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THE DYNAMIC PROPERTIES OF ENDOGENOUS GROWTH MODELS

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This paper explores the dynamics of semiendogenous versus fully endogenous growth models in "lab equipment" specifications of the models with expanding sectors. Capital is allowed to accumulate and is used, together with other inputs, to produce new knowledge. The stability of the steady state path is found to be determined by the inequality and/or knife-edge restrictions needed to produce steady state growth. This paper takes the ratio of the shadow price of capital to knowledge and the level of consumption as jump variables. Semiendogenous growth models lead to a 4×4 dynamic system where the sign of the coefficient matrix of the log linearized dynamic system is indefinite, leading to a potential for both stable and unstable equilibria. The knife-edge restrictions needed to generate policy influences on growth are shown to be restrictions that reduce the system to 3×3 with a positive definite coefficient matrix, thereby guaranteeing a globally stable equilibrium. Implications for empirical testing are addressed.

Keywords: Endogenous Growth, Dynamic Properties, Transitional Dynamics, Stability

1. INTRODUCTION

The dynamics outside of the steady state and the stability of the steady state growth rate have been studied for first-generation versus semiendogenous growth models by Eicher and Turnovsky (1999, 2001). The purpose of this paper is to study the dynamic properties of semiendogenous versus fully endogenous growth models and the relationship between the stability of the equilibrium steady state rate of growth and the inequality restrictions and/or knife-edge restrictions needed to produce semiendogenous and fully endogenous growth when the economy accumulates capital and ideas.

The results of the model developed here have strong implications for empirical testing of the various models that allows the use of the full time series properties of the data. To date, most testing of endogenous growth models has assumed a steady state, which ignores the non–steady state dynamics of the various models, making empirical testing problematic (Zachariadis, 2003, 2004; Ha and Howitt,

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2007). One important outcome of the model is a new avenue for testing the various models, based on the restrictions necessary to generate a stable path to the steady state.

This paper makes constant reference to first-generation, semiendogenous, and fully endogenous growth models. It is worthwhile to explain what, for the purposes of this paper, qualifies a particular theoretical formulation for categorization with a particular label. Our categorization is based on the steady state equilibrium properties of the model. For our purposes, a first-generation model is any model that possesses a scale effect, i.e., in which the steady state growth rate is a function of a scale metric of the economy, commonly the level of population. These models also predict that government policies that encourage savings or direct resources to R&D will boost the steady state growth rate of the economy. Authors such as Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992) pioneered these first generation models in which the private incentive to innovate results from profit opportunities in the imperfectly competitive sector of the economy. For ease of exposition, these models typically ignore capital accumulation and look at steady state predictions within a one-factor model where the only rival input is labor.

Semiendogenous models follow Jones (1995a, 1995b), Kortum (1997), and Segertrom (1998). These models predict that the steady state growth rate is proportional to the growth rate of a scale metric of the economy, commonly the growth rate of population. The scale effect is absent and is eliminated by imposing the most general parameter restrictions upon the model that are consistent with a steady state [see Eicher and Turnovsky (1999); Jones (1999)]. Many semiendogenous growth models, such as those developed in Jones (1995a, 1995b, 1999, 2001, 2003), Kortum (1997), Dinopoulas and Thompson (1998), Segerstrom (1998), and Giordani and Luca (2008) assume that labor is the only rival input in the production of output and new ideas. Clearly, transitional dynamics and stability properties are not an issue in a one-factor model. The general parameter restrictions common to semiendogenous growth models also eliminate policy's impact on the steady state growth rate.[1]

Fully endogenous growth models follow Aghion and Howitt (1998, Ch. 12), Peretto (1998), and Young (1998). These models start from the general parameter restrictions of a semiendogenous growth model and then impose restrictions, usually in the form of knife-edge conditions, to bring back a role for policy in influencing the long-run rate of growth in the steady state. Labor is often assumed to be the only factor of production, as in Aghion and Howitt (1992), Peretto (1998), Young (1998), Jones (1999), Segerstrom (2000), and Li (2001).[2]

Some attempts have been made to expand semiendogenous and fully endogenous growth models to include capital as a factor of production. A handful of new models[3] generalize by assuming that final output can be used for consumption, saved and used to produce capital, or used as an input into the production of new ideas. Thus part of final output can be diverted and used as "lab equipment" in the research sector [Aghion and Howitt (1998, Ch. 12); Jones (2001 and 2003)].

Our work is closely related to that of Eicher and Turnovsky (1999, 2001), who work strictly within a semiendogenous growth framework. Eicher and Turnovsky (2001) allow for capital in the production of final output, but assume that only labor is used in the production of new ideas. They develop the transitional dynamics for this simple version of the semiendogenous growth model. Eicher and Turnovsky (1999) relax the assumption of a single rival input in the knowledge sector and investigate the steady state properties of a more general semiendogenous growth model, demonstrating the generality of nonscale growth. Their model does not allow comparison of semiendogenous and fully endogenous growth models, as it is not constructed in a way that allows the key knife-edge conditions that distinguish fully endogenous growth models from semiendogenous growth models to be explicitly imposed. A full investigation of the dynamic properties of a more general "lab equipment" version of the semiendogenous growth model and newer fully endogenous growth models has not been accomplished.

2. THEORETICAL STRUCTURE

2.1. The Environment

Although the micro foundations of new growth models differ slightly from one author to the next, they lead to the same basic reduced form. We follow Eicher and Turnovsky (1999, 2001) and formulate an optimal control problem for a central planner. This allows our work to be directly compared with their work as well as the "lab equipment" model developed by Jones and Williams (2000). We allow output and intermediate goods to be produced with the production functions

$$Y_t = C_t + I_t + N_t = L_t^{1-\alpha} Q_t^{\alpha-1} \sum_{i=0}^{Q} A_t^{\psi} x_{i,t}^{\alpha} \qquad \textbf{(1)}$$

$$x_{i,t} = x_{j,t} = x_t = \frac{K_{i,t}}{A_t} \quad \text{for all } i \text{ and } j, \qquad \textbf{(2)}$$

where Y is aggregate output, C is consumption, I is investment, N is resources devoted to the R&D process, L is the labor force, Q is the number of sectors over which R&D resources are spread, A is a measure of the stock of knowledge, x_i is the input of the ith intermediate good, and K is the capital stock. The t subscript denotes time and equation (2), which shows the complementarity between capital and knowledge, imposes the symmetry of demand for intermediates as suggested by all models, utilizing the Dixit–Stiglitz approach.

It is easy to show that equations (1) and (2) imply an aggregate Cobb–Douglas production function. Straightforward substitution of equation (2) into equation (1) leads to

$$Y_t = A_t^{\psi-\alpha} L_t^{1-\alpha} K_t^{\alpha}. \qquad \textbf{(3)}$$

The form of technological neutrality depends on the value of ψ. Typically the existing new-growth literature sets this parameter equal to 1, $\psi = 1$, generating

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a production function of the form $Y_t = A_t^{1-\alpha} L_t^{1-\alpha} K_t^{\alpha} = (A_t L_t)^{1-\alpha} K_t^{\alpha}$. Thus the assumptions of labor-augmenting technological change and Cobb–Douglas production are made. If ψ takes the value of 2α, then equation (3) becomes $Y_t = A_t^{2\alpha-\alpha} L_t^{1-\alpha} K_t^{\alpha} = L_t^{1-\alpha}(A_t K_t)^{\alpha}$ and technology is capital-augmenting or Solow-neutral. Clearly a Hicks-neutral specification is obtained with $\alpha + 1$. Equation (3) becomes $Y_t = A_t^{\alpha+1-\alpha} L_t^{1-\alpha} K_t^{\alpha} = A_t L_t^{1-\alpha} K_t^{\alpha}$.

It is assumed that the number of sectors evolves according to

$$Q_t = L_t^{\beta} \quad \beta \geq 0. \tag{4}$$

We assume a constant growth rate of the labor force such that $L(t) = L(o)e^{nt}$.

The dynamics of the economy depends, in part, on the dynamics of the two producible factors of production, capital and knowledge. The evolution of these stocks is described by the equations[4]

$$\dot{A}_t = \sigma \left(\frac{N_t}{Q_t} \right)^{\theta} A_t^{\varepsilon} \quad \theta \leq 1, \varepsilon \leq 1 \tag{5}$$

$$\dot{K}_t = Y_t - C_t - N_t. \tag{6}$$

For tractability, equation (6) ignores the depreciation of capital.[5] C is consumption and N is final output diverted to the R&D sector of the economy. From equation (6) we see that this is a "lab equipment" version of a new-growth model.

The assumptions that the production of new ideas is related to resources *per sector* and that the number of sectors can grow in relation to the size of our scale variable, population, are what allows us to easily compare semiendogenous and fully endogenous growth models along the lines of Jones (1999). In this sense our model is more general than that of Eicher and Turnovsky (1999). However, Eicher and Turnovsky (1999) is more general in the sense that Cobb–Douglas forms are not imposed, at least not in the earlier part of their paper. Our paper becomes a special case of their model if and only if we set $\beta = 0$ in equation (4). In this case the dynamics of our model follows each of Eicher and Turnovsky's propositions exactly. Actually, most of the propositions of Eicher and Turnovsky are robust to our specification even with $\beta > 0$. It should be pointed out that Eicher and Turnovsky (1999) only investigate steady state properties. Eicher and Turnovsky (2001) do look at transitional dynamics, but Cobb–Douglas is imposed throughout and the specification allows only labor to be used in the knowledge–producing sector of the economy.

2.2. Dynamics of Knowledge and Capital Accumulation

The social planner's problem is to maximize an intertemporal utility function,

$$\frac{1}{1-\gamma} \int_0^{\infty} \left(\frac{C}{L} \right)^{1-\gamma} e^{-\rho t} dt, \rho > 0, \gamma > 0, \tag{7}$$

where C/L is per capita consumption. The optimization is subject to the constraints represented by equations (3), (5), and (6). The control variables are C and N. The state variables are K and A. The equations of motion are equations (5) and (6). Optimization and transversality conditions lead to the equations

$$C^{-\gamma} = \nu L^{1-\gamma} \tag{8a}$$

$$-\nu + \mu\theta \dot{A} N^{-1} = 0 \tag{8b}$$

$$\alpha\left(\frac{Y}{K}\right) = \rho - \frac{\dot{\nu}}{\nu} \tag{8c}$$

$$\frac{\nu}{\mu}(\Psi - \alpha)\frac{Y}{A} + \varepsilon\frac{\dot{A}}{A} = \rho - \frac{\dot{\mu}}{\mu} \tag{8d}$$

$$\lim_{t\to\infty} \nu K e^{-\rho t} = \lim_{t\to\infty} \mu A e^{-\rho t} = 0, \tag{8e}$$

where ν and μ are the shadow prices of capital and knowledge, respectively.

In this paper the convention of denoting the growth rate of some variable x as γ_x is adopted. In balanced growth equilibrium the ratios Y/K, C/K, and N/K are constant. ${}^{Y}\!/_{K} = A^{\Psi-\alpha} L^{1-\alpha} K^{\alpha-1}$ and $\gamma_K = \gamma_N$ in a balanced growth equilibrium. Log differentiation of the expression for the Y/K ratio and equation (5) lead to

$$\gamma_K = \frac{(\Psi - \alpha)}{(1 - \alpha)}\gamma_A + \gamma_L \tag{9}$$

$$\gamma_K = \frac{(1 - \varepsilon)}{\theta}\gamma_A + \beta\gamma_L. \tag{10}$$

The remainder of the analysis in this paper is developed by imposing parameter restrictions and/or knife-edge restrictions on the model to generate semiendogenous and fully endogenous growth models. These restrictions specify the definition of neutrality of technological change through the specification of ψ.

3. THE SEMIENDOGENOUS GROWTH MODEL: $0 \leq \theta \leq 1$, $\varepsilon \leq 1$, $\beta \geq 0$, $\beta \neq 1$

The parameter restrictions consistent with a semiendogenous growth model are $0 \leq \theta \leq 1$, $\varepsilon \leq 1$, $\beta \geq 0$, and $\beta \neq 1$. The semiendogenous growth model predicts that the growth rate of capital, γ_K, and the growth rate of knowledge, γ_A, are constant in the steady state. Furthermore, these growth rates are proportional to population growth, γ_L. Denote these balanced growth rates as $\gamma_A = B_A\gamma_L$ and $\gamma_K = B_K\gamma_L$. B_A and B_K are constant and are functions of exogenous parameters. These are solved for explicitly later.

To develop the dynamics of the model around the steady state, define variables that are constant in balanced growth equilibrium. They are $n = {}^{N}\!/_{L^{B_K}}$, $y = {}^{Y}\!/_{L^{B_K}}$, $k = {}^{K}\!/_{L^{B_K}}$, $c = {}^{C}\!/_{L^{B_K}}$, $a = {}^{A}\!/_{L^{B_A}}$, and $q = \frac{({}^{\nu}\!/_{\mu})}{L^{(B_A - B_K)}}$. The variable q is the scale-adjusted ratio of the shadow price of capital to the shadow price of labor. This

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ratio must be constant in a steady state, because a steady state requires constant proportions of resources used in the capital and knowledge-producing sectors. In the steady state, therefore, $\gamma_\nu - \gamma_{\mu=(B_A-B_K)\gamma_L}$.

Use equation (8b) and the scale-adjusted variables listed previously to solve for n as a function of q:

$$n = \left(\frac{\theta\sigma a^{\varepsilon}}{q}\right)^{\frac{1}{1-\theta}}. \tag{11}$$

Next rewrite the production functions in terms of scale-adjusted variables:

$$y = a^{\Psi-\alpha}k^{\alpha}, \tag{12a}$$

$$\dot{a} = \sigma n^{\theta}a^{\varepsilon} - B_A\gamma_L a. \tag{12b}$$

Equations (8c) and (8d) are differential equations for the shadow prices of capital and knowledge. It is also necessary to express these in terms of the scale-adjusted variables:

$$\frac{\dot{\nu}}{\nu} = \rho - \alpha\left(a^{\Psi-\alpha}k^{\alpha-1}\right) \tag{13a}$$

$$\frac{\dot{\mu}}{\mu} = \rho - \sigma n^{\theta}a^{\varepsilon}\left((\varepsilon - \theta(\Psi-\alpha)\,a^{\Psi-\alpha}\left(k^{\alpha}/n\right)\right). \tag{13b}$$

Equations (13a) and (8a) together provide the growth rate of consumption, $\frac{\dot{C}}{C} = \frac{1}{\gamma}\left(\alpha a^{\Psi-\alpha}k^{\alpha-1} - (1-\gamma)\,\gamma_L - \rho\right)$. Using equation (6) together with the Euler equation for consumption and equations (12a) through (13b), the dynamic system is expressed as

$$\dot{k} = k\left[a^{\Psi-\alpha}k^{\alpha-1} - c/k - n/k - B_K\gamma_L\right] \tag{14a}$$

$$\dot{a} = a\left[\sigma n^{\theta}a^{\varepsilon-1} - B_A\gamma_L\right] \tag{14b}$$

$$\dot{q} = q\left[\sigma n^{\theta}a^{\varepsilon-1}\left(\varepsilon - n^{-1}(\Psi-\alpha)a^{\Psi-\alpha}k^{\alpha}\right) - \alpha a^{\Psi-\alpha}k^{\alpha-1} - (B_A - B_K)\gamma_L\right] \tag{14c}$$

$$\dot{c} = \frac{c}{\gamma}\left[\alpha a^{\Psi-\alpha}k^{\alpha-1} - \rho + (\gamma(1-B_K)-1)\,\gamma_L\right]. \tag{14d}$$

PROPOSITION 1. *In the steady state, the growth rate of capital is* $\gamma_K = \frac{\theta(1-\beta)(\Psi-\alpha)+(1-\alpha)(1-\varepsilon)-\theta(\Psi-\alpha)}{(1-\alpha)(1-\varepsilon)-\theta(\Psi-\alpha)}\gamma_L$ *and the growth rate of knowledge and per capita output is* $\gamma_A = \gamma_{\frac{Y}{L}} = \frac{\theta(1-\alpha)(1-\beta)}{(1-\alpha)(1-\varepsilon)-\theta(\Psi-\alpha)}\gamma_L$. *The parameter restrictions needed to generate growth depend on the value of* β. *With* $\beta < 1$ *the restriction is* $\theta(\psi-\alpha)+\varepsilon(1-\alpha) < (1-\alpha)$. *When* $\beta > 1$ *the restriction is* $\theta(\psi-\alpha)+\varepsilon(1-\alpha) > (1-\alpha)$.

Proof. In the steady state, equations (9) and (10) can be solved simultaneously to recover the steady state growth rate. Neither equation is a function of scale or policy variables. Thus, producing a semiendogenous growth model is equivalent to imposing restrictions that generate an intersection of equations (9) and (10) in the first quadrant of the Cartesian plane. There is no reason to assume that the rate of imitation grows more slowly than the population; thus β can be greater than or less than 1. β, however cannot equal one, or the intercepts of the equations will coincide, ruling out semiendogenous growth equilibrium.

With $\beta < 1$, the restriction that the slope of equation (10) must be greater than the slope of equation (9) produces the restriction $\theta(\psi - \alpha) + \varepsilon(1 - \alpha) < (1 - \alpha)$. With $\psi = 1$ and labor-augmenting or Harrod-neutral technological change, the restriction reduces to $\theta + \varepsilon < 1$. With $\psi = 2\alpha$ and capital-augmenting or Solow-neutral technological change, the restriction reduces to $\frac{\theta\alpha}{(1-\alpha)(1-\varepsilon)} < 1$.[6] With $\psi = \alpha + 1$and Hicks-neutral technological change, the restriction reduces to $\frac{\theta}{(1-\alpha)(1-\varepsilon)} < 1$. If the direction of technological change is other than labor-augmenting, the restriction involves capital's share.

With $\beta > 1$ the restriction that the slope of equation (10) must be less than the slope of equation (9) produces the restriction $\theta(\psi - \alpha) + \varepsilon(1 - \alpha) > (1 - \alpha)$. With $\psi = 1$ and labor-augmenting or Harrod-neutral technological change, the restriction reduces to $\theta + \varepsilon > 1$. With $\psi = 2\alpha$ and capital-augmenting or Solow-neutral technological change, the restriction reduces to $\frac{\theta\alpha}{(1-\alpha)(1-\varepsilon)} > 1$. With $\psi = \alpha + 1$ and Hicks-neutral technological change, the restriction reduces to $\frac{\theta}{(1-\alpha)(1-\varepsilon)} > 1$.

Using equations (9) and (10) to solve for expressions for the growth rates of capital and knowledge along the growth path provides the equations

$$\gamma_A = \gamma_{\frac{Y}{L}} = \frac{\theta(1-\alpha)(1-\beta)}{(1-\alpha)(1-\varepsilon) - \theta(\Psi - \alpha)}\gamma_L = B_A\gamma_L \tag{15a}$$

$$\gamma_K = \frac{\theta(1-\beta)(\Psi - \alpha) + (1-\alpha)(1-\varepsilon) - \theta(\Psi - \alpha)}{(1-\alpha)(1-\varepsilon) - \theta(\Psi - \alpha)}\gamma_L = B_K\gamma_L. \tag{15b}$$

■

With labor-augmenting technology, $\gamma_A = \gamma_{K/L} = \frac{\theta(1-\beta)}{1-\theta-\varepsilon}n$. With capital-augmenting technology, the growth rate is $\gamma_A = \gamma_{K/L} = \frac{\theta(1-\alpha)(1-\beta)}{(1-\alpha)(1-\varepsilon)-\theta\alpha}n$. A Hicks-neutral specification leads to $\gamma_A = \gamma_{K/L} = \frac{\theta(1-\alpha)(1-\beta)}{(1-\alpha)(1-\varepsilon)-\theta}n$. In each case, positive growth requires that the denominator in the expression for the growth rate be positive or negative if $\beta < 1$ or $\beta > 1$, respectively. It is easy to see that scale effects in growth rates do not exist and policy variables, represented by thrift or ρ and γ in this model, do not have any impact on steady state growth rates.

PROPOSITION 2. *A steady state equilibrium in the lab equipment semiendogenous growth model requires that equations (9) and (10) be linearly independent and intersect in the first quadrant of the Cartesian plane and* $\theta(\psi - \alpha) + \varepsilon(1 - \alpha) \neq$

$(1-\alpha)$. *The resulting equilibrium can be globally stable, globally unstable, or saddlepath stable.*

Proof. Equations (9) and (10) are not functions of the policy parameters (ρ and γ in this model). Any unique solution to this two-equation system in the first quadrant of the Cartesian plane will define γ_A and γ_K as positive-equilibrium steady state growth rates. Furthermore, these growth rates will be functions of n and not of the policy variables in the model.

Denoting steady state values with an asterisk and performing a log linearization of the system defined by equations (14a) through (14d) yields

$$\begin{pmatrix} \dot{k} \\ \dot{a} \\ \dot{q} \\ \dot{c} \end{pmatrix} = D \times \begin{pmatrix} k-k^* \\ a-a^* \\ q-q^* \\ c-c^* \end{pmatrix} \tag{16}$$

$$D = \begin{pmatrix} \alpha a^{\psi-\alpha}k^{\alpha-1} - B_K\gamma_L, & (\psi-\alpha)a^{\psi-\alpha-1}k^{\alpha-1} - \frac{\partial n/\partial a}{k}, & -\frac{\partial n/\partial q}{k}, & -1 \\ 0, & \sigma a^{\varepsilon-1}\left(\varepsilon n^{\theta} + \theta n^{\theta-1}\partial n/\partial a\right) - B_A\gamma_L, & \sigma a^{\varepsilon}\theta n^{\theta-1}\partial n/\partial q, & 0 \\ a_{31} & a_{32}, & a_{33}, & 0 \\ \frac{(\alpha-1)}{k}\frac{c}{\gamma}(a_{11}+B_K\gamma_L), & \frac{\alpha c}{\gamma a}\left(a_{12} + \frac{\partial n/\partial a}{k}\right), & 0, & 0 \end{pmatrix}$$

$$a_{31} = q\left[-\frac{\gamma}{c}a_{41} - \frac{\sigma n^{\theta-1}(\Psi-\alpha)}{a^{1-\varepsilon}}a_{11} - \frac{\sigma n^{\theta-1}(\Psi-\alpha)}{a^{1-\varepsilon}}B_K\gamma_L\right]$$

$$\begin{aligned} a_{32} = q\Bigg[&\frac{1-\varepsilon}{a}(-a_{22} - B_A\gamma_L) \\ &+ \left[\left(\frac{1-\varepsilon}{a} + \varepsilon\right)\theta a^{\varepsilon-1} + \frac{(1-\theta)(\Psi-\alpha)}{n}k^{\alpha}a^{\Psi-\alpha+\varepsilon-1}\right]\sigma n^{\theta-1}\frac{\partial n}{\partial a} \\ &- \left(\frac{\Psi-\alpha+\varepsilon-1}{\alpha}n^{\theta-1}a^{\varepsilon-1}k^{1-\alpha} + 1\right)\frac{\gamma}{c}a_{42}\Bigg] \end{aligned}$$

$$a_{33} = q\left[\varepsilon/_a a_{23} + (1-\theta)n^{\theta-2}(\Psi-\alpha)a^{\Psi-\alpha+\varepsilon-1}k^{\alpha}\sigma\frac{\partial n}{\partial q}\right].$$

The determinant $|D|$ is equal to the product of four eigenvalues that determine the stability of the dynamic system. It is straightforward to show that $|D|$ is ambiguous in sign. If the value is positive then it is possible to have zero, two, or four unstable positive roots.

Use of the correspondence principle to rule out four positive roots and global instability is standard. Following common practice and allowing q and c to be jump variables produces a globally stable two-dimensional system with a direct path to equilibrium, in which equilibrium is globally stable. This case is depicted in Figure 1.

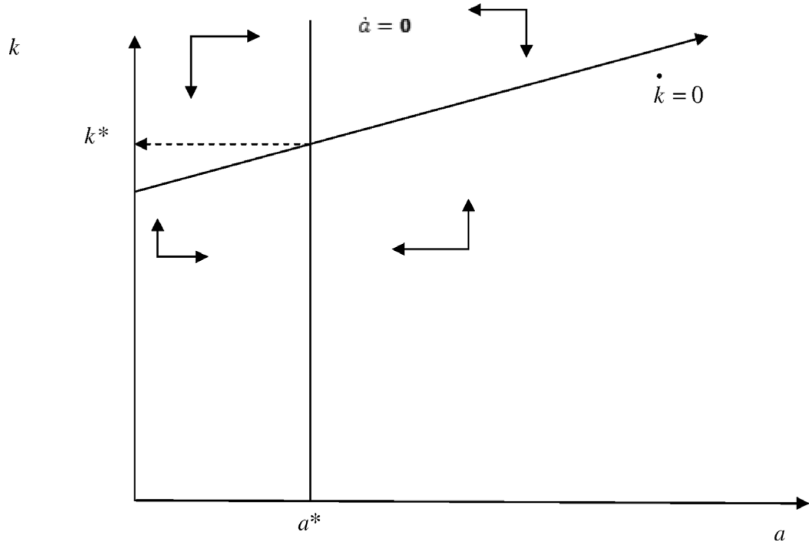

FIGURE 1. Semiendogenous growth model with two negative roots.

If, on the other hand, the value is negative, there are one or three unstable roots. Continuing to assume that q and c are jump variables and focusing on the case of three unstable roots produces a saddlepath equilibrium, as depicted in Figure 2. In this case the out-of-steady state behavior follows a saddlepath to equilibrium. Any position off of the saddlepath leads away from steady state growth. ■

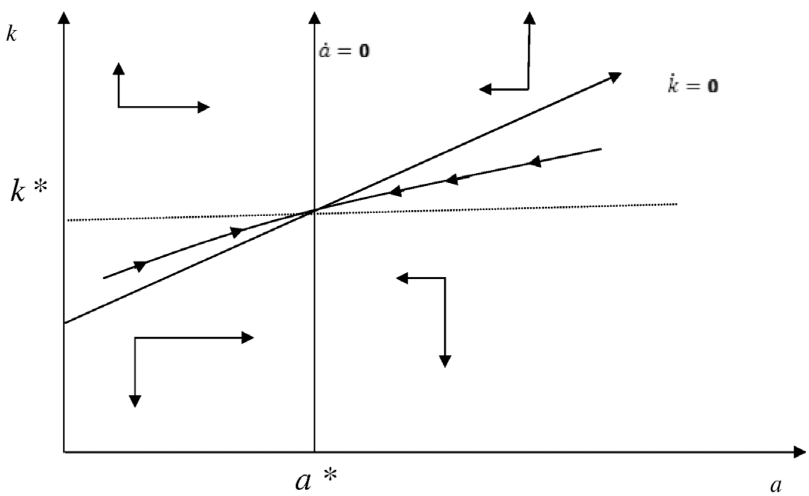

FIGURE 2. Semiendogenous growth model with one negative root.

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Proposition 2 is equivalent to Proposition 1 in Eicher and Turnovsky (1999)[7] for the case where $\beta < 1$. This parallel is not surprising, because our model is a special case of their model if and only if $\beta = 0$. Because, in this case, the intercept of equation (10) is zero and less than the intercept of equation (9), this case requires that equation (10) be steeper than equation (9), which can only hold if $[\theta(\psi - \alpha) - (1 - \varepsilon)] < 0$, consistent with a globally stable equilibrium.

A new result arises if $\beta > 1$, which is permissible in a completely general semiendogenous growth model with lab equipment and expanding sectors. In this case Eicher and Turnovsky's Proposition 1 is overturned and $\theta(\psi - \alpha) + \varepsilon(1 - \alpha) > (1 - \alpha)$ is required for balanced growth. A rate of expansion of sectors higher than the rate of population growth is perfectly consistent with positive scale-free per capita growth if returns to scale in knowledge production are increasing in lab equipment and previously discovered knowledge taken together. This result is consistent with diminishing returns to each of these factors taken separately, and this equilibrium requires no knife-edge restrictions on any parameter. The manner in which our model is distinctly different from that of Eicher and Turnovsky (1999) and the relationship between the two models should be clear from comparison of our Proposition 2 with Proposition 1 from their paper.

4. THE FULLY ENDOGENOUS GROWTH MODEL: $0 \leq \theta \leq 1$, $\varepsilon \leq 1$, $\beta = 1$

Eliminating the scale effect by setting $\beta = 1$ without losing the policy implications on growth rates amounts to forcing the intercepts of the equations (9) and (10) to be equal. In the original first-generation growth models this is accomplished by setting $n = 0$. An alternative means of eliminating the scale effect in the present model is to allow the number of sectors over which research is spread to grow at the same rate as the growth rate of the scale variable, L. Jones (1999) shows that this amounts to setting $\beta = 1$ in equation (4). This follows Young (1998), who allows firms to innovate along the intensive margin (quality improvements) and along the extensive margin (introducing new brands of existing products). It is assumed that only innovation along the intensive margin impacts the level of technology and has desirable social spillovers. As population grows, scarce resources are spread across both types of innovation. This crowding effect offsets the positive impact of the scale effect. Peretto (1998) and Segerstrom (1998) extend Young's treatment beyond a simple two-period model and reintroduce uncertainty into the growth model.

To develop the dynamics of the fully endogenous growth system, follow the same approach used in studying the semiendogenous model. The scale-adjusted variables are defined, in this case, as $\tilde{y} = {}^{Y}/_{A^{\frac{\Psi-\alpha}{1-\alpha}}L}$, $\tilde{k} = {}^{K}/_{A^{\frac{\Psi-\alpha}{1-\alpha}}L}$, $\tilde{n} = {}^{N}/_{A^{\frac{\Psi-\alpha}{1-\alpha}}L}$, $\tilde{c} = {}^{C}/_{A^{\frac{\Psi-\alpha}{1-\alpha}}L}$, and $\tilde{q} = \frac{\nu}{\mu}\left(\frac{Y}{A}\right)$. In the steady state $\gamma_\nu - \gamma_\mu = \gamma_A - \gamma_K = \gamma_A - \gamma_Y$.

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Use equation (8b) and the scale-adjusted variables to solve for $\tilde{n}$ as a function of $\tilde{q}$:

$$\tilde{n} = \left(\frac{\theta\sigma\tilde{k}^{\alpha}}{\tilde{q}}\right)^{\frac{1}{1-\theta}}. \qquad \textbf{(17)}$$

Next, rewrite the production functions in terms of scale-adjusted variables:

$$\tilde{y} = \tilde{k}^{\alpha}, \qquad \textbf{(18a)}$$

$$\gamma_A = \sigma\tilde{n}^{\theta}. \qquad \textbf{(18b)}$$

Again, express equations (8c) and (8d) in terms of the scale-adjusted variables.

$$\frac{\dot{\nu}}{\nu} = \rho - \alpha(\tilde{k}^{\alpha-1}). \qquad \textbf{(19a)}$$

$$\frac{\dot{\mu}}{\mu} = \rho - \sigma\tilde{n}^{\theta}\left(\varepsilon - \theta(\Psi - \alpha)\tilde{k}^{\alpha}/_{\tilde{n}}\right) \qquad \textbf{(19b)}$$

Equation (19a) and equation (8a) together provide the growth rate of consumption, $\frac{\dot{C}}{C} = \frac{1}{\gamma}(\alpha\tilde{k}^{\alpha-1} - (1-\gamma)\gamma_L - \rho)$. Using equation (6) together with the Euler equation for consumption and equations (18a) through (19b), the dynamic system can be expressed as

$$\dot{\tilde{k}} = \tilde{k}\left[\tilde{k}^{\alpha-1} - \left(\gamma_L + \frac{\Psi-\alpha}{1-\alpha}\sigma\tilde{n}^{\theta}\right) - \tilde{c}/_{\tilde{k}} - \tilde{n}/_{\tilde{k}}\right] \qquad \textbf{(20a)}$$

$$\dot{\tilde{n}} = \tilde{n}\frac{1}{1-\theta}\left[\alpha\tilde{k}^{\alpha-1} - \left(\left(\varepsilon + \frac{\Psi-1}{1-\alpha}\right) - \theta(\Psi-\alpha)\frac{\tilde{k}^{\alpha}}{\tilde{n}}\right)\sigma\tilde{n}^{\theta} - \gamma_L\right] \qquad \textbf{(20b)}$$

$$\dot{\tilde{q}} = \tilde{q}\left[\left\{(\varepsilon + \Psi - \alpha - 1) - \theta(\Psi-\alpha)\frac{\tilde{k}^{\alpha}}{\tilde{n}}\right\}\sigma\tilde{n}^{\theta} + (1-\alpha)\gamma_L - \alpha\tilde{c}/_{\tilde{k}} - \alpha\tilde{n}/_{\tilde{k}}\right] \qquad \textbf{(20c)}$$

$$\dot{\tilde{c}} = \frac{\tilde{c}}{\gamma}\left[\alpha\tilde{k}^{\alpha-1} - \gamma_L - \rho - \frac{\Psi-\alpha}{1-\alpha}\sigma\tilde{n}^{\theta}\gamma\right]. \qquad \textbf{(20d)}$$

Log linearization of this system leads to a singular-coefficient matrix. The restrictions reduce the system to a 3×3 system that can be evaluated in $\tilde{k}$, $\tilde{n}$, and $\tilde{c}$.

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PROPOSITION 3. *In the fully endogenous growth model, the growth rate of the economy is implicitly defined as a function of policy variables and parameters,*

$$\gamma_A = \sigma S_N^{\theta} \left[\frac{\gamma_L + \frac{\psi - \alpha}{1-\alpha} \gamma_A}{S_I} \right]^{(1-\alpha)\theta\alpha}.$$

In addition to the restriction that $\beta = 1$, the model requires the razor's-edge restriction $\theta(\Psi - \alpha) + \varepsilon(1 - \alpha) = (1 - \alpha)$.

Proof. If $\beta = 1$, we produce a fully endogenous growth model, as this razor's-edge restriction restricts sector growth to the growth rate of the economy's scale metric, population. Equations (15) and (17) show that balanced growth cannot be achieved without an additional restriction. Fully endogenous growth requires that equations (9) and (10) be linearly dependent. The additional razor's-edge restriction required for fully endogenous growth, therefore, is $\theta(\Psi - \alpha) + \varepsilon(1 - \alpha) = (1 - \alpha)$.

Log differentiation of the definition of $\tilde{k}$ gives $\gamma_{\tilde{k}} = (\frac{Y}{K} - \frac{C}{K} - \frac{N}{K}) - \frac{\Psi - \alpha}{1-\alpha} \gamma_A - \gamma_L = 0$. This equation and equations (9) and (10), together with the parameter restrictions of the model, allow the steady state growth rate of the economy to be expressed, implicitly, as

$$\gamma_A = \sigma S_N^{\theta} \left[\frac{\gamma_L + \frac{\psi - \alpha}{1-\alpha} \gamma_A}{S_I} \right]^{(1-\alpha)\theta\alpha}, \tag{21}$$

where S_N and S_I are endogenous and are functions of any policy variables included in the model. Thus in this case the steady state growth rate is a function of policy variables and the population growth rate. ■

PROPOSITION 4. *The equilibrium resulting from the "lab equipment" specification of the fully endogenous growth model is globally stable.*

Proof. Because equations (9) and (10) are linear and share a common intercept, there is no steady state growth rate other than zero if the equations do not share the same slope and coincide. A log linearization of the system defined by equations (19a), (19b), and (19d) yields:

$$\begin{pmatrix} \dot{\tilde{k}} \\ \dot{\tilde{n}} \\ \dot{\tilde{c}} \end{pmatrix} = D \times \begin{pmatrix} \tilde{k} - \tilde{k}^* \\ \tilde{n} - \tilde{n}^* \\ \tilde{c} - \tilde{c}^* \end{pmatrix} \tag{22}$$

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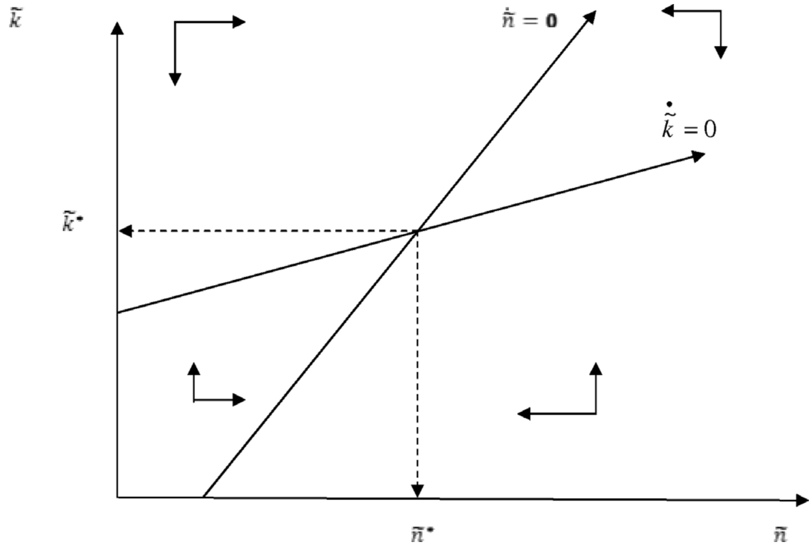

FIGURE 3. Fully endogenous growth model with two negative roots.

$$D = \begin{pmatrix} \alpha\tilde{k}^{\alpha-1} - \gamma_L - \frac{(\Psi-\alpha)}{(1-\alpha)}\sigma\tilde{n}^{\theta}, & \theta\tilde{k}\,\frac{\Psi-\alpha}{1-\alpha}\sigma\tilde{n}^{\theta-1}, & -1 \\ \frac{\tilde{n}}{1-\theta}\tilde{k}^{\alpha-1}\alpha\left(\tilde{k}^{-1}(\alpha-1) + \frac{\theta}{1-\theta}(\Psi-\alpha)\tilde{n}^{\theta-1}\sigma\right), & \frac{\tilde{n}}{1-\theta}\left(-\left(\varepsilon + \frac{\Psi-\alpha}{1-\alpha}\right)\sigma\theta\tilde{n}^{\theta-1} + \left(\theta(\Psi-\alpha)\tilde{k}^{\alpha}\sigma(\theta-1)\tilde{n}^{\theta-2}\right)\right), & 0 \\ \frac{\tilde{c}}{\gamma}\alpha(\alpha-1)\tilde{k}^{\alpha-2}, & -\tilde{c}\,\frac{\Psi-\alpha}{1-\alpha}\sigma\theta\tilde{n}^{\theta-1}, & 0 \end{pmatrix}.$$

The determinant $|D|$ is equal to the product of three eigenvalues that determine the stability of the dynamic system. It is straightforward to show that $|D|$ is unambiguously positive in sign. The system can have one or three unstable positive roots. We use the correspondence principle to rule out three positive roots. We follow common practice and allow $\tilde{c}$ to be a jump variable to produce a globally stable two-dimensional system. This case is depicted in Figure 3. ■

The model shows that in a "lab equipment" setting, generating scale-free fully endogenous growth requires two razor's-edge restrictions. The first, $\beta = 1$, eliminates the scale effect. This is equivalent to equating the intercepts of the equilibrium loci. A second razor's-edge restriction, $\theta(\Psi - \alpha) + \varepsilon(1 - \alpha) = (1 - \alpha)$, is needed to generate a constant positive growth rate in the steady state. This is equivalent to setting the slopes of equations (9) and (10) equal. This restriction becomes $\theta + \varepsilon = 1$ with labor-augmenting technology, $\frac{\theta\alpha}{(1-\alpha)(1-\varepsilon)} = 1$ with capital-augmenting technology, and $\frac{\theta}{(1-\alpha)(1-\varepsilon)} = 1$ with Hicks-neutral technology.

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5. EMPIRICAL IMPLICATIONS

This framework allows us to distinguish between the full dynamics of the fully endogenous and semiendogenous growth models. The differences are interesting from a purely theoretical point of view, as they provide a full account of two competing frameworks that have very different properties in the steady state. However, perhaps the most important implications are for empirical testing, given that the model motivates simple empirical distinctions between fully endogenous and semiendogenous growth frameworks.

Studies that have attempted to evaluate the importance of transition dynamics vs. steady state dynamics in the growth of per capita income empirically find that transition dynamics dominates [Jones (2002); Sedgley and Elmslie (2010)]. This implies a difficulty in testing various models solely by steady state predictions as in Zachariadis (2003, 2004), Ha and Howitt (2007), and Madsen (2008). For example, Zachariadis (2004) uses U.S. manufacturing data to test for a positive relation between R&D intensity and growth (a steady state prediction of the fully endogenous model) versus the null of no relationship (a steady state prediction of the semiendogenous model). The concern is that a positive relation is predicted by the semiendogenous model as the economy transcends to a new higher steady state income, blurring the test if not fully controlling for transitional growth. The approach implied by this analysis does not require the assumption of a steady state in the data. It allows tests based on the parameter restrictions required to generate a stable equilibrium.

Suppose, for example, we allow labor-augmenting technological change and make the simplifying assumption that the savings rate and the proportion of output used in knowledge production are constant. It is possible to show that the dynamics of the general model can be described by

$$\gamma_{\gamma_A} = [\theta(1-\alpha) - 1]\,\gamma_A + \theta\alpha\gamma_K + \theta\,[(1-\alpha) - \beta]\,n \tag{23}$$

$$\gamma_{\gamma_K} = (1-\alpha)\gamma_A + (\alpha - 1)\gamma_K + (1-\alpha)n. \tag{24}$$

Adding stochastic error terms, these equations can be expressed in an error-correction form:

$$\Delta\gamma_{\gamma_A} = -\theta\alpha[\gamma_A - \gamma_{K/L}] + (\theta - 1)\left[\gamma_A - \frac{\theta(1-\beta)}{(1-\theta)}n\right] + \varepsilon_1 \tag{25}$$

$$\Delta\gamma_{\gamma_K} = (1-\alpha)[\gamma_A - \gamma_{K/L}] - \Delta n + \varepsilon_2. \tag{26}$$

Δ signifies the first difference. Equations (25) and (26) are in the error-correction form. The long-run cointegrating vectors are in square brackets and the coefficients $-\theta\alpha$, $(\theta - 1)$, and $(1-\alpha)$ are speed-of-adjustment parameters. The fully endogenous model is a nested model within the semiendogenous growth model. The restriction for fully endogenous growth is $\theta = \beta = 1$, where equations (25) and (26) become

$$\Delta\gamma_{\gamma_A} = (-\alpha)[\gamma_A - \gamma_{K/L}] + \varepsilon_1 \tag{27}$$

$$\Delta\gamma_{\gamma K/L} = (1-\alpha)[\gamma_A - \gamma_{K/L}] - \Delta n + \varepsilon_2. \tag{28}$$

Notice that the number of cointegrating vectors collapses to 1 in the fully endogenous model. Therefore a meaningful test of fully endogenous growth in a framework that works as well outside the steady state as in the steady state could be a test of the number of cointegrating vectors between a measure of total factors of production, population or labor force growth, and growth of the capital-to-labor ratio. Of course other extensions and iterations are possible. The approach taken here is more consistent with the true data-generating process and will lead to sharper tests of alternative growth theories. Sedgley and Elmslie (2010) utilize this approach with time-series data from the United States covering the period from 1950 to 2000. They find evidence in favor of a single cointegrating vector, suggesting some support for the Schumpeterian approach.

6. CONCLUSION

This paper provides for a missing analysis of out-of-steady state behavior and stability of steady state growth rates in semiendogenous versus fully endogenous growth models in a "lab equipment" and expanding–product variety version of the innovation-driven growth model. Growth economists depend strongly on the steady state predictions of these models; thus a deeper understanding of the dynamic nature of the steady state equilibria is warranted. With expanding sectors, it is possible that the semiendogenous model growth will exhibit saddlepath stability or global stability. The fully endogenous model is shown to exhibit global stability, because equilibrium requires the slopes and intercepts of the equilibrium loci to coincide. In each case, the familiar restrictions needed to generate a positive-equilibrium growth rate with the desired properties in the steady state are enough to fully determine the nature of the path to the steady state and the stability of the system.

More attention should be paid to the transitional dynamics of semiendogenous and fully endogenous growth models. If a simple approach, such as the one used here, is adopted in empirical work, economists can make better use of available data to test both semiendogenous and fully endogenous growth models. Any attempt at choosing one of these frameworks based on empirical evidence will have to include the possibility that the economy is operating along a path of transition to the steady state.

NOTES

1. Attempts have been made to restore policy effectiveness in a semiendogenous growth framework without imposing knife-edge restrictions. For example, Jones (2003) shows that an R&D subsidy lowers the steady state growth rate if fertility is endogenous. Giordani and Luca (2008) aim to restore policy impacts on steady state growth by building a Jones model without razor's-edge conditions,

but where industries differ in innovative potential. They show that an industrial policy of lump-sum taxation of low-potential industries to fund R&D subsidies in high-potential industries can boost steady state growth. Although interesting, it is unlikely that this type of policy has significant real-world implications.

2. Human capital has been incorporated into the Jones model in several papers, including Dalgaard and Kreiner (2001), to determine the extent of human capital accumulation's role in determining both income levels and growth rates. Hall and Jones (1999) conduct an empirical study of human capital's importance in answering the question of why income levels differ so dramatically across countries. In their "level accounting" exercise, they find that levels of human and physical capital cannot be the main explanation for income differences. Jones (2002) builds on this framework to show in a growth-accounting exercise that 80% of growth is transitional dynamics and that only one-third of this is due to human capital. The Dalgaard and Kreine (2001) model is classified as a fully endogenous growth model, as it employs a razor's-edge condition to produce a policy impact on growth. We do not explicitly model human capital in this paper, because leaving human capital out makes our model easier to compare to the bulk of the literature. Such an expansion might be an important area for future research.

3. Aghion and Howitt (1998, Ch. 12) argue that the key distinction between their Schumpeterian model and Jones's model (1995a and 1995b) rests in the fact that Jones assumes that labor is the only rival factor used in the production of new ideas. They claim to show how policy implications for steady state growth, a hallmark of the Schumpeterian approach, can be restored in a scale-free Schumpeterian framework when capital is included in the model. In fact, their model qualifies as a fully endogenous growth model according to our categorization, because it depends on knife-edge restrictions that can be fully understood in our model.

4. Jones's specification is much simpler. He specifies $\dot{A} = \delta L_A A^{\phi}$ for knowledge production and $Y = A^{\sigma} L_Y$ for final goods production. Equation (4) is identical to Jones's equation (7).

5. With depreciation, equation (8) would have to be log linearized with a Taylor series expansion around the steady state. Because depreciation would not be multiplied by any dynamic variable after being divided through by K_t, it would drop out of the Taylor series approximation. A differencing would lead to the same dynamics as in the model without depreciation. Thus the additional mathematics provides no additional economic insight. This assumption is utilized, for example, by Eicher and Turnovsky (1999).

6. Boskin and Lau (2000) argue that post–World War II growth has been Solow-neutral.

7. Thus the model developed here is a special case of Eicher and Turnovsky's model if and only if a knife-edge condition is imposed on our model. The necessary restriction that makes our model a special case of their general model is one that eliminates the link between the number of sectors over which research is spread and the population of the economy.

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