J. Appl. Prob. **55,** 823–833 (2018)
*doi:*10.1017/jpr.2018.52

ON THE MINIMAL NUMBER OF DRIVING LÉVY MOTIONS IN A MULTIVARIATE PRICE MODEL

JEAN JACOD,* *Université Pierre et Marie Curie*
MARK PODOLSKIJ,** *Aarhus University*

Abstract

In this paper we consider the factor analysis for Lévy-driven multivariate price models with stochastic volatility. Our main aim is to provide conditions on the volatility process under which we can possibly reduce the dimension of the driving Lévy motion. We find that these conditions depend on a particular form of the multivariate Lévy process. In some settings we concentrate on nondegenerate symmetric α-stable Lévy motions.

Keywords: Factor model; Lévy process; semimartingale

2010 Mathematics Subject Classification: Primary 60H05; 60G51
Secondary 60G18

1. Introduction

Multivariate Itô semimartingales are nowadays the commonly used models for multidimensional price processes in economics and finance. Much effort has been devoted to understanding the dependence structure between the price components. The most prominent approaches include the factor models, which distinguish between the common and endogenous shocks, in addition to principal component analysis, and the theory of cointegration. The common feature of these methods is the dimension reduction, which is extremely useful for practical calibration and simulation of the model.

The aim of this paper is to provide a simple factor analysis for multivariate Lévy-driven price models that exhibit stochastic volatility. More specifically, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and a d-dimensional process $X = (X^j)_{1\leq j\leq d}$, which takes the form

$$X_t = \int_0^t \sigma_s \,\mathrm{d}Z_s, \tag{1}$$

where Z is a symmetric Lévy motion with dimension q and σ is an $\mathbb{R}^{d\times q}$-valued predictable process (we often write $X = \sigma \cdot Z$ for notational simplicity). We aim determine the minimal dimension of the driving Lévy motion such that (1) holds. Mathematically speaking, the central question of this paper is the following:

> what is the smallest integer r for which there exist two predictable processes γ and σ' with values in $\mathbb{R}^{r\times q}$ and $\mathbb{R}^{d\times r}$ respectively, satisfying
>
> $$Z' = \gamma \cdot Z \text{ is a Lévy process}, \qquad X = \sigma' \cdot Z'. \tag{2}$$

Received 16 November 2017; revision received 18 July 2018.
* Postal address: Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4 Place Jussieu, 75 005 Paris, France. Email address: jean.jacod@gmail.com
** Postal address: Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus, Denmark. Email address: mpodolskij@math.au.dk

In relation to this problem, we would also like to know whether the Lévy process Z' is of the same 'type' as Z (for example, does Z' have independent components if Z does?). There are several examples in finance where the minimal dimension r can be expected to be smaller than d. Jacod *et al.* [5] mentioned the case of baskets of energy prices, where the required amount of driving Brownian motions is much smaller than d; see [1, Chapter 5] for the detailed empirical analysis. Similar conclusions can be found in the London interbank offered rate (LIBOR) market models, where the principal component analysis suggests a sparse amount of Brownian motions that are accurately describing the dynamics; see [2, Section 6.19] for a more detailed discussion. Another example are bond prices with different maturities. In the situation of economic stability, these prices have very similar trajectories and a rather minimal amount of driving Lévy motions might suffice to model the bonds. However, things can change drastically during an economic crisis.

When Z and Z' are Brownian motions, the formulated problem is indeed elementary. The answer is formulated in the theorem below. This statement can be found in [5] or, alternatively, it follows directly from the results of this paper, which will be presented later.

Theorem 1. *Let $\overline{\mathbb{P}}$ denote the product measure $\mathbb{P}(\mathrm{d}\omega) \otimes \mathrm{d}t$ on $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$ and assume that Z is a standard d-dimensional Brownian motion. The minimal dimension r for which (2) holds (and Z' is a r-dimensional Brownian motion) is the smallest integer such that the rank of the $\mathbb{R}^{d\times q}$-valued matrix $\sigma(\omega)_t$ is not larger than r, outside a $\overline{\mathbb{P}}$-null set of $\Omega \times \mathbb{R}_+$.*

This characterization was used in [3]–[5] to construct statistical tests for the minimal dimension of the Brownian motion in the setting of continuous diffusion models possibly contaminated by microstructure noise. As we will see later, the result of Theorem 1 extends to all *isotropic* Lévy processes Z (meaning that ΠZ has the same law as Z for any orthogonal $q \times q$ matrix Π). Otherwise, except under some very special conditions, typically on both σ and Z, the minimal r for which (2) holds is $r = q$, and no dimension reduction is available. This is probably not surprising when Z is an arbitrary Lévy process without any special structure. A more surprising analysis appears in what we call the *independent and identically distributed (i.i.d.) case*, where all components Z^i have the same law and are independent. We will study this setting in more detail and characterize the minimal r in (2) when the process Z^1 is symmetric α-stable for any d, and when Z^1 is an arbitrary symmetric Lévy process and $d = 1$. Even in these cases, the characterization is indeed somewhat involved.

The paper is structured as follows. In Section 2 we present some basics from the theory of Itô semimartingales and Lévy processes. Section 3 is devoted to the study of the isotropic setting, where the obtained result is similar to Theorem 1. We investigate the one-dimensional case, which turns out to be not quite trivial, in Section 4. Finally, in Section 5 we provide a complete answer to problem (2) in the framework of the i.i.d. stable motion Z.

2. Preliminaries

In this section we introduce some notation and recall some basic facts from the theory of semimartingales and Lévy processes.

We denote by $\mathcal{M}_{r,q}$ the vector space of $r \times q$ matrices. For any $A \in \mathcal{M}_{r,q}$, we write $A^\top \in \mathcal{M}_{q,r}$ for the transpose of A. The space of all orthogonal $\mathbb{R}^{q\times q}$-valued matrices is denoted by $\mathcal{P}_q$. For any $q \geq 1$, we write S_{q-1} for the unit sphere in $\mathbb{R}^q$.

First, we recall that a q-dimensional Lévy process Z is uniquely characterized by the *Lévy triplet* (b, c, F), where b is an $\mathbb{R}^q$-valued drift, c is a positive semidefinite volatility matrix in $\mathcal{M}_{q,q}$, and F is a measure on $\mathbb{R}^q$ satisfying $\int_{\mathbb{R}^q} \min(1, \|x\|^2) F(\mathrm{d}x) < \infty$. In particular, the characteristic function of Z_t is $\mathbb{E}[\exp(\mathrm{i}u^\top Z_t)] = \exp(t\phi^Z(u))$, where the characteristic exponent ϕ^Z has the representation

$$\phi^Z(u) = \mathrm{i}u^\top b - \frac{1}{2}u^\top cu + \int_{\mathbb{R}^q} (\exp(\mathrm{i}u^\top x) - 1 - \mathrm{i}u^\top x\mathbf{1}_{\{\|x\|\le 1\}})F(\mathrm{d}x).$$

We refer the reader to [6] for a detailed exposition of Lévy processes. A particular class of Lévy processes are the nondegenerate q-dimensional symmetric α-stable processes ($\alpha \in (0, 2)$) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, which we denote by SSS_q^α. Recall that a q-dimensional Lévy process is in SSS_q^α if and only if its characteristic triple is $(0, 0, F)$, with the Lévy measure factorizing as

$$F(\mathrm{d}x) = \frac{1}{C_\alpha \rho^{1+\alpha}}\,\mathrm{d}\rho H(\mathrm{d}\theta) \tag{3}$$

with $C_\alpha = \int_0^\infty (1 - \cos x)x^{1+\alpha}\,\mathrm{d}x$, where $x = (\rho, \theta) \in \mathbb{R}_+ \times S_{q-1}$ and H is a finite positive measure on the unit sphere S_{q-1} of $\mathbb{R}^q$ (the *directional measure*), which is symmetric (i.e. invariant by the map $\theta \mapsto -\theta$) and satisfies $H(\{\theta : \lambda^\top\theta \ne 0\}) > 0$ for all $\lambda \in S_{q-1}$ (the latter reflects the nondegeneracy of the process). In this setting, the characteristic exponent of Z takes the following simple form:

$$\phi^Z(u) = -\int_{S_{q-1}} |u^\top\theta|^\alpha H(\mathrm{d}\theta). \tag{4}$$

We note that the process $Z \in \mathrm{SSS}_q^\alpha$ is isotropic if and only if H is proportional to the Lebesgue measure on S_{q-1}, or, equivalently, if and only if $\phi^Z(u) = -a\|u\|^\alpha$ for some constant $a > 0$. On the other hand, $Z \in \mathrm{SSS}_q^\alpha$ has i.i.d. components if and only if $H = \frac{1}{2}a\sum_{i=1}^q(\varepsilon_{e_i} + \varepsilon_{-e_i})$, where $(e_i)_{1\le i\le q}$ is the canonical basis of $\mathbb{R}^q$, ε_x denotes the Dirac measure at x, and $a > 0$ or, equivalently, if and only if $\phi^Z(u) = a\sum_{i=1}^q |u^i|^\alpha$. We remark that these two types of SSS_q^α-processes are different when $\alpha \in (0, 2)$, but they coincide for $\alpha = 2$, which corresponds to the case of Brownian motion.

Before we start with the preliminary analysis, we make the following simple observation. The ‘true’ dimension of the driving Lévy motion Z in (1) may very well be smaller than q. Indeed, there is a minimal linear subspace V of $\mathbb{R}^q$ in which Z really lives, which is the linear subspace spanned by the support of the law of Z_t (it is the same for all $t > 0$ due to the properties of a Lévy process). Let q' be the dimension of V. Then, if $q' < q$, the true dimension of Z is q' in the following sense: there is a $\Pi \in \mathcal{P}_q$ such that Πx belongs to the subspace spanned by the first q' elements of the canonical basis of $\mathbb{R}^q$, for all $x \in H$. Then $\overline{Z} = \Pi Z$ is a Lévy process with components $\overline{Z}^i$ for $i > q'$ being identically vanishing, and if $Z' = (\overline{Z}^1, \dots, \overline{Z}^{q'})^\top$, (1) implies $X = \sigma' \cdot Z'$ for the process σ'_t, which is the $d \times q'$ left block of the matrix $\sigma_t\Pi^\top$. It is, of course, natural to do this trivial dimension reduction before starting to solve the original problem. Hence, it is no restriction to assume that Z is *nondegenerate* in the sense that the linear subspace generated by the support of Z_1 is $\mathbb{R}^q$.

Next, we recall some well-known facts about Itô semimartingales. A d-dimensional semimartingale Y on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ is called an Itô semimartingale if its characteristics are absolutely continuous (in time) with respect to Lebesgue measure, and their derivatives with respect to time are called the spot characteristics. More specifically, they consist of a spot triple (b_t, c_t, F_t), where b_t is the drift (a d-dimensional predictable process), c_t is the diffusion coefficient (predictable with values in the set of symmetric nonnegative definite $d \times d$ matrices),

https://doi.org/10.1017/jpr.2018.52 Published online by Cambridge University Press

and F_t is the spot Lévy measure on $\mathbb{R}^d$ (predictable with $\int_{\mathbb{R}^d} \min(1, \|x\|^2) F_t(\mathrm{d}x) < \infty$). The 'spot characteristic exponent' at time t is the function $\phi_t = \phi_{\omega,t}$ on $\mathbb{R}^d$ defined by

$$\phi_t^Y(u) = \mathrm{i}u^\top b_t - \frac{1}{2} u^\top c_t u + \int_{\mathbb{R}^p} (\exp(\mathrm{i}u^\top x) - 1 - \mathrm{i}u^\top x \mathbf{1}_{\{\|x\| \le 1\}}) F_t(\mathrm{d}x),$$

and it characterizes (b_t, c_t, F_t) for any (ω, t). In fact, $\phi_t^Y(u)$ is uniquely determined up to a $\overline{\mathbb{P}}$-null set. If G is an $\mathcal{M}_{d',d}$-valued predictable process such that $Y' = G \cdot Y$ exists, then Y' is also an Itô semimartingale and

$$\phi_t^{Y'}(u) = \phi_t^Y(G_t^\top u) \quad \text{for all } u \in \mathbb{R}^{d'}. \tag{5}$$

Moreover, Y is Lévy if and only if there exists a version of ϕ_t^Y (equivalently, of (b_t, c_t, F_t)) that is independent of (ω, t). Then $\phi^Y(u)$ is the characteristic exponent and (b, c, F) is the characteristic triplet. The Lévy process Y is nondegenerate if and only if $\phi^Y(u) \neq 0$ for all nonvanishing $u \in \mathbb{R}^d$, and symmetric if and only if $\phi^Y(u) = \phi^Y(-u)$.

We are now ready to state a necessary and sufficient condition for having (2) (recall that an integral process $\psi \cdot Z$ is unchanged if we modify ψ on a $\overline{\mathbb{P}}$-null set).

Lemma 1. *Consider model (1) driven by a nondegenerate Lévy process Z. Let γ and σ' be predictable processes with values in $\mathcal{M}_{r,q}$ and $\mathcal{M}_{d,r}$, respectively. Then (2) holds if and only if, outside an $\overline{\mathbb{P}}$-null set, we have*

$$\sigma(\omega)_t = \sigma'(\omega)_t \gamma(\omega)_t, \qquad \phi^Z(\gamma(\omega)_t^* v) \text{ is independent of } (\omega, t) \text{ for all } v \in \mathbb{R}^r. \tag{6}$$

Proof. The second condition of (6) implies that the stochastic integral process $Z' = \gamma \cdot Z$ is well defined and it is a Lévy process with characteristic exponent $\phi^{Z'}(u) = \phi^Z(\Gamma^\top u)$, where $\Gamma = \gamma(\omega)_t$ for any (ω, t) outside the above-mentioned $\overline{\mathbb{P}}$-null set. Then the first condition of (6) implies $\sigma' \cdot Z' = (\sigma'\gamma) \cdot Z = X$ and (2) holds.

Conversely, assume that (2) holds. Clearly, the second part of (6) holds. As for the first part, observe that a combination of (1) and (2) yields $G \cdot Z = 0$, where $G_t = \sigma_t - \sigma_t' \gamma_t$. Then we must have $\phi_t^{G \cdot Z}(u) = 0$; hence, $\phi^Z(G_t^\top u) = 0$ outside a $\overline{\mathbb{P}}$-null set and for all $u \in \mathbb{R}^d$. Since Z is nondegenerate, this implies $G_t^\top u = 0$ for all u. Hence, $G_t = 0$. □

When the first part of (6) holds, the rank of $\sigma(\omega)_t$ is not more than r. Then a simple consequence of the previous lemma leads to the following result.

Corollary 1. *Assume that (1) holds with Z being a nondegenerate Lévy process. If (2) holds for some integer r then outside a $\overline{\mathbb{P}}$-null set, the rank of $\sigma(\omega)_t$ is not greater than r.*

On the other hand, if the rank of $\sigma(\omega)_t$ is not greater than r, one can find $\sigma'(\omega)_t \in \mathcal{M}_{d,r}$ and $\gamma(\omega)_t \in \mathcal{M}_{r,q}$ such that the first part of (6) holds (see, for example, the proof of Theorem 2 below), and when $\mathrm{rank}(\sigma(\omega)_t) \le r$ for all (ω, t), it is no problem to find predictable versions for σ_t' and γ_t. However, there is no reason why the second part of (6) should hold, except, of course, when $\sigma(\omega)_t$; hence, $\sigma'(\omega)_t \in \mathcal{M}_{d,r}$ and $\gamma(\omega)_t \in \mathcal{M}_{r,q}$ as well, are independent of (ω, t). So we have a corollary.

Corollary 2. *Assume that (1) holds with Z being a nondegenerate Lévy process, and $\sigma(\omega)_t = \sigma$ a constant matrix. The minimal integer r for which (2) holds is the rank m of σ.*

Unfortunately, for the reason mentioned above, this result (with m the $\overline{\mathbb{P}}$-essential supremum of $\mathrm{rank}(\sigma(\omega)_t)$) fails in general, except in the isotropic case.

https://doi.org/10.1017/jpr.2018.52 Published online by Cambridge University Press

3. The isotropic case

In this section we treat the isotropic case. As mentioned earlier, when Z is a nondegenerate isotropic Lévy process, the result of Theorem 1 remains valid.

Theorem 2. *We consider model (1) with Z being a nondegenerate isotropic Lévy process. The minimal dimension r for which (2) holds is the smallest integer such that the rank of the matrix $\sigma(\omega)_t$ is not greater than r outside an $\overline{\mathbb{P}}$-null set. Furthermore, we can choose Z' in (2) to be isotropic as well.*

Proof. Let $m(\omega)_t = \operatorname{rank}(\sigma(\omega)_t)$. Up to a modification on a $\overline{\mathbb{P}}$-null set, we can assume that there exists a $m \in \mathbb{N}$ such that

$$m_t \leq m \quad \text{and} \quad m_t = m \text{ on a set } A \text{ with } \overline{\mathbb{P}}(A) > 0.$$

Omitting the dependence on ω, we can decompose σ_t as $\sigma_t = \Pi_t \Lambda_t \Pi'_t$, where Π_t, Π'_t, and Λ_t are predictable processes with respective values in $\mathcal{P}_d$, $\mathcal{P}_q$, and $\mathcal{M}_{d,q}$, with all entries of Λ_t equal to 0 except possibly for $\lambda_t^i = \Lambda_t^{ii}$ when $i \leq m$. We also consider the $\mathcal{M}_{d,m}$-valued process Λ'_t consisting in the upper left $d \times m$ block of Λ_t.

The isotropy of Z implies the identity $\phi^Z(\Pi'' u) = \phi^Z(u)$ for any $u \in \mathbb{R}^q$ and $\Pi'' \in \mathcal{P}_q$. Then $\phi^Z(\Pi'_t u) = \phi^Z(u)$, which shows, by (5), that the process $\overline{Z} = \Pi' \cdot Z$ is well defined and has the same law as Z itself. Thus, the m-dimensional process Z' whose components are the first m components of $\overline{Z}$ is obviously an isotropic Lévy process. Moreover, we have

$$X = \sigma \cdot Z = (\Pi \Lambda \Pi') \cdot Z = (\Pi \Lambda) \cdot \overline{Z} = (\Pi \Lambda') \cdot Z',$$

where the last equality follows from $\Lambda_t^{ij} = 0$ when $j > m$. Hence, (2) holds with $r = m$ and $\sigma'_t = \Pi_t \Lambda'_t$.

Conversely, assume that (2) holds for some r. Then (6) implies the identity $\sigma_t = \sigma'_t \gamma_t$. This yields $m_t \leq r$ and, hence, $m \leq r$ as well. □

The characterization of Theorem 2 was the basis for statistical tests for the minimal number of Brownian motions in continuous diffusion models that were developed in [3] and [4]. In these papers the authors employed the following *matrix perturbation* method to uncover the rank r of a positive semi-definite matrix $A \in \mathcal{M}_{d,d}$. For an arbitrary positive definite matrix $B \in \mathcal{M}_{d,d}$, the multi-linearity of the determinant implies the asymptotic expansion

$$\det(A + \lambda B) = \lambda^{d-r} \gamma_r(A, B) + O(\lambda^{d-r+1}) \quad \text{as } \lambda \downarrow 0,$$

where the constant $\gamma_r(A, B)$ is given by $\gamma_r(A, B) = \sum_{C \in \mathbb{M}^r_{A,B}} \det(C)$ and the set $\mathbb{M}^r_{A,B}$ is defined via

$$\mathbb{M}^r_{A,B} := \{C \in \mathcal{M}_{d,d} \colon C_i = A_i \text{ or } C_i = B_i \text{ with } \#\{i \colon C_i = A_i\} = r\}.$$

This expansion is the key to identification of the unknown rank r of the matrix A. Indeed, when $\gamma_r(A, B) \neq 0$, we deduce the convergence

$$\frac{\det(A + 2\lambda B)}{\det(A + \lambda B)} \to 2^{d-r} \quad \text{as } \lambda \downarrow 0.$$

In [4], the latter idea was applied to random perturbation of high-frequency observations of the process X driven by a d-dimensional Brownian motion Z to perform hypothesis testing for the maximal rank of the squared volatility process $c_t = \sigma_t \sigma_t^\top$; hence, deriving statistical methods to answer the question formulated in (2). Later on, this approach was extended to continuous diffusion models observed with microstructure noise in [3]. However, due to substantial differences in the asymptotic theory, the extension of this idea to general isotropic Lévy processes seems to be out of reach.

4. The one-dimensional case

While in the isotropic case the question in problem (2) is fully answered in Theorem 2, things are less obvious if we drop this isotropy condition. In this section we explicitly concentrate on the $d = 1$ case, but keep $q \geq 2$. Hence, the process $\sigma^\top$ at (1) becomes $\mathbb{R}^q$-valued. Our first result arises from treating the stable case.

Theorem 3. *We consider model (1) with $d = 1$ and $Z \in \mathrm{SSS}_q^\alpha$. Then (2) holds with $r = 1$ and we can choose Z' in SSS_1^α.*

Proof. Let $e_1 = (1, 0, \dots, 0)^\top$ be the first element of the canonical basis of $\mathbb{R}^q$. Recalling (4), we see that both $a = \phi^Z(e_1^\top)$ and $\phi^Z(\sigma_t^\top)$ are negative. We define the predictable processes

$$\sigma'_t = (-\phi^Z(\sigma_t^\top))^{1/\alpha}, \qquad \gamma_t = \left(\frac{1}{\sigma'_t} \mathbf{1}_{\{\sigma'_t > 0\}}\right)\sigma_t + \left(\frac{1}{(-a)^{1/\alpha}} \mathbf{1}_{\{\sigma'_t = 0\}}\right) e_1^\top.$$

Note that $\sigma'_t \geq 0$ from (4), and $\sigma'_t = 0$ if and only if $\sigma_t = 0$. We obviously have $\sigma_t = \sigma'_t \gamma_t$, and for $v \in \mathbb{R}$, we deduce

$$\sigma_t \neq 0 \implies \phi^Z(\gamma_t^\top v) = \phi^Z\left(\frac{v}{\sigma'_t} \sigma_t^\top\right) = -|v|^\alpha,$$

$$\sigma_t = 0 \implies \phi^Z(\gamma_t^\top v) = \phi^Z\left(\frac{v}{a^{1/\alpha}} e_1\right) = -|v|^\alpha.$$

Hence, (3) is satisfied, and using Lemma 1, we obtain (2) with $Z' = \gamma \cdot Z$, which satisfies $\phi^{Z'}(v) = -|v|^\alpha$. Thus, $Z' \in \mathrm{SSS}_1^\alpha$ and the proof is complete. □

When Z is a Lévy process the same result does not hold in general, but we have a criterion under which it holds when, further, Z is a nondegenerate symmetric Lévy process without a Gaussian part. We need to introduce some additional notation: for any measure μ on $\mathbb{R}^p$ and any $y \in \mathbb{R}^p$, we denote by $\mu^{(y)}$ the measure on $\mathbb{R}$, which is the image of μ by the map $x \mapsto y^\top x$.

Theorem 4. *We consider model (1) with $d = 1$ and Z being a q-dimensional nondegenerate symmetric Lévy process without a Gaussian part. Then (2) holds with $r = 1$ if and only if there exist a measure μ on $\mathbb{R}$ with $\mu(\{0\}) = 0$ and a nonnegative predictable process a_t such that, outside of a $\overline{\mathbb{P}}$-null set, we have*

$$F^{(\sigma_t^\top)} = \mu^{(a_t)}, \tag{7}$$

where F is the Lévy measure of Z.

Proof. The nondegeneracy of Z and $F(\{0\}) = 0$ imply that $F^{(y)} = 0$ holds if and only if $y = 0$. We have two cases: either $\sigma_t = 0$ $\overline{\mathbb{P}}$-almost surely and (2) holds with $\sigma' \equiv 0$ and, say $Z' = Z^1$ and (7) holds with $\mu = 0$; or $\overline{\mathbb{P}}(\{\sigma \neq 0\}) > 0$ and if (7) holds the measure μ is nonvanishing. In the sequel we discard the first trivial case.

https://doi.org/10.1017/jpr.2018.52 Published online by Cambridge University Press

First, we prove that (7) implies (2) with $r = 1$. Upon a modification of the process σ on an $\overline{\mathbb{P}}$-null set, we may assume that (7) holds for all (ω, t). We pick a vector $\delta \in \mathbb{R}^q$ such that $F^{(\delta)} = \mu$ (such a vector exists by (7)) and set

$$\gamma_t = \frac{1}{a_t}\sigma_t \mathbf{1}_{\{a_t>0\}} + \delta^\top \mathbf{1}_{\{a_t=0\}}, \qquad \sigma'_t = a_t.$$

Then, for $v \in \mathbb{R}$, we obtain the identities

$$\begin{aligned}
\phi^Z(v\gamma_t^\top) &= \int_{\mathbb{R}^q} (\exp(\mathrm{i}v\gamma_t x) - 1)F(\mathrm{d}x) \\
&= \mathbf{1}_{\{a_t>0\}} \int_{\mathbb{R}^q} \bigg(\exp\bigg(\frac{\mathrm{i}v\sigma_t x}{a_t}\bigg) - 1\bigg)F(\mathrm{d}x) + \mathbf{1}_{\{a_t=0\}} \int_{\mathbb{R}^q} (\exp(\mathrm{i}v\delta x) - 1)F(\mathrm{d}x) \\
&= \mathbf{1}_{\{a_t>0\}} \int_{\mathbb{R}} \bigg(\exp\bigg(\frac{\mathrm{i}vz}{a_t}\bigg) - 1\bigg)F^{(\sigma_t^\top)}(\mathrm{d}z) + \mathbf{1}_{\{a_t=0\}} \int_{\mathbb{R}} (\exp(\mathrm{i}vz) - 1)F^{(\delta)}(\mathrm{d}z) \\
&= \mathbf{1}_{\{a_t>0\}} \int_{\mathbb{R}} (\exp(\mathrm{i}vz) - 1)\mu(\mathrm{d}z) + \mathbf{1}_{\{a_t=0\}} \int_{\mathbb{R}} (\exp(\mathrm{i}vz) - 1)\mu(\mathrm{d}z) \\
&= \int_{\mathbb{R}} (\exp(\mathrm{i}vz) - 1)\mu(\mathrm{d}z).
\end{aligned}$$

This implies the second part of (6). Moreover, $\mu^{(0)} = 0$ since $\mu(\{0\}) = 0$. So, if $a_t = 0$ we have $F^{(\sigma_t^\top)} = 0$, which as mentioned earlier implies that $\sigma_t = 0$. Therefore, the first part of (6) is obvious, and (2) holds with $r = 1$.

Conversely, assume that (2) holds with $r = 1$. Hence, we can assume that (6) holds identically for some predictable processes γ, σ', and $Z' = \gamma \cdot Z$. Let μ be the Lévy measure of Z'. Then $X = \sigma \cdot Z = \sigma' \cdot Z'$ implies that the spot Lévy measure of X is $F_t^X = F^{(\sigma_t^\top)}$ and also $F_t^X = \mu^{(\sigma'_t)}$. Hence, (7) holds with this μ and $a_t = \sigma'_t$. □

When Z is isotropic, so is F, and it is simple to check (7) for any process σ_t: one may take $\mu = F^{(e_1)}$ and $a_t = \|\sigma_t\|$. When $Z \in \mathrm{SSS}_q^\alpha$, (7) again holds for any process σ_t: we take for μ the Lévy measure of a one-dimensional symmetric α-stable process, and $F^{(y)}$ is necessarily of the same type for any $y \in \mathbb{R}^q \setminus \{0\}$.

One may then wonder whether, for a given Z as in the previous theorem, (7) holds for all $\mathcal{M}_{1,q}$-valued process σ (with a_t depending on σ_t, of course). We do not know a criterion for this question except in the i.i.d. case.

Theorem 5. *Let Z be an i.i.d. symmetric nondegenerate Lévy process without a Gaussian part, and $q \geq 2$. Then (1) with $d = 1$ implies that (2) holds with $r = 1$ for any predictable $\mathcal{M}_{1,q}$-valued process σ_t if and only if $Z \in \mathrm{SSS}_q^\alpha$.*

Proof. The sufficient condition follows from Theorem 3. For the necessary condition, we denote by ν the Lévy measure of all components of Z and (e_j) the canonical basis of $\mathbb{R}^q$. Then $F(A) = \sum_{j=1}^d \int_{\mathbb{R}} \mathbf{1}_A(ze_j)\nu(\mathrm{d}z)$ for $A \subset \mathbb{R}^q$ and, hence,

$$F^{(\sigma_t^\top)} = \sum_{j=1}^q \nu^{(\sigma_t^j)}.$$

If we take the process with components $\sigma_t^1 = 1$ and $\sigma_t^2 = \mathbf{1}_{\{t>1\}}$, and $\sigma_t^j = 0$ for $3 \leq j \leq q$, Theorem 4 yields a measure μ on $\mathbb{R}$ and a nonnegative process a_t such that

$$t \leq 1 \quad \Longrightarrow \quad \nu = \mu^{(a_t)}, \qquad t > 1 \quad \Longrightarrow \quad 2\nu = \mu^{(a_t)}.$$

https://doi.org/10.1017/jpr.2018.52 Published online by Cambridge University Press

Then we can take $\mu = \nu$ and $a_t = 1$ for $t \leq 1$, and the second part above yields $2\nu = \nu^{(b)}$ for some $b > 0$ and all $s \geq 0$. If $\overline{\nu}(x) = \nu((x, \infty))$ is the tail of the symmetric measure ν, we thus have $2\overline{\nu}(x) = \overline{\nu}(x/b)$ for all $x > 0$, which implies that $\overline{\nu}$ has the form $\overline{\nu}(x) = Cx^{-\alpha}$ for some constants C, α, with necessarily $\alpha \in (0, 2)$ since $\overline{\nu}$ is the tail of a Lévy measure. This implies that indeed the components Z^j are symmetric α-stable. □

5. The stable i.i.d. case

In this last section we dealt with $d \geq 2$. In view of the previous results for the general Lévy motions, finding the minimal r in (2) seems out of reach. Hence, we assume here that $Z \in \mathrm{SSS}_q^\alpha$. As a matter of fact, in the general nonisotropic stable setting, the problem also seems difficult to analyze, except in the i.i.d. case. So, we restrict our attention to this framework.

The interpretation of the minimal r for which (2) holds as the number of factors is clearer if we require Z' to be i.i.d. as well, and we begin with this case. We need some additional notation. Omitting ω, we introduce for $j = 0, \dots, q$ the predictable integer-valued processes M_t^j by induction on j ($\sigma_t^{\cdot,j}$ below denotes the jth column vector of the matrix σ_t), i.e.

$$M_t^0 = 0, \qquad M_t^{j+1} = \begin{cases} M_t^j & \text{if } \sigma_t^{\cdot,j+1} \text{ vanishes,} \\ & \text{or is proportional to } \sigma_t^{\cdot,l} \text{ for some } 1 \leq l \leq j, \\ M_t^j + 1 & \text{otherwise.} \end{cases} \tag{8}$$

Then $m_t = M_t^q$ is the number of column vectors of the matrix σ_t, which are pairwise noncollinear (with the convention $m_t = 0$ when $\sigma_t = 0$). If r_t is the rank of σ_t, we have $m_t = r_t$ if $r_t = 0, 1$, and if $r_t \geq 2$ then m_t can attain all values between r_t and q.

Theorem 6. *We consider model (1) with Z being an i.i.d. symmetric α-stable motion. The minimal dimension r for which (2) holds with Z' being an i.i.d. symmetric α-stable motion is the smallest integer m such that $m(\omega)_t \leq m$ outside an $\overline{\mathbb{P}}$-null set.*

Proof. As in the proof of Theorem 2, we may assume that $m_t \leq m$ identically, and $m(\omega)_t = m$ on a subset A with $\overline{\mathbb{P}}(A) > 0$. We also assume that $m < q$; otherwise there is nothing to prove. We deal with the proof in three parts.

Part 1. In order to simplify the analysis, we perform a type of permutation of the coordinates of Z, according to a random and time-dependent scheme. Let $I = \{1, \dots, q\}$. With the notation of (8), we denote by $J_t^1, \dots, J_t^{m_t}$ the ordered indices j in I such that $M_t^j = M_t^{j-1} + 1$, and by $J_t^{m_t+1}, \dots, J_t^q$ the ordered indices j in I such that $M_t^j = M_t^{j-1}$, so $j \mapsto J_t^j$ is a bijection from I onto itself, and its inverse is denoted by $j \mapsto N_t^j$ (so $J_t^{N_t^j} = j$). Note that J_t^j and N_t^j are predictable, as well as the invertible matrix $\zeta_t^{ij} = \mathbf{1}_{\{j=J_t^i\}} = \mathbf{1}_{\{i=N_t^j\}}$. So the q-dimensional process $\overline{Z} = \zeta \cdot Z$ is well defined and, since for $u = (u_i)_{i \in I} \in \mathbb{R}^q$ the components of $\zeta_t^\top u$ are $u_{N_t^i}$, its spot characteristic exponent is

$$\phi_t^{\overline{Z}}(u) = \phi^Z(\zeta_t^\top u) = -a \sum_{i=1}^q |(\zeta_t^\top u)_i|^\alpha = -a \sum_{i=1}^q |u_{N_t^i}|^\alpha = -a \sum_{i=1}^q |u_i|^\alpha \quad \text{for some } a > 0.$$

Therefore, $\overline{Z}$ is an i.i.d. symmetric α-stable Lévy motion. Moreover, for any predictable $\mathcal{M}_{r,q}$-valued process ψ_t, we have $\psi \cdot Z = (\psi_t \zeta_t^{-1}) \cdot \overline{Z}$. So, upon replacing σ_t by $\overline{\sigma}_t = \sigma_t \zeta^{-1}$, which amounts to doing so for each t, a permutation of the columns (hence, letting the number m_t be unchanged), it is enough to prove the result for $X = \overline{\sigma} \cdot \overline{Z}$.

Equivalently, this amounts to solving the original problem under the additional assumption that $J_t^j = j$ identically for all $j \in I$. In other words, the columns $\sigma_t^{\cdot,j}$ for $j = 1, \dots, m_t$ are pairwise noncollinear, and each other column vector is a multiple of one of the first m_t columns vectors. Hence, since $m_t \le m$, for $j = m+1, \dots, q$ there are predictable processes β_t^j and L_t^j with values in $\mathbb{R}$ and $\{1, \dots, m\}$ such that

$$j > m \quad \Longrightarrow \quad \sigma_t^{\cdot,j} = \beta_t^j \sigma_t^{\cdot,L_t^j}.$$

Part 2. We define the predictable processes ρ_t^i and γ_t^{ij} for $1 \le i \le m$ and $j \in I$ by

$$\rho_t^i = 1 + \sum_{l=m+1}^{q} |\beta_t^l|^\alpha \mathbf{1}_{\{L_t^l = i\}}, \qquad \gamma_t^{ij} = \frac{1}{(\rho_t^i)^{1/\alpha}} \big(\mathbf{1}_{\{i=j\le m\}} + \beta_t^j \mathbf{1}_{\{L_t^j = i,\, j>m\}}\big).$$

Then, for any $u = (u_i) \in \mathbb{R}^m$, we see that

$$\begin{aligned}
\sum_{j=1}^{q} |(\gamma_t{}^\top u)_j|^\alpha &= \sum_{j=1}^{q} \bigg| \sum_{i=1}^{m} \frac{u_i}{(\rho_t^i)^{1/\alpha}} \big(\mathbf{1}_{\{i=j\le m\}} + \beta_t^j \mathbf{1}_{\{L_t^j = i,\, j>m\}}\big) \bigg|^\alpha \\
&= \sum_{j=1}^{m} \frac{|u_j|^\alpha}{\rho_t^j} + \sum_{j=m+1}^{q} |\beta_t^j|^\alpha \frac{|u_{L_t^j}|^\alpha}{\rho_t^{L_t^j}} \\
&= \sum_{i=1}^{m} \frac{|u_i|^\alpha}{\rho_t^i} \bigg(1 + \sum_{j=m+1}^{q} |\beta_t^j|^\alpha \mathbf{1}_{\{L_t^j = i\}}\bigg) \\
&= \sum_{i=1}^{m} |u_i|^\alpha,
\end{aligned}$$

from which we deduce that $\phi^Z(\gamma_t{}^\top u) = -a \sum_{i=1}^m |u_i|^\alpha$. Thus, γ is integrable with respect to Z, and $Z' = \gamma \cdot Z$ is an i.i.d. symmetric α-stable Lévy motion. A simple computation shows that $\sigma_t = \sigma_t' \gamma_t$ if

$$\sigma_t'^{ij} = (\rho_t^i)^{1/\alpha} \sigma_t^{ij} \quad \text{for } 1 \le i \le d,\ 1 \le j \le m.$$

We thus have (2) with $r = m$.

Part 3. Conversely, suppose that (2) holds for some r, and Z' is an i.i.d. symmetric α-stable Lévy motion. We will prove that necessarily $m \le r$.

The Lévy measure of each component of Z (respectively, Z') is $\nu(\mathrm{d}x) = (1/2C_\alpha |x|^{1+\alpha})\,\mathrm{d}x$, and those components have no common jumps. Hence, since $X = \sigma \cdot Z = \sigma' \cdot Z'$, the spot Lévy measure F_t^X of X is, outside a $\overline{\mathbb{P}}$-null set,

$$F_t^X(B) = \sum_{j=1}^{q} \int_{\mathbb{R}} \mathbf{1}_B(x\sigma_t^{\cdot,j}) \frac{1}{C_\alpha |x|^{1+\lambda}}\,\mathrm{d}x = \sum_{j=1}^{r} \int_{\mathbb{R}} \mathbf{1}_B(x\sigma_t'^{\cdot,j}) \frac{1}{C_\alpha |x|^{1+\lambda}}\,\mathrm{d}x. \tag{9}$$

If, for $y \in \mathbb{R}^d$, we denote by $D(y)$ the one-dimensional linear subspace of $\mathbb{R}^d$ spanned by y (with the convention $D(0) = \{0\}$), (9) implies that the support of F_t^X is $D_t = \bigcup_{1\le j\le q} D(\sigma_t^{\cdot,j})$ and also $D_t' = \bigcup_{1\le j\le r} D(\sigma_t'^{\cdot,j})$. Having $D_t = D_t'$ yields that all $\sigma^{\cdot,j}$ for $j = 1, \dots, q$ are proportional to at most r vectors of $\mathbb{R}^d$; hence, necessarily $m_t \le r$ on the $\overline{\mathbb{P}}$-full set on which (9) holds, yielding the claim. □

https://doi.org/10.1017/jpr.2018.52 Published online by Cambridge University Press

If we now relax the additional condition in (2) that the process Z' is symmetric i.i.d. α-stable, we do not have a general characterization of the minimal r for which this holds. However, this minimal r can still be equal to q when $d \geq 2$ (since when $d = 1$ this number is always $r = 1$).

Theorem 7. *Let Z be q-dimensional symmetric i.i.d. α-stable, and $d \geq 2$. One can find a d-dimensional processes $X = \sigma \cdot Z$ for which (2) with $Z' \in \mathrm{SSS}^\alpha_r$ holds for $r \geq q$ only.*

Proof. By looking at the first two components of X, it is enough to prove it when $d = 2$ and, of course, when $q \geq 2$.

We will indeed exhibit an $\mathcal{M}_{2,q}$-valued process (σ_t) which does not satisfy (2) for $r = q-1$, and with a very simple structure. We choose q pairwise noncollinear vectors w_j in $\mathbb{R}^2 \setminus \{0\}$ and use the notation $D(w_j)$ of the previous proof, and let $D = \bigcup_{j=1}^{q-1} D(w_j)$ and $D' = D \cup D(w_q)$. The process σ_t is defined column-wise by

$$\sigma_t^{\cdot,j} = \begin{cases} w_j & \text{if } 1 \leq j \leq q-1, \\ w_q \mathbf{1}_{\{t>1\}} & \text{if } j = q, \end{cases}$$

and we also set $D_t = D$ if $t \leq 1$ and $D_t = D'$ if $t > 1$.

The rank of σ_t is 2 for all t and the process $X = \sigma \cdot Z$ is, of course, well defined. Suppose now that we can find Z' in $\mathrm{SSS}^\alpha_{q-1}$ and a predictable $\mathcal{M}_{2,q-1}$-valued process σ'_t such that $X = \sigma' \cdot Z'$. If F' is the Lévy measure of Z', a version of the spot Lévy measure of X is (instead of (9))

$$F_t^X(B) = \int_{\mathbb{R}^{q-1}} \mathbf{1}_B(\sigma'_t x) F'(\mathrm{d}x) = \begin{cases} \displaystyle\sum_{j=1}^{q-1} \int_{\mathbb{R}} \mathbf{1}_B(x w_j) \frac{1}{C_\alpha |x|^{1+\lambda}} \,\mathrm{d}x & \text{if } t \leq 1, \\ \displaystyle\sum_{j=1}^{q} \int_{\mathbb{R}} \mathbf{1}_B(x w_j) \frac{1}{C_\alpha |x|^{1+\lambda}} \,\mathrm{d}x & \text{if } t > 1. \end{cases}$$

Therefore, on the one hand, the support of F_t^X is the image S_t of F' by the map $x \mapsto f_t(x) = \sigma'_t x$ from $\mathbb{R}^{q-1}$ into $\mathbb{R}^2$ and, on the other hand, it is D_t.

Suppose first that $q = 2$. Then $\sigma'_t \in \mathbb{R}^2$ and S_t is contained in $D(\sigma'_t)$, which, when $t > 1$, contradicts the fact that $S_t = D'$ contains two noncollinear vectors: so Z' as above cannot exist.

Suppose now that $q \geq 3$. The linear space spanned by $S_t = D_t$ is of dimension 2 for all t, so the matrix σ'_t has rank 2 and, thus, f_t is a bijection from a two-dimensional subspace E_t of $\mathbb{R}^{q-1}$ into $\mathbb{R}^2$, and the lines $\overline{D}_t^j = f_t^{-1}(D(w_j))$ in E_t are pairwise distinct since the lines $D(w_j)$ are such in $\mathbb{R}^2$, whereas $\overline{D}_t = f_t^{-1}(D_t)$ is $\bigcup_{j=1}^{q-1} \overline{D}_t^j$ if $t \leq 1$ and $\bigcup_{j=1}^{q} \overline{D}_t^j$ if $t > 1$. Moreover, $S_t = D_t$ implies that $F'(\mathbb{R}^{q-1} \setminus \overline{D}_t) = 0$ and F' puts a positive mass on all lines $\overline{D}_t^j$ in $\overline{D}_t$. Since there are $q-1$ such distinct lines when $t \geq 1$ and q of them when $t > 1$, this is clearly impossible. So again Z' as above cannot exist. □

References

[1] Bardou, O. (2005). Contrôle dynamique des erreurs de simulation et d'estimation de processus de diffusion. Doctoral Thesis, Université Nice Sophia Antipolis.

[2] Brigo, D. and Mercurio, F. (2006). *Interest Rate Models–Theory and Practice*, 2nd edn. Springer, Berlin.

[3] Fissler, T. and Podolskij, M. (2017). Testing the maximal rank of the volatility process for continuous diffusions observed with noise. *Bernoulli* **23,** 3021–3066.
[4] Jacod, J. and Podolskij, M. (2013). A test for the rank of the volatility process: the random perturbation approach. *Ann. Statist.* **41,** 2391–2427.
[5] Jacod, J., Lejay, A. and Talay, D. (2008). Estimation of the Brownian dimension of a continuous Itô process. *Bernoulli* **14,** 469–498.
[6] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.