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# ON PERPETUITIES WITH GAMMA-LIKE TAILS

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**Abstract**

An infinite convergent sum of independent and identically distributed random variables discounted by a multiplicative random walk is called perpetuity, because of a possible actuarial application. We provide three disjoint groups of sufficient conditions which ensure that the right tail of a perpetuity $\mathbb{P}\{X > x\}$ is asymptotic to $ax^c e^{-bx}$ as $x \to \infty$ for some $a, b > 0$, and $c \in \mathbb{R}$. Our results complement those of Denisov and Zwart (2007). As an auxiliary tool we provide criteria for the finiteness of the one-sided exponential moments of perpetuities. We give several examples in which the distributions of perpetuities are explicitly identified.

*Keywords:* Distribution tail; selfdecomposable distribution; perpetuity; exponential moment

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## 1. Introduction

Let $(A_n, B_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed $\mathbb{R}^2$-valued random vectors with generic copy $(A, B)$. Set $\Pi_0 := 1$ and $\Pi_n := A_1 \cdots A_n$ for $n \in \mathbb{N}$. The random discounted sum

$$X := \sum_{k\geq 1} \Pi_{k-1} B_k,$$

provided that $|X| < \infty$ almost surely (a.s.), is called *perpetuity* and is of interest in various fields of applied probability. The term perpetuity stems from the fact that such random series occur in the realm of insurance and finance as sums of discounted payment streams. Detailed information about various aspects of perpetuities, including applications, can be found in [5] and [19].

A number of authors have investigated the asymptotics of $-\log \mathbb{P}\{|X| > x\}$ as $x \to \infty$ in the situations when $\mathbb{P}\{|X| > x\}$ exhibits exponential or superexponential decrease; see [1], [2], [12], [17], [18], and [27]. In this paper we are interested in the precise (nonlogarithmic) asymptotics of $\mathbb{P}\{X > x\}$ as $x \to \infty$. Specifically, our main concern is finding conditions

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which ensure $\mathbb{P}\{X > x\} \sim ax^c \mathrm{e}^{-bx}$ as $x \to \infty$ for some positive $a$, $b$, and real $c$. Distribution tails which exhibit such asymptotics may be called *gamma-like tails*, hence the title of this paper. To the best of the authors' knowledge, work in this direction is rare; see [9], [28], [29], and [31]. The first three papers are concerned with exponential tails of perpetuities which correspond to nonnegative and independent $A$ and $B$. The results obtained in [31] cover the situation when $A = \gamma \in (0, 1)$ a.s., and $B$ is not necessarily nonnegative and satisfies $\mathbb{P}\{B > x\} \sim ax^c \mathrm{e}^{-x^p}$ as $x \to \infty$ for some positive $a$, $p$, and real $c$. Under additional technical assumptions in the $p > 1$ case, the author of [31] obtained the asymptotics of $\mathbb{P}\{X > x\}$ as $x \to \infty$.

We note that perpetuities with heavy tails have received much more attention than those with light tails, [11], [15], [16], and [24] being classical articles in this area. A nonexhaustive list of very recent contributions includes [7], [8], [10], [25], and [26].

## 2. Main results

The following result is from [9, Proposition 4.1] under the assumptions that $A$ and $B$ are a.s. nonnegative and that $\mathbb{P}\{A = 1\} = 0$, which are partially dispensed with here. For $s \in \mathbb{R}$, define $\psi(s) := \mathbb{E}\mathrm{e}^{sX}$ and $\varphi(s) := \mathbb{E}\mathrm{e}^{sB}$ finite or infinite.

**Proposition 1.** *Let $A$ and $B$ be independent and $r > 0$. Suppose that $\mathbb{P}\{A = 1\} \in [0, 1)$ and that either of the following cases applies:*

(a) $\mathbb{P}\{A \in (0, 1]\} = 1$, *or*

(b) $\mathbb{P}\{A \in (-1, 0)\} = 1$, *or*

(c) $\mathbb{P}\{|A| \in (0, 1]\} = 1$ *and* $\mathbb{P}\{A = -1\} \in (0, 1)$.

(i) *Assume that $\mathbb{P}\{B = 0\} < 1$.*

*Let case (a) prevail. If $\mathbb{P}\{A = 1\} = 0$ then $\mathbb{E}\psi(rA) < \infty$ if and only if $\mathbb{E}\varphi(rA) < \infty$. If $\mathbb{P}\{A = 1\} \in (0, 1)$ then $\mathbb{E}\psi(rA) < \infty$ if and only if $\varphi(r)\mathbb{P}\{A = 1\} < 1$.*

*Under case (b), $\mathbb{E}\psi(rA) < \infty$ if and only if*

$$\mathbb{E}\mathrm{e}^{rA_1(B_2+A_2B_3)} < \infty. \tag{1}$$

*Under case (c), $\mathbb{E}\psi(rA) < \infty$ if and only if*

$$\mathbb{E}\mathrm{e}^{-rB}\mathbb{E}\mathrm{e}^{rB}[\mathbb{P}\{A = -1\}]^2 < (1 - \mathbb{E}\mathrm{e}^{-rB}\mathbb{P}\{A = 1\})(1 - \mathbb{E}\mathrm{e}^{rB}\mathbb{P}\{A = 1\}).$$

(ii) *Suppose that $\mathbb{P}\{B > x\} \sim g(x)\mathrm{e}^{-bx}$ as $x \to \infty$ for some $b > 0$ and some function $g$ such that $g(\log x)$ is slowly varying at $\infty$, and*

$$\limsup_{x\to\infty} \frac{\sup_{1\le y\le x} g(y)}{g(x)} < \infty.$$

*Then*

$$\mathbb{P}\{X > x\} \sim \mathbb{E}\psi(bA)\mathbb{P}\{B > x\}, \qquad x \to \infty,$$

*provided that $\mathbb{E}\psi(bA) < \infty$.*

**Remark 1.** We now comment on inequality (1). If $B \ge 0$ or $B \le 0$ a.s. then (1) is equivalent to $\mathbb{E}\varphi(rA_1A_2) < \infty$ and $\mathbb{E}\varphi(rA) < \infty$, respectively. If $A = -\gamma$ a.s. for some $\gamma \in (0, 1)$ and $B$ takes values of both signs with positive probability, then (1) is equivalent to $\varphi(-r\gamma) < \infty$ and $\varphi(r\gamma^2) < \infty$. In the general case, (1) imposes restrictions on both tails of $B$ and entails, but is not equivalent to, $\mathbb{E}\varphi(rA) < \infty$ and $\mathbb{E}\varphi(rA_1A_2) < \infty$.

The argument of [9] applied to Proposition 1(ii) remains valid in the extended situation treated here. Our contribution consists in proving Proposition 1(i), that is, a criterion for $\mathbb{E}\psi(bA)$ to be finite which is actually a consequence of Theorems 3 and 4, and Remark 6.

Next we present the more complicated result in which $A$ and $B$ are allowed to be dependent in a certain way, and the right tail of a possibly two-sided $B$ is gamma like. Throughout the paper we will use the standard notation $x^+ := \max(x, 0)$ and $x^- := -\min(x, 0)$ for $x \in \mathbb{R}$.

**Theorem 1.** *Assume that* $\mathbb{P}\{A \in (0, 1]\} = 1$*;*

$$\mathbb{P}\{B > x\} \sim a x^c \mathrm{e}^{-bx}, \qquad x \to \infty, \tag{2}$$

*for some* $a, b > 0$ *and* $c < -1$*;*

$$\mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{A=1\}} < 1; \tag{3}$$

$$\mathbb{P}\{Ay + B > x\} \sim f(y)\mathbb{P}\{B > x\}, \qquad x \to \infty, \tag{4}$$

*for each* $y \in \mathbb{R}$ *and a nonnegative measurable function* $f$*; and*

$$\mathbb{E}\log(1 + B^-) < \infty. \tag{5}$$

*Then* $\mathbb{E}f(X) < \infty$ *and*

$$\mathbb{P}\{X > x\} \sim \frac{\mathbb{E}f(X)}{1 - \mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{A=1\}}}\mathbb{P}\{B > x\}, \qquad x \to \infty. \tag{6}$$

**Remark 2.** Recall that the distribution of a nonnegative random variable $Y$ belongs to the class $\mathcal{S}(\alpha)$ for $\alpha \geq 0$ if

(i) $\lim_{x\to\infty} \mathbb{P}\{Y > x - y\}/\mathbb{P}\{Y > x\} = \mathrm{e}^{\alpha y}$ for each $y \in \mathbb{R}$;

(ii) $\lim_{x\to\infty} \mathbb{P}\{Y + Y^* > x\}/\mathbb{P}\{Y > x\} = 2\mathbb{E}\mathrm{e}^{\alpha Y} < \infty$, where $Y^*$ is an independent copy of $Y$.

Condition (2) with $c < -1$ ensures that the distribution of $B^+$ belongs to $\mathcal{S}(b)$. While point (i) above is easily checked, point (ii) follows from [31, Lemma 7.1(iii)]. Theorem 1 is closely related to [9, Proposition 4.2] in which a similar asymptotic result was proved under the assumptions that $A$ and $B$ are independent, that $\mathbb{P}\{A = 1\} = 0$ and $\mathbb{P}\{B \geq 0\} = 1$, and that the distribution of $B$ belongs to the class $\mathcal{S}(b)$. Theorem 3.2 of [28] is another result in this vein. A perusal of the proof given below reveals that (6) remains valid if (2) is replaced by the assumption that the distribution of $B^+$ belongs to $\mathcal{S}(b)$. However, we refrain from formulating Theorem 1 in this way, for our focus here is on the gamma-like tails.

**Remark 3.** Here we provide more details on functions $f$ arising in (4), assuming that the assumptions of Theorem 1 are in force. It is clear that $f(y) = \mathbb{E}\mathrm{e}^{byA}$, $y \in \mathbb{R}$, whenever $A$ and $B$ are independent. The last equality does not necessarily hold when $A$ and $B$ are dependent. For instance, if $A = \zeta_1\,\mathbf{1}_{\{B>q\}} + \zeta_2\,\mathbf{1}_{\{B\leq q\}}$ for some $\zeta_1, \zeta_2 \in (0, 1)$, $\zeta_1 \neq \zeta_2$, and some $q > 0$, then $f(y) = \mathrm{e}^{by\zeta_1} \neq \mathbb{E}\mathrm{e}^{byA}$, $y \in \mathbb{R}$.

We note that a condition of form (4) appears in [30, Theorem 3] in a setting quite different from ours. The cited result gives sufficient conditions under which the right tail of $\sup_{k\geq 1} \Pi_{k-1} B_k$ is heavy. One of the anonymous referees has kindly informed us that our method of proof of Theorem 1 is rather similar to that of [30, Theorem 3]. We provide more details on this point at the end of Section 6.

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Proposition 1 and Theorem 1 cover the situation when a gamma-like tail of $X$ is inherited from a gamma-like tail of $B$, the influence of the distribution of $A$ being small as it is seen only in the multiplicative constant. In Example 1 below we see that the distributions of both $A$ and $B$ may give principal contributions to a gamma-like tail of $X$.

To proceed we need more notation. Denote by $\gamma(a, b)$ and $\beta(c, d)$ a gamma distribution with parameters $a, b > 0$ and a beta distribution with parameters $c, d > 0$, respectively. Recall that

$$\gamma(a, b)(\mathrm{d}x) = \frac{b^a x^{a-1} \mathrm{e}^{-bx}}{\Gamma(a)} \mathbf{1}_{(0,\infty)}(x)\,\mathrm{d}x,$$

where $\Gamma(\cdot)$ is the Euler gamma function, and

$$\beta(c, d)(\mathrm{d}x) = \frac{1}{\mathrm{B}(c, d)} x^{c-1}(1-x)^{d-1}\,\mathbf{1}_{(0,1)}(x)\,\mathrm{d}x,$$

where $\mathrm{B}(\cdot, \cdot)$ is the Euler beta function. The following example is well known; see, for instance, [33, Example 3.8.2].

**Example 1.** Assume that $A$ and $B$ are independent, $A$ has a $\beta(c, 1)$ distribution, and $B$ has a $\gamma(1, b)$ (exponential) distribution. Then $X$ has a $\gamma(c + 1, b)$ distribution (this can be checked in several ways, for instance, via the argument in Example 3). In particular,

$$\mathbb{P}\{X > x\} \sim \frac{(bx)^c}{\Gamma(c+1)} \mathrm{e}^{-bx}, \qquad x \to \infty. \tag{7}$$

Our next result, Theorem 2, provides an extension of Example 1 in that $B$ is allowed to take values of both signs with positive probability and that the right tail of $B$ is approximately, rather than precisely, exponential. Our Theorem 2 is close in spirit to [29, Theorem 6.1] since in both results it is assumed that while one of the independent input random variables $A$ and $B$ obeys a particular distribution ($A$ has a $\beta(1, \lambda)$ distribution in our Theorem 2; $B$ has a $\gamma(1, \lambda)$ distribution in [29, Theorem 6.1]), the distribution of the other random variable follows a prescribed tail behavior.

**Theorem 2.** *Assume that $A$ and $B$ are independent, $A$ has a $\beta(\lambda, 1)$ distribution for some $\lambda > 0$, condition (5) holds, and*

$$\mathbb{P}\{B > x\} = C\mathrm{e}^{-bx} + r(x) \tag{8}$$

*for some $C, b > 0$, all $x \geq 0$, and a function $r$ such that*

$$\lim_{x\to\infty} \mathrm{e}^{bx} r(x) = 0, \tag{9}$$

$$\int_1^\infty \frac{\mathrm{e}^{by}}{y} r^+(y)\,\mathrm{d}y < \infty, \quad \text{and} \quad \int_1^\infty \frac{\mathrm{e}^{(b+\varepsilon)y}}{y} r^-(y)\,\mathrm{d}y < \infty \quad \text{for some } \varepsilon > 0. \tag{10}$$

*Then*

$$\mathbb{P}\{X > x\} \sim K x^{\lambda C} \mathrm{e}^{-bx}, \qquad x \to \infty, \tag{11}$$

*where*

$$K := \frac{C b^{C\lambda}}{\Gamma(C\lambda + 1)} \exp\biggl(\lambda\biggl[\int_0^\infty \frac{\mathrm{e}^{by} - 1}{y} r(y)\,\mathrm{d}y - \int_{-\infty}^0 \frac{\mathrm{e}^{by} - 1}{y} \mathbb{P}\{B \leq y\}\,\mathrm{d}y\biggr]\biggr) < \infty.$$

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The remainder of the paper is organized as follows. In Section 3 we provide several examples intended to illustrate Proposition 1 and Theorem 2. Also in this section is a discussion of an interesting connection between perpetuities arising in Theorem 2 and certain selfdecomposable distributions. It is exactly this link which makes the proof of Theorem 2 relatively simple. In Section 4 we provide criteria for the existence of the one-sided exponential moments of perpetuities, the results of which are needed for the proof of Proposition 1. The picture is not yet complete since finding a criterion remains a challenge in the case where both $A$ and $B$ take values of both signs with positive probability. All the proofs can be found in Sections 5–7.

## 3. Illustrating examples

We now provide an example to illustrate Proposition 1.

**Example 2.** Denote by $\theta_a$ and $\theta_b$ independent random variables with a $\gamma(1, a)$ and $\gamma(1, b)$ distribution, respectively. Let $A = \gamma \in (0, 1)$ a.s. and

$$\mathbb{E}\mathrm{e}^{sB} = \frac{a - \gamma s}{a - s}\frac{b + \gamma s}{b + s} \quad \text{for } -b < s < a.$$

Then $X = \theta_a - \theta_b$ or, equivalently,

$$\mathbb{E}\mathrm{e}^{sX} = \frac{a}{a - s}\frac{b}{b + s}.$$

A standard calculation yields $\mathbb{P}\{X > x\} = (b/(a+b))\mathrm{e}^{-ax}$ for $x > 0$. The distribution of $B$ is a mixture of the atom at zero with weight $\gamma^2$, the distribution of $\theta_a$ with weight $\gamma(1-\gamma)$, the distribution of $-\theta_b$ with weight $\gamma(1-\gamma)$ and the distribution of $\theta_a - \theta_b$ with weight $(1-\gamma)^2$. Hence,

$$\mathbb{P}\{B > x\} = (1 - \gamma)\frac{b + a\gamma}{a + b}\mathrm{e}^{-ax} \quad \text{for } x > 0$$

in full agreement with Proposition 1.

It is well known that the explicit distributions of the perpetuities are rarely available. Below we provide several examples of distributions of $B$ satisfying the assumptions of Theorem 2 for which distributions of the corresponding perpetuities $X$ can be identified. Among others, this allows us to check the validity of (11). We start with a trivial observation that the distributions of $A$ and $B$ as given in Example 1 satisfy the assumptions of Theorem 2 with $\lambda = c$, $C = 1$, and $r(x) \equiv 0$, in which case (11) amounts to (7) as it must be.

Throughout the rest of this section we assume, without further notice, that $B$ is independent of $A$ and that $A$ has a $\beta(1, \lambda)$ distribution. We first point out an interesting connection with special selfdecomposable distributions which enables us to obtain a useful representation:

$$\Psi(t) := \mathbb{E}\mathrm{e}^{\mathrm{i}tX} = \Phi(t)\exp\biggl(\lambda\int_0^t \frac{\Phi(u) - 1}{u}\,\mathrm{d}u\biggr), \qquad t \in \mathbb{R}, \tag{12}$$

where $\Phi(t) := \mathbb{E}\mathrm{e}^{\mathrm{i}tB}$, $t \in \mathbb{R}$. This connection is implicit in [32] and [33] and perhaps some other work.

The class $L$ of selfdecomposable distributions is comprised of all possible limit distributions for the sums, properly normalized and centered, of independent (not necessarily identically distributed) random variables satisfying an infinitesimality condition. It was proved in [23] that the

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class $L$ coincides with the class of distributions of the random variables $J := \int_{(0,\infty)} \mathrm{e}^{-s}\,\mathrm{d}Y(s)$, where $(Y(t))_{t\geq 0}$ is a Lévy process with $\mathbb{E}\log(1+|Y(1)|) < \infty$. It is known (see, for example, [23, Equation (4.4)]) that

$$\log \mathbb{E}\mathrm{e}^{\mathrm{i}tJ} = \int_0^t \frac{\log \mathbb{E}\mathrm{e}^{\mathrm{i}sY(1)}}{s}\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

If $(Y(t))_{t\geq 0}$ is a compound Poisson process of intensity $\lambda$ with jumps $B_k$ satisfying the assumptions of Theorem 2, then

$$\log \mathbb{E}\mathrm{e}^{\mathrm{i}tJ} = \lambda \int_0^t \frac{\Phi(s)-1}{s}\,\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{13}$$

as a consequence of $\log \mathbb{E}\mathrm{e}^{\mathrm{i}tY(1)} = \lambda(\Phi(t)-1)$ for $t \in \mathbb{R}$. Recalling that the function $x \mapsto \log(1+x)$ is subadditive on $[0,\infty)$, we conclude that conditions (5) and (8) ensure that $\mathbb{E}\log(1+|B|) \leq \mathbb{E}\log(1+B^+) + \mathbb{E}\log(1+B^-) < \infty$, whence $\mathbb{E}\log(1+|Y(1)|) \leq \mathbb{E}N\mathbb{E}\log(1+|B|) < \infty$, where $N$ is a Poisson distributed random variable with parameter $\lambda$. The latter inequality secures the convergence of the integral in (13). The selfdecomposable distributions with the characteristic functions of form (13) were investigated in [20] and [21]. Equation (12) is a consequence of (13) and a representation $X = B_1 + A_1(B_2 + A_2B_3 + \cdots)$ a.s., and the fact that $A_1(B_2 + A_2B_3 + \cdots)$ is independent of $B_1$ and has the same distribution as $J$ in (13).

**Example 3.** Let $B = \xi/b - \eta/a$ for $a, b > 0$ and independent random variables $\xi$ and $\eta$ with a $\gamma(1,1)$ distribution (exponential distribution of unit mean). Then $\mathbb{P}\{B > x\} = (a/(a+b))\mathrm{e}^{-bx}$ for $x > 0$ and

$$\mathbb{P}\{B \leq x\} = \frac{b}{a+b}\mathrm{e}^{ax} \quad \text{for } x < 0, \tag{14}$$

so that the assumptions of Theorem 2 are satisfied with $C = a/(a+b)$ and $r(x) \equiv 0$. Since

$$\Phi(t) = \mathbb{E}\mathrm{e}^{\mathrm{i}tB} = \frac{b}{b-\mathrm{i}t}\frac{a}{a+\mathrm{i}t}, \qquad t \in \mathbb{R},$$

we infer that, with the help of (12),

$$\mathbb{E}\mathrm{e}^{\mathrm{i}tX} = \left(\frac{b}{b-\mathrm{i}t}\right)^{a\lambda/(a+b)+1}\left(\frac{a}{a+\mathrm{i}t}\right)^{b\lambda/(a+b)+1}, \qquad t \in \mathbb{R}.$$

Thus, $X$ has the same distribution as $Y - Z$, where $Y$ and $Z$ are independent random variables with $\gamma(a\lambda/(a+b)+1, b)$ and $\gamma(b\lambda/(a+b)+1, a)$ distributions, respectively. Noting that the function $x \mapsto \mathbb{P}\{\mathrm{e}^Y > x\}$ is regularly varying at $\infty$ of index $-b$ and applying Breiman's lemma (see [4, Proposition 3] and [6, Corollary 3.6(iii)]), we conclude that

$$\begin{aligned}\mathbb{P}\{X > x\} &= \mathbb{P}\{\mathrm{e}^{Y-Z} > \mathrm{e}^x\} \\ &\sim \mathbb{E}\mathrm{e}^{-bZ}\mathbb{P}\{Y > x\} \\ &= \left(\frac{a}{a+b}\right)^{b\lambda/(a+b)+1}\mathbb{P}\{Y > x\}, \qquad x \to \infty.\end{aligned}$$

In view of (7), this entails

$$\mathbb{P}\{X > x\} \sim \left(\frac{a}{a+b}\right)^{b\lambda/(a+b)+1}\frac{(bx)^{a\lambda/(a+b)}}{\Gamma(a\lambda/(a+b)+1)}\mathrm{e}^{-bx}, \qquad x \to \infty. \tag{15}$$

To check that (11) yields the same answer, we have to calculate $K$ appearing in that formula. Using (14), we obtain

$$\exp\biggl(-\lambda\int_{-\infty}^{0}\frac{\mathrm{e}^{by}-1}{y}\mathbb{P}\{B\le y\}\,\mathrm{d}y\biggr)=\exp\biggl(-\frac{b\lambda}{a+b}\int_{0}^{\infty}\frac{\mathrm{e}^{-ay}-\mathrm{e}^{-(a+b)y}}{y}\,\mathrm{d}y\biggr)$$
$$=\exp\biggl(-\frac{b\lambda}{a+b}\log\biggl(\frac{a+b}{a}\biggr)\biggr)$$
$$=\biggl(\frac{a}{a+b}\biggr)^{b/(a+b)} \tag{16}$$

having observed that the last integral is a Frullani integral. Thus,

$$K=\frac{(a/(a+b))b^{a\lambda/(a+b)}}{\Gamma(a\lambda/(a+b)+1)}\biggl(\frac{a}{a+b}\biggr)^{b\lambda/(a+b)}=\biggl(\frac{a}{a+b}\biggr)^{b\lambda/(a+b)+1}\frac{b^{a\lambda/(a+b)}}{\Gamma(a\lambda/(a+b)+1)},$$

which is in line with (15).

**Example 4.** Set $B:=\xi-\eta$ for independent positive random variables $\xi$ and $\eta$. Assume that

$$\mathbb{P}\{\xi>x\}=C_1\mathrm{e}^{-bx}+r_1(x),\qquad x\ge 0, \tag{17}$$

and that $r_1$ satisfies (9) and (10). Then

$$\mathbb{P}\{B>x\}=\int_0^{\infty}\mathbb{P}\{\xi>x+y\}\mathbb{P}\{\eta\in\mathrm{d}y\}=C_1(\mathbb{E}\mathrm{e}^{-b\eta})\mathrm{e}^{-bx}+\mathbb{E}r_1(x+\eta)=:C\mathrm{e}^{-bx}+r(x).$$

By the Lebesgue dominated convergence theorem, $\lim_{x\to\infty}\mathrm{e}^{bx}r(x)=0$. Further, by Fubini's theorem and the fact that $y\mapsto y^{-1}\mathrm{e}^{by}$ is nondecreasing on $[1/b,\infty)$, we obtain

$$\int_{1/b}^{\infty}\frac{\mathrm{e}^{by}}{y}r^{+}(y)\,\mathrm{d}y\le\mathbb{E}\int_{1/b}^{\infty}\frac{\mathrm{e}^{by}}{y}r_1^{+}(y+\eta)\,\mathrm{d}y$$
$$\le\mathbb{E}\int_{1/b}^{\infty}\frac{\mathrm{e}^{b(y+\eta)}}{y+\eta}r_1^{+}(y+\eta)\,\mathrm{d}y$$
$$=\mathbb{E}\int_{1/b+\eta}^{\infty}\frac{\mathrm{e}^{by}}{y}r_1^{+}(y)\,\mathrm{d}y$$
$$\le\int_{1/b}^{\infty}\frac{\mathrm{e}^{by}}{y}r_1^{+}(y)\,\mathrm{d}y$$
$$<\infty.$$

Analogously,

$$\int_1^{\infty}\frac{\mathrm{e}^{(b+\varepsilon)y}}{y}r^{-}(y)\,\mathrm{d}y<\infty.$$

Hence, under (17), the right tail of the distribution of $B$ satisfies the assumptions of Theorem 2 with $C:=C_1\mathbb{E}\mathrm{e}^{-b\eta}$ and $r(x):=\mathbb{E}r_1(x+\eta)$ whatever the distribution of $\eta$.

To give a concrete example, let $\xi$ and $\eta$ be independent with $\mathbb{P}\{\xi>x\}=\mathbb{P}\{\eta>x\}=p\mathrm{e}^{-bx}+(1-p)\mathrm{e}^{-cx}$ for $x\ge 0$, $c>b>0$, and $p\in(0,1)$. Condition (17) holds with $C_1=p$ and $r_1(x)=(1-p)\mathrm{e}^{-cx}$, which trivially satisfies (9) and (10). Further,

$$\mathbb{P}\{B>x\}=\mathbb{P}\{B\le -x\}=\frac{c_1\mathrm{e}^{-bx}+c_2\mathrm{e}^{-cx}}{2},\qquad x\ge 0, \tag{18}$$

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where

$$c_1 := p^2 + \frac{2p(1-p)c}{b+c}, \qquad c_2 := (1-p)^2 + \frac{2p(1-p)b}{b+c},$$

which immediately implies that condition (5) holds and that $B = \xi - \eta$ has the characteristic function

$$\Phi(t) = \mathbb{E}e^{\mathrm{i}tB} = c_1 \frac{b^2}{b^2+t^2} + c_2 \frac{c^2}{c^2+t^2}, \qquad t \in \mathbb{R}.$$

Observing that

$$\exp\biggl(\alpha \int_0^\infty (e^{\mathrm{i}ut} - 1)\frac{e^{-\beta u}}{u}\,\mathrm{d}u\biggr) = \biggl(\frac{\beta}{\beta - \mathrm{i}t}\biggr)^{\alpha}, \qquad t \in \mathbb{R}$$

for $\alpha, \beta > 0$, we obtain, with the help of (18),

$$\begin{aligned}
&\exp\biggl(\lambda \int_0^t \frac{\Phi(u)-1}{u}\,\mathrm{d}u\biggr) \\
&\quad = \exp\biggl(\lambda \int_0^\infty (e^{\mathrm{i}ut} - 1)\frac{\mathbb{P}\{B>u\}}{u}\,\mathrm{d}u\biggr) \exp\biggl(\lambda \int_0^\infty (e^{-\mathrm{i}ut} - 1)\frac{\mathbb{P}\{-B>u\}}{u}\,\mathrm{d}u\biggr) \\
&\quad = \biggl(\frac{b^2}{b^2+t^2}\biggr)^{c_1\lambda/2} \biggl(\frac{c^2}{c^2+t^2}\biggr)^{c_2\lambda/2}.
\end{aligned}$$

This entails

$$\mathbb{E}e^{\mathrm{i}tX} = \Phi(t)\biggl(\frac{b^2}{b^2+t^2}\biggr)^{c_1\lambda/2} \biggl(\frac{c^2}{c^2+t^2}\biggr)^{c_2\lambda/2},$$

from which we conclude that $X$ has the same distribution as $\xi - \eta + Y_1 - Y_2 + Z_1 - Z_2$, where the latter random variables are independent, $Y_1$ and $Y_2$ have a $\gamma(c_1\lambda/2, b)$ distribution, and $Z_1$ and $Z_2$ have a $\gamma(c_2\lambda/2, c)$ distribution. Note that

$$\mathbb{E}e^{-b\eta} = \frac{p}{2} + \frac{(1-p)c}{b+c} = \frac{c_1}{2p}, \qquad \mathbb{E}e^{-bY_2} = \biggl(\frac{1}{2}\biggr)^{c_1\lambda/2},$$

$$\mathbb{E}e^{b(Z_1-Z_2)} = \biggl(\frac{c^2}{c^2-b^2}\biggr)^{c_2\lambda/2},$$

and that the exponential moments of order $b+\varepsilon$ for $\varepsilon \in (0, c-b)$ are finite. Invoking Breiman's lemma yields

$$\begin{aligned}
\mathbb{P}\{X > x\} &\sim \mathbb{E}e^{b(-\eta - Y_2 + Z_1 - Z_2)}\mathbb{P}\{\xi + Y_1 > x\} \\
&= \frac{c_1}{2p}\biggl(\frac{1}{2}\biggr)^{c_1\lambda/2}\biggl(\frac{c^2}{c^2-b^2}\biggr)^{c_2\lambda/2}\mathbb{P}\{\xi + Y_1 > x\} \quad \text{as } x \to \infty.
\end{aligned}$$

In view of the equality $\gamma(c_1\frac{1}{2}\lambda, b) * \gamma(1, b) = \gamma(c_1\frac{1}{2}\lambda + 1, b)$ and the asymptotic relation

$$\gamma\bigl(\tfrac{1}{2}c_1\lambda, b\bigr) * \gamma(1, c)((x,\infty)) = o\bigl(\gamma\bigl(\tfrac{1}{2}c_1\lambda + 1, b\bigr)((x,\infty))\bigr), \qquad x \to \infty,$$

we have

$$\mathbb{P}\{\xi + Y_1 > x\} \sim p\gamma\bigl(c_1\bigl(\tfrac{1}{2}\lambda\bigr) + 1, b\bigr)((x,\infty)), \qquad x \to \infty.$$

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Combining these results and applying (7), we obtain

$$\mathbb{P}\{X > x\} \sim \frac{c_1}{2}\left(\frac{1}{2}\right)^{c_1\lambda/2}\left(\frac{c^2}{c^2-b^2}\right)^{c_2\lambda/2}\frac{b^{\lambda c_1/2}}{\Gamma(c_1(\lambda/2)+1)}x^{c_1\lambda/2}\mathrm{e}^{-bx}, \qquad x\to\infty. \quad (19)$$

We now show that (19) follows from Theorem 2 with $C = c_1/2$ and $r(x) = (c_2/2)\mathrm{e}^{-cx}$. To this end, we have to calculate only $K$ appearing in (11). Using a formula for Frullani's integrals (see (16)), we obtain

$$\begin{aligned} K &= \frac{c_1}{2}\frac{b^{c_1\lambda/2}}{\Gamma(c_1(\lambda/2)+1)}\exp\bigg(\lambda\bigg(\int_0^\infty \frac{\mathrm{e}^{by}-1}{y}\frac{c_1}{2}\mathrm{e}^{-by}\,\mathrm{d}y \\ &\qquad\qquad - \int_0^\infty \frac{1-\mathrm{e}^{-by}}{y}\left(\frac{c_1}{2}\mathrm{e}^{-by}+\frac{c_2}{2}\mathrm{e}^{-cy}\right)\mathrm{d}y\bigg)\bigg) \\ &= \frac{c_1}{2}\left(\frac{1}{2}\right)^{c_1\lambda/2}\left(\frac{c^2}{c^2-b^2}\right)^{c_2\lambda/2}\frac{b^{\lambda c_1/2}}{\Gamma(c_1(\lambda/2)+1)}, \end{aligned}$$

which is in agreement with (19).

**Example 5.** Let $B$ be a positive random variable with the distribution tail

$$\mathbb{P}\{B > x\} = \frac{1}{\lambda}\frac{\mathrm{e}^{-bx}(1-\mathrm{e}^{-\lambda x})}{1-\mathrm{e}^{-x}}, \qquad x > 0,$$

where $b, \lambda > 0$, and $2b + \lambda > 1$. The last assumption warrants that the right-hand side is a decreasing function. Writing

$$\mathbb{P}\{B > x\} = \frac{1}{\lambda}\mathrm{e}^{-bx} + \frac{1}{\lambda}\frac{\mathrm{e}^{-bx}(\mathrm{e}^{-x}-\mathrm{e}^{-\lambda x})}{1-\mathrm{e}^{-x}} =: C\mathrm{e}^{-bx} + r(x),$$

we conclude that if $\lambda > 1$ then $r^+(x) = r(x) \to (\lambda-1)/\lambda$ as $x \to 0^+$ and $r^+(x) = O(\mathrm{e}^{-(b+1)x})$ as $x \to \infty$, whereas if $\lambda \in (0, 1)$ then $r^-(x) = -r(x) \to (1-\lambda)/\lambda$ as $x \to 0^+$ and $r^-(x) = O(\mathrm{e}^{-(b+\lambda)x})$ as $x \to \infty$. Thus, in both cases conditions (9) and (10) are satisfied.

Let $Y$ be a random variable which is independent of $B$ and has a $\beta(b, \lambda)$ distribution. It can be checked that

$$\mathbb{E}\mathrm{e}^{-\mathrm{i}tY} = \frac{\Gamma(b-\mathrm{i}t)\Gamma(b+\lambda)}{\Gamma(b)\Gamma(b+\lambda-\mathrm{i}t)}, \qquad t\in\mathbb{R}.$$

On the other hand, [14, Equation 3.413(1)] yields

$$\begin{aligned} \exp\left(\lambda\int_0^t \frac{\Phi(u)-1}{u}\,\mathrm{d}u\right) &= \exp\left(\lambda\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}ut}-1}{u}\mathbb{P}\{B>u\}\,\mathrm{d}u\right) \\ &= \exp\left(\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}ut}-1}{u}\frac{\mathrm{e}^{-bu}(1-\mathrm{e}^{-\lambda u})}{1-\mathrm{e}^{-u}}\,\mathrm{d}u\right) \\ &= \frac{\Gamma(b-\mathrm{i}t)\Gamma(b+\lambda)}{\Gamma(b)\Gamma(b+\lambda-\mathrm{i}t)}, \end{aligned}$$

whence

$$\exp\left(\lambda\int_0^t \frac{\Phi(u)-1}{u}\,\mathrm{d}u\right) = \mathbb{E}\mathrm{e}^{-\mathrm{i}tY}.$$

This representation can be read off from [3, Example 9.2.3], but both the setting and the proof of [3] are slightly different from ours. Using (12), we conclude that $X$ has the same distribution as $-\log Y + B$. This representation enables us to find the asymptotics

$$\begin{aligned}\mathbb{P}\{X > x\} &= \mathbb{P}\{-\log Y > x\} + \mathbb{P}\{-\log Y + B > x, -\log Y \leq x\}\\ &= o(x\mathrm{e}^{-bx}) + \frac{1}{\lambda \mathrm{B}(b, \lambda)} \int_0^x \mathrm{e}^{-b(x-y)-by}(1-\mathrm{e}^{-y})^{\lambda-1}\,\mathrm{d}y\\ &\sim \frac{1}{\lambda \mathrm{B}(b, \lambda)} x\mathrm{e}^{-bx} \quad \text{as } x \to \infty.\end{aligned}$$

An application of Theorem 2 in combination with the already used [14, Equation 3.413(1)] yields the same asymptotics. We omit the details.

## 4. Criteria for the finiteness of the one-sided exponential moments

Throughout the rest of the paper we will often assume that the following nondegeneracy conditions hold:

$$\mathbb{P}\{A = 0\} = 0, \qquad \mathbb{P}\{B = 0\} < 1, \tag{20}$$

and

$$\mathbb{P}\{B + Ac = c\} < 1 \quad \text{for all } c \in \mathbb{R}. \tag{21}$$

Also, we will make repeated use of the following well known decomposition:

$$\begin{aligned}X &= B_1 + A_1 B_2 + \cdots + A_1 \cdots A_{\tau-1} B_\tau + A_1 \cdots A_\tau (B_{\tau+1} + A_{\tau+1} B_{\tau+2} + \cdots)\\ &=: X_\tau + \Pi_\tau X^{(\tau)},\end{aligned} \tag{22}$$

where $\tau \geq 1$ is either deterministic or a stopping time with respect to the filtration generated by $(A_k, B_k)_{k\in\mathbb{N}}$. Observe that $X^{(\tau)} = B_{\tau+1} + A_{\tau+1} B_{\tau+2} + \cdots$ has the same distribution as $X$ and is independent of $(\Pi_\tau, X_\tau)$. This particularly demonstrates that $X$ is a perpetuity generated by $(\Pi_\tau, X_\tau)$.

Some of our subsequent arguments will rely upon Proposition 2 given below, which is a criterion for the finiteness of $\mathbb{E}\mathrm{e}^{r|X|}$. Parts (i) and (ii) of Proposition 2 are Theorems 1.6 and 1.7 of [2], respectively.

**Proposition 2.** (i) *Suppose that (20) and (21) hold,* $\mathbb{P}\{|A| = 1\} = 0$*, and let* $r > 0$*. Then* $\mathbb{E}\mathrm{e}^{r|X|} < \infty$ *if and only if*

$$\mathbb{P}\{|A| < 1\} = 1 \quad \text{and} \quad \mathbb{E}\mathrm{e}^{r|B|} < \infty.$$

(ii) *Suppose that (20) and (21) hold,* $\mathbb{P}\{|A| = 1\} \in (0, 1)$*, and let* $r > 0$*. Then* $\mathbb{E}\mathrm{e}^{r|X|} < \infty$ *if and only if*

$$\mathbb{P}\{|A| \leq 1\} = 1, \qquad \mathbb{E}\mathrm{e}^{r|B|} < \infty,$$

*and*

$$\mathbb{E}\mathrm{e}^{-rB}\,\mathbf{1}_{\{A=-1\}}\,\mathbb{E}\mathrm{e}^{rB}\,\mathbf{1}_{\{A=-1\}} < (1 - \mathbb{E}\mathrm{e}^{-rB}\,\mathbf{1}_{\{A=1\}})(1 - \mathbb{E}\mathrm{e}^{rB}\,\mathbf{1}_{\{A=1\}}).$$

Next we provide necessary and sufficient conditions for the finiteness of the one-sided moments $\mathbb{E}\mathrm{e}^{rX}$ which is a somewhat more delicate problem. First, we state a criterion for positive $A$.

**Theorem 3.** *Suppose that (20) and (21) hold,* $\mathbb{P}\{A > 0\} = 1$*,* $|X| < \infty$ *a.s., and let* $r > 0$*. The conditions*

$$\mathbb{P}\{A \le 1\} = 1, \tag{23}$$

$$\mathbb{E}e^{rB} < \infty, \quad \textit{and} \quad \mathbb{E}e^{rB}\,\mathbf{1}_{\{A=1\}} < 1 \tag{24}$$

*are sufficient for*

$$\mathbb{E}e^{rX} < \infty \tag{25}$$

*to hold.*

*Conversely, if the support of the distribution of* $X$ *is unbounded from the right, then (25) entails (23) and (24), whereas if the support of the distribution of* $X$ *is bounded from the right, then* $\mathbb{E}e^{sB} < \infty$ *for all* $s > 0$*.*

**Remark 4.** As far as condition (23) is concerned, the assumption about unboundedness of the support of the distribution of $X$ is indispensable. For a trivial counterexample, just take a.s. nonpositive $B$, so that $X \in [-\infty, 0]$ a.s. Then $\mathbb{E}e^{rX} < \infty$ for each $r > 0$, irrespective of whether $\mathbb{P}\{A > 1\}$ is positive or is equal to 0. More interestingly, the support of the distribution of $X$ can be bounded from the right even if $\mathbb{P}\{B > 0, A \neq 1\} > 0$ and $\mathbb{P}\{A > 1\} > 0$. Indeed, assume that the last two inequalities hold, that $\mathbb{P}\{A > 0\} = 1$, and that

$$\Pi_\tau m + X_\tau = A_1 \cdots A_\tau m + B_1 + \cdots + B_\tau \le m \quad \text{a.s.}$$

for some real $m$, where $\tau := \inf\{k \in \mathbb{N}\colon \Pi_k \neq 1\}$ (here, we have used decomposition (22) with the particular $\tau$). Then $X \le m$ a.s. (see [5, Lemma 2.5.7 and Figure 2.4(c)]), whence $\mathbb{E}e^{rX} < \infty$ for each $r > 0$ yet $\mathbb{P}\{A > 1\} > 0$.

**Remark 5.** A perusal of the proof of Theorem 3 reveals that $\mathbb{E}e^{rX} < \infty$ in combination with $\mathbb{P}\{A \in (0, 1]\} = 1$ entails $\mathbb{E}e^{rB}\,\mathbf{1}_{\{A=1\}} < 1$, irrespective of whether the support of the distribution of $X$ is bounded or not.

**Remark 6.** Passing to the case where $A$ is negative with positive probability, we first single out a simpler situation in which $\mathbb{P}\{A = -1\} > 0$. Then $\mathbb{E}e^{rX} < \infty$ if and only if $\mathbb{E}e^{r|X|} < \infty$. Assume that $\psi(r) = \mathbb{E}e^{rX} < \infty$. Decomposition (22) with $\tau = 1$ is equivalent to

$$\psi(r) = \mathbb{E}e^{rB}\psi(rA). \tag{26}$$

Now we use (26) to obtain

$$\psi(r) = \mathbb{E}e^{rB}\psi(rA) \ge \mathbb{E}e^{rB}\,\mathbf{1}_{\{A=-1\}}\,\psi(-r),$$

which shows that $\psi(-r) < \infty$, whence $\mathbb{E}e^{r|X|} \le \psi(r) + \psi(-r) < \infty$. This proves the '$\Longrightarrow$' implication, the implication '$\Longleftarrow$' being trivial. Thus, whenever $\mathbb{P}\{A = -1\} > 0$, a criterion for the finiteness of $\mathbb{E}e^{rX}$ coincides with that for the finiteness $\mathbb{E}e^{r|X|}$. The latter can be found in Proposition 2.

When $A$ takes values of both signs with positive probability and $\mathbb{P}\{A = -1\} = 0$, we can only prove a criterion under the additional assumption that $B$ is a.s. nonnegative.

**Theorem 4.** *Suppose that (20) and (21) hold,* $\mathbb{P}\{A = -1\} = 0$*,* $|X| < \infty$ *a.s., and let* $r > 0$*. Assume that* $\mathbb{P}\{A < 0\}\mathbb{P}\{A > 0\} > 0$ *and* $\mathbb{P}\{B \ge 0\} = 1$*. Then (25) holds if and only if*

$$\mathbb{P}\{|A| \le 1\} = 1 \tag{27}$$

*and condition (24) holds.*

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*Assume that* $\mathbb{P}\{A < 0\} = 1$. *Then (25) holds if and only if condition (27) holds and*

$$\mathbb{E}e^{r(B_1+A_1B_2)} < \infty. \tag{28}$$

## 5. Proofs of Theorems 3 and 4, and Proposition 1

*Proof of Theorem 3.* The proof comprises three parts.

(i) *Proof of (23) and (24)* $\Longrightarrow$ *(25).* Assume first that $A \in (0, 1)$ a.s., that is, $\mathbb{P}\{A = 1\} = 0$, so that we have to show that $\mathbb{E}e^{rB} < \infty$ entails $\mathbb{E}e^{rX} < \infty$ or, equivalently, that

$$\mathbb{E}e^{rB^+} < \infty \quad \Longrightarrow \quad \mathbb{E}e^{rX^+} < \infty. \tag{29}$$

Since the function $x \mapsto x^+$ is subadditive on $\mathbb{R}$ and satisfies $(\alpha x)^+ = \alpha x^+$ for $\alpha > 0$ and $x \in \mathbb{R}$, we infer that

$$\exp[rX^+] = \exp\bigg(r\bigg(\sum_{k\ge1}\Pi_{k-1}B_k\bigg)^+\bigg) \le \exp\bigg(r\sum_{k\ge1}\Pi_{k-1}B_k^+\bigg) =: \exp(rX^*).$$

The random variable $X^* \ge 0$ is a perpetuity generated by $(A, B^+)$. Hence, by Proposition 2, $\mathbb{E}e^{rB^+} < \infty$ entails $\mathbb{E}e^{rX^*} < \infty$ and thereupon (29).

Assuming that $A \in (0, 1]$ a.s. and that $\mathbb{P}\{A = 1\} \in (0, 1)$, we must check that $\mathbb{E}e^{rB} < \infty$ together with $\mathbb{E}e^{rB}\,\mathbf{1}_{\{A=1\}} < 1$ guarantee $\mathbb{E}e^{rX} < \infty$. Set

$$\widehat{T}_0 := 0, \qquad \widehat{T}_k := \inf\{n > \widehat{T}_{k-1} : A_n < 1\} \quad \text{for } k \in \mathbb{N},$$

$$\widehat{A}_k := A_{\widehat{T}_{k-1}+1}\cdots A_{\widehat{T}_k}, \qquad \widehat{B}_k = B_{\widehat{T}_{k-1}+1} + A_{\widehat{T}_{k-1}+1}B_{\widehat{T}_{k-1}+2} + \cdots + A_{\widehat{T}_{k-1}+1}\cdots A_{\widehat{T}_k-1}B_{\widehat{T}_k}$$

for $k \in \mathbb{N}$. The vectors $(\widehat{A}_1, \widehat{B}_1), (\widehat{A}_2, \widehat{B}_2), \ldots$ are independent and identically distributed and $X = \widehat{B}_1 + \sum_{n\ge1}\widehat{A}_1\cdots\widehat{A}_{n-1}\widehat{B}_n$. Since

$$\begin{aligned}
\mathbb{E}e^{r\widehat{B}_1} &= \sum_{n\ge1}\mathbb{E}e^{r(B_1+A_1B_2+\cdots+A_1\cdots A_{n-1}B_n)}\,\mathbf{1}_{\{A_1=\cdots=A_{n-1}=1,\,A_n<1\}}\\
&= \mathbb{E}e^{rB}\,\mathbf{1}_{\{A<1\}}\sum_{n\ge1}(\mathbb{E}e^{rB}\,\mathbf{1}_{\{A=1\}})^{n-1}\\
&= \frac{\mathbb{E}e^{rB}\,\mathbf{1}_{\{A<1\}}}{1-\mathbb{E}e^{rB}\,\mathbf{1}_{\{A=1\}}}\\
&< \infty
\end{aligned}$$

and $\mathbb{P}\{\widehat{A}_1 = 1\} = 0$, we conclude that $\mathbb{E}e^{rX} < \infty$ by the previous part of the proof.

(ii) *Proof of (25)* $\Longrightarrow$ *(23).* Assuming that the support of the distribution of $X$ is unbounded from the right, we intend to prove that $\mathbb{P}\{A > 1\} > 0$ entails $\mathbb{E}e^{rX} = \infty$ for any $r > 0$, thereby providing a contradiction.

In view of $\mathbb{P}\{A > 1\} > 0$, there exist positive constants $\delta$ and $c$, and $\gamma \in (0, 1)$ such that

$$\mathbb{P}\{A > 1+\delta, B > -c\} = \gamma.$$

Let $(a_i)_{i\in\mathbb{N}}$ be any sequence satisfying $a_i > 1+\delta$ for all $i \in \mathbb{N}$. Pick now a large enough $m$ such that $m/(m-1) \le 1+\delta$. For the subsequent proof, we need the following inequality:

$$1 + a_1 + a_1a_2 + \cdots + a_1\cdots a_n \le m a_1\cdots a_n, \tag{30}$$

which will be proved by mathematical induction. For $n = 1$, (30) holds due to $m - 1 \geq 1/(1+\delta) \geq 1/a_1$. Assuming that (30) holds for $n = k$, we have

$$\begin{aligned} 1 + a_1 + a_1 a_2 + \cdots + a_1 \cdots a_k + a_1 \cdots a_k a_{k+1} &\leq a_1 \cdots a_k (m + a_{k+1}) \\ &\leq m a_1 \cdots a_k a_{k+1} \left( \frac{1}{a_{k+1}} + \frac{1}{m} \right) \\ &\leq m a_1 \cdots a_{k+1}, \end{aligned}$$

by our choice of $m$. Thus, (30) holds for $n = k + 1$.

Using (22) with $\tau = n$ yields $X = X_n + \Pi_n X^{(n)}$. By assumption, $X$ takes arbitrarily large values with positive probability, which implies that $\mathbb{P}\{X^{(n)} > mc + 1\} = \mathbb{P}\{X > mc + 1\} = \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. With this at hand, we have, for any $n \in \mathbb{N}$ and any $r > 0$,

$$\begin{aligned} \mathbb{E} e^{rX} &= \mathbb{E} e^{r(X_n + \Pi_n X^{(n)})} \\ &\geq \mathbb{E} e^{r(X_n + \Pi_n X^{(n)})} \mathbf{1}_{\{A_i > 1+\delta,\, B_i > -c \text{ for } i=1,\dots,n\}} \mathbf{1}_{\{X^{(n)} > mc+1\}} \\ &\geq \mathbb{E} e^{r(-c(1 + A_1 + \cdots + A_1 \cdots A_{n-1}) + \Pi_n X^{(n)})} \\ &\qquad \times \mathbf{1}_{\{A_i > 1+\delta,\, B_i > -c \text{ for } i=1,\dots,n\}} \mathbf{1}_{\{X^{(n)} > mc+1\}} \\ &\overset{(30)}{\geq} \mathbb{E} e^{r(-mc\Pi_{n-1} + \Pi_n X^{(n)})} \mathbf{1}_{\{A_i > 1+\delta,\, B_i > -c \text{ for } i=1,\dots,n\}} \mathbf{1}_{\{X^{(n)} > mc+1\}} \\ &\geq \mathbb{E} e^{r(\Pi_{n-1}(A_n X^{(n)} - mc))} \mathbf{1}_{\{A_i > 1+\delta,\, B_i > -c \text{ for } i=1,\dots,n\}} \mathbf{1}_{\{X^{(n)} > mc+1\}} \\ &\geq e^{r(1+\delta)^{n-1}} \gamma^n \varepsilon. \end{aligned}$$

Letting $n$ tend to $\infty$, we obtain $\mathbb{E} e^{rX} = \infty$.

(iii) *Proof of (25)* $\Longrightarrow$ *(24).* Assume that $\psi(r) = \mathbb{E} e^{rX} < \infty$ for some $r > 0$ and that the support of the distribution of $X$ is unbounded from the right. Then $\mathbb{P}\{A \leq 1\} = 1$ by the previous part of the proof. Set $c := \min_{0 \leq t \leq r} \psi(t)$ and note that $c > 0$. Since $\mathbb{E} e^{rB} \psi(rA) \geq c \mathbb{E} e^{rB}$, the proof is complete in the $\mathbb{P}\{A = 1\} = 0$ case in view of (26). Suppose now that $\mathbb{P}\{A = 1\} \in (0, 1)$. In order to check the second inequality in (24), we use once again (26) to infer that

$$\psi(r) = \mathbb{E} e^{rB} \psi(rA) \mathbf{1}_{\{A<1\}} + \psi(r) \mathbb{E} e^{rB} \mathbf{1}_{\{A=1\}} > \psi(r) \mathbb{E} e^{rB} \mathbf{1}_{\{A=1\}},$$

where the strict inequality follows from $\mathbb{P}\{A < 1\} > 0$. Now $\mathbb{E} e^{rB} \mathbf{1}_{\{A=1\}} < 1$ is a consequence of the last displayed equation.

It remains to show that $\mathbb{E} e^{sB} < \infty$ for all $s > 0$, provided that the support of the distribution of $X$ is bounded from the right. If $X \leq 0$ a.s. then $B \leq 0$ a.s., whence $\mathbb{E} e^{sB} \leq 1$ for all $s > 0$. Assume now that $\mathbb{P}\{X > 0\} > 0$. This implies that $\lim_{s \to \infty} \psi(s) = \infty$. The latter together with the log-convexity of $\psi$ and its finiteness for all positive arguments ensure the existence of $s_0 \geq 0$ such that $\psi(s_0) = 1$ and $\psi(t) > 1$ for any $t > s_0$ (note that $s_0 = 0$ if $\mathbb{P}\{X > 0\} = 1$, and $s_0 > 0$ if $\mathbb{P}\{X > 0\} \in (0, 1)$). Using (26), we obtain, for $t > s_0$,

$$\psi(t) = \mathbb{E} e^{tB} \psi(tA) \mathbf{1}_{\{A \leq 1\}} + \mathbb{E} e^{tB} \psi(tA) \mathbf{1}_{\{A > 1\}} \geq c_1 \mathbb{E} e^{tB} \mathbf{1}_{\{A \leq 1\}} + \mathbb{E} e^{tB} \mathbf{1}_{\{A > 1\}} \geq c_1 \mathbb{E} e^{tB},$$

where $c_1 := \min_{0 \leq s \leq t} \psi(s) \in (0, 1)$. The proof of Theorem 3 is complete. □

*Proof of Theorem 4.* The proof comprises three parts.

(i) We start by showing that (25) in combination with $\mathbb{P}\{A < 0\} > 0$ entails (27). Indeed, as a consequence of (26), we infer that

$$\psi(r) \geq \mathbb{E}\mathrm{e}^{rB}\psi(rA)\,\mathbf{1}_{\{A<0\}},$$

whence $\psi(-s) < \infty$ for some $s \in (0, r]$ and thereupon $\mathbb{E}\mathrm{e}^{s|X|} \leq \psi(s) + \psi(-s) < \infty$. Hence, (27) holds by Proposition 2.

Assume now that $\mathbb{P}\{A \in (-1, 0)\} = 1$. Then $\mathbb{P}\{A_1 A_2 \in (0, 1)\} = 1$. Using now decomposition (22) with $\tau = 2$, we conclude that $\mathbb{E}\mathrm{e}^{rX_2} = \mathbb{E}\mathrm{e}^{r(B_1+A_1B_2)} < \infty$ ensures (25) by Theorem 3. In the converse direction, assuming merely that $A$ is a.s. negative, so that $A_1A_2$ is a.s. positive, we use again (22) with $\tau = 2$ to see that (25) entails (28).

Throughout the rest of the proof we assume that $A$ takes values of both signs with positive probability and that $B$ is a.s. nonnegative.

(ii) *Proof of (24) and (27) $\Longrightarrow$ (25).* We will use representation (22) with

$$\tau := \inf\{k \in \mathbb{N} \colon \Pi_k > 0\}.$$

Observe that $\mathbb{P}\{\tau = 1\} = \mathbb{P}\{A > 0\} =: p$ and $\mathbb{P}\{\tau = k\} = p^{k-2}(1 - p)^2$ for $k \geq 2$, whence $\tau < \infty$ a.s. In view of the first condition in (24),

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{rX_\tau} &= \mathbb{E}\mathrm{e}^{r(B_1+\Pi_1B_2+\cdots+\Pi_{\tau-1}B_\tau)}\\
&= \mathbb{E}\mathrm{e}^{rB_1}\,\mathbf{1}_{\{A_1>0\}} + \sum_{n\geq 2}\mathbb{E}\mathrm{e}^{r(B_1+\cdots+\Pi_{n-1}B_n)}\,\mathbf{1}_{\{A_1<0,\,A_2>0,\ldots,A_{n-1}>0,\,A_n<0\}}\\
&\leq \mathbb{E}\mathrm{e}^{rB} + \mathbb{E}\mathrm{e}^{rB}\sum_{n\geq 2}\mathbb{P}\{A_2 > 0, \ldots, A_{n-1} > 0, A_n < 0\}\\
&\leq 2\mathbb{E}\mathrm{e}^{rB}\\
&< \infty.
\end{aligned}$$

Further, $\mathbb{E}\mathrm{e}^{rX_\tau}\,\mathbf{1}_{\{\Pi_\tau=1\}} = \mathbb{E}\mathrm{e}^{rB_1}\,\mathbf{1}_{\{A_1=1\}} < 1$ according to the second condition in (24). Since $\mathbb{P}\{\Pi_\tau \in (0, 1]\} = 1$, we conclude that (25) holds by Theorem 3, which applies since $X$ is also the perpetuity generated by $(\Pi_\tau, X_\tau)$.

(iii) *Proof of (25) $\Longrightarrow$ (24) and (27).* We will use $\tau$ as above. Recall that we have already proved that (25) ensures (27) and thereupon $\mathbb{P}\{\Pi_\tau \in (0, 1]\} = 1$. Hence, $\mathbb{E}\mathrm{e}^{rX} < \infty$ entails $\mathbb{E}\mathrm{e}^{rB_1}\,\mathbf{1}_{\{A_1=1\}} = \mathbb{E}\mathrm{e}^{rX_\tau}\,\mathbf{1}_{\{\Pi_\tau=1\}} < 1$ by Remark 5 and $\mathbb{E}\mathrm{e}^{rX_\tau} < \infty$ by Theorem 3. In particular,

$$\infty > \mathbb{E}\mathrm{e}^{rX_\tau}\,\mathbf{1}_{\{\tau=1\}} = \mathbb{E}\mathrm{e}^{rB}\,\mathbf{1}_{\{A>0\}},$$
$$\infty > \mathbb{E}\mathrm{e}^{rX_\tau}\,\mathbf{1}_{\{\tau=2\}} = \mathbb{E}\mathrm{e}^{r(B_1+A_1B_2)}\,\mathbf{1}_{\{A_1<0,\,A_2<0\}} \geq \mathbb{E}\mathrm{e}^{rB}\,\mathbf{1}_{\{A<0\}}\,\mathbb{E}\mathrm{e}^{-rB}\,\mathbf{1}_{\{A<0\}},$$

whence $\mathbb{E}\mathrm{e}^{rB} < \infty$. The proof of Theorem 4 is complete. □

*Proof of Proposition 1.* In view of our remark in the introduction, we prove only part (i).

For $k \in \mathbb{N}$, set $(A_k^*, B_k^*) := (A_k, A_kB_{k+1})$. The vectors $(A_1^*, B_1^*), (A_2^*, B_2^*), \ldots$ are independent and identically distributed, and

$$A_1B_2 + A_1A_2B_3 + \cdots = B_1^* + A_1^*B_2^* + A_1^*A_2^*B_3^* + \cdots =: X^*,$$

which shows that the left-hand side is a perpetuity generated by $(A_k^*, B_k^*)_{k\in\mathbb{N}}$. This implies that $\mathbb{E}\psi(rA) = \mathbb{E}\mathrm{e}^{r(A_1B_2+A_1A_2B_3+\cdots)} = \mathbb{E}\mathrm{e}^{rX^*}$.

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*Case (a).* By Theorem 3, $\mathbb{E}e^{rX^*} < \infty$ if and only if $\infty > \mathbb{E}e^{rB_1^*} = \mathbb{E}\varphi(rA)$ and $1 > \mathbb{E}e^{rB_1^*}\mathbf{1}_{\{A_1^*=1\}} = \mathbb{E}e^{rB}\mathbb{P}\{A=1\}$. If $\mathbb{P}\{A=1\}=0$, the last inequality holds automatically, whereas if $\mathbb{P}\{A=1\} \in (0,1)$ it entails $\varphi(r) < \infty$ and thereupon $\mathbb{E}\varphi(rA) < \infty$ since $A \in (0,1]$ a.s.

*Case (b).* By Theorem 4, $\mathbb{E}e^{rX^*} < \infty$ if and only if $\infty > \mathbb{E}e^{r(B_1^*+A_1^*B_2^*)} = \mathbb{E}e^{rA_1(B_2+A_2B_3)}$.

*Case (c).* According to Remark 6 and Proposition 2, $\mathbb{E}e^{rX^*} < \infty$ if and only if $\infty > \mathbb{E}e^{r|B_1^*|} = \mathbb{E}e^{r|A_1B_2|}$ and

$$\mathbb{E}e^{-rB_1^*}\mathbf{1}_{\{A_1^*=-1\}}\,\mathbb{E}e^{rB_1^*}\mathbf{1}_{\{A_1^*=-1\}} < (1-\mathbb{E}e^{-rB_1^*}\mathbf{1}_{\{A_1^*=1\}})(1-\mathbb{E}e^{rB_1^*}\mathbf{1}_{\{A_1^*=1\}}).$$

The latter is equivalent to

$$\mathbb{E}e^{-rB}\mathbb{E}e^{rB}[\mathbb{P}\{A=-1\}]^2 < (1-\mathbb{E}e^{-rB}\mathbb{P}\{A=1\})(1-\mathbb{E}e^{rB}\mathbb{P}\{A=1\}), \tag{31}$$

which entails

$$\mathbb{E}e^{r|A_1B_2|} \le \mathbb{E}e^{r|B|} \le \mathbb{E}e^{-rB} + \mathbb{E}e^{rB} < \infty.$$

Thus, $\mathbb{E}e^{rX^*} < \infty$ if and only if (31) holds. □

## 6. Proof of Theorem 1

Our proof of Theorem 1 is based on two auxiliary results.

**Lemma 1.** *Suppose that (2) with $c < -1$, (3) and (4) hold, and $\mathbb{P}\{A \in (0,1]\} = 1$. Let $Y$ be a random variable independent of $(A, B)$, which satisfies*

$$\mathbb{P}\{Y > x\} \sim c_Y\mathbb{P}\{B > x\}, \qquad x \to \infty \tag{32}$$

*for some constant $c_Y > 0$. Then $\mathbb{E}f(Y) < \infty$ and*

$$\mathbb{P}\{AY + B > x\} \sim (c_Y\mathbb{E}e^{bB}\mathbf{1}_{\{A=1\}} + \mathbb{E}f(Y))\mathbb{P}\{B > x\}, \qquad x \to \infty.$$

*Proof.* Fix $\delta \in (0,1)$. In view of

$$\mathbb{P}\{B > (1-\delta)x, Y > \delta x\} \sim a^2c_Ye^{-bx}x^{2c}(\delta(1-\delta))^c = o(e^{-bx}x^c), \qquad x \to \infty$$

and

$$\{AY + B > x, B \le (1-\delta)x\} \subseteq \{AY > \delta x\} \subseteq \{Y > \delta x\} \quad \text{a.s.},$$

we have

$$\begin{aligned}\mathbb{P}\{AY+B>x\} &= \mathbb{P}\{AY+B>x, Y \le \delta x\} + \mathbb{P}\{AY+B>x, B \le (1-\delta)x\} + o(e^{-bx}x^c)\\ &=: I_1(x) + I_2(x) + o(e^{-bx}x^c), \qquad x \to \infty.\end{aligned}$$

We claim that

$$\frac{I_1(x)}{\mathbb{P}\{B > x\}} = \int_{(-\infty,\,\delta x]} \frac{\mathbb{P}\{B > x - Ay\}}{\mathbb{P}\{B > x\}}\mathbb{P}\{Y \in dy\} \to \mathbb{E}f(Y) < \infty, \qquad x \to \infty.$$

Indeed, this is a consequence of (4) and the Lebesgue dominated convergence theorem in combination with the following two facts.

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First,

$$\frac{\mathbb{P}\{B > x - Ay\}}{\mathbb{P}\{B > x\}} \le \frac{\mathbb{P}\{B > x - y\}}{\mathbb{P}\{B > x\}} \le M\mathrm{e}^{by}\left(\frac{x-y}{x}\right)^c \le M\mathrm{e}^{by}(1-\delta)^c$$

for large enough $x$, $y \in [0, \delta x]$ and an appropriate $M > 0$;

$$\frac{\mathbb{P}\{B > x - Ay\}}{\mathbb{P}\{B > x\}} \le 1$$

for all $x > 0$ and all $y < 0$.

Second, $\mathbb{E}\mathrm{e}^{bY} < \infty$, which is an easy consequence of (2) and (32).

Passing to the analysis of $I_2(x)$, we observe that

$$\lim_{x\to\infty} \frac{\mathbb{P}\{uY > x - v\}}{\mathbb{P}\{B > x\}} = c_Y\mathrm{e}^{bv}\,\mathbf{1}_{\{u=1\}}$$

for $u \in (0, 1]$ and $v \in \mathbb{R}$. Further,

$$\frac{\mathbb{P}\{uY > x - v\}}{\mathbb{P}\{B > x\}} \le \frac{\mathbb{P}\{Y > x - v\}}{\mathbb{P}\{B > x\}} \le M\mathrm{e}^{bv}\left(\frac{x-v}{x}\right)^c \le M\mathrm{e}^{bv}\delta^c$$

for large enough $x$, all $u \in [0, 1]$, $v \in [0, (1-\delta)x]$, and some appropriate $M > 0$. Also

$$\frac{\mathbb{P}\{uY > x - v\}}{\mathbb{P}\{B > x\}} \le M_1$$

for large enough $x$, all $u \in [0, 1]$, $v < 0$, and appropriate $M_1 > 0$.

Recalling $\mathbb{E}\mathrm{e}^{bB} < \infty$, we infer that

$$\frac{I_2(x)}{\mathbb{P}\{B > x\}} = \int_{[0,1]\times(-\infty,(1-\delta)x]} \frac{\mathbb{P}\{uY > x - v\}}{\mathbb{P}\{B > x\}}\mathbb{P}\{A \in \mathrm{d}u, B \in \mathrm{d}v\} \to c_Y\mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{A=1\}}$$

as $x \to \infty$ by the dominated convergence theorem.

Combining our results completes the proof of the lemma. □

Apart from Lemma 1 we will use a technique of stochastic bounds which is a quite commonly used method nowadays. In the area of perpetuities, this approach, to the best of the authors' knowledge, originates from [15]. For random variables $U$ and $V$, we will write $U \le_{\mathrm{st}} V$ to indicate that $V$ stochastically dominates $U$, that is, $\mathbb{P}\{U > x\} \le \mathbb{P}\{V > x\}$ for all $x \in \mathbb{R}$.

**Lemma 2.** *Suppose that (2) with $c < -1$, (3) and (4) hold, and $\mathbb{P}\{A \in (0, 1]\} = 1$. On a possibly enlarged probability space, there exists a nonnegative random variable $Z$ independent of $(A, B)$ such that*

$$\mathbb{P}\{Z > x\} \sim c_Z\mathbb{P}\{B > x\}, \qquad x \to \infty \tag{33}$$

*for a positive constant $c_Z$ and $AZ + B \le_{\mathrm{st}} Z$.*

*Proof.* Pick large enough $q > 0$ satisfying

$$\mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{A=1\}} + \mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{B>q\}} < 1$$

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and then large enough $d > 0$ satisfying

$$e^{bd} > \frac{\mathbb{P}\{B \le q\}}{1 - \mathbb{E}e^{bB}\,\mathbf{1}_{\{A=1\}} - \mathbb{E}e^{bB}\,\mathbf{1}_{\{B>q\}}}.$$

Let $B'$ be a copy of $B$ independent of $(A, B)$. Setting $Y := (B' + d)\,\mathbf{1}_{\{B'>q\}}$, we infer that

$$\mathbb{P}\{Y > x\} \sim e^{bd}\mathbb{P}\{B > x\}, \qquad x \to \infty.$$

Using Lemma 1 with $c_Y = e^{bd}$ yields

$$\mathbb{P}\{AY + B > x\} \sim (e^{bd}\mathbb{E}e^{bB}\,\mathbf{1}_{\{A=1\}} + \mathbb{E}f(Y))\mathbb{P}\{B > x\}, \qquad x \to \infty. \tag{34}$$

Since, for each $y \ge 0$,

$$\frac{\mathbb{P}\{B > x - Ay\}}{\mathbb{P}\{B > x\}} \le \frac{\mathbb{P}\{B > x - y\}}{\mathbb{P}\{B > x\}} \to e^{by}, \qquad x \to \infty,$$

in view of (2), we conclude that

$$f(y) \le e^{by}, \qquad y \ge 0. \tag{35}$$

This implies that

$$\mathbb{E}f(Y) \le \mathbb{E}e^{bY} = e^{bd}\mathbb{E}e^{bB}\,\mathbf{1}_{\{B>q\}} + \mathbb{P}\{B \le q\},$$

whence

$$e^{bd}\mathbb{E}e^{bB}\,\mathbf{1}_{\{A=1\}} + \mathbb{E}f(Y) \le e^{bd}\mathbb{E}e^{bB}\,\mathbf{1}_{\{A=1\}} + e^{bd}\mathbb{E}e^{bB}\,\mathbf{1}_{\{B>q\}} + \mathbb{P}\{B \le q\} < e^{bd}, \tag{36}$$

by the choice of $d$ and $q$. Now (34) and (36) together imply that there exists $x_0 > 0$ such that $\mathbb{P}\{AY + B > x\} \le \mathbb{P}\{Y > x\}$ whenever $x \ge x_0$.

Let $Z$ be a random variable independent of $(A, B)$ with the distribution

$$\mathbb{P}\{Z > x\} = \mathbb{P}\{Y > x \mid Y \ge x_0\}.$$

For $x \ge x_0$, we have

$$\mathbb{P}\{AZ+B > x\} = \mathbb{P}\{AY+B > x \mid Y \ge x_0\} \le \frac{\mathbb{P}\{AY + B > x\}}{\mathbb{P}\{Y \ge x_0\}} \le \frac{\mathbb{P}\{Y > x\}}{\mathbb{P}\{Y \ge x_0\}} = \mathbb{P}\{Z > x\}.$$

For $x < x_0$, $\mathbb{P}\{Z > x\} = 1$, so that $\mathbb{P}\{AZ + B > x\} \le \mathbb{P}\{Z > x\}$ holds for all $x \in \mathbb{R}$. The proof of Lemma 2 is complete. □

*Proof of Theorem 1.* Let $X_0$ be a nonnegative random variable independent of $(A_n, B_n)_{n\in\mathbb{N}}$. The sequence $(X_n)_{n\in\mathbb{N}_0}$, recursively defined by the random difference equation

$$X_n = A_n X_{n-1} + B_n, \qquad n \in \mathbb{N},$$

forms a Markov chain. Occasionally, we write $X_n(X_0)$ for $X_n$ to bring out the dependence on $X_0$.

While condition (2) entails $\mathbb{E}\log(1 + B^+) < \infty$, which in combination with (5) ensures that $\mathbb{E}\log(1 + |B|) < \infty$ (see the paragraph following (13)), condition $\mathbb{P}\{A \in (0, 1]\} = 1$ together with (3) guarantee that $\mathbb{E}\log A \in [-\infty, 0)$. Further, condition (21) obviously holds.

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Invoking now [13, Theorem 3.1(c)], we conclude that $X_n$ converges in distribution to the a.s. finite $X = \sum_{k\geq 1} \Pi_{k-1} B_k$ as $n \to \infty$ whatever the distribution of $X_0$. Our plan is to approach the distribution of $X$ from above and from below by the distributions of $X_n(X_0^{(i)})$, $n \in \mathbb{N}_0$, $i = 1, 2$. By picking appropriate distributions of $X_0^{(i)}$, we are be able to provide tight bounds on the distribution tail of $X$.

*Upper bound.* Set $X_0 = Z$ for a random variable $Z$ as defined in Lemma 2, which is also independent of $(A_n, B_n)_{n\in\mathbb{N}}$. Then

$$X_1 = A_1 X_0 + B_1 \leq_{\text{st}} X_0$$

and thereupon

$$X_{n+1} = A_{n+1} X_n + B_{n+1} \leq_{\text{st}} A_n X_{n-1} + B_n = X_n, \qquad n \in \mathbb{N},$$

since $A_k > 0$ a.s. for $k \in \mathbb{N}$.

Define a sequence $(c_{X_n})_{n\in\mathbb{N}_0}$ recursively by

$$c_{X_0} = c_Z, \qquad c_{X_{n+1}} = c_{X_n} \mathbb{E} \mathrm{e}^{bB}\, \mathbf{1}_{\{A=1\}} + \mathbb{E} f(X_n), \quad n \in \mathbb{N}_0.$$

Note that

$$\mathbb{E} f(X) \leq \mathbb{E} f(X_n) \leq \mathbb{E} f(Z) \leq \mathbb{E} \mathrm{e}^{bZ} < \infty,$$

where the first two inequalities hold since $f$ is nondecreasing and $(X_n)_{n\in\mathbb{N}_0}$ is a stochastically nonincreasing sequence, the third inequality is a consequence of (35), and the fourth inequality follows from (2) and (33). Starting with

$$\mathbb{P}\{X_0 > x\} \sim c_{X_0} \mathbb{P}\{B > x\}, \qquad x \to \infty,$$

we use mathematical induction to obtain

$$\mathbb{P}\{X_n > x\} = \mathbb{P}\{A_n X_{n-1} + B_n > x\} \sim c_{X_n} \mathbb{P}\{B > x\}, \qquad x \to \infty$$

with the help of Lemma 1. The latter limit relation together with the stochastic monotonicity imply that $(c_{X_n})_{n\in\mathbb{N}_0}$ is a nonincreasing sequence of positive numbers which must have a limit $c_X$, say, given by

$$c_X = \frac{\mathbb{E} f(X)}{1 - \mathbb{E} \mathrm{e}^{bB}\, \mathbf{1}_{\{A=1\}}}.$$

The form of the limit is justified by the fact that the distributional convergence of $X_n$ to $X$, together with continuity of the distribution of $X$ (see [19, Theorem 2.1.2] or [2, Theorem 1.3]), ensures that $f(X_n)$ converges in distribution to $f(X)$ as $n \to \infty$, whence $\lim_{n\to\infty} \mathbb{E} f(X_n) = \mathbb{E} f(X)$ by the Lévy monotone convergence theorem.

Since $X \leq_{\text{st}} X_n$ for each $n \in \mathbb{N}_0$, we infer that

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{X > x\}}{\mathbb{P}\{B > x\}} \leq \limsup_{x\to\infty} \frac{\mathbb{P}\{X_n > x\}}{\mathbb{P}\{B > x\}} = c_{X_n} \quad \text{for each } n \in \mathbb{N}_0$$

and thereupon

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{X > x\}}{\mathbb{P}\{B > x\}} \leq c_X. \tag{37}$$

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*Lower bound*. We start by noting that

$$\mathbb{P}\{X > x\} = \mathbb{P}\{AX+B > x\} \ge \mathbb{P}\{AX+B > x,\ X > 0\} \ge \mathbb{P}\{X > 0\}\mathbb{P}\{B > x\}, \qquad x \in \mathbb{R}.$$

Therefore, denoting by $X_0$ a random variable which is independent of $(A_n, B_n)_{n\in\mathbb{N}_0}$ and has distribution $\mathbb{P}\{X_0 > x\} = \mathbb{P}\{X > 0\}\mathbb{P}\{B > x\}$ for $x \ge 0$ and $\mathbb{P}\{X_0 > x\} = \mathbb{P}\{X > x\}$ for $x < 0$, and arguing in the same way as in the previous part of the proof, we obtain a sequence $(X_n)_{n\in\mathbb{N}_0}$ approaching $X$ in distribution such that $X_n \le_{\mathrm{st}} X$ for $n \in \mathbb{N}_0$. It is worth stating explicitly that $(X_n)_{n\in\mathbb{N}_0}$ is not necessarily stochastically monotone.

Define a sequence $(c'_{X_n})_{n\in\mathbb{N}_0}$ recursively by

$$c'_{X_0} = \mathbb{P}\{X > 0\}, \qquad c'_{X_{n+1}} = c'_{X_n}\mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{A=1\}} + \mathbb{E}f(X_n), \quad n \in \mathbb{N}_0.$$

We claim that

$$\lim_{n\to\infty} \mathbb{E}f(X_n) = \mathbb{E}f(X) < \infty, \tag{38}$$

where the finiteness follows from the previous part of the proof. Mimicking the argument given in the treatment of the upper bound, we conclude that $f(X_n)$ converges in distribution to $f(X)$ as $n \to \infty$. Therefore, $\liminf_{n\to\infty} \mathbb{E}f(X_n) \ge \mathbb{E}f(X)$ by Fatou's lemma. On the other hand, we have $\mathbb{E}f(X_n) \le \mathbb{E}f(X)$ for $n \in \mathbb{N}_0$, and (38) follows.

Now (38), together with

$$c'_{X_n} = (\mathbb{P}\{X > 0\}\mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{A=1\}})^n + \sum_{k=0}^{n-1}(\mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{A=1\}})^{n-k-1}\mathbb{E}f(X_k) \quad \text{for } n \in \mathbb{N},$$

ensures that $c'_X := \lim_{n\to\infty} c'_{X_n}$ exists and

$$c'_X = \frac{\mathbb{E}f(X)}{1 - \mathbb{E}\mathrm{e}^{bB}\,\mathbf{1}_{\{A=1\}}} = c_X.$$

The same argument as in the previous part of the proof enables us to conclude that

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{X > x\}}{\mathbb{P}\{B > x\}} \ge c'_{X_n} \quad \text{for each } n \in \mathbb{N}_0,$$

whence

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{X > x\}}{\mathbb{P}\{B > x\}} \ge c_X. \tag{39}$$

A combination of (37) and (39) yields (6). The proof of Theorem 1 is complete. □

As was announced in Remark 3, we now discuss the similarities between the preceding proof and the proof of Theorem 3 of [30]. First, our Lemma 1 resembles Lemma 2 of [30]. Second, the random variables $Z$ and $Y_1^{\downarrow}$ appearing in our Lemma 2 and the proof of Theorem 3 of [30], respectively, serve analogous purposes.

## 7. Proof of Theorem 2

Recall that $\Psi(t) = \mathbb{E}\mathrm{e}^{\mathrm{i}tX}$, $t \in \mathbb{R}$, satisfies (12). Using

$$\int_0^\infty \mathrm{e}^{\mathrm{i}ty}\mathbb{P}\{B > y\}\,\mathrm{d}y - \int_{-\infty}^0 \mathrm{e}^{\mathrm{i}ty}\mathbb{P}\{B \le y\}\,\mathrm{d}y$$
$$= \mathbb{E}\biggl(\biggl(\int_0^B \mathrm{e}^{\mathrm{i}ty}\,\mathrm{d}y\biggr)\mathbf{1}_{\{B>0\}}\biggr) - \mathbb{E}\biggl(\biggl(\int_B^0 \mathrm{e}^{\mathrm{i}ty}\,\mathrm{d}y\biggr)\mathbf{1}_{\{B\le 0\}}\biggr)$$

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$$= \mathbb{E}\bigg(\frac{\mathrm{e}^{\mathrm{i}tB}-1}{\mathrm{i}t}\mathbf{1}_{\{B>0\}}\bigg) - \mathbb{E}\bigg(\frac{1-\mathrm{e}^{\mathrm{i}tB}}{\mathrm{i}t}\mathbf{1}_{\{B\le 0\}}\bigg)$$
$$= \frac{\Phi(t)-1}{\mathrm{i}t},$$

we obtain an equivalent form of (12):

$$\Psi(t) = \Phi(t)\exp\bigg(\mathrm{i}\lambda\int_0^t\bigg(\int_0^\infty \mathrm{e}^{\mathrm{i}uy}\mathbb{P}\{B>y\}\,\mathrm{d}y - \int_{-\infty}^0 \mathrm{e}^{\mathrm{i}uy}\mathbb{P}\{B\le y\}\,\mathrm{d}y\bigg)\,\mathrm{d}u\bigg)$$
$$= \Phi(t)\exp\bigg(\lambda\bigg(\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}ty}-1}{y}\mathbb{P}\{B>y\}\,\mathrm{d}y - \int_{-\infty}^0 \frac{\mathrm{e}^{\mathrm{i}ty}-1}{y}\mathbb{P}\{B\le y\}\,\mathrm{d}y\bigg)\bigg) \quad \text{for } t\in\mathbb{R}.$$

In view of (8), this can be further represented as

$$\Psi(t)\exp\bigg(\lambda\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}ty}-1}{y}r^-(y)\,\mathrm{d}y\bigg) = \Phi(t)\bigg(\frac{b}{b-\mathrm{i}t}\bigg)^{C\lambda}\exp\bigg(\lambda\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}ty}-1}{y}r^+(y)\,\mathrm{d}y\bigg)$$
$$\times\exp\bigg(\lambda\int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}ty}-1}{y}\mathbb{P}\{-B\ge y\}\,\mathrm{d}y\bigg) \tag{40}$$

for $t\in\mathbb{R}$. Let $Z_1$, $Z_2$, and $Z_3$ be infinitely divisible nonnegative random variables with zero drift and Lévy measures $\nu_1(\mathrm{d}y) = y^{-1}r^-(y)\,\mathbf{1}_{(0,\infty)}(y)\,\mathrm{d}y$, $\nu_2(\mathrm{d}y) = y^{-1}r^+(y)\,\mathbf{1}_{(0,\infty)}(y)\,\mathrm{d}y$, and $\nu_3(\mathrm{d}y) = y^{-1}\mathbb{P}\{-B\ge y\}\,\mathbf{1}_{(0,\infty)}(y)\,\mathrm{d}y$, respectively. Let $V$ be a random variable with a $\gamma(C\lambda, b)$ distribution. Assume that $Z_1$ is independent of $X$ and that $B$, $V$, $Z_2$, and $Z_3$ are mutually independent. Equation (40) tells us that $X+Z_1$ has the same distribution as $B+V+Z_2-Z_3$. We claim that

$$\mathbb{P}\{X+Z_1>x\} = \mathbb{P}\{B+V+Z_2-Z_3>x\} \sim C\mathbb{E}\mathrm{e}^{b(Z_2-Z_3)}\frac{(bx)^{C\lambda}}{\Gamma(C\lambda+1)}\mathrm{e}^{-bx} \tag{41}$$

as $x\to\infty$, where $\mathbb{E}\mathrm{e}^{b(Z_1-Z_2)}\le\mathbb{E}\mathrm{e}^{bZ_1}<\infty$ by virtue of the first condition in (10).

We now prove (41). By (8) and (9), $\mathbb{P}\{B>x\}\sim C\mathrm{e}^{-bx}$ as $x\to\infty$. Hence,

$$\mathbb{P}\{B+V>x\}\sim\frac{C(bx)^{C\lambda}}{\Gamma(C\lambda+1)}\mathrm{e}^{-bx}, \qquad x\to\infty,$$

by [31, Lemma 7.1(iii)] (in the notation of [31], we set $Y_1 := bB$ and $Y_2 := bZ$). According to an extension of Breiman's lemma (see [9, Proposition 2.1]), (41) follows, provided that the following conditions hold:

(a) $\mathbb{E}\mathrm{e}^{b(Z_1-Z_2)}<\infty$;

(b) $x^b\mathbb{P}\{\mathrm{e}^{Z_1-Z_2}>x\} = o(h(x))$ as $x\to\infty$, where $h(x) := x^b\mathbb{P}\{\mathrm{e}^{B+V}>x\}$ for $x\ge 0$;

(c) $\limsup_{x\to\infty}\sup_{1\le y\le x} h(y)/h(x)<\infty$.

We already know that (a) holds which, in particular, implies that

$$\lim_{x\to\infty} x^b\mathbb{P}\{\mathrm{e}^{Z_1-Z_2}>x\} = 0.$$

While this, in combination with $h(x)\sim(Cb^{C\lambda}/\Gamma(C\lambda+1))(\log x)^{C\lambda}$, proves (b), the last asymptotic relation alone secures (c). The proof of (41) is complete.

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With (41) at hand, we infer that

$$\mathbb{P}\{X > x\} \sim \frac{\mathbb{E}\mathrm{e}^{b(Y_2-Y_3)}}{\mathbb{E}\mathrm{e}^{bY_1}} \frac{C(bx)^{\lambda C}}{\Gamma(\lambda C+1)}\mathrm{e}^{-bx} = Kx^{\lambda C}\mathrm{e}^{-bx}, \qquad x \to \infty,$$

by [22, Corollary 4.3(ii)], which is applicable since the distribution of $Y_1$ is infinitely divisible and $\mathbb{E}\mathrm{e}^{(b+\varepsilon)Y_1} < \infty$ by the second part of (10). The proof of Theorem 2 is complete.

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