d}t = \int_0^T [f - v - \Phi_{yy}]^2\,\mathrm{d}t,$$

which implies that

$$\int_0^T \{\|v_{xx}\|^2(t) + \|v_{yy}\|^2(t) + 2\|v_{xy}\|^2(t) + 2\|\nabla v\|^2(t) + \|v\|^2(t)\}\,\mathrm{d}t \leqslant CI_0.$$

This gives

$$\int_0^T \|v\|^2_{H^2(\mathcal{D})}(t)\,\mathrm{d}t \leqslant CI_0$$

with C independent of $B>0$.

Taking into account (3.23) and (3.24), we complete the proof of Lemma 3.4. □

Lemma 3.5. *A weak solution given by Lemma 3.3 satisfies*

$$\|u\|^2_{H^2(\mathcal{D})}(t) + \|\Delta u_x\|^2(t) \leqslant CI_0 \tag{3.26}$$

with C independent of $B>0$.

Proof. The proof is similar to the proof of Lemma 3.2. □

In the regularization process we have imposed suitable smoothness and consistency conditions upon u_0 that are actually defined by (3.15). In the final steps these excessive restrictions may clearly be weakened by standard compactness arguments.

Making use of Lemmas 3.3–3.5, we complete the proof of the existence part of Theorem 3.1.

4. Uniqueness

Let u_1 and u_2 be two distinct solutions of (2.1)–(2.4) and let $\alpha=1$. Then $z=u_1-u_2$ solves the following IBVP:

$$Az \equiv z_t + z_x + \tfrac{1}{2}(u_1^2 - u_2^2)_x + \Delta z_x = 0 \quad \text{in } \mathcal{Q}_T, \tag{4.1}$$

$$z(0,y,t) = z(L,y,t) = z_x(L,y,t) = z(x,\pm B,t) = 0, \quad t>0, \tag{4.2}$$

$$z(x,y,0) = 0, \quad (x,y)\in\mathcal{D}. \tag{4.3}$$

From

$$2(Az,(1+x)z)(t) = 0$$

we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x),z^2)(t)+3\|z_x\|^2(t)+\|z_y\|^2(t)+\int_{\mathcal{D}} z^2\,\mathrm{d}x\,\mathrm{d}y+\int_{-B}^{B} z_x^2(0,y,t)\,\mathrm{d}y = -\int_{\mathcal{D}}[(u_1+u_2)z]_x z(1+x)\,\mathrm{d}x\,\mathrm{d}y. \quad (4.4)$$

Consider

$$\begin{aligned} I &= -\int_{\mathcal{D}}[(u_1+u_2)z]_x z(1+x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathcal{D}}(u_1+u_2)z^2\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathcal{D}}(u_1+u_2)(1+x)zz_x\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \sup_{\mathcal{Q}_T}|u_1+u_2|\|z\|^2(t)+\|z_x\|^2(t)+(1+L)^2\sup_{\mathcal{Q}_T}|u_1+u_2|^2\|z\|^2(t). \end{aligned}$$

Due to (3.26) and Proposition 3.2,

$$\sup_{\mathcal{Q}_T}|u_1+u_2|^2(x,y,t)\leqslant CI_0,$$

whence

$$I\leqslant \|z_x\|^2(t)+C\|z\|^2(t).$$

This and (4.4) give

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x),z^2)(t)\leqslant C((1+x),z^2)(t). \quad (4.5)$$

Gronwall's lemma and (4.3) then imply

$$\|z\|^2(t)\equiv 0 \quad \text{for all } t>0.$$

The proof of uniqueness and, consequently, the proof of Theorem 3.1 is therefore completed. □

Remark 4.1. The estimate (4.5) partly implies that the data-solution map is continuous. More precisely, let u_0 and $\bar{u}_0$ satisfy the conditions of Theorem 3.1 with I_0 and $\bar{I}_0$, respectively, and let u and $\bar{u}$ be corresponding solutions to (2.1)–(2.4). Then, for all ε, there exists $\delta=\delta(\varepsilon,T,\max\{I_0,\bar{I}_0\})$ such that

$$\|u_0-\bar{u}_0\|<\delta \quad \Longrightarrow \quad \|u-\bar{u}\|(t)<\varepsilon \quad \text{for all } 0<t<T.$$

5. Problem on a strip

Taking into account that the estimates of Theorem 3.1 do not depend on $B>0$, one can expand a bounded domain $\mathcal{D}$ to a strip

$$\mathcal{S}_L=\{(x,y)\in\mathbb{R}^2\colon x\in(0,L),\ y\in\mathbb{R}\}.$$

The IBVP to be considered reads

$$A_\alpha u \equiv u_t + (\alpha + u)u_x + u_{xxx} + u_{xyy} = 0 \quad \text{in } \mathcal{S}_L \times (0,T), \tag{5.1}$$

$$u(0,y,t) = u(L,y,t) = u_x(L,y,t) = 0, \quad y \in \mathbb{R},\ t > 0, \tag{5.2}$$

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \mathcal{S}_L. \tag{5.3}$$

The following result then holds.

Theorem 5.1. *Let $\alpha = 1$ and let u_0 be a given function such that*

$$\|u_0\|^2_{H^1(\mathcal{S}_L)} + \|\partial_y^2 u_0\|^2_{L^2(\mathcal{S}_L)} + \|u_0 u_{0x} + \Delta u_{0x}\|^2_{L^2(\mathcal{S}_L)} < \infty$$

and

$$u_0(0,y) = u_0(L,y) = u_{0x}(L,y) = 0.$$

Then, for all finite positive L and T, there exists a unique regular solution to (5.1)–(5.3) such that

$$u \in L^\infty(0,T;H^2(\mathcal{S}_L)) \cap L^2(0,T;H^3(\mathcal{S}_L)),$$
$$\Delta u_x \in L^\infty(0,T;L^2(\mathcal{S}_L)) \cap L^2(0,T;H^1(\mathcal{S}_L)),$$
$$u_t \in L^\infty(0,T;L^2(\mathcal{S}_L)) \cap L^2(0,T;H^1(\mathcal{S}_L))$$

and this solution depends continuously on the initial data.

6. Spectral analysis

In this section we provide explicit conditions defining critical sizes of bounded rectangles and unbounded strip-like domains of $\mathbb{R}^2$ in which the stabilization of solutions may not hold, at least in a linear case. Our considerations are based on spectral-type arguments and may be viewed as motivation for posterior nonlinear studies, as well as a two-dimensional generalization of the critical lengths from [**29**].

We start with the linearization of (2.1)–(2.4) with $\alpha = 1$:

$$Pu \equiv u_t + u_x + u_{xxx} + u_{xyy} = 0 \quad \text{in } \mathcal{Q}_T, \tag{6.1}$$

$$u(x,-B,t) = u(x,B,t) = 0, \quad x \in (0,L),\ t > 0, \tag{6.2}$$

$$u(0,y,t) = u(L,y,t) = u_x(L,y,t) = 0, \quad y \in (-B,B),\ t > 0, \tag{6.3}$$

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \mathcal{D}. \tag{6.4}$$

The related eigenvalue problem for the stationary part of P becomes the following: find $L > 0$, $B > 0$ and a non-trivial $v\colon \mathcal{D} \to \mathbb{C}$ such that

$$v_x + v_{xxx} + v_{xyy} = \lambda v \quad \text{in } \mathcal{D},\ \lambda \in \mathbb{C}, \tag{6.5}$$

$$v(x,-B) = v(x,B) = 0, \quad x \in (0,L), \tag{6.6}$$

$$v(0,y) = v(L,y) = v_x(0,y) = v_x(L,y) = 0, \quad y \in (-B,B). \tag{6.7}$$

To derive (6.5)–(6.7) see, for example, [**29**, **30**] for the direct approach, and [**8**, **14**] for the duality arguments.

Separating variables as $v(x,y) = p(x)q(y)$, we infer that

$$q'' + \xi q = 0, \quad q(-B) = q(B) = 0, \tag{6.8}$$

and

$$(1-\xi)p' + p''' = \lambda p, \quad p(0) = p(L) = p'(0) = p'(L) = 0. \tag{6.9}$$

Therefore,

$$\xi = \left(\frac{\pi n}{2B}\right)^2, \quad n \in \mathbb{N},$$

and

$$\lambda = \mathrm{i}\beta, \quad \beta \in \mathbb{R}.$$

To find β from (6.9), let $\mu_j = \mu_j(B,\beta)$, $j = 1,2,3$, be the roots of the characteristic equation

$$(1-\xi)\mu + \mu^3 = \mathrm{i}\beta. \tag{6.10}$$

Then a function

$$p(x) = \sum_{j=1}^{3} C_j x^{k_j} \mathrm{e}^{\mu_j x}$$

solves the ODE in (6.9). Here, C_j are constants to be determined and k_j depends on the multiplicity of μ_j. Observe that double roots of (6.10) give only $p(x) \equiv 0$, and therefore $k_j = 0$, $j = 1,2,3$. Boundary conditions in (6.9) yield

$$\sum_{j=1}^{3} C_j = 0, \qquad \sum_{j=1}^{3} \mu_j C_j = 0, \qquad \sum_{j=1}^{3} C_j \mathrm{e}^{\mu_j L} = 0, \qquad \sum_{j=1}^{3} \mu_j C_j \mathrm{e}^{\mu_j L} = 0.$$

Solving this system, we conclude that $C_j \neq 0$ if and only if

$$\mathrm{e}^{\mu_j L} = \mathrm{e}^{\mu_i L}, \quad i \neq j. \tag{6.11}$$

Next, taking $\mu_j = \mathrm{i}s_j$, (6.10) becomes

$$s^3 - (1-\xi)s + \beta = 0. \tag{6.12}$$

This is the real coefficients equation, which always possesses at least one real root; call it $s_1 \in \mathbb{R}$. Then (6.11) implies that*

$$s_2 = s_1 + \frac{2\pi}{L}k, \quad s_3 = s_1 + \frac{2\pi}{L}(k+l), \quad k,l \in \mathbb{N}.$$

* In general, $k,l \in \mathbb{Z} \setminus \{0\}$, but one can set s_j to be $s_1 < s_2 < s_3$.

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Furthermore, Viète's formulae for (6.12) read

$$s_1 + s_2 + s_3 = 0,$$
$$s_1 s_2 + s_1 s_3 + s_2 s_3 = -(1 - \xi),$$
$$s_1 s_2 s_3 = -\beta.$$

Simple computations give

$$s_1 = -\frac{2\pi}{3L}(2k + l) \quad \text{and} \quad L = \frac{2\pi}{\sqrt{3}}\sqrt{\frac{k^2 + kl + l^2}{1 - \xi}},$$

and finally

$$\left(\frac{2\pi}{L\sqrt{3}}\sqrt{k^2 + kl + l^2}\right)^2 + \left(\frac{\pi n}{2B}\right)^2 = 1. \tag{6.13}$$

Remark 6.1. If the size of a rectangle $\mathcal{D} = (0, L) \times (-B, B)$ satisfies (6.13), there are solutions to (6.1)–(6.4) that do not decay; if both L and B are sufficiently small, one can expect decay (in time) of the solutions. Once either L or B is small, we expect decay of solutions to problems posed on domains unbounded in one of the variables; namely, those problems that are posed on a strip and/or on a half-strip.

Remark 6.2. If $\alpha = 0$, (6.12) reads $s^3 + (\pi n/2B)^2 s + \beta = 0$. This equation does not possess three distinct real roots, and therefore (6.11) fails for all $L > 0$. This means that decay (in time) of solutions to (6.1)–(6.4) holds for all sizes of rectangle $\mathcal{D}$.

7. Decay of small solutions

In this section, we provide sufficient conditions to prove the exponential decay rate of small regular solutions to problems (2.1)–(2.4) and (5.1)–(5.3).

We start with a bounded rectangle $\mathcal{D} = (0, L) \times (-B, B)$.

Theorem 7.1. *Let $\alpha = 1$ and let B and L be positive real numbers such that*

$$\pi^2\left[\frac{3}{L^2} + \frac{1}{4B^2}\right] - 1 = 2A^2 > 0. \tag{7.1}$$

If

$$\|u_0\|^2 < \frac{9A^4L^2B^2}{4\pi^2(4B^2 + L^2)},$$

then regular solutions of (2.1)–(2.4) satisfy the inequality

$$\|u\|^2(t) \leqslant ((1 + x), u^2)(t) \leqslant \mathrm{e}^{-(A^2/(1+L))t}((1 + x), u_0^2).$$

Proof. It is easy to see that the following equality

$$(A_1 u, u)(t) = 0$$

https://doi.org/10.1017/S0013091514000248 Published online by Cambridge University Press

can be reduced to the form

$$\|u\|^2(t) + \int_{-B}^{B} u_x^2(0,y,t)\,\mathrm{d}y = \|u_0\|^2.$$

From here,

$$\|u\|^2(t) \leqslant \|u_0\|^2. \tag{7.2}$$

The following proposition is crucial for our proof.

Proposition 7.2. *Let $L > 0$ and $B > 0$ be finite numbers and let $w \in H_0^1(\mathcal{D})$. Then the following inequalities hold:*

$$\int_0^L \int_{-B}^{B} w^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{4B^2}{\pi^2} \int_0^L \int_{-B}^{B} w_y^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{7.3}$$

and

$$\int_0^L \int_{-B}^{B} w^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{L^2}{\pi^2} \int_0^L \int_{-B}^{B} w_x^2(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{7.4}$$

Proof. Since $u(0,y,t) = u(L,y,t) = u(x,-B,t) = u(x,B,t) = 0$, fixing one variable, we can use the following Steklov inequality with respect to the other one: if $f(s) \in H_0^1(0,\pi)$, then

$$\int_0^\pi f^2(x)\,\mathrm{d}x \leqslant \int_0^\pi |f_x(x)|^2\,\mathrm{d}x.$$

After an appropriate scaling, we prove Proposition 7.2. □

Next, consider the inner product

$$((1+x)A_1 u, u)(t) = 0$$

and write it as

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x),u^2)(t) + \int_{-B}^{B} u_x^2(0,y,t)\,\mathrm{d}y + 3\|u_x\|^2(t) + \|u_y\|^2(t) - \|u\|^2(t) = \tfrac{2}{3}(1,u^3)(t). \tag{7.5}$$

Making use of (2.5), we compute

$$\begin{aligned} I_1 &= \tfrac{2}{3}(1,u^3)(t) \\ &\leqslant \tfrac{2}{3}(2^{1/3}\|\nabla u\|^{1/3}(t)\|u\|^{2/3}(t))^3 \\ &\leqslant \tfrac{4}{3}\|\nabla u\|(t)\|u\|^2(t) \\ &\leqslant \delta\|u\|^2(t) + \frac{4}{9\delta}\|u\|^2(t)\|\nabla u\|^2(t) \\ &= \delta\|u\|^2(t) + \frac{4}{9\delta}\|u\|^2(t)(\|u_x\|^2(t) + \|u_y\|^2(t)) \end{aligned}$$

with an arbitrary $\delta > 0$ and, in addition,

$$\begin{aligned} I_2 &= 3\|u_x\|^2(t) + \|u_y\|^2(t) \\ &= (3-\epsilon)\|u_x\|^2(t) + (1-\epsilon)\|u_y\|^2(t) + \epsilon\|u_x\|^2(t) + \epsilon\|u_y\|^2(t) \end{aligned}$$

with an arbitrary $\epsilon > 0$. By Proposition 7.2 and (7.2), (7.5) reduces to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}((1+x),u^2)(t) + \left[\pi^2\left(\frac{3}{L^2}+\frac{1}{4B^2}\right) - 1 - \delta - \epsilon\pi^2\left(\frac{1}{L^2}+\frac{1}{4B^2}\right)\right]\|u\|^2(t) \\ + \left[\epsilon - \frac{4}{9\delta}\|u_0\|^2(t)\right]\|u_x\|^2(t) + \left[\epsilon - \frac{4}{9\delta}\|u_0\|^2(t)\right]\|u_y\|^2(t) \leqslant 0. \quad (7.6) \end{aligned}$$

Define

$$2A^2 = \pi^2\left(\frac{3}{L^2}+\frac{1}{4B^2}\right) - 1 > 0$$

and take

$$\delta = \frac{A^2}{2}, \qquad \epsilon = \frac{A^2}{2\pi^2((1/L^2)+(1/4B^2))}.$$

With this choice of ϵ and δ, (7.6) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x),u^2)(t) + A^2\|u\|^2(t) + \left[\epsilon - \frac{4}{9\delta}\|u_0\|^2\right]\|\nabla u\|^2(t) \leqslant 0. \quad (7.7)$$

If $\|u_0\|^2 < 9\epsilon\delta/4$, then (7.7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+x),u^2)(t) + \frac{A^2}{1+L}((1+x),u^2)(t) \leqslant 0,$$

which has a solution

$$\|u\|^2(t) \leqslant ((1+x),u^2)(t) \leqslant \mathrm{e}^{-(A^2/(1+L))t}((1+x),u_0^2).$$

The proof of Theorem 7.1 is complete. □

In the case of a strip (see § 5) the existence result is given by Theorem 5.1, and for $\mathcal{S}_L = \{(x,y) \in \mathbb{R}^2 \colon x \in (0,L),\ y \in \mathbb{R}\}$ the following assertion holds.

Theorem 7.3. *Let $\alpha = 1$ and let $L > 0$ be a finite number such that*

$$3\pi^2/L^2 - 1 = 2A^2 > 0$$

and

$$\|u_0\|^2 < \frac{9A^4L^2}{16\pi^2}.$$

Then a regular solution to (5.1)–(5.3) satisfies

$$\|u\|^2(t) \leqslant ((1+x),u^2)(t) \leqslant \mathrm{e}^{-\varrho t}((1+x),u_0^2),$$

where $\varrho = (3\pi^2 - L^2)/2L^2(1+L)$.

It is clear from (7.6) that restrictions on B and L appear due to the presence of the term u_x, i.e. $\alpha = 1$ in (2.1). If $\alpha = 0$, then there are no restrictions on $B > 0$ and $L > 0$, and the following results hold.

Theorem 7.4. *Let B and L be any finite positive numbers and let $\alpha = 0$. If*

$$\|u_0\|^2 < \frac{9A^4L^2B^2}{4\pi^2(4B^2 + L^2)},$$

where

$$A^2 = \pi^2\left[\frac{3}{2L^2} + \frac{1}{8B^2}\right],$$

then regular solutions to (2.1)–(2.4) satisfy the inequality

$$\|u\|^2(t) \leqslant ((1+x), u^2)(t) \leqslant \mathrm{e}^{-\sigma t}((1+x), u_0^2)$$

with $\sigma = \pi^2(12B^2 + L^2)/8B^2L^2(1+L)$.

Theorem 7.5. *Let L be any finite positive number and let $\alpha = 0$. If*

$$\|u_0\|^2 < \frac{81\pi^2}{64L^2},$$

then regular solutions to (5.1)–(5.3) satisfy the inequality

$$\|u\|^2(t) \leqslant ((1+x), u^2)(t) \leqslant \mathrm{e}^{-\nu t}((1+x), u_0^2)$$

with $\nu = 3\pi^2/2L^2(1+L)$.

8. Conclusions

As a conclusion, we provide a comparison between conditions (6.13) and (7.1), i.e. a comparison between size restrictions for linear and nonlinear models. Taking $k = l = m = 1$, (6.13) becomes

$$\frac{4\pi^2}{L^2} + \frac{\pi^2}{4B^2} = 1 \tag{8.1}$$

and recall that (7.1) reads

$$\frac{3\pi^2}{L^2} + \frac{\pi^2}{4B^2} > 1. \tag{8.2}$$

Suppose that $L^* > 0$ and $B^* > 0$ solve (8.1) and define

$$\mathcal{D}^* = (0, L^*) \times (-B^*, B^*) \subset \mathbb{R}^2.$$

Call this set the *minimal critical rectangle*. If $L < L^*$ and $B < B^*$ satisfy (8.2), then $\mathcal{D} \subset \mathcal{D}^*$. This means that if $\mathcal{D}$ is located inside the *minimal critical rectangle*, then a sufficiently small solution to the nonlinear problem (2.1)–(2.4) necessarily stabilizes. In this sense, restrictions (7.1) are stipulated by (6.13) and, therefore, smallness conditions (7.1) can be interpreted not as only technical ones, but as close to sharp ones. In particular, stabilizability holds for all rectangles $\mathcal{D}$ either with the width $L < \sqrt{3}\pi$ or with the height $2B < \pi$.

Furthermore, a small solution for problems posed on a sufficiently narrow strip $\mathcal{S}_L$ stabilizes as well. This partly responds to Remark 3.7 from [**34**]. Also observe that (8.1) fits well with the stabilization result from [**22**].

Acknowledgements. The authors very much appreciate the constructive comments and suggestions of the reviewers.

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