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ARTICLE

Endogenous working hours, overlapping generations, and balanced neoclassical growth

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Abstract

A balanced growth path that accounts for a decline in hours worked per worker approximates the evolution of today's industrialized countries since 1870. This stylized fact is explained in an overlapping generations (OLG) model featuring two-period lived individuals equipped with per-period utility functions of the generalized log-log type proposed by Boppart and Krusell (2020) and a neoclassical production sector. Technological progress drives real wages up and expands the amount of consumption goods. The value of leisure increases, and the supply of hours worked declines. Technological progress moves a poor economy out of a regime with low wages and an inelastic supply of hours worked into a regime with high wages and a declining supply of hours worked. The balanced growth path is unique and stable. In the high wage regime, the equilibrium difference equation is available in closed form. A balanced growth path with declining hours worked may also be obtained with endogenous technological progress as in Romer (1986).

Keywords: Technological change; comparative economic development; endogenous labor supply; neoclassical endogenous growth; OLG model

JEL classifications: D15; J22; O33; O41

1. Introduction

Many of today's industrialized countries have seen a significant decline in the amount of hours worked per worker at least since 1870. According to recent estimates by Huberman (2004) and Huberman and Minns (2007), a full-time job of a US production worker in 1870 required an annual workload of 3096 hours of work. In the year 2000 this had come down to 1878 hours of work, an absolute decline of roughly 40%. A similar tendency can be found for Australia, Belgium, Canada, Denmark, France, Germany, Ireland, Italy, the Netherlands, Spain, Sweden, Switzerland, and the UK.

For this group of countries and the time span 1870–2000, Boppart and Krusell (2020) argue that the decline in annual hours worked per worker is a fairly stable trend with an estimated average annual rate of decline of roughly 0.57%.[1] At the same time, these countries evolve in line with Kaldor's growth facts (Kaldor (1961), pp. 177–178, Herrendorf, et al. (2019), Boppart and Krusell (2020)). Taken together, these stylized observations suggest an interpretation of the decline of hours worked as a balanced growth phenomenon.

The present paper develops a simple theory that is consistent with the balanced-growth-path interpretation of these stylized facts. Key elements include an OLG model with two-period lived individuals and a neoclassical production sector. The novel feature is the individual lifetime utility with periodic utility functions of the generalized log-log, henceforth BK-gll, type. This preference representation was recently proposed by Boppart and Krusell (2020), Section V, for applications in the Ramsey–Cass–Koopmans model with exogenous neoclassical growth. The present paper shows that its scope of application extends to a household sector with overlapping generations.

The mechanics of the theory emphasizes a dual role of technological progress. On the one hand, technological progress drives productivity and the growth of real wages and real incomes. On the other hand, it expands the supply of consumption goods that individuals buy and enjoy during their leisure time. Hence, the value individuals attach to leisure time increases.[2]

These mechanics are shown to function on a balanced growth path with a time-invariant real return on savings and real wages growing at a constant rate, g_w, equal to the exogenous growth rate of labor-augmenting technological knowledge, $g_A > 0$, that is, $g_w = g_A$. The individual supply of hours worked declines approximately at rate $-\nu g_w = -\nu g_A$ where $(-\nu) < 0$ is the wage elasticity of the individual supply of hours worked that reflects preferences.

Since a growing stock of technological knowledge applies to an ever declining amount of hours worked, the economy's growth rate of per-capita variables on a balanced growth path is $g_A - \nu g_w = (1 - \nu)\, g_A$ and falls with ν. Hence, even though g_A is exogenous, the economy's growth rate is endogenous through the preference parameter ν.[3]

The scope of the theory is not confined to the balanced growth path under exogenous neoclassical growth. There are at least two relevant additional features. The first concerns the global transitional dynamics, the second the compatibility with neoclassical endogenous growth à la Romer (1986).

The transitional dynamics give rise to two regimes. Roughly speaking, in Regime 0 wages are low, and individuals are poor. The prospect of a low wage income induces individuals to supply their entire time endowment to the labor market. Moreover, the supply of hours worked does not respond to an increase in the real wage. In contrast, wages are high in Regime 1. Here, the individual supply of hours worked declines in response to a wage hike. This behavioral pattern makes intuitive sense. For poor people the additional purchasing power of a higher real wage is spent on consumption goods that satisfy basic needs rather than on leisure. The demand for leisure becomes only positive once these needs are adequately satisfied. This marks the switch from Regime 0 into Regime 1.

This interpretation suggests that the common assumption of an inelastic labor supply made in the literature on modern economic growth applies best to poor economies. In contrast, a plausible growth theory of rich economies ought to include a mechanism by which the supply of hours worked declines in response to higher incomes.

The neoclassical endogenous growth model of Romer (1986) is meant to describe economic growth in modern industrialized economies. Yet, it assumes an inelastic labor supply. The present paper shows how this model may be amended to possess a balanced growth path with endogenously declining hours worked.

On the household side, this possibility arises for a constant workforce of two-period lived overlapping generations with periodic BK-gll utility functions. On the production side, this requires the relationship describing the process of decentralized knowledge creation via capital investments, K, to account for the decline in the supply of hours worked. In fact, the linear specification between technological knowledge and capital, $A = K$, stipulated by Romer has to be replaced by $A = K^{1/(1-\nu)}$. Then, with g_K denoting the growth rate of capital, the balanced growth path has approximately $g_A = g_K/(1 - \nu)$ and $g_w = g_A$ so that the growth rates of per-capita variables and of capital are equal, that is, $g_A - \nu g_A = g_K$. This modification leaves the "scale effect" intact, that is, an economy with more workers grows faster.

The present paper is related to at least two strands of the literature. First, it adds an analytically tractable variant to the literature on discrete-time models with overlapping generations (de la Croix and Michel (2002), Nourry and Venditti (2006)).

Key to the tractability is the BK-gll preference representation. It implies that neither the individual demand for leisure nor the supply of savings hinges on the real return on savings. In addition, the wage elasticity of the individual supply of hours worked is a negative constant.

Section 2 reveals that alternative preference representations used in the literature do not possess these three properties.

In conjunction with a neoclassical production function of the Cobb–Douglas type and exogenous technical change, the BK-gll preference representation implies a unique and stable balanced growth path. More surprisingly, for Regime 1 the equilibrium difference equation is available in closed form.[4]

Second, the present paper helps toward filling a gap in the modern growth literature that largely neglects the long-lasting decline in hours worked per worker and, instead, sticks to the assumption of an inelastic labor supply. Notable exceptions to this trend include Duranton (2001), Iong and Irmen (2021), and Irmen (2021).

Duranton (2001) explores the role of an endogenous labor supply under a wage elasticity different from zero in a model with two-period-lived overlapping generations and endogenous neoclassical growth as in Romer (1986). Abstracting from a consumption-savings trade-off on the household side, this author follows the then common assessment that a zero wage elasticity of the demand for leisure is a necessary condition for a balanced growth path (Duranton (2001), p. 297) and, therefore, focuses on "unbalanced" paths. In contrast, Section 5 of the present paper shows that balanced growth with a positive wage elasticity of the demand for leisure is possible if, on the household side, individuals trade-off consumption and savings when young, BK-gll utility functions represent preferences, and, on the production side, the relationship between technological knowledge and the capital stock is strictly convex as discussed above.

Iong and Irmen (2021) and Irmen (2021) study, respectively, endogenous fluctuations between growth regimes and the incentives to automate production processes in aging economies when rising wages induce a declining supply of hours worked. Similar to the present paper, these contributions feature a household sector with two-period lived individuals equipped with per-period utility functions of the BK-gll type. Yet, these studies differ in at least two respects from the present one.

First, the present paper extends and generalizes the treatment of the household sector. In particular, here, the analysis accounts for an additional preference parameter reflecting the utility weight attached to leisure. Moreover, I allow for the individual choice set to coincide with the entire domain of the lifetime utility function and not to be restricted to the subset of this domain over which the lifetime utility function is strictly concave.[5]

Second, the production sector of these contributions features endogenous technological change, respectively, along the lines of Romer (1990) and of Irmen (2017, 2020). In contrast, the focus of the present paper is on exogenous and endogenous neoclassical growth.

This paper is organized as follows. Section 2 compares the behavioral implications of some popular preference representations involving a labor-leisure trade-off to those of a BK-gll preference representation. It highlights why the latter is both reasonable and analytically convenient. Section 3 presents the model. Section 3.1 derives and characterizes the optimal plan of each cohort. Section 3.2 introduces the neoclassical production sector. The main results on balanced growth paths and transitional dynamics are contained in Section 4 and 5. Section 4 studies the intertemporal general equilibrium under the assumption of exogenous technological change. Section 4.1 states its definition and proves existence and uniqueness. Section 4.2 sets up the dynamical system in Regime 1 and provides the analysis of the steady state. The focus of Section 4.3 is on the global dynamics. Section 5 explores the role of declining hours of work in the neoclassical endogenous growth model of Romer (1986). Section 6 concludes. All proofs are contained in Section A, the Appendix.

2. Two-period lived individuals and the demands for consumption and leisure

The overlapping generations economy of the following sections is populated by identical, two-period lived individuals. When young they consume, enjoy leisure, supply labor, and save. When

Table 1. Preference representations and key properties of the demands for consumption, leisure, and savings when young

	η_1 and η_3	U_{12}	comp. statics: w	comp. statics: R
KPR-ll	$\eta_1=\eta_3=1$	$U_{12}=0$	$\frac{dc^y}{dw}>0$, $\frac{dl}{dw}=0$	$\frac{dc^y}{dR}=\frac{dl}{dR}=\frac{ds}{dR}=0$
$\bar{c}>0$	$\eta_1>1, \eta_3>1$	$U_{12}=0$	$\frac{dc^y}{dw}>0$, $\frac{dl}{dw}>0$	$\frac{dc^y}{dR}>0$, $\frac{dl}{dR}>0$, $\frac{ds}{dR}<0$
MaCurdy	$\eta_1=\eta_3=\sigma>1$	$U_{12}=0$	$\frac{dc^y}{dw}>0$, $\frac{dl}{dw}>0$	$\frac{dc^y}{dR}>0$, $\frac{dl}{dR}>0$, $\frac{ds}{dR}<0$
BK-gll	$\eta_1>1, \eta_3=1$	$U_{12}>0$	$\frac{dc^y}{dw}>0$, $\frac{dl}{dw}>0$	$\frac{dc^y}{dR}=\frac{dl}{dR}=\frac{ds}{dR}=0$

old, they have to retire and consume the proceeds of their savings. The purpose of this section is to highlight the behavioral implications of popular preference representations and to compare them to a BK-gll preference representation.

To address this issue I first identify the determinants of an individual's demands for consumption and leisure when young under a general preference representation in Section 2.1. Based on this, Section 2.2 compares specific preference representations used in the literature to a BK-gll utility representation. Table 1 summarizes the key results.

2.1. *Price responses for general preference representations*

Let y and o indicate the two periods of life. Then, c^y and c^o denote consumption when young and old. Leisure when young is l. A time constraint requires $l \in [0,1]$. The individual assesses bundles $\left(c^y, l, c^o\right)$ according to a preference relation $\succsim$ on $\left(c^y, l, c^o\right) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}_+$. The utility function U represents these preferences, that is, $\left(c^y, l, c^o\right) \mapsto U\left(c^y, l, c^o\right)$ where U:$\mathbb{R}_+ \times [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, strongly increasing, strictly quasi-concave, and satisfies the Inada conditions at the origin.

Consumption serves as numéraire. With w denoting the real wage, R the real return factor on savings, s, the periodic budget constraints read $c^y + wl + s \leq w$ and $c^o \leq Rs$. Since U is strongly increasing the latter will hold as equalities and can be consolidated. This gives the intertemporal budget constraint in present value terms as $c^y + wl + c^o/R = w$. Moreover, maximizing $U\left(c^y, l, c^o\right)$ subject to the latter constraint delivers the demand functions:

$$c^y = c^y(w,R), \quad l = l(w,R), \quad \text{and} \quad c^o = c^o(w,R). \tag{2.1}$$

To bring the analysis closer to the remaining parts of this paper I assume henceforth that $U_{13} = U_{23} = 0$, that is, there is time separability between c^y and c^o and between l and c^o. Moreover, let me introduce the following elasticities

$$\eta_1 \equiv -c^y \frac{U_{11}}{U_1} > 0 \quad \text{and} \quad \eta_3 \equiv -c^o \frac{U_{33}}{U_3} > 0, \tag{2.2}$$

which measure the curvature of U with respect to c^y and c^o at the demands (2.1). Then, the demands for consumption and leisure when young of (2.1) satisfy[6]

$$\begin{aligned}
\frac{dc^y}{dw} \gtreqless 0 \quad &\Leftrightarrow \quad 1 - \frac{U_{12}}{U_1}\frac{1}{\eta_3}\frac{c^o}{Rw} + \left(\frac{-U_{22}}{U_2} + \frac{U_{12}}{U_1}\right)(1-l) \gtreqless 0, \\
\frac{dl}{dw} \gtreqless 0 \quad &\Leftrightarrow \quad -1 - \frac{\eta_1}{\eta_3}\frac{c^o}{Rc^y} + \left(\eta_1 \frac{w}{c^y} + \frac{U_{21}}{U_1}\right)(1-l) \gtreqless 0, \\
\frac{dc^y}{dR} \gtreqless 0 \quad &\Leftrightarrow \quad (1-\eta_3)(U_{22} - wU_{12}) \gtreqless 0, \\
\frac{dl}{dR} \gtreqless 0 \quad &\Leftrightarrow \quad (1-\eta_3)(wU_{11} - U_{21}) \gtreqless 0.
\end{aligned}$$

The sign of the comparative statics dc^y/dw and dl/dw hinge on how the substitution effect, represented by the first two terms in the respective expressions, relates to the sum of the ordinary and the endowment income effect, represented by the third term.

Since changing R does not affect the value of the labor endowment, the comparative statics dc^y/dR and dl/dR show only the tension between the substitution and the ordinary income effect. The strength of the latter hinges on the elasticity η_3.[7]

2.2. Price responses for specific preference representations

With the expressions above it is possible to compare the implications of three frequently used preference representations to the BK-gll representation. Throughout, $\beta \in (0, 1)$ is the discount factor.

First, consider a preference representation with periodic utility functions of the King–Plosser–Rebelo log-log type (King, et al. (1988)), henceforth KPR-ll. Then,

$$U\left(c^y, l, c^o\right) = \ln c^y + \kappa \ln\left(1 - \phi\left(1 - l\right)\right) + \beta \ln c^o, \tag{2.3}$$

where $\kappa > 0$ and $\phi > (1 + \beta) / (\kappa + 1 + \beta) \equiv \phi_c$. Here, U satisfies $\eta_1 = \eta_3 = 1$ and $U_{12} = 0$. This implies that i) the demand for leisure does not hinge on the real wage and ii) neither the demands for consumption when young and leisure nor savings depend on the real return factor. Hence, the joint evolution of increasing real wages and declining hours of work over the long run cannot be replicated with this preference representation.

To generate a demand for leisure that increases in the real wage one may extend (2.3) and allow for subsistence consumption, $\bar{c} > 0$, in both periods of life (see, e.g., Ohanian, et al. (2008) or Bick, et al. (2018)), that is,

$$U\left(c^y, l, c^o\right) = \ln\left(c^y - \bar{c}\right) + \kappa \ln\left(1 - \phi\left(1 - l\right)\right) + \beta \ln\left(c^o - \bar{c}\right). \tag{2.4}$$

Then, $\eta_1 = c^y / (c^y - \bar{c}) > 1$, $\eta_3 = c^o / (c^o - \bar{c}) > 1$, $U_{12} = 0$, and $dl/dw > 0$. Yet, the wage elasticity of the supply of hours worked is not constant. Moreover, the demands for consumption when young and leisure increase whereas savings fall in the real return factor. The latter dependency complicates the derivation of analytical results and may not be consistent with the evidence on the determinants of life cycle savings (see, e.g., Bloom, et al. (2003)).

Finally, consider periodic utility functions of the MaCurdy type (MaCurdy (1981)). Then, the lifetime utility function reads

$$U\left(c^y, l, c^o\right) = \frac{\left(c^y\right)^{1-\sigma} - 1}{1 - \sigma} - \frac{(1 - l)^{1+\epsilon}}{1 + \epsilon} + \beta \frac{\left(c^o\right)^{1-\sigma} - 1}{1 - \sigma}, \tag{2.5}$$

where $\sigma > 1$ and $\epsilon > 0$. Here, $\eta_1 = \eta_3 = \sigma > 1$ and $U_{12} = 0$. With the same argument as for subsistence consumption the demand for leisure increases in the real wage. Moreover, the wage elasticities of the supply of hours worked and of savings are constant. Hence, MaCurdy preferences are a special case of Boppart–Krusell preferences (see, Boppart and Krusell (2020), p. 138). Yet, as under (2.4), the demands for consumption when young and leisure increase whereas savings fall in the real return factor.

Table 1 collects the findings derived so far. Its last line states the corresponding properties for a utility representation with periodic BK-gll utility functions that will be derived below. Here, the elasticities (2.2) are constant and differ. From the expressions derived in Section 2.1, it is then immediate that (i) the demand for consumption and leisure when young as well as savings are independent of R since $\eta_3 = 1$. Moreover, (ii) the demands for consumption and leisure increase in the wage since $\eta_1 > 1$ and $U_{12} > 0$. Finally, (iii) the implied supply of hours worked declines in the real wage at a constant proportionate rate. Observe that the BK-gll preference representation is the only functional form that satisfies the properties (i), (ii), and (iii).

3. The model

The economy has a household sector and a production sector in an infinite sequence of periods $t = 1, 2, \ldots, \infty$. The household sector comprises overlapping generations of individuals who live for two periods, youth and old age. Individual preferences are represented by a lifetime utility function that features periodic BK-gll utility functions.

The production sector has competitive firms producing a single good using physical capital, technology, and labor hours as inputs. This good may be either consumed or invested. In the latter case, it serves as future capital. Henceforth, I shall refer to the single produced good as the *manufactured good*. If consumed it is referred to as the *consumption good*, if invested as *capital*.

In all periods, there are three objects of exchange, the *consumption good*, *labor,* and *capital*. Capital at t is built from the savings of period $t-1$ and depreciates at rate $\delta \in [0, 1]$ after use. Households supply labor and capital. Labor is "owned" by the young; the old own the capital stock. Each period has markets for all three objects of exchange. Capital is the only asset in the economy. The manufactured good serves as numéraire.

3.1. The household sector

The population at t consists of L_t young (cohort t) and L_{t-1} old individuals (cohort $t-1$). Due to birth and other demographic factors the number of young individuals between two adjacent periods grows at rate $g_L > (-1)$. For short, I shall refer to g_L as the population growth rate.

When young, individuals supply labor, earn wage income, save, and enjoy leisure as well as the consumption good. When old, they retire and consume their wealth.

3.1.1. Preferences, utility, and the optimal plan of cohort t

For cohort t, denote consumption when young and old by c_t^y and c_{t+1}^o, and leisure time enjoyed when young by l_t. I normalize the maximum per-period time endowment supplied to the labor market to unity. Then, $1 - l_t = h_t$, where $h_t \in [0, 1]$ is hours worked when young. Individuals of all cohorts assess bundles $\left(c_t^y, l_t, c_{t+1}^o\right)$ according to a lifetime utility function:

$$U\left(c_t^y, l_t, c_{t+1}^o\right) = \ln c_t^y + \kappa \ln\left(1 - \phi\left(1 - l_t\right)\left(c_t^y\right)^{\frac{\nu}{1-\nu}}\right) + \beta \ln c_{t+1}^o, \tag{3.1}$$

where $\kappa \in (0, 1]$, $\phi > 0$, and $\nu \in (0, 1)$ are parameters to be interpreted below, and $\beta \in (0, 1)$ is the discount factor. Let $\mathcal{D}$ denote the domain of U. Since the natural logarithmic function requires a strictly positive argument, $\mathcal{D}$ cannot include the set $\mathcal{B} = \{\left(c_t^y, l_t\right) | 1 - \phi\left(1 - l_t\right)\left(c_t^y\right)^{\frac{\nu}{1-\nu}} \leq 0\}$. Hence, $\mathcal{D} = \{\left(c_t^y, l_t, c_{t+1}^o\right) \in \mathbb{R}_{++} \times [0, 1] \times \mathbb{R}_{++} \setminus \mathcal{B}\}$.[8]

The function U evaluates consumption and leisure in both periods of life according to a BK-gll utility function. Retirement means that leisure when old, l_{t+1}^o, is equal to unity. Accordingly, the term $\beta\kappa \ln\left(1 - \phi\left(1 - l_{t+1}^o\right)\left(c_{t+1}^o\right)^{\frac{\nu}{1-\nu}}\right)$ disappears from U. For ease of notation, I follow Boppart and Krussell and use henceforth

$$x_t \equiv \left(1 - l_t\right)\left(c_t^y\right)^{\frac{\nu}{1-\nu}}. \tag{3.2}$$

The term $\kappa \ln\left(1 - \phi x_t\right)$ reflects the disutility of labor when young. It is more pronounced the greater κ or ϕ. These parameters may be associated with institutional, cultural, or geographical features of the labor market that affect the disutility of labor in addition to the amount of hours worked and the level of consumption. For instance, in an economy with demanding occupational safety regulations κ and ϕ may be lower than in an economy without such regulations. Similarly, these parameters should be low if the labor market gives rise to a good matching between individual career aspirations and actual occupations. Alternatively, as suggested respectively by Weber (1930) and Landes (1998), κ and ϕ may reflect a prevailing work ethic or the climatic conditions under which labor is done. Finally, the parameter ν determines how the disutility of labor increases with the level of consumption.[9] In the context of Proposition 1 below we will see that $\nu > 0$ is key for the income effect of a wage hike on the demand for leisure to dominate the substitution effect. In the limit $\nu \to 0$, U boils down to the KPR-ll preference representation (2.3) for which the income and the substitution effect cancel.

The following lemma summarizes relevant properties of the lifetime utility function (3.1).

Lemma 1. *(Properties of U)*
The lifetime utility function U of (3.1) has the following properties:

1. *The marginal utility of consumption when young, U_1, satisfies $\lim_{c_t^y \to 0} U_1 = \infty$ and is strictly positive for pairs $\left(c_t^y, l_t\right) \in \mathcal{D}$ that also satisfy*
$$\frac{1-\nu}{\phi\,(1-\nu(1-\kappa))} > x_t. \tag{3.3}$$
Moreover, $U_{11} < 0$.
2. *The marginal utility of leisure when young, U_2, satisfies $\lim_{l_t \to 0} U_2 < \infty$, and is strictly positive. Moreover, $U_{22} < 0$.*
3. *$U\left(c_t^y, l_t, c_{t+1}^o\right)$ is strictly concave on $\mathcal{D}$ if*
$$\frac{1-2\nu+(1-\kappa)\nu^2}{\phi(1-\nu)\,(1-\nu(1-\kappa))} > x_t. \tag{3.4}$$

Hence, as to consumption when young, U satisfies the Inada condition at the origin. However, the interaction between consumption and leisure implies that U_1 is negative if c_t^y becomes too large. As to leisure when young, U is monotonically increasing without satisfying the Inada condition at the origin. Moreover, U is concave in c_t^y and l_t though not necessarily jointly concave in $\left(c_t^y, l_t\right)$.

Let $w_t > 0$ denote the real wage per hour worked and $r_t > (-1)$ the real rental rate per unit of capital. Then, $R_{t+1} \equiv 1 + r_{t+1} - \delta > (-1)$ is the perfect foresight real return factor on savings, that is, the total amount of consumption that an investment of one unit of capital at $t-1$ generates at t. I refer to $\left(c_t^y, l_t, c_{t+1}^o, s_t\right)$ as the plan of cohort t. The optimal plan of cohort t solves

$$\max_{\left(c_t^y, l_t, c_{t+1}^o, s_t\right) \in \mathcal{D}\times\mathbb{R}} \ln c_t^y + \kappa \ln\left(1 - \phi\left(1-l_t\right)\left(c_t^y\right)^{\frac{\nu}{1-\nu}}\right) + \beta \ln c_{t+1}^o \tag{3.5}$$

subject to the per-period budget constraints:

$$c_t^y + s_t \leq w_t(1-l_t) \quad \text{and} \quad c_{t+1}^o \leq R_{t+1} s_t. \tag{3.6}$$

The following assumption assures the existence of a unique optimal plan for all $w_t > 0$ and $R_{t+1} > (-1)$.

Assumption 1. *(Upper Bound on ν)*
It holds that

$$0 < \nu < \bar{\nu}\,(\beta,\kappa) \equiv \frac{3+\beta-\sqrt{(1+\beta)^2+4\kappa}}{2\,(2+\beta-\kappa)}.$$

The function $\bar{\nu}:[0,1]^2 \to \mathbb{R}_+$ takes on strictly positive values. Moreover, it is monotonically declining in β and in κ with $\bar{\nu}\,(0,0) = 1/2$ and $\bar{\nu}\,(1,1) = 1 - 1/\sqrt{2} \approx 0.2923$. In other words, Assumption 1 says that ν must not be too large.

The following proposition reveals that the optimal plan involves a corner solution $l = 0$ if the real wage is lower than the following critical value:

$$w_c \equiv \left(\frac{(1+\beta)(1-\nu)}{(\phi(\kappa+(1+\beta)(1-\nu)))^{1-\nu}(1-\nu(1+\beta))^{\nu}} \right)^{\frac{1}{\nu}}.$$

Proposition 1. *(Optimal Plan of Cohort t)*

Suppose Assumption 1 holds. Then, for cohorts $t=1,2,\ldots,\infty$, prices $w_t>0$, and $R_{t+1}>(-1)$ the optimal plan involves continuous, piecewise defined functions:

$$h_t = h(w_t), \quad c_t^y = c^y(w_t), \quad c_{t+1}^o = c^o(w_t, R_{t+1}), \quad \textit{and} \quad s_t = s(w_t). \tag{3.7}$$

Moreover, there exists a critical wage, w_c, such that

1. *if $w_t \le w_c$ then $l_t = 0$, $h_t = 1$, and $c^y(w_t)$ is implicitly given by:*

$$c_t^y = \left(\frac{(1-\nu)\left(1-\phi\left(c_t^y\right)^{\frac{\nu}{1-\nu}}\right) - \nu\kappa\phi\left(c_t^y\right)^{\frac{\nu}{1-\nu}}}{(1-\nu)(1+\beta)\left(1-\phi\left(c_t^y\right)^{\frac{\nu}{1-\nu}}\right) - \nu\kappa\phi\left(c_t^y\right)^{\frac{\nu}{1-\nu}}} \right) w_t. \tag{3.8}$$

   Moreover, $s_t = w_t - c^y(w_t) = s(w_t)$ and $c_{t+1}^o = R_{t+1}s(w_t) = c^o(w_t, R_{t+1})$;
2. *if $w_t \ge w_c$ then $l_t \ge 0$, $h_t \le 1$ and*

$$h_t = w_c^{\nu} w_t^{-\nu}, \quad c_t^y = \frac{1-\nu(1+\beta)}{(1+\beta)(1-\nu)} w_c^{\nu} w_t^{1-\nu},$$

$$s_t = \frac{\beta}{(1+\beta)(1-\nu)} w_c^{\nu} w_t^{1-\nu}, \quad c_{t+1}^o = \frac{\beta R_{t+1}}{(1+\beta)(1-\nu)} w_c^{\nu} w_t^{1-\nu}.$$

Finally, for members of cohort 0, we have $c_1^o = R_1 s_0 > 0$ where $s_0 > 0$ is given.

Proposition 1 makes two important points. First, it establishes that the optimal plan hinges on the level of the real wage. If the real wage is below the critical level w_c, then individuals supply their entire time endowment to the labor market. Henceforth, I call this case *Regime 0*. I refer to *Regime 1* if the real wage exceeds w_c. Then individuals supply less than their time endowment to the labor market. Hence, the individual labor supply is indeed piecewise defined, yet, as established in the proposition, continuous at $w = w_c$. As the real wage increases above its critical level, the supply of hours worked declines at the constant proportionate rate $\nu \in (0,1)$. This finding suggests that the standard assumption of an inelastic labor supply made in almost all growth models is in fact most plausible for low-wage, i.e., poor economies.

The implied behavioral pattern makes intuitive sense. When wages and incomes are low then the individual demand for consumption goods satisfies basic needs. The demand for leisure is zero since the only way to earn a decent income is by working the maximum of available hours. Rising wages and incomes allow people to adequately satisfy their basic needs, to consume beyond these needs and, eventually, to demand leisure.

The critical wage level, w_c, and, hence, the consumption level $c^y(w_c)$ above which the demand for leisure becomes positive, reflects an intricate relationship among preference parameters. As mentioned above, these parameters may depend on institutional, cultural, or geographical factors that differ across countries. For instance, w_c and $c^y(w_c)$ decline in κ and ϕ. Hence, of two otherwise identical economies the readiness to reduce the labor supply in response to an increasing wage begins at a lower wage level and a correspondingly lower consumption level in the economy with lower occupational safety regulations. Mutatis mutandis, a similar argument can be made for economies that differ in their work ethic or in their climatic conditions.

To understand why the individual labor supply is piecewise defined recall that the utility-maximizing plan involving $\left(c_t^y, l_t, c_{t+1}^o\right) \gg 0$ satisfies the first-order condition $U_2/U_1 = w_t$. However, when the optimal plan involves $l_t = 0$ then $U_2/U_1 \le w_t$. Hence, in Regime 0 it holds that $U_2/U_1 \leq w_t \leq w_c$ with equality only if $w_t = w_c$.[10] In other words, w_c has an interpretation as the marginal rate of substitution, U_2/U_1, evaluated at the optimal plan at $w_t = w_c$.

Second, Proposition 1 shows that the individual supply of hours worked, consumption when young, and individual savings are independent of the real return factor on savings. This follows from Section 2. Since the lifetime utility function (3.1) features $\eta_3 = 1$ the substitution and the income effect in the comparative statics for c_t^y and l_t vanish. Then, the budget constraint when young implies that s_t becomes also independent of R_{t+1}. Observe that the relevant comparative statics are independent of whether $l_t = 0$ or $l_t > 0$. Hence, they apply to the optimal plan under both regimes. An immediate implication of these findings is that consumption and leisure when young are demand complements in Regime 1, that is, $dl/dw > 0$ and $dc^y/dw > 0$.[11]

The intertemporal trade-off between consumption when young and consumption when old is governed by the first-order condition $U_1/U_3 = R_{t+1}$. Evaluated at optimal pairs $\left(c_t^y, l_t\right)$ this trade-off delivers the Euler equation in Regime 1 as:

$$\frac{c_{t+1}^o}{c_t^y} = \frac{\beta R_{t+1}}{1 - \nu(1+\beta)}. \tag{3.9}$$

The latter states the desired consumption growth factor of a member of cohort t. The parameter ν reflects the disutility of consumption when young associated with the labor supply that shows up in the second term of U. Its presence weakens the tendency to smooth consumption over the life cycle.

Regime 1 of Proposition 1 exhibits another intuitive property of the optimal plan: c_t^y, s_t, and c_{t+1}^o are proportionate to the wage income, $w_t h_t$. In particular, one finds that

$$c_t^y = \frac{1 - \nu(1+\beta)}{(1+\beta)(1-\nu)} w_t h_t \quad \text{and} \quad s_t = \frac{\beta}{(1+\beta)\,(1-\nu)} w_t h_t. \tag{3.10}$$

Hence, ceteris paribus, the marginal (and average) propensity to consume when young declines in ν whereas the marginal propensity to save out of wage income increases in ν.

Next, consider the role of ν for the response of leisure when young to changes in the real wage. Inspection of the expression for dl/dw from Section 2.1 reveals that leisure is a normal good under the lifetime utility function (3.1) since

$$\eta_1 = \frac{\kappa\,(1 - \nu\,(1 + (1+\beta)\,(1-2\nu))) + (1+\beta)^2\,(1-\nu)\,\nu^2}{\kappa(1-\nu)(1-\nu(1+\beta))} > 1,$$

$\eta_3 = 1$, and $U_{12} > 0$ (see Footnote 9). Using the consolidated budget constraint one readily verifies that

$$\frac{dl}{dw} \gtreqless 0 \quad \Leftrightarrow \quad -1 + \eta_1 + \frac{U_{21}}{U_1}\left(1 - l\right) \gtreqless 0,$$

where

$$\frac{U_{21}}{U_1}(1-l) = \frac{\nu(1+\beta)(\kappa + (1+\beta)(1-\nu))}{\kappa\,(1-\nu(1+\beta))} > 0.$$

As $\lim_{\nu\to 0}\eta_1 = 1$ and $\lim_{\nu\to 0} U_{21}(1-l)/U_1 = 0$, it is clear that $dl/dw > 0$ results since $\nu > 0$ implies both $\eta_1 > 1$ and $U_{21} > 0$. As mentioned above, in the limit $\nu \to 0$, the lifetime utility function U becomes part of the KPR class and the demand for leisure will no longer respond to changes in the real wage.[12]

One readily verifies that the response of the optimal plan to changes in factor prices satisfies

$$h'\,(w_t) \leq 0, \quad \left(c^y\right)'\,(w_t) > 0, \quad s'\,(w_t) > 0,$$
$$c_1^o\,(w_t, R_{t+1}) > 0, \quad c_2^o\,(w_t, R_{t+1}) > 0. \tag{3.11}$$

For Regime 0, this is to be expected. As $h'(w_t) = 0$, a higher real wage increases real income one-to-one. Then, consumption smoothing requires that the higher income is used to increase consumption when young and old, hence savings. For Regime 1, a similar intuition holds since

$$\frac{d \ln (w_t h_t)}{d \ln w_t} = 1 - \nu > 0,$$

that is, the proportionate increase in the wage income induced by a higher wage is still positive even though the labor supply declines. Clearly, only c^o_{t+1} increases in response to a higher R_{t+1}.

Finally, let me turn to the comparative statics of the optimal plan of Proposition 1 with respect to the preference parameters ϕ and β.[13]

Corollary 1. *(Comparative Statics of the Optimal Plan)*
Consider the optimal plan of Proposition 1 at given prices (w_t, R_{t+1}).
For Regime 0, it holds that

$$\frac{\partial c^y_t}{\partial \phi} < 0, \quad \frac{\partial c^o_{t+1}}{\partial \phi} > 0, \quad \frac{\partial s_t}{\partial \phi} > 0,$$
$$\frac{\partial c^y_t}{\partial \beta} < 0, \quad \frac{\partial c^o_{t+1}}{\partial \beta} > 0, \quad \frac{\partial s_t}{\partial \beta} > 0.$$

For Regime 1, it holds that

$$\frac{\partial h_t}{\partial \phi} < 0, \quad \frac{\partial c^y_t}{\partial \phi} < 0, \quad \frac{\partial c^o_{t+1}}{\partial \phi} < 0, \quad \frac{\partial s_t}{\partial \phi} < 0,$$
$$\frac{\partial h_t}{\partial \beta} > 0, \quad \frac{\partial c^y_t}{\partial \beta} < 0, \quad \frac{\partial c^o_{t+1}}{\partial \beta} > 0, \quad \frac{\partial s_t}{\partial \beta} > 0.$$

Corollary 1 shows that the comparative statics properties of the optimal plan hinge on whether the supply of hours worked responds to the respective parameter change or not. First, consider Regime 1. For a greater ϕ the disutility of labor is more pronounced. Accordingly, the labor supply falls. Consumption smoothing dictates that the concomitant decline in the wage income reduces consumption in both periods of life, hence, savings decline. Consumption when young is further reduced since the marginal utility of c^y_t falls in ϕ. A greater β increases the value of consumption when old. Therefore, c^o_{t+1} increases at the expense of the demand for leisure and for consumption when young. Accordingly, the labor supply and savings increase.

In Regime 0 parameter changes do not affect the labor supply. However, a greater ϕ reduces the marginal utility of consumption when young whereas a greater β increases the value of consumption when old. Hence, unlike in Regime 1, for *both* parameter changes, c^y_t falls whereas s_t and c^o_{t+1} increase.

3.1.2. Some orders of magnitude

The validity of Proposition 1 hinges on the parameter restriction of Assumption 1. The purpose of this section is to show by example that this assumption can be satisfied for reasonable magnitudes of key parameters of the model. To see this, set $\kappa = 1$ and let a period correspond to 30 years. Moreover, suppose that hours worked per worker and the real wage grow at constant annual rates denoted by g_h and g_w. Then, the economy is in Regime 1 and

$$\frac{h_{t+1}}{h_t} = (1 + g_h)^{30} \quad \text{and} \quad \frac{w_{t+1}}{w_t} = (1 + g_w)^{30}.$$

According to Proposition 1 the growth rates g_h and g_w are linked, that is,

$$\frac{h_{t+1}}{h_t} = \left(\frac{w_{t+1}}{w_t}\right)^{-\nu} \quad \text{or} \quad \left(1+g_h\right)^{30} = \left(1+g_w\right)^{-30\nu}.$$

The latter gives an estimate of ν as:

$$\nu = -\frac{\ln\left(1+g_h\right)}{\ln\left(1+g_w\right)}.$$

It follows that Assumption 1 is satisfied whenever

$$\nu = -\frac{\ln\left(1+g_h\right)}{\ln\left(1+g_w\right)} < \bar{\nu}\left(\beta, 1\right). \tag{3.12}$$

Boppart and Krusell (2020) estimate that $g_h = -0.57\%$. Since $\bar{\nu}\left(\beta, 1\right)$ is monotonically declining in β, condition (3.12) is easier to satisfy the smaller β and the larger g_w. Hence, for $\bar{\nu}\left(1, 1\right) = 1 - 1/\sqrt{2}$ a sufficient condition for (3.12) to hold is

$$\nu = -\frac{\ln .9943}{\bar{\nu}\left(1, 1\right)} < \ln\left(1+g_w\right) \quad \text{or} \quad 1.971\% < g_w.$$

If the annual discount factor is equal to 0.96 as suggested by Prescott (1986) then $\beta = 0.294$ and $\bar{\nu}\left(0.294, 1\right) = 0.352$ so that (3.12) is satisfied for all $g_w > 1.753\%$.[14]

Hence, for reasonable values of β and g_w sufficiently close to 2% and above, Assumption 1 will be satisfied.

3.2. Firms

At all t, the production sector can be represented by a single competitive firm with access to the production function

$$Y_t = \Gamma K_t^{\gamma}\left(A_t H_t\right)^{1-\gamma}, \quad \Gamma > 0, \quad 0 < \gamma < 1. \tag{3.13}$$

Here, K_t is physical capital and H_t the amount of hours of work employed by the firm. Technological knowledge is represented by A_t and advances exogenously at rate $g_A > 0$. Accordingly, $A_t = (1+g_A)^{t-1}A_1$, with $A_1 > 0$ given. The productivity parameter $\Gamma > 0$ may reflect cross-country differences in geography, technical and social infrastructure that affect the transformation of capital and efficient hours worked into the manufactured good.

In each period, the firm chooses the amounts of capital, K_t, and of hours of work, H_t, to maximize the net-present value of profits. Doing so, it takes the evolution of A_t as given. Void of intertemporal considerations, the respective first-order conditions read

$$w_t = \Gamma(1-\gamma)K_t^{\gamma}A_t^{1-\gamma}H_t^{-\gamma} \quad \text{and} \quad r_t = \Gamma\gamma K_t^{\gamma-1}\left(A_t H_t\right)^{1-\gamma}, \tag{3.14}$$

where r_t is the real rental rate of capital at t.

4. Intertemporal general equilibrium

4.1. Definition, existence, and uniqueness

A *price system* corresponds to a sequence $\{w_t, r_t\}_{t=1}^{\infty}$. An *allocation* is a sequence $\{c_t^y, l_t, c_t^o, s_t, Y_t, H_t, K_t\}_{t=1}^{\infty}$ that comprises a plan $\{c_t^y, l_t, c_{t+1}^o, s_t\}_{t=1}^{\infty}$ for all cohorts, consumption of the old at $t=1$, c_1^o, and a plan for the production sector $\{Y_t, H_t, K_t\}_{t=1}^{\infty}$.

For an exogenous evolution of the labor force, $L_t = L_1\left(1+g_L\right)^{t-1}$ with $L_1 > 0$, an exogenous evolution of technological knowledge, $A_t = A_1(1+g_A)^{t-1}$ with $A_1 > 0$, and a given initial level of capital, $K_1 > 0$, an intertemporal general equilibrium with perfect foresight corresponds to a price system and an allocation that satisfy the following conditions for all $t = 1, 2, \ldots, \infty$:

(E1) The plan of each cohort satisfies Proposition 1.

(E2) The production sector satisfies (3.14).

(E3) The market for the manufactured good clears, that is,

$$L_{t-1}c_t^o + L_t c_t^y + I_t = Y_t + (1-\delta) K_t, \tag{4.1}$$

where I_t is aggregate capital investment.

(E4) There is full employment of labor, that is,

$$H_t = L_t h_t. \tag{4.2}$$

(E1) guarantees the optimal behavior of the household sector under perfect foresight. Since the old own the capital stock, their consumption at $t = 1$ is $L_0 c_1^o = R_1 K_1 = (1 + r_1 - \delta) K_1$ and $s_0 = K_1/L_0$. (E2) assures the optimal behavior of the production sector and zero profits. (E3) states that the aggregate demand for the manufactured good at t is equal to its produced output plus the non-depreciated capital stock. According to (E4) the demand for hours worked must be equal to the supply.

The labor market requires a special treatment. Since both the aggregate demand for hours worked and the aggregate supply of hours worked are decreasing in the real wage there may be none, one, or multiple wage levels at which demand is equal to supply. To address this issue let me refer to the first condition of (3.14) as the firms' aggregate demand for hours worked at t and restate it as:

$$H_t^d = K_t A_t^{\frac{1-\gamma}{\gamma}} \left(\frac{\Gamma(1-\gamma)}{w_t} \right)^{\frac{1}{\gamma}} \equiv H_t^d(w_t). \tag{4.3}$$

Let $H_t^s = L_t h_t$ denote the aggregate supply of hours worked at t. Using Proposition 1 gives

$$H_t^s = \left\{ \begin{array}{ll} L_t w_c^{\nu} w_t^{-\nu} & \text{if } w_t \geq w_c \\ L_t \cdot 1 & \text{if } 0 < w_t \leq w_c \end{array} \right\} \equiv H_t^s(w_t). \tag{4.4}$$

Then, the labor market equilibrium, $\left(\hat{w}_t, \hat{H}_t\right)$, satisfies $\hat{H}_t = H_t^d\left(\hat{w}_t\right) = H_t^s\left(\hat{w}_t\right)$. To simplify the notation let $k_t \equiv K_t / \left(A_t^{1-\nu} L_t\right)$ and define the time-varying critical value:

$$k_{c,t} \equiv \left[\frac{w_c}{\Gamma(1-\gamma) A_t^{1-\gamma\nu}} \right]^{\frac{1}{\gamma}}. \tag{4.5}$$

Then, one readily verifies that at all t there is a unique labor market equilibrium where the equilibrium real wage is given by:

$$\hat{w}_t = \begin{cases} A_t \cdot \left(\frac{\Gamma(1-\gamma)}{w_c^{\gamma\nu}} \right)^{\frac{1}{1-\gamma\nu}} \cdot k_t^{\frac{\gamma}{1-\gamma\nu}} & \text{if } k_t \geq k_{c,t}, \\ A_t^{1-\nu\gamma} \cdot \Gamma(1-\gamma) \cdot k_t^{\gamma} & \text{if } k_t \leq k_{c,t}, \end{cases} \tag{4.6}$$

and the equilibrium amount of hours worked is

$$\hat{H}_t = \begin{cases} \frac{L_t}{A_t^{\nu}} \cdot \left(\frac{w_c}{\Gamma(1-\gamma)} \right)^{\frac{\nu}{1-\gamma\nu}} \cdot k_t^{\frac{-\gamma\nu}{1-\gamma\nu}} & \text{if } k_t \geq k_{c,t}, \\ L_t \cdot 1 & \text{if } k_t \leq k_{c,t}. \end{cases} \tag{4.7}$$

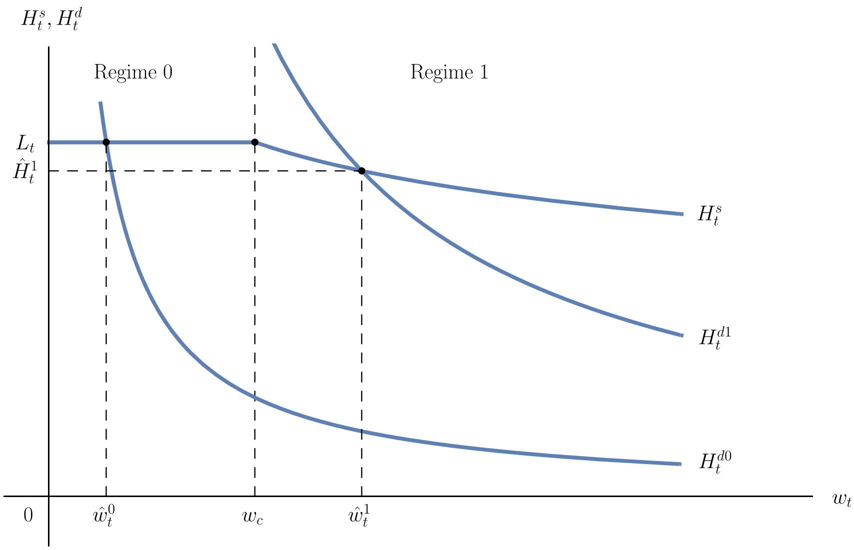

Figure 4.1. The labor market equilibrium of period t.

Note: If the aggregate demand for hours worked is H_t^{d1} then the labor market equilibrium is $\left(\hat{w}_t^1, \hat{H}_t^1\right)$. The individual supply of hours worked is in Regime 1, that is, it falls in w_t, and aggregate demand for hours worked is high. If the aggregate demand for hours worked is H_t^{d0} then the labor market equilibrium is $\left(\hat{w}_t^0, \hat{H}_t^0\right)$ where $\hat{H}_t^0 = L_t \cdot 1$. The individual supply of hours worked is in Regime 0, that is, it does not hinge on w_t, and the aggregate demand for hours worked is low.

Existence and uniqueness follow from two properties of the labor market that are illustrated in Figure 4.1. First, since ν is quite small the individual and the aggregate supply of hours worked, $h(w_t)$ and $H_t^s(w_t)$, is fairly flat. Second, the image of the aggregate demand for hours worked, $H_t^d(w_t)$, is $\mathbb{R}_{++}$ since the aggregate production function satisfies both Inada conditions.

If $k_t \geq k_{c,t}$, then $\hat{w}_1 \geq w_c$ and the labor market equilibrium is in Regime 1. Intuitively, for a given aggregate supply of hours worked this is the case if the aggregate demand for hours worked is large (see $\left(\hat{w}_t^1, \hat{H}_t^1\right)$ in Figure 4.1). From (4.3) the latter is more likely the greater K_t, A_t, or Γ. Intuitively, modern industrialized economies should possess these features. Conversely, economies with a low demand for hours worked would find their labor market equilibrium in Regime 0 (see $\left(\hat{w}_t^0, \hat{H}_t^0\right)$ where $\hat{H}_t^0 = L_t \cdot 1$ in Figure 4.1).

For a given aggregate demand for hours worked the labor market equilibrium is more likely to be in Regime 1 the smaller the total amount of workers, that is, the smaller L_t. Intuitively, when L_t falls then the aggregate supply of hours worked shifts downward. Labor becomes scarcer so that the equilibrium wage increases. Then, even for a low aggregate demand of hours worked such as H_t^{d0} an equilibrium wage in Regime 1 is possible.

Finally, observe that the equilibrium conditions (E1) – (E4) imply for all t and both regimes that aggregate saving equals capital investment, that is,

$$s_t L_t = I_t = K_{t+1}. \tag{4.8}$$

The following proposition states and proves the existence and the uniqueness of the intertemporal general equilibrium.

Proposition 2. *(Existence and Uniqueness of the Intertemporal General Equilibrium)*
For all $k_t > 0$ there exists a unique intertemporal general equilibrium.

4.2. Dynamical system for Regime 1 and steady-state analysis

Using Proposition 1, the labor market equilibrium, and the capital market equilibrium (4.8) reveals that in Regime 1 the intertemporal general equilibrium may be studied by means of the sequence $\{k_t\}_{t=1}^{\infty}$. This state variable will be constant in steady state and equal to

$$k^* \equiv w_c^{\nu}\left[\frac{\beta\left(\Gamma(1-\gamma)\right)^{\frac{1-\nu}{1-\gamma\nu}}}{(1+\beta)\,(1-\nu)\left(1+g_L\right)\left(1+g_A\right)^{1-\nu}}\right]^{\frac{1-\gamma\nu}{1-\gamma}}. \tag{4.9}$$

The following assumption serves the purpose of this section.

Assumption 2. *The initial conditions* (K_1, L_1, A_1) *satisfy*

$$A_1\left(\frac{K_1}{A_1L_1}\right)^{\gamma}\Gamma(1-\gamma) \geq w_c$$

and

$$A_1\left[\frac{\beta\left[\Gamma\left(1-\gamma\right)\right]^{\frac{1}{\gamma}}}{(1+\beta)\,(1-\nu)\left(1+g_L\right)\left(1+g_A\right)^{1-\nu}}\right]^{\frac{\gamma}{1-\gamma}} > w_c.$$

The first inequality guarantees that the equilibrium wage in $t=1$ satisfies $\hat{w}_1 \geq w_c$. Hence, the economy starts in Regime 1. In terms of the state variable, k_t, and its critical value, $k_{c,t}$, defined in (4.5) this means that $k_1 \geq k_{c,1}$. The second inequality ensures $k^* > k_{c,1}$, that is, over time the sequence $\{k_t\}_{t=1}^{\infty}$ remains in Regime 1.[15,16]

Proposition 3. *(Dynamical System - Regime 1)*

Suppose the initial conditions (K_1, L_1, A_1) *are such that Assumption 2 holds. Then, the transitional dynamics of the intertemporal general equilibrium is given by a unique and monotonous sequence* $\{k_t\}_{t=1}^{\infty}$*, generated by the difference equation:*

$$k_{t+1} = \frac{\beta\left[w_c^{\nu(1-\gamma)}\left(\Gamma(1-\gamma)\right)^{1-\nu}\right]^{\frac{1}{1-\gamma\nu}}}{(1+\beta)(1-\nu)(1+g_L)(1+g_A)^{1-\nu}}\cdot k_t^{\frac{\gamma(1-\nu)}{1-\gamma\nu}} \tag{4.10}$$

with

$$\lim_{t\to\infty} k_t = k^*. \tag{4.11}$$

Hence, in Regime 1, the evolution of the state variable is governed by the difference equation (4.10). Since $\gamma(1-\nu)/\left(1-\gamma\nu\right) < 1$ the sequence generated by this equation is monotonous and the steady state is stable. This is illustrated in Figure 4.2.

The key parameter in the difference equation (4.10) is ν. It affects the sequence $\{k_t\}_{t=1}^{\infty}$ through four channels. To see this let me write (4.8) using (3.10) and Proposition 1 as:

$$\frac{\beta}{(1+\beta)\,(1-\nu)}\hat{w}_t h\left(\hat{w}_t\right)L_t = K_{t+1}, \tag{4.12}$$

where $\hat{w}_t$ is the equilibrium wage for Regime 1. Hence, the factor $(1-\nu)$ in the denominator of (4.10) shows the effect of ν on the marginal propensity to save. A greater ν increases the fraction of the wage income that is saved and invested, hence, K_{t+1} increases. This is the first channel.

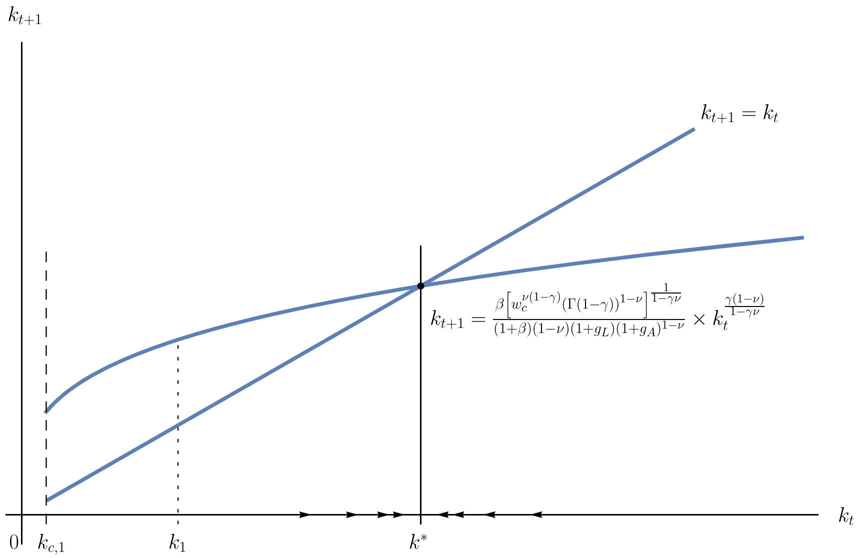

Figure 4.2. The dynamical system under Regime 1.
Note: Under Assumption 2 the steady state of the equilibrium difference equation (4.10), k^*, is unique and stable for $k_1 > k_{c,1}$.

Next, observe that Proposition 1 and (4.6) imply that the equilibrium individual wage income can be expressed as:

$$\hat{w}_t h\left(\hat{w}_t\right) = A_t^{1-\nu}\left[w_c^{\nu(1-\gamma)}\left(\Gamma(1-\gamma)\right)^{1-\nu}\right]^{\frac{1}{1-\gamma\nu}} k_t^{\frac{\gamma(1-\nu)}{1-\gamma\nu}}. \tag{4.13}$$

The remaining three channels show how ν affects the difference equation (4.10) through this expression.

The second channel is related to the factor $A_t^{1-\nu}$. It captures that, given k_t, a greater A_t increases the equilibrium wage and reduces the individual supply of hours worked. This channel shows up as $\left(1+g_A\right)^{1-\nu}$ in the denominator of (4.10) since the latter equation expresses (4.12) in units of $A_{t+1}^{1-\nu}L_{t+1}$. As $g_A > 0$ a greater ν implies a greater k_{t+1}.

The third channel reflects all effects of ν related to the bracketed term in the numerator of (4.10). From (4.13) it is clear that this term shows how preference and technology parameters affect the equilibrium individual wage income given (K_t, A_t, L_t).

Finally, ν impacts on how the state variable affects the equilibrium individual wage income. Increasing ν augments the exponent of k_t on the right-hand side of (4.10) and accelerates the process of convergence toward the steady state. Indeed, the speed of convergence defined as

$$-\frac{\partial \ln\left(\frac{k_{t+1}}{k_t}\right)}{\partial \ln k_t} = \frac{1-\gamma}{1-\gamma\nu} > 0$$

increases in ν.

The following proposition characterizes the steady state of Regime 1.

Proposition 4. *(Properties of the Steady State - Regime 1)*

Along the steady-state path the growth rate of the real wage is $g_w = g_A > 0$, and the real rental rate of capital is constant, that is, $\hat{r}_t = \hat{r}$. Moreover, it holds that

$$a) \quad \frac{h_{t+1}}{h_t} = \left(1+g_A\right)^{-\nu}, \quad \frac{\hat{H}_{t+1}}{\hat{H}_t} = \left(1+g_A\right)^{-\nu}\left(1+g_L\right),$$

$$b) \quad \frac{c^y_{t+1}}{c^y_t} = \frac{c^o_{t+1}}{c^o_t} = \frac{s_{t+1}}{s_t} = \left(1+g_A\right)^{1-\nu},$$

$$c) \quad \frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = \left(1+g_A\right)^{1-\nu}\left(1+g_L\right),$$

$$d) \quad \frac{\partial\left(1+g_A\right)^{1-\nu}}{\partial \nu} < 0.$$

Hence, in steady state the individual supply of hours worked declines at an approximate rate νg_A since $(-\nu)$ is the wage elasticity of $h(w_t)$. The steady-state growth rate of the aggregate supply of hours worked is approximately equal to $-\nu g_A + g_L$. It reflects the intensive and the extensive margin of the labor supply. Depending on which margin dominates it may be positive or negative. The growth rates under b) follow from Proposition 1 as the wage elasticity of c^y_t, c^o_{t+1}, and s_t is $1-\nu$.

The findings under a) and b) highlight why the optimal plan of Proposition 1 is consistent with a steady state equilibrium. The steady-state growth factor of individual hours worked is $\left(1+g_h\right) = \left(1+g_A\right)^{-\nu}$, the one of c^y_t, c^o_{t+1}, and s_t is $\left(1+g_A\right)^{1-\nu}$. In steady state, individual wage income, $w_t h_t$, grows at a factor $\left(1+g_h\right)\left(1+g_w\right) = \left(1+g_A\right)^{1-\nu}$ that coincides with the growth factor of c^y_t, c^o_{t+1}, and s_t. Therefore, these growth patterns are consistent with the individual and the economy-wide budget constraints. As to c), we obtain from (4.8) that in steady state $\left(1+g_K\right) = \left(1+g_A\right)^{1-\nu}\left(1+g_L\right)$. Then, the production function delivers $g_Y = g_K$.

Overall, the rule is that the steady-state growth factor of economic aggregates like Y_t, K_t, or aggregate consumption, $L_t c^y_t + L_{t-1} c^o_t$, is the growth factor of aggregate efficient hours worked, $A_t H_t = A_t L_t h_t$. The growth factor of per-capita variables like c^y_t, c^o_{t+1}, s_t, or output per worker, Y_t/L_t, is the one of efficient individual hours worked, $A_t H_t/L_t = A_t h_t$. The latter growth factor is $\left(1+g_A\right)^{1-\nu}$ and reflects the attenuating effect of a declining individual supply of hours worked on the growth rate of per-capita variables. Hence, all growth factors under a) – c) are endogenous.

According to d), the attenuation of the growth factor is more pronounced the greater is ν. Hence, the growth rate of per-capita variables declines in ν, and, ceteris paribus, an economy with a greater ν is predicted to grow slower in per-capita terms.

Observe that for all adjacent periods t and $t+1$ hours worked per worker and hours worked per capita grow at the same rate. To see this, denote the population at t by $N_t = L_t + L_{t-1}$. Then, hours worked per capita at t is the product of hours worked per worker and the labor market participation rate, that is,

$$\frac{H_t}{N_t} = h_t \times \frac{L_t}{L_t + L_{t-1}} = h_t \times \frac{1+g_L}{2+g_L}.$$

Hence, in line with the cross-country evidence over the long run the participation rate is constant (Boppart and Krusell (2020)). Moreover, the growth factor of hours worked is h_{t+1}/h_t and, in steady state, equal to $\left(1+g_A\right)^{-\nu}$.

Define a *Boppart–Krusell Balanced Growth Path*, as an allocation that satisfies Kaldor's growth facts (Kaldor (1961)), that is, the capital-output ratio, the real rental rate of capital, and factor shares remain constant whereas output per worker grows at a constant rate, *and* has the supply of hours worked declining at a constant rate.

Corollary 2. *(Boppart–Krusell Balanced Growth Path of Regime 1)*
The steady-state path of Proposition 4 is a Boppart–Krusell Balanced Growth Path.

Corollary 2 follows as the allocation described by Proposition 4 implies indeed that the labor share, $\hat{w}_t\hat{H}_t/Y_t$, and the capital share, $\hat{r}K_t/Y_t$, are time-invariant. Hence, this path is consistent with the stylized facts discussed in the Introduction.

Before closing this section two remarks are in order. First, observe that Proposition 3 and 4 do not hinge on the depreciation rate, that is, they hold for any $\delta \in [0, 1]$. This contrasts with the discrete-time neoclassical growth model where a closed-form solution requires $\delta = 1$ (Boppart and Krusell (2020), Appendix B.2). This difference is due to the fact that savings of cohort t do not hinge on R_{t+1}.

Second, consider the limit $\nu \to 0$ in which the lifetime utility function (3.1) converges to (2.3). Then, $\lim_{\nu\to 0} w_c = 0$ implies $\lim_{\nu\to 0} k_{c,1} = 0$. Moreover, the shape of the difference equation (4.10) depends on whether $\phi > \phi_c$ or $\phi = \phi_c$. In the former case, the demand for leisure is strictly positive since $\lim_{\nu\to 0} h(w_t) = \lim_{\nu\to 0} w_c^{\nu} = (1+\beta) \,/\, (\phi(\kappa + 1 + \beta)) < 1$. Accordingly, (4.10) becomes

$$k_{t+1} = \frac{\beta \left(\frac{1+\beta}{\phi(\kappa+1+\beta)}\right)^{1-\gamma} \Gamma(1-\gamma)}{(1+\beta)(1+g_L)(1+g_A)} \cdot k_t^{\gamma}.$$

If $\phi = \phi_c$ then the demand for leisure vanishes since $\lim_{\nu\to 0} h(w_t) = \lim_{\nu\to 0} w_c^{\nu} = (1+\beta) \,/\, (\phi_c(\kappa + 1 + \beta)) = 1$ and (4.10) boils down to

$$k_{t+1} = \frac{\beta\Gamma(1-\gamma)}{(1+\beta)(1+g_L)(1+g_A)} \cdot k_t^{\gamma}.$$

The latter coincides with the difference equation of the canonical OLG model with ($g_A > 0$) or without ($g_A = 0$) technological progress.

4.3. Global dynamics: technological progress as an engine of liberation

This section studies the global dynamics of the economy of Section 3. The analysis reveals that sustained technological progress is the main cause for why workers have enjoyed more and more leisure over time. It liberated poor individuals from the necessity to supply long hours of work to assure a subsistence income. In this sense, technological progress has been an engine of liberation.[17]

On the supply side, technological progress increases the marginal product of total hours worked. Accordingly, equilibrium real wages increase. During the transition to the steady state the growth rate of real equilibrium wages also reflects the growth rates of the physical capital stock and of the total supply of hours worked. However, in the long run it is technological progress alone that determines the growth rate of equilibrium wages. In addition, with technological progress aggregate output of the manufactured good increases.

On the household side, individuals who see their real income increase want to buy more of the consumption good. Additional purchases of the consumption good become feasible since technological progress allows for the total output of the consumption good to increase. Preferences exhibit a latent desire to work less. As consumption per capita increases the valuation of leisure increases. Eventually, individuals decide to enjoy more and more leisure and to supply less labor.

Section 4.3.1 starts out with the analysis of an economy where capital accumulation is the only source of economic growth. There is no technological progress. Initially, the economy is in Regime 0. Hence, real wages are low and individuals are poor. As a consequence, they supply their entire time endowment to the labor market. For the chosen parameter values, I establish that this economy converges toward a steady state with a constant real wage below w_c. Hence, while real wages may grow over time due to capital accumulation individuals remain poor and supply their entire time endowment to the labor market.[18] Section 4.3.2 adds sustained technological progress to an otherwise identical economy. The initial state of the economy is again in Regime 0. However, due to technological progress the economy evolves in finite time from Regime 0 into Regime 1 where the supply of individual hours worked continuously declines. Eventually, there is convergence to the steady state of Proposition 3.

To straighten the presentation I choose particular parameter values and make the following simplifying assumptions. On the household side, I set

$$\nu = \frac{1}{4}, \quad \beta = \frac{1}{3}, \quad \kappa = 1, \quad \text{and} \quad \phi = \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{1}{3}}. \tag{4.14}$$

While the values for ν and β are not far away from those discussed in Section 3.1.2, this calibration involves a judicious choice of κ and ϕ so that $w_c = 1$. Assumption 1 is satisfied since $1/4 < 0.352$, and the optimal plan for Regime 1 involves

$$h_t = w_t^{-\frac{1}{4}}, \quad c_t^y = \frac{2}{3} w_t^{\frac{3}{4}}, \quad c_{t+1}^o = \frac{R_{t+1}}{3} w_t^{\frac{3}{4}}, \quad \text{and} \quad s_t = \frac{w_t^{\frac{3}{4}}}{3}. \tag{4.15}$$

Without loss of generality for my qualitative results, I simplify further and set $L_t = 1$ for all t. Then, the evolution of the capital stock (4.8) becomes

$$s\left(w_t\right) = K_{t+1}. \tag{4.16}$$

On the production side, let $\Gamma = 3/2$ and $\gamma = 1/3$. Then, from (3.14) the inverse aggregate demand for hours worked is

$$w_t = A_t^{\frac{2}{3}} \left(\frac{K_t}{H_t^d}\right)^{\frac{1}{3}}. \tag{4.17}$$

Throughout, it proves convenient to describe the evolution of the economy in terms of its equilibrium real wage, $\hat{w}_t$.

4.3.1. No technological progress: the equilibrium dynamics in Regime 0

Consider the intertemporal general equilibrium of Section 4 void of technological progress, that is, $A_t = 1$ for all t. The initial state of the economy is in Regime 0. Hence, the aggregate supply of hours worked is $H_t^s = 1 \cdot 1$. Then, from (4.17) the equilibrium real wage is $\hat{w}_t = K_t^{1/3}$. Combining the latter with (4.16) delivers the evolution of the equilibrium real wage in Regime 0 as

$$\hat{w}_{t+1} = \left[s\left(\hat{w}_t\right)\right]^{\frac{1}{3}} \quad \text{for} \quad \hat{w}_t \leq s^{-1}\left(w_c^3\right), \tag{4.18}$$

where the latter inequality assures that $\hat{w}_{t+1} \leq w_c$.

The following proposition characterizes the steady state and the transitional dynamics of Regime 0.

Proposition 5. *(Dynamical System - Regime 0)*

The difference equation (4.18) gives rise to a unique, strictly positive steady-state equilibrium real wage $\hat{w}^{**} < w_c$ given by:

$$\hat{w}^{**} = \left[s\left(\hat{w}^{**}\right)\right]^{\frac{1}{3}}. \tag{4.19}$$

Suppose $0 < \hat{w}_1 < w_c$ then the sequence $\{\hat{w}_t\}_{t=1}^{\infty}$ generated by (4.18) converges monotonically with $\lim_{t\to\infty} \hat{w}_t = \hat{w}^{**}$.

The point of Proposition 5 is that a poor economy may not escape from poverty without technological progress but remain forever stuck in Regime 0. The reason is that wage growth is driven by the process of capital accumulation alone. Due to a declining impact of additional capital on equilibrium wages the latter eventually peters out and the growth of wages comes to a halt.

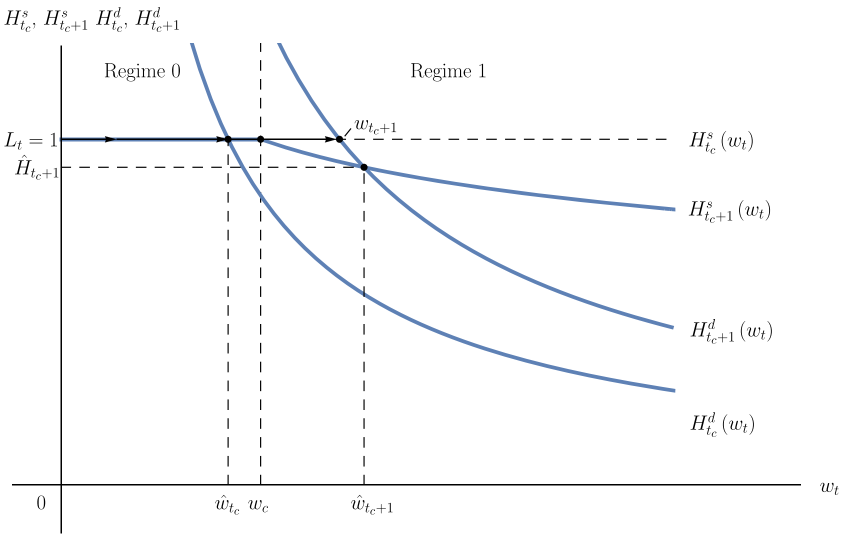

Figure 4.3. The switch between Regime 0 and 1.
Note: At t_c the equilibrium in the labor market is $(\hat{w}_{t_c}, 1)$. Then, the difference equation (4.20) delivers $w_{t_c+1} > w_c$ which is the intersection between $H^d_{t_c+1}$ and $H^s_{t_c} = 1$. However, for wage levels greater than w_c the equilibrium expression for the aggregate supply of hours worked is $H^s_{t_c+1}$. Accordingly, the labor market equilibrium in period $t_c + 1$ is $(\hat{w}_{t_c+1}, \hat{H}_{t_c+1})$ where $\hat{w}_{t_c+1} > w_{t_c+1}$.

This tendency cannot be outweighed by the utility interaction between consumption and leisure when young that reduces the marginal utility of consumption when young and, thus, implies higher savings per worker.[19] The following section shows that sustained technological progress annihilates the possibility of a steady state involving a stationary real wage.

4.3.2. Global dynamics with sustained exogenous technological progress
Sustained technological progress means that A_t grows over time at a constant rate $g_A > 0$. Let the economy start in Regime 0 with an equilibrium real wage $\hat{w}_1 < \hat{w}^{**} < w_c$. Then, equations (4.16) and (4.17) deliver the evolution of the equilibrium real wage in Regime 0 as:

$$\hat{w}_{t+1} = A_t^{\frac{2}{3}} \left[s\left(\hat{w}_t\right)\right]^{\frac{1}{3}} \quad \text{for} \quad \hat{w}_t \le s^{-1}\left(\frac{w_c^3}{A_t^2}\right), \tag{4.20}$$

where the latter inequality assures that $\hat{w}_{t+1} \le w_c$.

As seen above, $\hat{w}_t$ increases over time in Regime 0 even without technological progress. However, technological progress prevents the economy from converging to the steady state of Proposition 5 since the right-hand side of the difference equation (4.20) shifts up by a factor $\left(1 + g_A\right)^{2/3}$ between any pair of periods t and $t + 1$.

Instead, there is a finite t_c at which $\hat{w}_{t_c} > s^{-1}\left(w_c^3/A_{t_c}^2\right)$ so that (4.20) prescribes a real wage $w_{t_c+1} > w_c$. This is illustrated in Figure 4.3. However, since w_{t_c+1} falls into Regime 1 it is not the equilibrium wage of period $t_c + 1$. Intuitively, at $t_c + 1$ individuals realize that the real wage is so high that they want to reduce their supply of hours worked. Accordingly, the aggregate supply of hours worked becomes $H^s_{t_c+1}$ of (4.4) for $w_{t_c+1} > w_c$. Equating the latter with the aggregate demand for hours worked, $H^d_{t_c+1}$ of (4.3), delivers the labor market equilibrium $\left(\hat{w}_{t_c+1}, \hat{H}_{t_c+1}\right)$. Here, $\hat{w}_{t_c+1} > w_{t_c+1}$ since the decline in the supply of hours worked increases the equilibrium wage.

At $t_c + 1$ the capital stock and the level of technological knowledge will be K_{t_c+1} and A_{t_c+1}, respectively. Hence, $k_{t_c+1} = K_{t_c+1} / \left(A_{t_c+1}^{1-\nu} \right)$ as $L_t = 1$ for all t. It satisfies $k_{t_c+1} > k_{c,t_c+1}$ and serves as the initial condition for the dynamical system of Regime 1 as outlined in Proposition 3. Using (4.6) the latter can be expressed as:

$$\frac{\hat{w}_{t+1}}{A_{t+1}} = \left(\frac{1}{3(1+g_A)^{3/4}} \right)^{\frac{4}{11}} \cdot \left(\frac{\hat{w}_t}{A_t} \right)^{\frac{3}{11}}, \quad t = t_c + 1, t_c + 2, t_c + 3 \dots \tag{4.21}$$

Proposition 6. *(Dynamical System with Exogenous Technological Progress)*
Consider the intertemporal general equilibrium of Section 4 under (4.14) - (4.17). If the economy starts in Regime 0 with initial conditions such that $\hat{w}_1 < \hat{w}^{**} < w_c$ then it switches in finite time into Regime 1 and remains there. The sequence $\{\hat{w}_t / A_t\}_{t=t_c+1}^{\infty}$ generated by (4.21) converges monotonically with

$$\lim_{t \to \infty} \left(\frac{\hat{w}_t}{A_t} \right) = \frac{1}{\sqrt{2\left(1+g_A\right)^{3/4}}}. \tag{4.22}$$

Hence, technological progress drives the economy out of the poverty Regime 0. The advantages of productivity growth are not confined to the possibility to buy larger amounts of the consumption good. They also open the opportunity to enjoy more leisure.

5. Neoclassical endogenous economic growth

Romer (1986) argues that endogenous steady-state growth of per-capita variables is consistent with the neoclassical growth model if the accumulation of technological knowledge occurs as a byproduct of capital accumulation and the labor supply is time-invariant. Key to the argument is a linear relationship linking the level of technological knowledge to the contemporaneous stock of capital, a constant population, and an exogenous supply of hours worked.[20]

This section asks whether Romer's argument still delivers endogenous steady-state growth if individuals reduce their supply of hours worked in response to a rising wage as described by Regime 1 of Proposition 1. Does this behavioral feature interact with the so-called "scale effect"? To address these issues, the production sector of Section 3.2 needs to be adapted to Romer's setting.

Consider a continuum $[0, 1]$ of identical competitive firms. At all t, firm $i \in [0, 1]$ produces output $Y_t(i)$ according to the production function

$$Y_t(i) = \Gamma K_t(i)^{\gamma} \left(A_t H_t(i)\right)^{1-\gamma}, \quad \Gamma > 0, \quad 0 < \gamma < 1, \tag{5.1}$$

where $K_t(i)$ is physical capital and $H_t(i)$ the amount of hours of work employed by this firm. In each period, firms choose $K_t(i)$ and $H_t(i)$ to maximize the net-present value of profits taking $\{A_t\}_{t=1}^{\infty}$, the evolution of technological knowledge, as given. The corresponding first-order conditions are

$$w_t = \Gamma(1-\gamma) K_t(i)^{\gamma} A_t^{1-\gamma} H_t(i)^{-\gamma} \quad \text{and} \quad r_t = \Gamma \gamma K_t(i)^{\gamma-1} \left(A_t H_t(i)\right)^{1-\gamma}. \tag{5.2}$$

Technological knowledge at t, A_t, is a function of the aggregate capital stock

$$A_t = K_t^{\zeta}, \quad \zeta > 0. \tag{5.3}$$

The latter generalizes the discussion in Romer (1986) allowing for values of $\zeta \neq 1$.

The labor market requires again a special treatment since both the labor demand and the labor supply of hours worked are decreasing in the real wage. To see this here, evaluate the first condition for hours worked in (5.2) at (5.3) and sum over firms. This gives the firms' aggregate demand for hours worked at t as:

$$H_t^d = K_t^{1+\zeta\frac{1-\gamma}{\gamma}} \left(\frac{\Gamma(1-\gamma)}{w_t}\right)^{\frac{1}{\gamma}}. \tag{5.4}$$

The aggregate supply of hours worked, H_t^s, is still given by (4.4). Let

$$\underline{K}_c \equiv \left[\frac{w_c L^\gamma}{\Gamma(1-\gamma)}\right]^{\frac{1}{\gamma+\zeta(1-\gamma)}}. \tag{5.5}$$

Then, the labor market equilibrium at t, $\left(\hat{w}_t, \hat{H}_t\right)$, is given by:

$$\hat{w}_t = \begin{cases} \left(\frac{\Gamma(1-\gamma)}{(w_c^\nu L)^\gamma}\right)^{\frac{1}{1-\gamma\nu}} \cdot K_t^{\frac{\gamma+\zeta(1-\gamma)}{1-\gamma\nu}} & \text{if } K_t \geq \underline{K}_c, \\ \frac{\Gamma(1-\gamma)}{L^\gamma} \cdot K_t^{\gamma+\zeta(1-\gamma)} & \text{if } K_t \leq \underline{K}_c, \end{cases} \tag{5.6}$$

and

$$\hat{H}_t = \begin{cases} \left(\frac{w_c^\nu L}{(\Gamma(1-\gamma))^\nu}\right)^{\frac{1}{1-\gamma\nu}} \cdot K_t^{-\nu\left(\frac{\gamma+\zeta(1-\gamma)}{1-\gamma\nu}\right)} & \text{if } K_t \geq \underline{K}_c, \\ L \cdot 1 & \text{if } K_t \leq \underline{K}_c. \end{cases} \tag{5.7}$$

Hence, for $K_t \geq \underline{K}_c$ the aggregate demand for hours worked induces an equilibrium wage $\hat{w}_t \geq w_c$. Accordingly, the expressions for $\hat{w}_t$ and $\hat{H}_t$ reflect a supply of hours worked that declines in the real wage. For $K_t \leq \underline{K}_c$ the equilibrium wage satisfies $\hat{w}_t \leq w_c$ and the equilibrium level of employment is L.

The accumulation of capital is described by (4.8). Let $g_K \equiv K_{t+1}/K_t - 1$ denote the time-invariant growth rate of the capital stock and assume that

$$\frac{\beta}{(1+\beta)(1-\nu)} \cdot \left[\left(w_c^\nu \cdot L\right)^{1-\gamma} \cdot (\Gamma(1-\gamma))^{1-\nu}\right]^{\frac{1}{1-\gamma\nu}} > 1. \tag{5.8}$$

Proposition 7. *(Neoclassical Endogenous Steady-State Growth)*
Suppose that (5.8) holds and let $K_1 > \underline{K}_c$. Then, for all $t = 1, 2, \ldots, \infty$, the

1. *growth rate of the capital stock is time-invariant and strictly positive if and only if*

$$\zeta = \frac{1}{1-\nu},$$

2. *growth factors of individual and aggregate variables satisfy*

$$a) \quad \frac{\hat{w}_{t+1}}{\hat{w}_t} = \frac{A_{t+1}}{A_t} = (1+g_K)^{\frac{1}{1-\nu}} \text{ and } \quad \hat{r}_t = \hat{r},$$

$$b) \quad \frac{h_{t+1}}{h_t} = \frac{\hat{H}_{t+1}}{\hat{H}_t} = (1+g_K)^{\frac{-\nu}{1-\nu}},$$

$$c) \quad \frac{Y_{t+1}}{Y_t} = \frac{c_{t+1}^y}{c_t^y} = \frac{c_{t+1}^o}{c_t^o} = \frac{s_{t+1}}{s_t} = 1+g_K.$$

The first claim of Proposition 7 highlights a key contrast to Romer's argument. If individuals reduce their supply of hours worked in response to a rising wage as in Regime 1 of Proposition 1 then endogenous steady-state growth requires the relationship between the level of technological knowledge and the contemporaneous stock of capital to be strictly convex. More precisely, steady-state growth is possible if and only if

$$A_t = K_t^{\frac{1}{1-\nu}}. \tag{5.9}$$

Hence, the level of technological knowledge has to grow faster than the capital stock. This suggest the presence of complementarities in the process of decentralized knowledge creation.

On the one hand, $\zeta = 1/(1-\nu)$ assures that the growth factor of capital is equal to the growth factor of aggregate savings. Indeed, absent of population growth the growth factors of aggregate and individual savings coincide and are equal to the growth factor of the individual wage income, $\hat{w}_t h_t$. Then, with (5.6) and Proposition 1 one readily verifies that

$$\frac{\hat{w}_{t+1}h_{t+1}}{\hat{w}_t h_t} = \left(\frac{\hat{w}_{t+1}}{\hat{w}_t}\right)^{1-\nu} = (1+g_K)^{\left(\frac{\gamma+\zeta(1-\gamma)}{1-\gamma\nu}\right)(1-\nu)}$$

The exponent is only equal to 1 if $\zeta = 1/(1-\nu)$.

On the other hand, $\zeta = 1/(1-\nu)$ assures that aggregate output is linked to the capital stock through a time-invariant factor of proportionality. To see this consider the neoclassical production function with $L_t = L$

$$Y_t = F\left(K_t, A_t h_t L\right) = K_t F\left(1, \frac{A_t h_t}{K_t} L\right).$$

Hence, Y_t is linear in K_t if $A_t h_t / K_t$ remains constant over time. Using (5.3), Proposition 1, and (5.6) the growth factor of this ratio can be expressed as:

$$\left(\frac{A_{t+1}}{A_t}\right)\left(\frac{h_{t+1}}{h_t}\right)\left(\frac{K_{t+1}}{K_t}\right)^{-1} = (1+g_K)^{\zeta}\,(1+g_K)^{-\nu\left(\frac{\gamma+\zeta(1-\gamma)}{1-\gamma\nu}\right)}\,(1+g_K)^{-1}.$$

The exponents vanish only if $\zeta = 1/(1-\nu)$. Hence, technological knowledge has to grow faster than capital to outweigh the decline in hours worked. As a consequence, aggregate output grows at the same rate as capital.

Observe that

$$g_K = \frac{\beta}{(1+\beta)(1-\nu)} \cdot \left[\left(w_c^{\nu} \cdot L\right)^{1-\gamma} \cdot (\Gamma(1-\gamma))^{1-\nu}\right]^{\frac{1}{1-\gamma\nu}} - 1. \tag{5.10}$$

This expression reflects the time-invariant parameters that determine the individual propensity to save as well the equilibrium wage level at all t. Condition (5.8) assures that $g_K > 0$. Moreover, the scale effect survives: g_K increases in L.

As shown in Claim 2, $\zeta = 1/(1-\nu)$ also assures a common growth rate of wages and technological knowledge and a constant rental rate of capital. Hence, the labor share, $\hat{w}_t \hat{H}_t / Y_t$, as well as the capital share, $\hat{r} K_t / Y_t$ are constant.

Corollary 3. *(Boppart–Krusell Balanced Growth Path in Romer (1986))*
The steady-state path of Proposition 7 is a Boppart–Krusell Balanced Growth Path.

These findings extend and complement those of Duranton (2001). In particular, they reveal that BK-gll preferences in conjunction with $\zeta = 1/(1-\nu)$ delivers positive and sustained growth in spite of the complementarity between leisure and consumption.

6. Concluding remarks

To a first approximation the growth performance of today's industrialized countries since 1870 may be described as an evolution along a Kaldorian balanced growth path (Kaldor (1961)). Yet, as suggested by Boppart and Krusell (2020), this notion should be extended to include the decline in the amount of hours worked per worker observed in these countries. The present paper accomplishes this with an OLG model featuring two-period lived individuals equipped with per-period utility functions of the generalized log-log type of Boppart and Krusell (2020) and a neoclassical production sector that features either exogenous or endogenous technological progress.

My analysis suggests several directions for future research. One concerns the role of government policies that may affect the supply of hours worked through payroll taxes, pension schemes, or differential labor market regulations (Prescott (2004)). A second concerns the recent literature on the structural properties of balanced growth paths (see, e.g., Grossman, et al. (2017), Grossman, et al. (2021), or Casey and Horii (2024)). These contributions focus on capital-augmenting technical change and human capital accumulation. None of them includes a declining amount of hours worked as a balanced growth phenomenon. Future research may reveal whether these phenomena can be combined in a unified framework.

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Competing interests. The author declares none.

Notes

1 Notwithstanding, for shorter time spans the evolution of hours of work may deviate in some countries from this negative trend. See, for example, the discussion of the US post World War II experience in McGrattan and Rogerson (2004), Ramey and Francis (2009), or Boppart and Krusell (2020) suggesting that there is no trend in labor hours.
2 Gordon (2016), p. 9, illustrates this mechanism with technological progress in home entertainment: "Added household equipment, such as TV sets, and technological change, such as the improvement in the quality of TV-set pictures, increase the marginal product of home time devoted to household production and leisure. For instance, the degree of enjoyment provided by an hour of leisure spent watching a TV set in 1955 is greater than that provided by an hour listening to the radio in the same living room in 1935."
3 Overall, these results underline that, mutatis mutandis, the properties of the balanced growth path derived for the Ramsey–Cass–Koopmans model by Boppart and Krusell (2020) carry over to a setting with two-period lived overlapping generations.
4 Appendix B.2 of Boppart and Krusell (2020) sketches a closed-form solution for the planner's problem in a discrete-time Ramsey model with BK-gll utilty, Cobb–Douglas production, and a rate of capital depreciation equal to 100%. The existence of this solution hinges crucially on the restrictive assumption that capital fully depreciates. In contrast, the closed-form solution with overlapping generations derived in the present paper obtains for any depreciation rate.
5 To be precise, the first generalization refers to the introduction of the parameter $\kappa \in (0, 1]$ in the lifetime utility function (see equation (3.1) below). This parameter is set equal to unity in Iong and Irmen (2021) and Irmen (2021). Both generalizations manifest themselves in Proposition 1 which, in the stated sense, generalizes Proposition 2.1 in Iong and Irmen (2021) and Proposition 2.4 in Irmen (2021).
6 See, Irmen (2023) for a derivation of these expressions.
7 To be precise, consider dc^y/dR. Then, $U_{22} - wU_{12}$ represents the substitution effect whereas $-\eta_3 \left(U_{22} - wU_{12}\right)$ captures the income effect; analogously for dl/dR.
8 Accordingly, U is not a utility function that represents a preference relation $\succeq$ over all bundles $\left(c_t^y, l_t, c_{t+1}^o\right) \in \mathbb{R}_{++} \times [0, 1] \times \mathbb{R}_{++}$.
9 In fact, $\nu > 0$ implies that the cross derivative U_{12} is strictly positive, that is,

$$U_{12} = \frac{\nu\phi\kappa}{(1-\nu)\left(c_t^y\right)^{\frac{1-2\nu}{1-\nu}}(1-\phi x_t)^2} > 0.$$

Hence, under the utility representation (3.1), consumption when young and leisure are "utility complements." Yet, the sign of U_{12} is not an ordinal property of the utility function U. To see this, consider parameter values $\kappa = 1$, $\phi = \sqrt[3]{3/2}/2$, and $\nu = 1/4$. U can take on all values in $\mathbb{R}$. Then, $V = 1 - \exp\left[-U\right]$ is a strictly increasing transformation of U, that is, both V and U represent the underlying preferences. Consider the bundle $\left(c_t^y, l_t, c_{t+1}^o\right) = (2, 1/4, 1) \in \mathcal{D}$. Then, one finds $U_{12}\,(2, 1/4, 1) \approx 0.57 > 0$ and $V_{12}\,(2, 1/4, 1) \approx -0.01 < 0$, that is, consumption and leisure appear as "utility substitutes" under V.
10 To see this analytically, use Lemma 1 to express the marginal rate of substitution between consumption when young and leisure in Regime 0 at the optimal plan. This gives

$$\frac{U_2\left(c^y(w_t), 0, c^o\left(w_t, R_{t+1}\right)\right)}{U_1\left(c^y(w_t), 0, c^o\left(w_t, R_{t+1}\right)\right)} = \frac{\kappa\phi(1-\nu)\left(c^y(w_t)\right)^{\frac{1}{1-\nu}}}{(1-\nu)\left(1-\phi\left(c^y(w_t)\right)^{\frac{1}{1-\nu}}\right) - \nu\kappa\phi\left(c^y(w_t)\right)^{\frac{1}{1-\nu}}}.$$

The right-hand side is increasing in w_t since $(c^y)'(w_t) > 0$ and converges to w_c as $w_t \uparrow w_c$ (see Lemma 3 in the proof of Proposition 1).

11 It is worth noting that the influence of the real return factor on savings for intertemporal decision-making is limited here since individuals are required to retire when old. Allowing for a supply hours of work when old introduces the possibility of intertemporal substitution in the labor supply, as suggested by Lucas and Rapping (1969). As this generalization would significantly complicate the mathematical analysis I leave it for future research.

12 As $\nu \to 0$, the optimal plan under Regime 0 converges, respectively, to $c_t^y = w_t/(1+\beta)$, $s_t = \beta w_t/(1+\beta)$, and $c_{t+1}^o = \beta R_{t+1} w_t/(1+\beta)$, which coincides with the canonical OLG model (see, e.g., Acemoglu (2009), Chapter 9.3).

13 Comparative statics with respect to κ mimic those of ϕ and are therefore omitted.

14 The value of β is sensitive to the chosen value of the annual discount factor. For instance, if the latter is 0.97 or 0.98 then one has, respectively, $\beta = 0.40$ or $\beta = 0.55$. These modifications impact on the corresponding values of $\bar{\nu}$ which are equal to 0.342 and 0.33. Yet, $g_w > 1.753\%$ remains sufficient for (3.12) to hold.

15 Note that (4.5) and exponential growth of A_t imply that $\lim_{t\to\infty} k_{c,t} = 0$, that is, asymptotically Regime 0 vanishes. Hence, if a steady state exists it is in Regime 1. With this in mind, the second inequality of Assumption 2 excludes a constellation $k_1 > k_{c,1} > k^*$ that may give rise to transitional dynamics involving switching back and forth between Regime 1 and Regime 0.

16 Notice that there are indeed plausible parameter constellations that satisfy Assumption 2. Consider, for example, the preference parameters of (4.14) below and the corresponding optimal plan (4.15). In addition, set $\Gamma = 3/2$ and $\gamma = 1/3$. Then, $k^* > k_{c,1}$ is satisfied whenever $A_1 > \sqrt{3(1+g_L)(1+g_A)^{3/4}}$. Moreover, $k_1 \geq k_{c,1}$ requires $K_1/L_1 > (A_1)^{-2}$.

17 In a related sense, the metaphor "engine of liberation" is also used by Greenwood, et al. (2005) to describe the role of technological change for the liberation of women from the home.

18 For alternative parameter constellations, for example, by allowing for a larger Γ, the process of capital accumulation may actually lead the economy in finite time out of Regime 0 into Regime 1. Upon arrival in Regime 1, individuals will start to reduce their supply of hours worked. However, due to a declining impact of additional capital per worker on equilibrium wages the economy will converge to a stationary steady state with constant levels of per-capita variables. Since this prediction contradicts the stylized facts set out in the Introduction I neglect this case.

19 To be precise, it is not difficult to show that in Regime 0 consumption when young, $c^y(w_t)$, is strictly smaller than $c^y(w_t) = w_t/(1+\beta)$ that results for $\nu = 0$ (see Footnote 9). Then, the budget constraint when young implies that $s(w_t)$ for $\nu > 0$ must exceed $s(w_t) = \beta w_t/(1+\beta)$ obtained for $\nu = 0$.

20 Roughly speaking the argument is as follows. Consider a neoclassical aggregate production function $Y_t = F\left(K_t, A_t h_t L_t\right)$. If technological knowledge obeys $A_t = K_t$ and $h_t = h$ then $Y_t = K_t F\left(1, hL_t\right)$. Moreover, the equilibrium factor prices paid in the competitive production sector are $\hat{r}_t = F_1\left(1, hL_t\right)$ and $\hat{w}_t = K_t F_2\left(1, hL_t\right)$. Hence, if $L_t = L$ then the marginal product of capital is constant whereas Y_t and w_t grow at the same rate as K_t.

21 The computations were supported by *Mathematica*. The notebook is available upon request.

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A. Appendix

A.1. Proof of Lemma 1

Claim 1 The marginal utility of consumption when young is given by:

$$U_1 = \frac{(1-\nu)(1-\phi x) - \nu\kappa\phi x}{c^y(1-\nu)(1-\phi x)}. \tag{A.1}$$

Since $\lim_{c^y\to 0} x = \lim_{c^y\to 0}(1-l)\,(c^y)^{\frac{\nu}{1-\nu}} = 0$ it holds that $\lim_{c^y\to 0} U_1 = \infty$. The denominator of (A.1) is positive for all $\left(c^y, l\right) \in \mathcal{D}$. The stated condition for $U_1 > 0$ is then necessary and sufficient for the numerator to be strictly positive. Moreover, from (A.1) one readily verifies that

$$U_{11} = -\frac{1-\nu-\phi x(1-\nu(1-\kappa))}{(c^y)^2\,(1-\nu)(1-\phi x)} - \frac{\kappa\nu^2\phi x}{(c^y)^2\,(1-\nu)^2(1-\phi x)^2} < 0. \tag{A.2}$$

Claim 2 The marginal utility of leisure when young is given by:

$$U_2 = \frac{\kappa\phi\,(c^y)^{\frac{\nu}{1-\nu}}}{1-\phi x}. \tag{A.3}$$

Since $\lim_{l\to 0} x = (c^y)^{\frac{\nu}{1-\nu}} > 0$ it holds that $\lim_{l\to 0} U_2 = \kappa\phi\,(c^y)^{\frac{\nu}{1-\nu}} \Big/ \left(1-\phi\,(c^y)^{\frac{\nu}{1-\nu}}\right) < \infty$. The denominator of (A.3) is strictly positive for all $\left(c^y, l\right) \in \mathcal{D}$. Hence, $U_2 > 0$. Moreover, from (A.3) one readily verifies that

$$U_{22} = -\frac{\kappa\phi^2\,(c^y)^{\frac{2\nu}{1-\nu}}}{(1-\phi x)^2} < 0. \tag{A.4}$$

Claim 3 Consider the leading principal minors of the Hessian matrix of $U\left(c^y, l, c^o\right)$, that is,

$$D_1\left(c^y, l, c^o\right) = -\frac{(1-\nu-\phi x)^2 + \nu\phi x\,(1-\phi x + (1-\kappa)\,(1+(1-\nu)\phi x))}{(c^y(1-\nu)\,(1-\phi x))^2},$$

$$D_2\left(c^y, l, c^o\right) = \frac{\kappa\phi^2\left(1-2\nu+(1-\kappa)\nu^2-(1-\nu)\,(1-(1-\kappa)\nu)\,\phi x\right)}{(c^y)^{\frac{2(1-2\nu)}{1-\nu}}\,(1-\nu)^2\,(1-\phi x)^3},$$

$$D_3\left(c^y, l, c^o\right) = -\frac{\beta}{(c^o)^2}D_2\left(c^y, l, c^o\right).$$

First, we have $-D_1\left(c^y, l, c^o\right) > 0$ since $1-\phi x > 0$ and $\kappa \leq 1$. Second, $D_2\left(c^y, l, c^o\right) > 0$ and $-D_3\left(c^y, l, c^o\right) > 0$ hold since the denominator in both expressions is strictly positive and the numerator is positive if and only if (3.4) holds. This implies strict concavity. □

A.2. Proof of Proposition 1

For ease of notation I shall suppress the time argument. First, I establish relevant properties of the solution to the constraint maximization problem (3.5) with (3.6) in Lemma 2. The latter implies that the solution to the constraint maximization problem can be derived as the solution to an unconstraint maximization problem. For this problem, I determine the candidate solution from the first-order conditions. Then, I prove in turn the findings stated in the proposition for Regime 0 and Regime 1 and establish the continuity of the piecewise defined optimal plan. Finally, I show that the second-order conditions for both regimes holds under Assumption 1.

Lemma 2. *(Properties of the Optimal Plan)*
Suppose $\left(c^y, l, c^o, s\right)$ is an optimal plan. Then,

$$c^y > 0, \quad l \in [0,1), \quad c^o > 0, \quad s > 0, \quad c^y + s = w(1-l), \quad \textit{and} \quad c^o = Rs.$$

Proof of Lemma 2
Since $\lim_{c^o\to 0} \partial U/\partial c^o = \infty$ the optimal plan involves $c^o > 0$. This requires $s > 0$ and $l < 1$. As $\partial U/\partial c^o > 0$ for all $c^o > 0$ the two budget constraints will hold as equalities. As shown in Lemma 1, the marginal utility of c^y satisfies $\lim_{c^y\to 0} \partial U/\partial c^y = \infty$. Hence, the optimal plan involves $c^y > 0$. □

Hence, the two budget constraints may be consolidated to $c^y + wl + c^o/R = w$. The latter pins down c^o for any given choice $\left(c^y, l\right)$. Accordingly, the choice set may be stated as $\mathcal{S} = \{(c^y, l, c^o) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}_+, c^y + wl + c^o/R = w\}$ which is a compact subset of $\mathbb{R}^3_+$. Then, in light of Lemma 2 the optimal plan may be characterized with the solution to the unconstrained problem

$$\max_{(c^y,l)\in\hat{\mathcal{D}}} \quad \hat{U}\left(c^y, l, R\left(w(1-l)-c^y\right)\right) = \ln c^y + \kappa \ln\left(1-\phi\left(1-l\right)\left(c^y\right)^{\frac{\nu}{1-\nu}}\right) \tag{A.5}$$

$$+\beta \ln R\left(w(1-l)-c^y\right),$$

where $\hat{\mathcal{D}} = \{(c^y, l) \in \mathbb{R}_{++} \times [0, 1] \setminus \mathcal{B}, c^y < w(1-l)\}$ is the domain of $\hat{U}$. Since $\hat{U}$ is continuous on $\hat{D}$ Weierstrass's Theorem assures the existence of a global maximum $\left(c^y, l\right) \in \hat{\mathcal{D}}$.

According to Lemma 2 a corner solution may only arise at $l = 0$. Accounting for this, the respective first-order necessary conditions to (A.5) read

$$\hat{U}_1 = U_1 - \frac{\beta}{w(1-l)-c^y} = 0, \tag{A.6}$$

$$\hat{U}_2 = U_2 - \frac{\beta w}{w(1-l)-c^y} \leq 0, \quad \text{with "<" only if } l = 0. \tag{A.7}$$

Observe that condition (A.6) may be expressed as

$$c^y = \left(\frac{(1-\nu)\,(1-\phi x) - \nu\kappa\phi x}{(1-\nu)\,(1+\beta)\,(1-\phi x) - \nu\kappa\phi x}\right) w(1-l). \tag{A.8}$$

Regime 0:

Suppose some $c^y > 0$ and $l = 0$ solve (A.5). Then, c^y is given by (3.8) which is (A.8) evaluated at $l = 0$. One readily verifies that (3.8) assigns to each value $w \in (0, w_c)$ a unique $c^y > 0$. An application of the implicit function theorem to (3.8) establishes the existence of a $\mathcal{C}^1$ function $w \mapsto c^y(w)$ where c^y: $(0, w_c) \to (0, w_c)$ with $(c^y)'(w) > 0$. Then, as indicated in the proposition, the functions $s\,(w_t)$ and $c^o\,(w_t, R_{t+1})$ follow from the respective periodic budget constraint (3.6).

At the boundaries of its domain the function $c^y(w)$ has the following properties.

Lemma 3. *(Limits of $c^y(w)$)*

It holds that

$$\lim_{w\downarrow 0} c^y(w) = 0 \quad \textit{and} \quad \lim_{w\uparrow w_c} c^y(w) = \left(\frac{(1+\beta)\,(1-\nu)}{\phi\,(\kappa + (1+\beta)\,(1-\nu))}\right)^{\frac{1-\nu}{\nu}} \equiv c^y(w_c). \tag{A.9}$$

Proof of Lemma 3

Consider (3.8). Then, the first limit follows since $x = (c^y)^{\frac{\nu}{1-\nu}}$, hence, $\lim_{c^y\to 0} x = 0$ and

$$\lim_{c^y\to 0} \frac{(1-\nu)\,(1-\phi x) - \nu\kappa\phi x}{(1-\nu)\,(1+\beta)\,(1-\phi x) - \nu\kappa\phi x} = \frac{1}{1+\beta} > 0. \tag{A.10}$$

The proof of the second limit is more involved. In addition to (3.8), Regime 0 requires (A.7) to hold for $l = 0$. The latter may be expressed as

$$c^y \geq w\left(1 - \frac{\beta}{\kappa} \cdot \frac{1-\phi x}{\phi x}\right). \tag{A.11}$$

Replacing c^y by (3.8) gives

$$\frac{(1-\nu)\,(1-\phi x) - \nu\kappa\phi x}{(1-\nu)\,(1+\beta)\,(1-\phi x) - \nu\kappa\phi x} \geq 1 - \frac{\beta}{\kappa} \cdot \frac{1-\phi x}{\phi x}. \tag{A.12}$$

The left-hand side of this inequality defines a $\mathcal{C}^1$ function $x \mapsto LHS(x)$ where LHS: $[0, \bar{x}] \to \mathbb{R}_+$. Here, $\bar{x} \equiv (1-\nu)\,/\,(\phi\,(1-\nu\,(1-\kappa)))$, that is, in accordance with condition (3.3), at $x = \bar{x}$ it holds that $\partial U/\partial c^y = 0$. Then, one readily verifies that

$$LHS(0) = \frac{1}{1+\beta}, \quad LHS(\bar{x}) = 0, \quad \text{and} \quad LHS'(x) < 0. \tag{A.13}$$

Similarly, the right-hand side of inequality (A.12) defines a C^1 function $x \mapsto RHS(x)$ where $RHS\colon (0, \bar{x}] \to \mathbb{R}_+$. With Assumption 1 the function RHS satisfies

$$\lim_{x \to 0} RHS(x) = -\infty, \quad RHS(\bar{x}) = \frac{1 - \nu\,(1+\beta)}{1-\nu} > 0, \quad \text{and} \quad RHS'(x) > 0. \tag{A.14}$$

Accordingly, there is a unique $x_c \in (0, \bar{x})$ such that $LHS(x_c) = RHS(x_c)$, that is,

$$x_c = \frac{(1+\beta)\,(1-\nu)}{\phi\,(\kappa + (1+\beta)\,(1-\nu))}. \tag{A.15}$$

Hence, for all $x \in [0, x_c]$ we have $LHS(x) \geq RHS(x)$ and conditions (A.6) and (A.7) are satisfied. From the definition of x, it follows that the level of consumption when young corresponding to x_c is

$$c_c^y = x_c^{\frac{1-\nu}{\nu}} = \left(\frac{(1+\beta)\,(1-\nu)}{\phi\,(\kappa + (1+\beta)\,(1-\nu))} \right)^{\frac{1-\nu}{\nu}}. \tag{A.16}$$

Using the latter in (3.8) reveals that the critical level of consumption, c_c^y, is attained at $w = w_c$. It follows that $c_c^y = c^y(w_c)$ and $\lim_{w \uparrow w_c} c^y(w) = c^y(w_c)$ as claimed. □

Regime 1:

Next, consider the interior solution involving $c^y > 0$ and $l > 0$. Then, conditions (A.6) and (A.7) have to hold as equality and determine c^y and l. The following algorithm delivers the closed-form solutions stated under Regime 1 in the proposition. Given x, (A.6) and (A.7) imply

$$c^y = \left(\frac{(1-\nu)(1-\phi x) - \nu \kappa \phi x}{\kappa (1-\nu) \phi x} \right) w(1-l). \tag{A.17}$$

The latter, in conjunction with either (A.6) or (A.7), determines

$$\phi x = \frac{(1+\beta)(1-\nu)}{\kappa + (1+\beta)\,(1-\nu)} \in (0, 1). \tag{A.18}$$

Hence, all pairs $(c^y, l) \gg 0$ that satisfy (A.6) and (A.7) also satisfy

$$c^y = \left(\frac{(1+\beta)(1-\nu)}{\phi\,(\kappa + (1+\beta)(1-\nu))\,(1-l)} \right)^{\frac{1-\nu}{\nu}}. \tag{A.19}$$

Next, use (A.18) to replace ϕx in (A.17). This gives the expression for c^y stated in (3.10). Combining the latter with (A.19) using $h = 1 - l$ gives

$$h = \frac{(1+\beta)(1-\nu)}{(\phi\,(\kappa + (1+\beta)(1-\nu)))^{1-\nu}\,(1-\nu(1+\beta))^{\nu}} w^{-\nu} \tag{A.20}$$

and

$$c^y = \left(\frac{1-\nu(1+\beta)}{\phi\,(\kappa + (1+\beta)(1-\nu))} \right)^{1-\nu} w^{1-\nu}. \tag{A.21}$$

Straightforward algebraic manipulations of (A.20) and (A.21) using w_c deliver the two expressions for c_t^y and h_t stated in the proposition. The expressions for s_t and c_{t+1}^o follow from the respective periodic budget constraints. Moreover, $l = 1 - h$.

Evaluation of (A.20) at $w = w_c$ gives $h = 1$, hence, $l = 0$. Moreover, evaluation of (A.21) at $w = w_c$ gives $c_c^y = c^y(w_c)$ of (A.16). Hence, the optimal plan is piecewise defined and the functions of (3.7) are indeed continuous.

Finally, consider the second order conditions for both regimes. One readily verifies that

$$\hat{U}_{11} = U_{11} - \frac{\beta}{\left(w(1-l) - c^y\right)^2} < 0,$$

$$\hat{U}_{22} = U_{22} - \frac{\beta w^2}{\left(w(1-l) - c^y\right)^2} < 0,$$

$$\hat{U}_{12} = U_{12} - \frac{\beta w}{\left(w(1-l) - c^y\right)^2},$$

where the two signs follow from Lemma 1. It remains to be shown that

$$\hat{U}_{11}\hat{U}_{22} - \left(\hat{U}_{12}\right)^2 > 0. \tag{A.22}$$

Using the above expressions, one readily verifies that (A.22) is satisfied whenever

$$U_{11}U_{22} - (U_{12})^2 > \frac{\beta}{\left(w(1-l) - c^y\right)^2}\left(w^2 U_{11} + U_{22} - 2wU_{12}\right). \tag{A.23}$$

Since $U_{12} > 0$ (see Footnote 12) the right-hand side of this inequality is negative.

Regime 0:

Under Assumption 1, each pair $\left(c^y, l\right) = (c^y(w), 0)$ that solves (A.5) for $w \in (0, w_c)$ satisfies inequality (3.4), that is, U is strictly concave. To see this consider (3.4) for $x = (c^y(w))^{\frac{\nu}{1-\nu}}$. Since $c^y(w)$ is increasing in w with $\lim_{w \uparrow w_c} c^y(w) = c^y(w_c)$ strict concavity is satisfied whenever

$$c^y(w_c)^{\frac{\nu}{1-\nu}} = \frac{(1+\beta)\,(1-\nu)}{\phi\,(\kappa + (1+\beta)\,(1-\nu))} < \frac{1 - 2\nu + (1-\kappa)\nu^2}{\phi(1-\nu)\,(1-\nu(1-\kappa))}. \tag{A.24}$$

One readily verifies that the latter inequality holds if and only if Assumption 1 holds, that is, $\nu \in (0, \bar{\nu}\,(\beta, \kappa)\,)$.

From Lemma 2 the solution to (A.5) must involve $c^y > 0$ and $c^o > 0$. Hence, alternative candidate solutions on the boundary of $\mathcal{S}$ can be excluded. Accordingly, under Assumption 1 any pair $(c^y(w), 0)$ that satisfies (A.6) and (A.7) for $w \in (0, w_c)$ identifies a global maximum of $\hat{U}$ on $\mathcal{S}$.

Regime 1:

Some tedious computations reveal that (A.22) when evaluated at (A.20) and (A.21) gives[21]

$$\hat{U}_{11}\hat{U}_{22} - \left(\hat{U}_{12}\right)^2 = \frac{\phi^2(1-\nu(1+\beta))(\kappa + (1+\beta)(1-\nu))^3}{(c^y)^{\frac{2(1-2\nu)}{1-\nu}}\;\beta\kappa(1-\nu)^2}.$$

The latter is strictly positive if and only if $\nu < 1/(1+\beta)$. Since $1/(1+\beta) > \bar{\nu}\,(\beta, \kappa)$, Assumption 1 assures that any pair $\left(c^y(w), l(w)\right)$ that satisfies (A.6) and (A.7) for $w > w_c$ identifies a global maximum of $\hat{U}$ on $\hat{\mathcal{D}}$.

From Lemma 2 the solution to (A.5) has $c^y > 0$ and $c^o > 0$. Hence, alternative candidate solutions that may lie on the boundary of $\mathcal{S}$ can be excluded. Accordingly, under Assumption 1 any pair $\left(c^y, l\right)$ that satisfies (A.6) and (A.7) for $w > w_c$ identifies a global maximum of $\hat{U}$ on $\mathcal{S}$. □

A.3. Proof of Corollary 1

For Regime 0 one obtains the comparative statics for ϕ and β from a straightforward application of the implicit function theorem.

Consider Regime 1. Since $\partial w_c/\partial\phi < 0$ it holds that $\partial h_t/\partial\phi < 0$. Moreover, since $\partial w_c/\partial\beta > 0$ it holds that $\partial h_t/\partial\beta > 0$. For c_t^y, $\partial c_t^y/\partial\phi < 0$ and $\partial c_t^y/\partial\beta < 0$ are immediate from (A.21).

As to s_t one has from (3.10) that $\partial s_t/\partial\phi < 0$ since $\partial h_t/\partial\phi < 0$. Moreover, since the marginal propensity to save and h_t increase in β we have $\partial s_t/\partial\beta > 0$. Finally, consider c_{t+1}^o. Since $c_{t+1}^o = R_{t+1}s_t$, the qualitative results of the comparative statics for s_t apply here, too. □

A.4. Proof of Proposition 2

If $k_t \leq k_{c,t}$ then the equilibrium wage of (4.6) satisfies $\hat{w}_t \leq w_c$. Hence, the economy is in Regime 0, $\hat{h}_t = 1$, and $\hat{H}_t = L_t$. From (3.13) and (3.14) the equilibrium output and the rental rate of capital are obtained as $\hat{Y}_t = \Gamma \cdot A_t^{1-\nu\gamma} \cdot L_t \cdot k_t^{\gamma}$ and $\hat{r}_t = \Gamma \cdot \gamma \cdot A_t^{1-\nu\gamma} \cdot k_t^{\gamma-1}$. Hence, $\hat{R}_t = 1 + \hat{r}_t - \delta$. Using $\hat{w}_t$ and $\hat{R}_t$ in Proposition 1 delivers $\hat{c}_t^y$, $\hat{s}_t$, and $\hat{c}_t^o$.

If $k_t \geq k_{c,t}$ then the equilibrium wage of (4.6) satisfies $\hat{w}_t \geq w_c$. Hence, the economy is in Regime 1, $\hat{h}_t \leq 1$, and $\hat{H}_t \leq L_t$. Using (3.14), (4.6), and $\hat{R}_t = 1 + \hat{r}_t - \delta$ in Proposition 1, as well as (4.7) and (3.13) it is readily verified that all endogenous variables pertaining to period t can be expressed as a function of k_t.

From (4.6), the equilibrium real wage is a piecewise defined, continuous, and increasing function of k_t. Let $\hat{w}_t = \hat{w}_t\left(k_t\right)$ denote this function. Then the definition of k_t, Proposition 1, and (4.8) deliver the equilibrium difference equation as:

$$s\left(\hat{w}\left(k_t\right)\right) = k_{t+1}\left(1+g_L\right)A_{t+1}^{1-\nu}. \tag{A.25}$$

From Proposition 1, $s_t = s\left(w_t\right)$ is a piecewise defined continuous function that satisfies $s\left(w_t\right) > 0$ for all $w_t > 0$. Hence, $s\left(\hat{w}\left(k_t\right)\right) > 0$ so that (A.25) associates a unique $k_{t+1} > 0$ with each $k_t > 0$. □

A.5. Proof of Proposition 3

To derive (4.10) use Proposition 1 and the equilibrium wage, $\hat{w}_t$, to express (4.8) as:

$$\frac{\beta\left[w_c^{\nu(1-\gamma)}\left(\Gamma(1-\gamma)\right)^{1-\nu}\right]^{\frac{1}{1-\gamma\nu}}}{(1+\beta)(1-\nu)}A_t^{1-\nu}L_t k_t^{\frac{\gamma(1-\nu)}{1-\gamma\nu}} = K_{t+1}.$$

Division by $A_{t+1}^{1-\nu}L_{t+1}$ delivers the desired result. Since $\gamma(1-\nu)/\left(1-\gamma\nu\right) < 1$ the sequence $\{k_t\}_{t=1}^{\infty}$ is monotonous and the steady state is stable. □

A.6. Proof of Proposition 4

Statements a) - c) follow from Proposition 1, the capital market equilibrium condition (4.8), and the production function (3.13). Statement d) follows since $g_A > 0$ implies $\partial g^*/\partial\nu < 0$. □

A.7. Proof of Corollary 2

From c) the capital-output ratio is constant. Since $\hat{r}_t = \hat{r}$ the capital share, $\hat{r}K_t/Y_t$, is constant. Moreover, the real return factor on savings is $\hat{R} = 1 - \hat{r} - \delta$ and constant. With $g_w = g_A$, a), and c) the labor share, $\hat{w}_t\hat{H}_t/Y_t$ is constant. The latter implies that output per hour worked, $Y_t/\hat{H}_t/$, and output per worker, $Y_t/L_t/$ grow at a constant rate. Finally, from a) the individual supply of hours worked grows at rate $(1+g_A)^{-\nu}-1<0$. □

A.8. Proof of Proposition 5

First, consider the existence of a unique steady state $\hat{w}^{**}\in(0,1)$ for equation (4.18). Consumption when young, $c^y(w)$, is implicitly given by (3.8), that is, for the indicated parameter values:

$$c^y\left(1+\frac{1-\frac{1}{2}\left(\frac{3c^y}{2}\right)^{\frac{1}{3}}}{3-2\left(\frac{3c^y}{2}\right)^{\frac{1}{3}}}\right)=w. \tag{A.26}$$

Hence, $\lim_{w\to 0} c^y(w)=0$, $\lim_{w\to 0}(c^y)'(w)=3/4$, and $c^y(1)=2/3$. For Regime 0 the budget when young dictates $s=w-c^y$. Accordingly, $s(w)$ satisfies

$$s(w)=c^y(w)\left(\frac{1-\frac{1}{2}\left(\frac{3c^y(w)}{2}\right)^{\frac{1}{3}}}{3-2\left(\frac{3c^y(w)}{2}\right)^{\frac{1}{3}}}\right) \tag{A.27}$$

with $\lim_{w\to 0} s(w)=0$, $\lim_{w\to 0} s'(w)=1/4$, and $s(1)=1/3$. Hence, with Proposition 1 and (3.11) the function $s(w)$ is continuous and strictly increasing on $w\in[0,1]$. It follows that equation (4.18) has at least one fixed point. To see that there is one and only one fixed point $\hat{w}^{**}>0$ write (4.18) as:

$$w^3=s(w). \tag{A.28}$$

The left-hand side of (A.28), $LHS(w)$, is strictly convex on $[0,1]$ with $LHS(0)=LHS'(0)=0$, $LHS(1)=1$, and $LHS'(1)=3$. With the properties of $s(w)$ established above there must be one and only one $w\in(0,1)$ that satisfies (A.28). Computations reveal that $\hat{w}^{**}=0.55<1(=w_c)$ and $s\left(\hat{w}^{**}\right)=0.167$.

Second, consider the local stability of the steady state. From (4.18), one has

$$\frac{d\hat{w}_{t+1}}{d\hat{w}_t}=\frac{1}{3}\left(\frac{s'\left(\hat{w}_t\right)}{\left[s\left(\hat{w}_t\right)\right]^{\frac{2}{3}}}\right).$$

Using (A.26) and the budget constraint when young, the function $s(w_t)$ satisfies

$$\frac{16(w_t-s)-5\cdot 2^{2/3}\cdot\sqrt[3]{3}\cdot(w_t-s)^{4/3}}{12-4\cdot 2^{2/3}\cdot\sqrt[3]{3}\cdot\sqrt[3]{w_t-s}}-w_t=0.$$

Then, implicit differentiation and evaluation at the steady state delivers $s'(\hat{w}^{**})=.339$. Hence, with (4.18)

$$\left.\frac{d\hat{w}_{t+1}}{d\hat{w}_t}\right|_{\hat{w}_t=\hat{w}^{**}}=\frac{1}{3}\left(\frac{s'(\hat{w}^{**})}{\left[s\left(\hat{w}_t\right)\right]^{\frac{2}{3}}}\right)=0.372<1.$$

Accordingly, the steady state is locally stable.

Finally, the global stability over the domain $w_t\in(0,1)$ follows since $s(0)=0$, $s(1)=1/3$, $s'(w)>0$, and

$$\lim_{\hat{w}_t\to 0}\frac{d\hat{w}_{t+1}}{d\hat{w}_t}=\infty.$$

□

A.9. Proof of Proposition 6

Given in the main text. □

A.10. Proof of Proposition 7

Claim 1 "$\Rightarrow$": Consider (4.8) in conjunction with Proposition 1 and (5.6). Then, the evolution of K_t obeys

$$K_{t+1} = \frac{\beta\left[\left(w_c^{\nu} \cdot L\right)^{1-\gamma} \left(\Gamma(1-\gamma)\right)^{1-\nu}\right]^{\frac{1}{1-\gamma\nu}}}{(1+\beta)(1-\nu)} \cdot K_t^{\frac{1-\nu}{1-\gamma\nu}(\gamma+\zeta(1-\gamma))}. \tag{A.29}$$

Hence, if $\zeta = 1/(1-\nu)$ then $K_{t+1}/K_t = 1 + g_K$ is time-invariant and given by (5.10). Moreover, under the stated condition g_K is strictly positive. "$\Leftarrow$": Consider some $g_K > 0$. Then, from (4.8) and Proposition 1 with $L_t = L$ it must be that

$$1 + g_K \equiv \frac{K_{t+1}}{K_t} = \frac{s_{t+1}}{s_t} = \left(\frac{\hat{w}_{t+1}}{\hat{w}_t}\right)^{1-\nu}.$$

Since $K_t > \underline{K}_c$, this requires

$$1 + g_K = (1+g_K)^{(1-\nu)\frac{\gamma+\zeta(1-\gamma)}{1-\gamma\nu}}. \tag{A.30}$$

The latter equation may have a solution for $g_K > 0$ only if

$$1 = (1-\nu)\frac{\gamma+\zeta(1-\gamma)}{1-\gamma\nu} \quad \Rightarrow \quad \zeta = \frac{1}{1-\nu}. \tag{A.31}$$

Claim 2 The growth factor of equilibrium wages follows from (4.6). The equilibrium rental rate follows from (5.2) and is equal to $\hat{r}_t = \Gamma \cdot \gamma \cdot K_t^{\frac{\nu(1-\gamma)}{1-\nu}} \cdot \hat{H}_t^{1-\gamma}$. Then, with (5.7) one finds that $\hat{r}_{t+1}/\hat{r}_t = 1$ and $\hat{r} = \Gamma \cdot \gamma \cdot K_1^{\frac{\nu(1-\gamma)}{1-\nu}} \cdot \hat{H}_1^{1-\gamma}$. Absent of population growth, the growth factors of h_t and $\hat{H}_t$ must coincide. They follow immediately from Proposition 1 and the growth factor of $\hat{w}_t$. The same is true for the growth factors of individual consumption and savings. The growth factor of Y_t is explained in the main text. □

A.11. Proof of Corollary 3

From 2.c) the capital-output ratio is constant. Since $\hat{r}_t = \hat{r}$ the capital share, $\hat{r}K_t/Y_t$, is constant. Moreover, the real return factor on savings is $\hat{R} = 1 - \hat{r} - \delta$ and constant. From 2.a) $g_w = \left(1+g_K\right)^{1/(1-\nu)}$. In conjunction with 2.b) and 2.c) the labor share, $\hat{w}_t\hat{H}_t/Y_t$, is constant. The latter implies that output per hour worked, $Y_t/\hat{H}_t$, and output per worker, Y_t/L, grow at a constant rate if $g_K > 0$, that is, the corresponding condition under 1. is satisfied. Finally, from 2.b) the individual supply of hours worked grows at rate $(1+g_K)^{-\nu/(1-\nu)} - 1 < 0$. □

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