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CHIRAL SMOOTHINGS OF KNOTS

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Abstract Can smoothing a single crossing in a diagram for a knot convert it into a diagram of the knot's mirror image? Zeković found such a smoothing for the torus knot $T(2,5)$, and Moore–Vazquez proved that such smoothings do not exist for other torus knots $T(2,m)$ with m odd and square free. The existence of such a smoothing implies that $K \# K$ bounds a Mobius band in B^4. We use Casson–Gordon theory to provide new obstructions to the existence of such chiral smoothings. In particular, we remove the constraint that m be square free in the Moore–Vazquez theorem, with the exception of $m = 9$, which remains an open case. Heegaard Floer theory provides further obstructions; these do not give new information in the case of torus knots of the form $T(2,m)$, but they do provide strong constraints for other families of torus knots. A more general question asks, for each pair of knots K and J, what is the minimum number of smoothings that are required to convert a diagram of K into one for J. The methods presented here can be applied to provide lower bounds on this number.

Keywords: Knot; Knot theory; Chiral smoothing

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1. Introduction

Given a knot $K \subset S^3$, does it have a diagram for which smoothing a single crossing results in a diagram of its mirror image? If so, the knot is said to *support a chiral smoothing*. The existence of such a smoothing is easily seen to be equivalent to the existence of a *chiral band move* [1, 10] or a chiral $H(2)$ move [9], and such a move is called *chirally cosmetic*. The problem naturally generalizes, asking if knots K and J are related by a single smoothing. Literature on this topic includes [1, 9–11, 13–16]. The paper [24] provides an overview of the role of studying smoothings in the knot theoretic study of DNA. Perhaps the starting point of this line of research was in the work of Lickorish [19], asking whether given knots could be unknotted with a single band move.

A basic example, first discovered by Zeković [29], is that the torus knot $T(2,5)$ supports a chiral smoothing. Figure 1 illustrates such a smoothing. In the reverse direction, Moore and Vazquez [23] showed that $T(2,5)$ is unique among positive torus knots $T(2,m)$, with m square-free, for which such a move exists. The paper [24] reports on extensive computer searches that have discovered chiral smoothings for the knots 8_8 and 8_{20}.

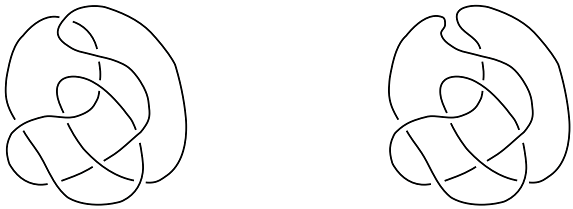

Figure 1. $T(2,-5)$ smoothed to $T(2,5)$.

If K supports a chiral smoothing, then we will see in Theorem 1 that a single band move converts $K \# K$ into $K \# -K$, where $-K$ denotes the mirror image of K with string orientation reversed. Notice that the band moves we are considering act on unoriented knots, but connected sums and knot concordance are well defined only in the oriented category. In § 2.1, we will clarify orientation issues.

The knot $-K$ is the inverse of K in the smooth concordance group, meaning that $K \# -K$ is a slice knot; it bounds a properly embedded smooth disk in B^4. In fact, $K \# -K$ is a ribbon knot. (Recall that a properly embedded smooth surface $F \subset B^4$ is called *ribbon* if the restriction of the radial height function to F is a Morse function that has no singularities of index 2.) Thus, if K supports a chiral smoothing, then $K \# K$ bounds a ribbon Mobius band in B^4. It follows that Casson–Gordon theory [2, 5] can be applied. We will follow this approach by using the application of Casson–Gordon theory to non-orientable ribbon surfaces in B^4 that was developed in [7, § 8]. One corollary is that the condition that m be square-free can be removed from the Moore–Vazquez result, with the exception of $T(2,9)$, which remains an unknown case.

We will concentrate on the case of $T(2,m)$, since these knots have two-fold branched covers that are lens spaces, $L(2m+1,1)$; Casson–Gordon invariants for these spaces can be computed in closed form. This work can be extended in a number of ways:

- Casson–Gordon invariants of two-bridge knots, $B(s,q)$, can be easily computed, lending themselves to produce further examples.
- Pairs of knots (K,J) rather than $(K,-K)$ can be considered. This is related to the $H(2)$ distance, studied, for instance, by Kanenobu in [14].
- The obstruction we focus on is based on the maximum of the absolute values of a set of sums, $\{|a_i + a_j|\}$, where the (a_i, a_j) are specified pairs taken from a set $\{a_i\}$. We use a weak bound on these sums, based on constraints on the set, given by $|\min\{a_i\} + \max\{a_i\}|$; this can be considerably improved.

We will provide a few examples of such extensions.

In the final section, we briefly discuss the application of Heegaard Floer techniques and results of Ozsváth–Stipsicz-Szabó [27] to the general problem.

2. Chiral smoothings, linking forms and metabolizers

We will be working in the smooth category throughout this paper. Homology groups are always with integer coefficients.

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Figure 2. Local view of crossing in K and tangle schematic for K.

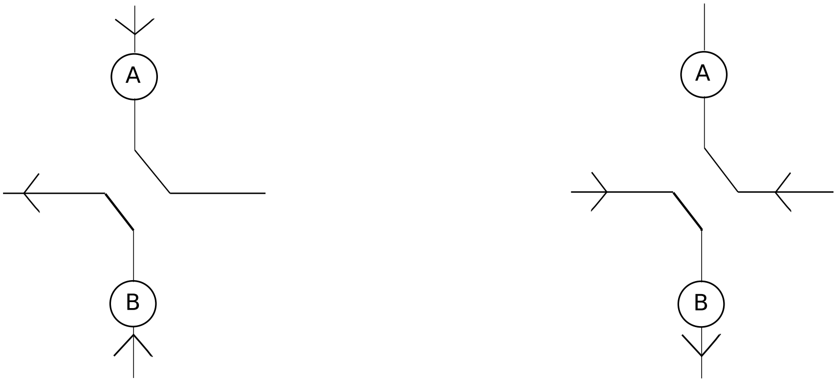

Figure 3. The two oriented smoothings of K.

2.1. Knot inversion via smoothings

The starting point for our work is the following result.

Theorem 1. *If K supports a chiral smoothing, then there is a smoothing that converts $K \# K$ into $K \# -K$.*

Proof. The illustration on the left in Figure 2 offers a schematic diagram of a crossing point in an oriented knot diagram for a knot K. The circled tangles A and B can initially be thought of as trivial tangles. On the right in the figure is a schematic tangle diagram for K (that is, if you connect the upper and lower endpoints with a trivial arc that misses the diagram, you get a diagram for K).

Figure 3 presents schematic diagrams of the smoothing of K with its two possible orientations. To say that the smoothing is chiral means that one of the two oriented knot diagrams is a representative of the concordance inverse, $-K$, that is, the mirror image of K with string orientation reversed.

Suppose that the diagram on the left in Figure 3 represents $-K$. Then if the tangle B is replaced with K, we have illustrated on the left a smoothing that converts $K \# K$ into $-K \# K$. On the other hand, if the diagram on the right in Figure 3 represents

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$-K$. Then if the tangle A is replaced with K, we have on the right an illustration of a smoothing that converts $K \# K$ into $-K \# K$. □

Remark 2. The reader might note that the proof above has an apparently stronger consequence than what has been stated: If K supports a chiral smoothing then $K \# K$ and $K \# K^r$ both bound ribbon Mobius bands in B^4. The author has written several papers concerning the role of reversal in knot concordance (for example, [17, 21]) but is unaware of any tools that could possibly exploit this added information. That is, the problem of finding a knot K for which exactly one of $K \# K$ and $K \# K^r$ bounds a ribbon Mobius band seems to be completely inaccessible using currently known techniques.

2.2. Linking forms and metabolizers

Let M_J denote the two-fold branched cover of S^3 branched over J. If J bounds an embedded surface $F \subset B^4$, let W_F denote the two-fold branched cover of B^4 branched over F. In [5, Theorem 2] and in [25], a result is proved which immediately implies the next theorem. In the statement of the theorem, we use $(H_1(M), lk)$ to denote the $\mathbb{Q}/\mathbb{Z}$–valued linking form on the first homology of a rational homology sphere M; more details can be found in [7, Appendix A].

Theorem 3. *If J bounds a Mobius band $F \subset B^4$, then the linking form on M_J splits as*

$$(H_1(M_J), lk) \cong (G_1, \beta_1) \oplus (G_2, \beta_2)$$

where G_1 is cyclic (possibly trivial) and β_2 vanishes on

$$\mathcal{M}_F = im\big(torsion\big(H_2(W_F, M_J)\big) \to H_1(M_J)\big) \subset G_2.$$

The order of $\mathcal{M}_F$ satisfies $|\mathcal{M}_F|^2 = |G_2|$.

By definition, the form (G_2, β_2) is called *metabolic* because it vanishes on a subgroup of order $\sqrt{|G_2|}$.

We now consider the case in which $J = K \# K$ and $H_1(M_K)$ is cyclic. This includes two-bridge knots $K = B(s, q)$ for which M_K is the lens space $L(s, q)$. Choose a prime divisor p of $|H_1(M_K)| = s$ so that $H_1(M_K) \cong \mathbb{Z}_{p^a} \oplus \mathbb{Z}_b$ for some $a > 0$ and $\gcd(p, b) = 1$.

Assume that $J = K \# K$ bounds a ribbon Mobius band and restrict the linking form to the *p–torsion subgroup* $\mathcal{H}_p \subset H_1(M_J)$ defined as the set of elements x such that $p^k x = 0$ for some $k > 0$. Corollary 5 is the first place in which the fact the F is ribbon becomes essential; details appear in [7]. The importance of the surface being ribbon is that this condition implies that the map $\pi_1(S^3 \setminus J) \to \pi_1(B^4 \setminus F)$ is surjective. This implies that $\pi_1(M_J) \to \pi_1(W_F)$ is surjective, which in turn places bounds on the rank of $H_2(\widetilde{W}_F)$ for the covers $\widetilde{W}_F$ of W_F that arise in Casson–Gordon theory.

Theorem 3 quickly implies that there are two possible splittings for the linking form on $\mathcal{H}_p$.

Corollary 4. *Suppose $J = K \# K$ and $H_1(M_K) \cong \mathbb{Z}_{p^a} \oplus \mathbb{Z}_b$, where p does not divide b. If J bounds an embedded Mobius band in B^4, then the linking form for $H_1(M_J)$ splits as one of two possibilities given by Theorem 3, with β_2 metabolic.*

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- $(\mathcal{H}_p, lk) \cong (\mathbb{Z}_{p^a}, \beta_1) \oplus (\mathbb{Z}_{p^a}, \beta_2)$, or
- $(\mathcal{H}_p, lk) \cong (0, \beta_1) \oplus (\mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^a}, \beta_2)$.

This has the following consequence.

Corollary 5. *Suppose that $H_1(M_K) \cong \mathbb{Z}_s$ and $K \# K$ bounds a ribbon Mobius band in B^4. If p is a prime satisfying $p \equiv 3 \mod 4$, then the exponent of p in s is even. If p is a prime having odd exponent in s, then $p \equiv 1 \mod 4$ and $(\mathcal{H}_p, lk) \cong (0, \beta_1) \oplus (\mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^a}, \beta_2)$.*

Proof. If a is odd, then a non-singular form on $\mathbb{Z}_{p^a}$ is not metabolic for any prime p. If $p \equiv 3 \mod 4$ and a is odd, the form $(\mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^a}, \beta_2)$ is a direct sum $(\mathbb{Z}_{p^a}, \beta_3) \oplus (\mathbb{Z}_{p^a}, \beta_3)$ for some non-singular form β_3. Such a form cannot be metabolic. (The proof of this number theoretic fact follows quickly from the theorem that -1 is a quadratic residue modulo p if and only if $p \equiv 2$ or $p \equiv 3 \mod 4$.) □

As a quick application that we use later, we have:

Corollary 6. *If $H_1(M_K) \cong \mathbb{Z}_s$ with $s \leq 100$ and K has a chiral smoothing, then*

$$s \in \{5, 9, 13, 17, 25, 29, 37, 41, 45, 49, 53, 61, 65, 73, 81, 85, 89, 97\}.$$

3. Casson–Gordon invariants and non-orientable surfaces

Let M be closed three-manifold with $H_1(M, \mathbb{Q}) = 0$, and let $\rho\colon H_1(M) \to \mathbb{Z}_m \subset \mathbb{Q}/\mathbb{Z}$. Casson and Gordon [2] defined an invariant $\sigma(M, \rho) \in \mathbb{Q}$. This is additive over connected sums and $\sigma(M, 0) = 0$. With regards to our work here, its key property is the following, proved in [7, Theorem 21].

Theorem 7. *Suppose that a closed three-manifold $M = \partial W$, where W is compact with $H_1(W, \mathbb{Q}) = 0$, and the inclusion $\pi_1(M) \to \pi_1(W)$ is surjective. For each $\rho\colon H_1(M) \to \mathbb{Z}_p$ that extends to $H_1(W)$,*

$$|\sigma(M, \rho)| \leq 2\beta_2(W) + 1 + \frac{1}{p-1}\beta_1(\widetilde{M}),$$

where $\widetilde{M}$ is the p–fold cover of M associated with ρ.

To apply this, we consider branched covers over surfaces. Suppose that J bounds a Mobius band $F \subset B^4$. Let $\overline{\rho} \in \mathcal{M}_F$. Linking with $\overline{\rho}$ defines a homomorphism $\rho\colon H_1(M_J) \to \mathbb{Z}_n$. Since $\overline{\rho} \in \mathcal{M}_F$, one can show that ρ extends to $H_1(W_F)$. If the order of ρ in $\mathcal{M}_F$ is m, then the non-singularity of the linking form implies that the image of ρ is cyclic of order m. To simplify our discussion, we will henceforth assume that m is a prime integer, which we will denote p. Theorem 7 has the following corollary.

Corollary 8. *If J bounds a ribbon Mobius band $F \subset B^4$, then for all $\overline{\rho} \in \mathcal{M}_F$ of prime order p,*

$$|\sigma(M_J, \rho)| \leq 3 + \frac{1}{p-1}\beta_1(\widetilde{M}_J).$$

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Proof. Apply the theorem to $M = M_J$ and $W = W_F$. Since F is ribbon, the map $\pi_1(M) \to \pi_1(W)$ is surjective. A result first proved by Massey [22] implies that $\beta_2(W_F) = 1$. $\square$

In general, determining $\beta_1(\widetilde{M}_J)$ might be difficult. For lens spaces, the value is easily computed.

Theorem 9. *Let $\rho\colon H_1(L(s,q)) \oplus H_1(L(s,q)) \to \mathbb{Z}_p$ be a surjection. Let $\widetilde{M}$ denote the induced p–fold cover of the connected sum.*

- *If ρ is non-trivial on both summands, then $\beta_1(\widetilde{M}) = p - 1$.*
- *If ρ is trivial on one of the two summands, then $\beta_1(\widetilde{M}) = 0$.*

Proof. Write $L(s,q) \# L(s,q)$ as $A \cup_S B$, where A and B are punctured lens spaces and $S \cong S^2$. Then the cover is of the form $\widetilde{A} \cup_{\widetilde{S}} \widetilde{B}$. Each of $\widetilde{A}$ and $\widetilde{B}$ are disjoint unions of (possibly multi-)punctured lens spaces having trivial first homology with rational coefficients. The cover $\widetilde{S}$ consists of p copies of S^2.

We have the Meyer–Vietoris sequence

$$0 \to H_1(\widetilde{M}, \mathbb{Q}) \to H_0(\widetilde{S}, \mathbb{Q}) \to H_0(\widetilde{A}, \mathbb{Q}) \oplus H_0(\widetilde{B}, \mathbb{Q}) \to H_0(\widetilde{M}, \mathbb{Q}) \to 0.$$

In the case that both maps to $\mathbb{Z}_p$ are non-trivial, each of $\widetilde{A}$ and $\widetilde{B}$ are connected. The sequence then becomes

$$0 \to H_1(\widetilde{M}, \mathbb{Q}) \to \mathbb{Q}^p \to \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \to 0,$$

and $H_1(\widetilde{M}, \mathbb{Q}) \cong \mathbb{Q}^{p-1}$, as desired.

In the case that, say, the map is non-trivial on $H_1(A)$ but trivial on $H_1(B)$, we have that $\widetilde{A}$ has one component and $\widetilde{B}$ has p components. Thus, the sequence becomes

$$0 \to H_1(\widetilde{M}, \mathbb{Q}) \to \mathbb{Q}^p \to \mathbb{Q} \oplus \mathbb{Q}^p \to \mathbb{Q} \to 0,$$

and $H_1(\widetilde{M}, \mathbb{Q}) \cong 0$, as desired. $\square$

4. Two-bridge knots and Casson–Gordon invariants of lens spaces

We begin by noting that Theorem 4 implies the following.

Corollary 10. *If K is a two-bridge knot $B(s,q)$ and $J = K \# K$ bounds a ribbon Mobius band $F \subset B^4$, then for every prime divisor of p of s there is an element $\overline{\rho} \in \mathcal{M}_F$ of order exactly p. Consequently, there is a surjective homomorphism $\rho\colon H_1(M_J) \to \mathbb{Z}_p$ that extends to $H_1(W_F)$.*

We will consider surjective homomorphisms $\rho\colon \pi_1(L(s,q)) \to \mathbb{Z}_p$, so will write $s = pn$. (In our main reference [2], the letter m was used instead of p, and m was not assumed to be prime. Stronger results could be obtained here by not restricting to the prime setting, but that generality is not required to generate interesting families of examples.)

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In [2], the values of $\sigma(L(s,q),\rho)$ were presented within a computation preceding the unnumbered corollary [2, p. 188]. We restate that result below as Theorem 11. There is a subtlety to the formula that appears there, as it depends on the choice of a particular generator of $H_1(L(s,q))$. We will not specify that choice here, but note that since we will be considering the set of all values, knowing the choice is not necessary for computation. That is, we are interested in the set of all values that arise for $\overline{\rho}$ and its multiples, so can avoid that technicality. Also, since $\sigma(L(s,q),\rho) = \sigma(L(s,q),-\rho)$ we can further restrict the set of values considered.

Theorem 11. *Let $\overline{\rho}$ be an element of order p in $H_1(L(pn,q))$. Then for $0 < r < p$,*

$$\sigma(L(pn,q), rq\rho) = 4\left(\mathit{area}\,\Delta(nr, \frac{qr}{p}) - \mathit{int}\,\Delta(nr, \frac{qr}{p})\right).$$

Here $\Delta(x,y)$ represents a triangle with vertices $\{(0,0),(x,0),(0,y)\}$. The value of $\operatorname{int}\Delta(x,y)$ is the weighted count of integer lattice points in the triangle, with interior points contributing 1, lattice points on the interiors of edges contributing 1/2, lattice points at non-zero vertices contributing 1/4, and the vertex at the origin contributing 0.

Focusing on the case of $L(s,1)$, we have the following.

Corollary 12. *Let p be a prime factor of $s = pn$, and let ρ be an element of $\mathcal{M}_F$ of order p. Then for all r, $0 < r < p$,*

$$\begin{aligned}\sigma(L(s,1), r\rho) &= 4\left(\text{area } \Delta\left(nr, \frac{r}{p}\right) - \mathit{int}\ \Delta\left(nr, \frac{r}{p}\right)\right)\\ &= \frac{2n}{p}r^2 - 2nr + 1.\end{aligned}$$

Proof. The computation of the lattice point count is simplified by the fact that $r/p < 1$, so that the only points in the count are on the left edge of the triangle. The rest of the computation is simple algebra. □

Corollary 13. *The maximum and minimum values of $-\sigma(L(pn,1), r\rho)$ for $0 < r < p-1$ are:*

- *$Minimum = 2n(1-1/p) - 1 > 0$; occurs at $r = 1$ and $r = p-1$.*
- *$Maximum = (n/2)(p - 1/p) - 1$; occurs at $r = (p \pm 1)/2$.*

5. Torus knots $T(2,m)$.

We can now use the results of Theorem 7 and Corollary 13 to restate the constraint from Corollary 8. This is sufficient to rule out chiral smoothings for torus knot $T(2,m) = B(m,1)$ for all odd m other than $m = 5$ and $m = 9$. As illustrated in § 1, $T(2,5)$ does support a chiral smoothing. The case of $T(2,9)$ is unknown.

Let $K = T(2,m)$ and suppose that K admits a chiral smoothing; in particular, assume that $K \# K$ bounds a ribbon Mobius band $F \subset B^4$. Note that $M(K \# K) = L(m,1) \# L(m,1)$. Let $m = pn$ for some odd prime p, and let $\overline{\rho}$ be an element of order p in $\mathcal{M}_F$. There are two cases to consider.

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Case 1. If ρ is non-trivial on both natural summands of $H_1(L(m,1)\mathbin{\#}L(m,1))$ then

$$\left(2n\left(1-\frac{1}{p}\right)-1\right)+\left(\frac{n}{2}\left(p-\frac{1}{p}\right)-1\right)\leq 4.$$

To see this, we observe that choosing the correct multiple of ρ, we can assure that one of the two values of the Casson–Gordon invariant is at the maximum.

Case 2. If ρ is trivial on one of the natural summands of $H_1(L(m,1)\mathbin{\#}L(m,1))$ then

$$\left(\frac{n}{2}\left(p-\frac{1}{p}\right)-1\right)\leq 3.$$

Again, this follows by choosing the multiple of ρ for which the Casson–Gordon invariant is at its maximum.

With these two bounds, the proof of the following theorem is immediate.

Theorem 14. *If the knot $K=T(2,m), m>1$, admits a chiral smoothing, then $m=5$ or $m=9$.*

Proof. If $p\geq 11$, then $\frac{1}{2}(p-1/p)-1>4$, so the inequality is violated regardless of n. By Corollary 6, the only remaining possibilities are $m=5$ and $m=9$. □

It is interesting to observe that in the one unknown case, $T(2,9)$, we would consider $p=3$ and $n=3$. Corresponding to the two cases, there are then two inequalities to consider, both of which can be seen to actually be equalities:

- $(3(1-\frac{1}{3})-1)+(\frac{3}{2}(3-\frac{1}{3})-1)=4.$
- $(\frac{3}{2}(3-\frac{1}{3})-1)=3.$

6. Further metabolizer constraints

6.1. Identifying metabolizing vectors

In the following discussion, we use the identification of $H_1(M)$ with $\mathbb{Q}/\mathbb{Z}$–valued characters on $H_1(M)$ that arises from the linking form. To generalize the examples of the previous section, we observe that Corollary 5 leads to the following result.

Theorem 15. *Suppose that K supports a chiral smoothing, that $H_1(M_K)\cong\mathbb{Z}_s$, and that p is a prime divisor of s with odd exponent a. Let ρ be a non-trivial $\mathbb{Z}_p$–valued character on $H_1(M_K)$. Then for some $\alpha\in\mathbb{Z}_p$ with $1+\alpha^2=0 \mod p$ and for all r :*

$$\left|\sigma(K,r\rho)+\sigma(K,r\alpha\rho)\right|\leq 4.$$

Proof. Let $a=2k+1$. For any abelian group G and prime p, let $G_p=\{g\in G \mid p^kg=0 \text{ for some } k\in\mathbb{Z}_+\}$. According to Corollary 5, if $H_1(M_K)_p\cong\mathbb{Z}_{p^a}$, then $H_1(M(K\mathbin{\#}K))_p\cong\mathbb{Z}_{p^a}\oplus\mathbb{Z}_{p^a}$ has a metabolizer $\mathcal{M}_F$ of order p^a, and thus $\mathcal{M}_F\cong\mathbb{Z}_{p^b}\oplus\mathbb{Z}_{p^c}$, where $b+c=a$. We can assume that $b\geq k+1$. In particular, $\mathcal{M}_F$ contains an element of order at least p^{k+1}. Some multiple of this element is of order exactly p^{k+1}. By taking a further multiple, we see that $\mathcal{M}_F$ contains an element $g = (p^k, \alpha p^j) \in Z_{p^a} \oplus \mathbb{Z}_{p^a}$, for some $j \geq k$, where $\gcd(\alpha, p) = 1$. This element has self-linking 0 if and only if $j = k$ and $1 + \alpha^2$ is divisible by p.

Suppose that the self-linking of $(1, 0) \in H_1(M(K \# K))_p$ is γ/p^a, where $\gcd(\gamma, p) = 1$. If g is multiplied by p^k, we get the element $(p^{a-1}, \alpha p^{a-1}) \in \mathcal{M}_F$, a metabolizing element of order p that takes value γ on the generator of the first summand and $\alpha\gamma$ on the second. The pairs of metabolizing characters, $(r\rho, \alpha r\rho)$ all arise as multiples of this element. □

Example 16. Consider the two-bridge knot $K = B(17, 2)$, also known as 10_1, the four-twisted double of the unknot. We have $H_1(M_K) \cong \mathbb{Z}_{17}$, and a calculation using Theorem 11 yields the following values of Casson–Gordon invariants, listed as pairs $(r,\ \sigma(M_K, 2r\rho))$:

$$\left\{\left(1, -\frac{13}{17}\right), \left(2, -\frac{35}{17}\right), \left(3, -\frac{49}{17}\right),\right.$$
$$\left.\left(4, -\frac{55}{17}\right), \left(5, -\frac{53}{17}\right), \left(6, -\frac{43}{17}\right), \left(7, -\frac{25}{17}\right), \left(8, \frac{1}{17}\right)\right\}.$$

For any non-singular linking form on $\mathbb{Z}_{17}$, the metabolizer for the direct sum, $\mathbb{Z}_{17} \oplus \mathbb{Z}_{17}$ consists of the multiples of $(1, \pm 4)$. Thus, the metabolizer contains $(3, \pm 5)$ and the absolute value of the sum of the two Casson–Gordon invariants is $102/17 > 4$. An obstruction to chirally smoothing this knot could not be derived by considering the maximum and minimum values of the Casson–Gordon invariants.

6.2. Non-prime order metabolizing elements

We have concentrated on the case of characters taking values in $\mathbb{Z}_p$ for some prime p. In the setting of ribbon surfaces, Theorem 7 applies without the constraint the p be prime (see [7] for details). However, the proof of Theorem 9 did require that p be prime. In general, the ranks of $H_0(\widetilde{A}, \mathbb{Q})$ and $H_0(\widetilde{B}, \mathbb{Q})$ might be any pair of divisors of p, say d_1 and d_2, except that surjectivity ensures that both cannot be p. Then, a simple modification of the proof of Theorem 9 shows $\beta_1(\widetilde{M}, \mathbb{Q}) = p + 1 - d_1 - d_2$. In all possible cases,

$$1 \leq p + 1 - d_1 - d_2 \leq p - 1,$$

so we continue to have

$$\frac{1}{p-1}\beta_1(\widetilde{M}) \leq 1.$$

Example 17. To find a knot in which the basic constraints do not apply, we need to consider a two-bridge knot $B(s, q)$, where s is composite and all prime factors that equal 3 modulo 4 have even exponent. Furthermore, we have already handled the case of $q = 1$, so must consider a larger value of q. A basic example is $K = B(3^2 \cdot 13, 20)$. In this case, if we consider only characters to $\mathbb{Z}_3$, the absolute value of the sum of the maximum and minimum Casson–Gordon invariant is 2. For characters to $\mathbb{Z}_{13}$, the sum is 4. However, for surjective characters to $\mathbb{Z}_{39}$ the sum is $\frac{53}{13} > 4$. Thus, this knot is obstructed from supporting a chiral smoothing.

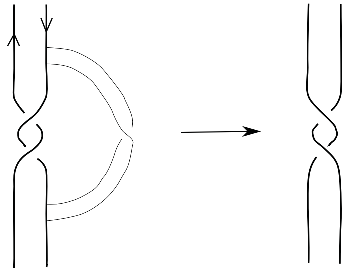

Figure 4. A band move yielding a 4-move.

6.3. Doubled knots

We have restricted to two-bridge knots, largely because this led to settings in which Casson–Gordon invariants are readily computed. In the case of doubled knots, the invariants can often be computed using results of Gilmer [6] and Litherland [20].

It was noticed by Kanenobu [14, § 11] that a so-called 4–move can be performed on a knot via a single band move. See Figure 4, taken from [14]. In particular, a single smoothing in a diagram for the positive clasped, k–twisted double of a knot J, $D_+(J, t)$ converts it into the negative clasped double, $D_-(J, t)$. In general, $-D_+(J, t) = D_-(-J, -t)$; thus, if J is amphicheiral, a single smoothing converts $D_+(K, 0)$ into $-D_+(J, 0)$, so these doubled knots support chiral smoothings. What if $t \neq 0$?

Example 18. The figure eight knot, $L = 4_1$, can be described as $D_+(U, -1)$ where U is the unknot. We have $M_{\mathrm{L}} = L(5, 2)$, a lens space. More generally, for any J, $H_1(M(D_+(J, -1))) \cong \mathbb{Z}_5$. According to the results of Gilmer and Litherland [6, 20], for any surjective character $\rho\colon H_1(M_{D_+(J,-1)}) \to \mathbb{Z}_5$, the value of $\sigma(M_{D_+(J,-1)}, \rho)$ is determined by the values of $\sigma(L(5, 2), \rho')$ (for some non-trivial ρ') and the classical Levine–Tristram signatures [18, 28] of J, as we now explain.

We first transcribe a formula from Litherland's paper [20, Corollary 2] and then translate into our situation:

$$\tau(S, \chi)[t] = \tau(K, \chi)[t] + \sum_{i=1}^{n} \alpha_C[\chi(x_i)t^{w/n}].$$

In this formula, the knot S is a *satellite* knot, in our case $D_+(J, -1)$. The knot K in Litherland's formula is the *orbit* knot, in our case the figure eight, L. Finally, C is the *companion*, in our case J. The character χ is our ρ. The invariant $\tau(S, \chi)[t]$ is a Casson–Gordon Witt class invariant, from which the invariant we are using, $\sigma(M, \rho)$, is derived. The integer w is a winding number, in our case 0. We will consider the case of two-fold branched covers, for which $n = 2$. Finally, the invariant α is an algebraic concordance invariant that determines the Levine–Tristram signature; here we view this signature function as a function on the unit complex circle, denoted $\sigma_K(\omega)$. The value of $\chi(x_i)$ will be a non-trivial value attained by ρ, which, without loss of generality, we can take as $e^{\pm 2\pi i/5}$. (As usual, we can view $\mathbb{Z}_5$ as contained in the unit complex circle.) To write the formula using this information, we change the summation index to j, so we can use

$i = \sqrt{-1}$. Putting this together we have in our setting:

$$\sigma(M_K, \rho) = \sigma(L(5,2), \rho) + \sigma_K(e^{2\pi i/5}) + \sigma_K(e^{-2\pi i/5}).$$

The signature function is conjugation invariant, so these last two summands are equal.

We can now apply Theorem 17 and find

$$\Big(\sigma(L(5,2), \rho) + \sigma_K(e^{2\pi i/5}) + \sigma_K(e^{-2\pi i/5})\Big) + \Big(\sigma(L(5,2), 2\rho) + \sigma_K(e^{4\pi i/5}) + \sigma_K(e^{-4\pi i/5})\Big) \leq 4.$$

The figure eight knot J is of order two, so there is a vanishing of a Casson–Gordon invariant of $J \# J$, yielding $\sigma(L(5,2), \rho) + \sigma(L(5,2), 2\rho) = 0$. The next result is then immediate.

Theorem 19. *If* $|\sigma_{1/5}(K) + \sigma_{2/5}(K)| > 2$, *then* $D_+(K, -1)$ *does not support a chiral smoothing.*

6.4. Smoothing distance

Rather than ask if a single smoothing can covert K into $-K$, one can more generally ask whether a single smoothing can convert a knot K into a knot J.

Example 20. The first knot that was shown to be algebraically slice but not slice by Casson and Gordon [2] was the two-bridge knot $B(25, 2)$. Since the appearance of a prime of even power in the first homology introduces challenges, we consider a related family of examples, the set of four two-bridge knots $\{B(25,1), B(25,24), B(25,2), B(25,23)\}$. The first two are mirror images, as are the last two. For each, up to conjugation, there are two characters to $\mathbb{Z}_5$ and the pair of values of the Casson–Gordon invariants is, for each knot, $\{(-7,-11), (7,11), (-3,-5), (3,5)\}$.

In general, if there is a smoothing that converts a knot K into a knot J, then $K \# -J$ bounds a Mobius band in the four-ball. The bounds based on Theorem 7 continue to apply. Using the additivity of the Casson–Gordon invariants, it is then clear that no two of these four knots differ by a single smoothing.

This example cannot be expanded to include all knots $B(25, q)$. For instance, consider the knot $B(25, 8)$. This knot has a diagram corresponding to each continued fraction expansion of $25/8$. Consider the particular expansion

$$\frac{25}{8} = 4 + \cfrac{1}{-2 + \cfrac{1}{2 + \cfrac{1}{-1 + \cfrac{1}{-5}}}}.$$

The corresponding diagram is illustrated schematically on the left in Figure 5, where the numbers in the boxes are the number of half-twists; recall that the sign of the twisting is opposite of that for the continued fraction for the second and fourth terms. Smoothing a

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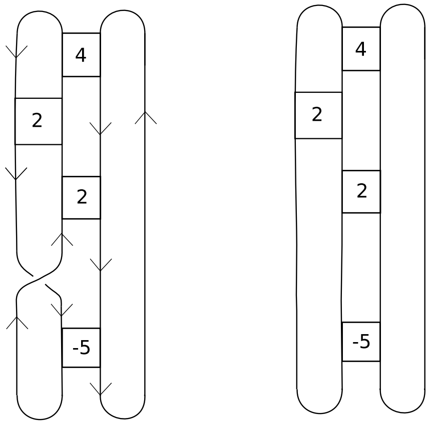

Figure 5. A band move yielding a 4-move.

single crossing yields a two-bridge knot with a diagram corresponding to the continued fraction

$$\frac{25}{7} = 4 + \cfrac{1}{-2 + \cfrac{1}{2 + \cfrac{1}{0 + \cfrac{1}{-5}}}}.$$

This is illustrated on the right in Figure 5, where it is evident that the knot does arise from a smoothing and that it is a two-bridge knot. Thus, the two-bridge knots $B(25,7)$ and $B(25,8)$ differ by a single smoothing. These are the knots $-11a364$ on the left and 8_9 on the right, in the standard notation [3].

7. Heegaard Floer obstructions and further examples of torus knots

In [27], Ozsváth–Stipcisz-Szabó developed a bound on the non-orientable four-ball genus of a knot K in terms of the *little upsilon function*, $\upsilon(K)$, and the classical signature [26], $\sigma(K)$. In brief, if K bounds a connected smooth surface $F \subset B^4$, then $\beta_1(F) \geq |\upsilon(K) - \sigma(K)/2|$. Both υ and σ are additive functions, so for knots of the form $K \# K$, the value of this difference is even, and in particular cannot equal 1. Thus, we have the following theorem.

Theorem 21. *If $\upsilon(K) \neq \sigma(K)/2$, then K does not support a chiral smoothing.*

Denote the difference $\upsilon(K) - \sigma(K)/2$ by $\phi(K)$.

For alternating knots K, $\phi(K) = 0$, so no obstruction arises. To briefly illustrate the application of this theorem, we summarize a few facts about torus knots $T(p,q)$, with $p, q > 2$, that follow quickly. For general torus knots, recursive formulas are available to compute ϕ: for the signature function, see [8], and for the upsilon function, see [4]. For a more general discussion, see [12]. As one example, one can show that $\phi(T(3, 4+6k)) = 1$ and $\phi(T(3, 5+6k)) = 1$ for all $k \geq 0$, and thus these admit no chiral smoothings. On the

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other hand, $\phi(T(3,7)) = \phi(T(3,8)) = 0$ and we do not know if these knots admit chiral smoothings. Similarly, $\phi(T(4,4k+3)) = 1$, but $\phi(T(4,4k+1)) = 0$. In general, one can find infinite families for which Theorem 23 provides effective obstructions, and find others for which we cannot at this time rule out the possibility of there being chiral smoothings.

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References

1. T. Abe and T. Kanenobu, Unoriented band surgery on knots and links, *Kobe J. Math.* **31**(12) (2014), 21–44.
2. A. J. Casson and C. McA. Gordon, *Cobordism of classical knots*, in À la recherche de la topologie perdue, Progr. Math., , pp. 181–199, Volume 62 (Birkhäuser Boston, Boston, MA, 1986). With an appendix by P. M. Gilmer.
3. J. C. Cha and C. Livingston, *Knotinfo: table of knot invariants*, http://www.indiana.edu/knotinfo (August 14, 2018)
4. P. Feller and D. Krcatovich, On cobordisms between knots, braid index, and the upsilon-invariant, *Math. Ann.* **369**(1–2) (2017), 301–329.
5. P. M. Gilmer, On the slice genus of knots, *Invent. Math.* **66**(2) (1982), 191–197.
6. P. M. Gilmer, Slice knots in S^3, *Quart. J. Math. Oxford Ser. (2)* **34**(135) (1983), 305–322.
7. P. M. Gilmer and C. Livingston, The nonorientable 4-genus of knots, *J. Lond. Math. Soc. (2)* **84**(3) (2011), 559–577.
8. C. McA. Gordon, R. A. Litherland and K. Murasugi, Signatures of covering links, *Can. J. Math.* **33**(2) (1981), 381–394.
9. J. Hoste, Y. Nakanishi and K. Taniyama, Unknotting operations involving trivial tangles, *Osaka J. Math.* **27**(3) (1990), 555–566.
10. K. Ichihara, I. D. Jong and H. Masai, Cosmetic banding on knots and links, ArXiv e-prints (February 2016), available at arxiv.org/abs/1602.01542.
11. K. Ichihara, T. Ito and T. Saito, Chirally cosmetic surgeries and Casson invariants, ArXiv e-prints (July 2017), available at arxiv.org/abs/1707.00152.
12. S. Jabuka and C. A. Van Cott, On a nonorientable analogue of the milnor conjecture, ArXiv e-prints (September 2018), available at arxiv.org/abs/1809.01779.
13. T. Kanenobu, Band surgery on knots and links, *J. Knot Theory Ramifications* **19**(12) (2010), 1535–1547.
14. T. Kanenobu, $H(2)$-Gordian distance of knots, *J. Knot Theory Ramifications* **20**(6) (2011), 813–835.
15. T. Kanenobu, Band surgery on knots and links, III, *J. Knot Theory Ramifications* **25**(10) (2016), 1650056, 12.
16. T. Kanenobu and Y. Miyazawa, $H(2)$-unknotting number of a knot, *Commun. Math. Res.* **25**(5) (2009), 433–460.
17. P. Kirk and C. Livingston, Twisted knot polynomials: inversion, mutation and concordance, *Topology* **38**(3) (1999), 663–671.
18. J. Levine, Invariants of knot cobordism, *Invent. Math.* **8** (1969), 98–110; addendum, ibid. **8** (1969), 355.
19. W. B. R. Lickorish, Unknotting by adding a twisted band, *Bull. London Math. Soc.* **18**(6) (1986), 613–615.

20. R. A. Litherland, Cobordism of satellite knots, in *Four-manifold theory (Durham, N.H., 1982)*, pp. 327–362. Contemp. Math., Volume 35 (Amer. Math. Soc., Providence, RI, 1984)
21. C. Livingston, Knots which are not concordant to their reverses, *Quart. J. Math. Oxford Ser. (2)* **34**(35) (1983), 323–328.
22. W. S. Massey, Proof of a conjecture of Whitney, *Pacific J. Math.* **31** (1969), 143–156.
23. A. H. Moore and M. Vazquez, A note on band surgery and the signature of a knot, (2018), available at arxiv.org/abs/1806.02440
24. A. H. Moore and M. Vazquez, Recent advances on the non-coherent band surgery model for site-specific recombination, (2018), available at arxiv.org/abs/1810.08751
25. H. Murakami and A. Yasuhara, Four-genus and four-dimensional clasp number of a knot, *Proc. Amer. Math. Soc.* **128**(12) (2000), 3693–3699.
26. K. Murasugi, On a certain numerical invariant of link types, *Trans. Amer. Math. Soc.* **117** (1965), 387–422.
27. P. S. Ozsváth, A. I. Stipsicz and Z. Szabó, Unoriented knot Floer homology and the unoriented four-ball genus, *Int. Math. Res. Not. IMRN* **17** (2017), 5137–5181.
28. A. G. Tristram, Some cobordism invariants for links, *Proc. Cambridge Philos. Soc.* **66** (1969), 251–264.
29. A. Zeković, Computation of Gordian distances and H_2-Gordian distances of knots, *Yugosl. J. Oper. Res.* **25**(1) (2015), 133–152.

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