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Bilinear forms on potential spaces in the unit circle

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In this paper we characterize the boundedness on the product of Sobolev spaces $H^s(\mathbb{T}) \times H^s(\mathbb{T})$ on the unit circle $\mathbb{T}$, of the bilinear form Λ_b with symbol $b \in H^s(\mathbb{T})$ given by

$$\Lambda_b(\varphi, \psi) := \int_{\mathbb{T}} ((-\Delta)^s + I)\,(\varphi\psi)(\eta) b(\eta)\, \mathrm{d}\sigma(\eta).$$

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1. Introduction

In [20], V.G. Maz'ya and I.E. Verbitsky characterize the class of measurable functions V such that the Schrödinger operator $-\Delta + V$ maps the homogeneous Sobolev space $\dot{W}^{1,2}(\mathbb{R}^n)$ to its dual, obtaining necessary and sufficient conditions for the classical inequality

$$\left| \int_{\mathbb{R}^n} (\varphi(x))^2 V(x)\, \mathrm{d}x \right| \leqslant C \int_{\mathbb{R}^n} |\nabla \varphi(x)|^2\, \mathrm{d}x, \quad u \in \mathcal{D}(\mathbb{R}^n),$$

to hold. They also obtained analogous characterizations for the non-homogeneous Sobolev space $W^{1,2}(\mathbb{R}^n)$. In this paper we will consider a similar problem on the unit circle $\mathbb{T}$ for the space $W^{s,2}(\mathbb{T})$, $0 < s < 1/2$.

The space $W^{s,2}(\mathbb{T})$ $s > 0$, is the space of functions $\varphi \in L^2(\mathbb{T})$ such that if $(\widehat{\varphi}(k))_{k\in\mathbb{Z}}$ is the sequence of its Fourier coefficients, then

$$\|\varphi\|_{W^{s,2}(\mathbb{T})} := \left(\sum_{k\in\mathbb{Z}} (1 + |k|^s)^2 |\widehat{\varphi}(k)|^2 \right)^{1/2} < \infty.$$

When $0 < s < 1$ and for functions in $\mathcal{C}^\infty(\mathbb{T})$, this norm is equivalent to $\|\varphi\|_{L^2(\mathbb{T})} + \|(-\Delta)^s\varphi\|_{L^2(\mathbb{T})}$, where $(-\Delta)^s$ is the fractional Laplacian defined, up to a constant, by

$$(-\Delta)^s(\varphi)(\zeta) = P.V. \int_{\mathbb{T}} \frac{\varphi(\zeta) - \varphi(\eta)}{|\zeta - \eta|^{1+2s}} \mathrm{d}\sigma(\eta)$$

and the space $W^{s,2}(\mathbb{T})$ coincides with the completion of $\mathcal{C}^\infty(\mathbb{T})$ with respect to this norm. In turn, this space coincides with the space of Riesz potentials, that we will denote by $H^s(\mathbb{T}) := I_s(L^2(\mathbb{T}))$, where I_s is the Riesz kernel defined by $I_s(\zeta,\eta) = ((\Gamma((1+s)/2)^2)/\Gamma(s))(1/(|1-\zeta\overline{\eta}|^{1-s})$ and $\|\varphi\|_{H^s(\mathbb{T})} = \|\psi\|_{L^2(\mathbb{T})}$, if $\varphi = I_s(\psi)$.

We are interested in the case where $0 < s < 1/2$. When $1/2 < s < 1$, $H^s(\mathbb{T})$ is an algebra and the problem that we will consider becomes trivial.

Let Λ_b be the bilinear form with symbol $b \in H^s(\mathbb{T})$ given by

$$\Lambda_b(\varphi,\psi) := \int_{\mathbb{T}} \left((-\Delta)^s + I\right)(\varphi\psi)(\eta) b(\eta)\, \mathrm{d}\sigma(\eta).$$

The main object of this paper is the characterization of the symbols $b \in H^s(\mathbb{T})$ for which the bilinear form Λ_b is bounded on $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, that is,

$$|\Lambda_b(\varphi,\psi)| \lesssim \|\varphi\|_{H^s(\mathbb{T})}\|\psi\|_{H^s(\mathbb{T})}. \tag{1.1}$$

This problem is equivalent (see proposition 4.6), to the characterization of the functions $c \in L^2(\mathbb{T})$ that are trace measures (that may change sign) for the space $H^s(\mathbb{T})$, i.e.,

$$\left|\int_{\mathbb{T}} |\varphi|^2 c\, \mathrm{d}\sigma\right| \lesssim \|\varphi\|^2_{H^s(\mathbb{T})},$$

In $\mathbb{R}^n$, V.G. Maz'ya and I.E. Verbitsky considered this problem for $s = 1$ (see [**20**]), showing that the inequality $\left|\int_{\mathbb{R}^n} |\varphi|^2 c\, \mathrm{d}\sigma\right| \lesssim \|\varphi\|^2_{H^1(\mathbb{R}^n)}$, is equivalent to the inequality $\left|\int_{\mathbb{R}^n} |\varphi|^2 |(-\Delta)^{-1/2}(c)|^2\, \mathrm{d}\sigma\right| \lesssim \|\varphi\|^2_{H^1(\mathbb{R}^n)}$, where $|(-\Delta)^{-1/2}(c)|^2$ is now a non-negative measure (see also [**21**] and [**15**] for related problems). In [**9**] it is considered the case $0 < s < 1/2$ in $\mathbb{R}$. We also recall that N. Arcozzi, R. Rochberg, E. Sawyer and B.D. Wick in [**5**] have considered a result on the boundedness of a holomorphic version of this problem on the Dirichlet space (see also [**8**] for a different proof).

Some of the main difficulties when dealing with fractional Laplacians arise from the fact that on one hand these operators are non-local and on the other hand, there is a complexity on the computation of fractional Laplacians when applied to products of functions. In order to avoid these difficulties, we will follow the ideas in [**9**] and consider an equivalent bilinear problem on a subspace of a weighted Sobolev space $\mathcal{W}^2_{1,1-2s}(\mathbb{D})$, of extensions of functions on $H^s(\mathbb{T})$ by a generalized Poisson operator P_s whose definition is given in § 2. For $\mathbb{R}^n$, a similar extension operator was considered by L. Caffarelli and L. Silvestre in [**7**].

Our main result is the following

THEOREM 1.1. *Let $0 < s < 1/2$ and let $b \in H^s(\mathbb{T})$. Then, the following assertions are equivalent:*

(i) *For any* $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{T})$,

$$|\Lambda_b(\varphi, \psi)| \lesssim \|\varphi\|_{H^s(\mathbb{T})} \|\psi\|_{H^s(\mathbb{T})};$$

(ii) *For any* $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{T})$,

$$\left| \int_{\mathbb{D}} \nabla(P_s(\varphi\psi)) \nabla(P_s(b))(1-|z|^2)^{1-2s} \, \mathrm{d}m(z) \right.$$
$$\left. + (1-2s)^2 \int_{\mathbb{D}} P_s(\varphi\psi) P_s(b)(1-|z|^2)^{-2s} \, \mathrm{d}m(z) \right| \lesssim \|\varphi\|_{H^s(\mathbb{T})} \|\psi\|_{H^s(\mathbb{T})};$$

(iii) *For any* $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{T})$,

$$\left| \int_{\mathbb{D}} \nabla(P_s(\varphi) P_s(\psi)) \nabla(P_s(b))(1-|z|^2)^{1-2s} \, \mathrm{d}m(z) \right.$$
$$\left. + (1-2s)^2 \int_{\mathbb{D}} P_s(\varphi) P_s(\psi) P_s(b)(1-|z|^2)^{-2s} \, \mathrm{d}m(z) \right| \lesssim \|\varphi\|_{H^s(\mathbb{T})} \|\psi\|_{H^s(\mathbb{T})};$$

(iv) *The measure* $\mathrm{d}\nu := \left|(-\Delta)^{s/2}(b)\right|^2 \mathrm{d}\sigma$ *is a trace measure for* $H^s(\mathbb{T})$*, that is,* $H^s(\mathbb{T}) \subset L^2(\mathrm{d}\nu)$*;*

(v) *The measure* $\mathrm{d}\mu := |\nabla(P_s(b))|^2 (1-|z|^2)^{1-2s} \, \mathrm{d}m(z)$ *is a Carleson measure for* $P_s(H^s(\mathbb{T}))$*, that is,* $P_s(H^s(\mathbb{T})) \subset L^2(\mathrm{d}\mu)$.

We observe that as it happens in the real case for $s = 1$ (see [**20**]) or $n = 1$ and $0 < s < 1/2$ (see [**9**]), the problem on traces in $H^s(\mathbb{T})$ for measures that may change sign is reduced to a problem of traces of non-negative measures on $H^s(\mathbb{T})$, whose characterization is well known.

The theorem answers the question posed at the beginning of the paper. Namely, the symbols $b \in H^s(\mathbb{T})$ that satisfy the bilinear problem given in (1.1) are the ones for which the non-negative measures $\mathrm{d}\nu = \left|(-\Delta)^{s/2}(b)\right|^2 \mathrm{d}\sigma$ are trace measures for $H^s(\mathbb{T})$ (which correspond to statements (i) and (iv)). Since those trace measures are non-negative, there are well-known characterizations.

The strategy of the proof of this equivalence is the following: the fact that (i) $\Rightarrow$ (iv) will be deduced from a delicate estimate of the norms in $H^s(\mathbb{T})$ of appropriate test functions, norms that will be estimated using the extension operator P_s. The implication (iv) $\Rightarrow$ (i) will be obtained observing firstly that, by the properties of the extension P_s and Stoke's theorem, (i), (ii), (iii) are equivalent. We then prove that (iv) $\Rightarrow$ (v), that is, we relate the traces measures on $\mathbb{T}$ with suitable Carleson measures on $\mathbb{D}$. Then, since trivially (v) $\Rightarrow$ (iii), we obtain the result.

The paper begins with the study of the extension kernel P_s which gives an isomorphism between the space $H^s(\mathbb{T})$ and a subspace of the weighted Sobolev space $\mathcal{W}^1_{1,1-2s}(\mathbb{D})$ defined in § 2. The Euler–Lagrange equation for a norm in this space is a partial differential equation whose solutions are given in terms of the so-called (α, α)-harmonic functions (see [**3**]). The solution of the corresponding Dirichlet problem for this PDE is given in terms of a kernel defined, up to a constant by $((1-|z|^2)^{2s})/(|1-z\overline{\zeta}|^{1+2s})$. The associated extension operator P_s, is studied in § 3. In particular, the above-mentioned isomorphism. We also obtain a formula that represents the fractional Laplacian of a function, $(-\Delta)^s\varphi(\zeta)$, in terms of the radial derivative of its extension, showing that, up to a constant, $\lim_{r\to 1^-}(1-r^2)^{1-2s}(\partial/\partial r)P_s(\varphi)(r\zeta)$ equals to $((-\Delta)^s+I)\varphi(\zeta)$.

In § 4, we consider the relation of the discrete Fourier multiplier operator $((-\Delta)^s+I)$ with the Riesz operator I_s on $\mathbb{T}$, obtaining a basic relation between these two operators, namely, $((-\Delta)^s+I)I_{2s}=I$. The family of Riesz kernels on $\mathbb{T}$ are not a semigroup with the convolution and, in particular, $I_{2s}^{1/2}\neq I_s$, defined as discrete Fourier multipliers. In order to deal with this fact, we prove that the Fourier multiplier operator $I_{2s}^{1/2}I_s^{-1}$ can be realized as a Calderón-Zygmund operator of type zero.

Sections 5 and 6 are technical, and they are necessary to obtain our main theorem, whose proof is finished in § 7. In this section we show that (iv) ⇒ (v) using two non-trivial facts: first that if p_E is the potential asssociated with an extremal capacitary measure, then there exists α such that p_E^α is in the Muckenhoupt class A_2 and $\|p_E^\alpha\|^2_{H^s(\mathbb{T})}\lesssim \mathrm{Cap}_s(E)$. The second one establishes a weighted estimate for an area function of a kernel related to a convolution of P_s with a Riesz kernel.

Finally, the proof that $|(-\Delta)^{s/2}(b)|^2\,\mathrm{d}\sigma$ is a trace measure for $H^s(\mathbb{T})$, that is, (i) ⇒ (iv), is proved applying the hypothesis (i) to the test functions $(I_{2s}^{1/2}(\chi_E I_{2s}^{-(1/2)}b))/p^\alpha_{E,\delta}$ and $p^\alpha_{E,\delta}$ where $p_{E,\delta}$ are regularizations of p_E. We use a delicate estimate of the norm in $H^s(\mathbb{T})$ of the first test function, deduced from the the technical §§ 5 and 6.

Throughout the paper, the letter C may denote various non-negative numerical constants, possibly different in different places. The notation $\varphi(x)\lesssim\psi(x)$ means that there exists $C>0$, which does not depend on x,φ and ψ, such that $\varphi(x)\leqslant C\psi(x)$. We will write $\varphi(x)\approx\psi(x)$ if $\varphi(x)\lesssim\psi(x)$ and $\psi(x)\lesssim\varphi(x)$. The fact that an estimate holds for $x\gg 1$, will mean that it holds for x big enough. All the function spaces considered will be real valued, the points in $\mathbb{T}$ will be denoted either by $\zeta\in\mathbb{C}$ or parametrized by $\mathrm{e}^{\mathrm{i}x}$, the points in the unit disc $\mathbb{D}$ will be denoted either by $z\in\mathbb{C}$ or $r\,\mathrm{e}^{\mathrm{i}x}$, $0<r<1$ and $|1-z\bar\zeta|=|z-\zeta|$ will denote the Euclidean distance from $z\in\overline{\mathbb{D}}$ and $\zeta\in\mathbb{T}$.

2. The weighted Sobolev space $W^2_{1,1-2s}$

If $0<s<1$, the space is defined by $\mathcal{W}^2_{1,1-2s}:=\mathcal{W}^2_{1,1-2s}(\mathbb{D})$, as the completion of functions Φ in the space of real-valued $\mathcal{C}^\infty$ functions on $\overline{\mathbb{D}}$, $\mathcal{C}^\infty(\overline{\mathbb{D}})$, with respect to the norm

$$\|\Phi\|_{\mathcal{W}^2_{1,1-2s}}:=\int_{\mathbb{D}}|\nabla\Phi(z)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z)\\ +\int_{\mathbb{D}}|\Phi(z)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z)<\infty.$$

This space coincides with the space of real-valued functions Φ defined a.e. on $\mathbb{D}$, such that Φ and its distributional derivatives are in $L^2((1-|z|^2)^{1-2s}\,\mathrm{d}m(z))$ (see, for instance, [22]).

The following lemma is well known (see [14] for the details of the proof).

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Lemma 2.1. *The trace operator* $T : \mathcal{C}^\infty(\overline{\mathbb{D}}) \to H^s(\mathbb{T})$ *defined by* $T(\Phi) = \Phi_{|\mathbb{T}}$ *extends by continuity to* $\mathcal{W}^2_{1,1-2s}$ *Hence we have that for all* $\Phi \in \mathcal{W}^2_{1,1-2s}$,

$$\|T(\Phi)\|^2_{H^s(\mathbb{T})} \lesssim \int_{\mathbb{D}} |\nabla\Phi(z)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z) + \int_{\mathbb{D}} |\Phi(z)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z).$$

From our next result we deduce an equivalent norm for $\mathcal{W}^2_{1,1-2s}$.

Lemma 2.2. *Let* $0 < s < 1/2$ *and let* $\Phi \in \mathcal{C}^\infty(\mathbb{D})$. *We then have that for any* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that*

$$\int_{\mathbb{D}} \Phi^2(z)(1-|z|^2)^{-2s}\,\mathrm{d}m(z) \leqslant C_\varepsilon \int_{\mathbb{D}} \Phi^2(z)(1-|z|^2)^{1-2s}\,\mathrm{d}m(z) + \varepsilon\int_{\mathbb{D}} |\nabla\Phi|^2(z)(1-|z|^2)^{1-2s}\,\mathrm{d}m(z). \tag{2.1}$$

In particular,

$$\int_{\mathbb{D}} |\nabla\Phi(z)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z) + \int_{\mathbb{D}} \Phi^2(z)(1-|z|^2)^{-2s}\,\mathrm{d}m(z) \approx \int_{\mathbb{D}} |\nabla\Phi(z)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z) + \int_{\mathbb{D}} \Phi^2(z)(1-|z|^2)^{1-2s}\,\mathrm{d}m(z).$$

Proof. As a consequence of Stokes's Theorem applied to the form

$$\omega = x\Phi^2(x,y)(1-x^2-y^2)^{1-2s}\,\mathrm{d}y - y\Phi^2(x,y)(1-x^2-y^2)^{1-2s}\,\mathrm{d}x$$

and the disc $D_r = \{z \in \mathbb{D};\ |z| \leqslant r\}$, $0 < r < 1$, we obtain

$$\int_{D_r} \left(2(2-2s)\Phi^2(x,y) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\Phi^2(x,y)\right)(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y - \int_{D_r} 2(1-2s)\Phi^2(x,y)(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \Phi^2(r\cos t, r\sin t)r^2(1-r^2)^{1-2s}\,\mathrm{d}t \geqslant 0.$$

Hence, Hölder's inequality gives that

$$\begin{aligned}
2(1-2s)&\int_{D_r} \Phi^2(x,y)(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \int_{D_r} \Phi^2(x,y)(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y + \int_{D_r} |\nabla\Phi^2(x,y)|(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \int_{D_r} \Phi^2(x,y)(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \varepsilon\int_{D_r} |\nabla\Phi(x,y)|^2(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

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and (2.1) is then a consequence of Lebesgue's Monotone Convergence Theorem. □

3. The generalized Poisson extensions P_s

The Euler–Lagrange equation associated with the functional of the equivalent norm in $\mathcal{W}^2_{1,1-2s}$, given by $\int_{\mathbb{D}} |\nabla u(z)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z) + (1-2s)^2\int_{\mathbb{D}} u^2(z)$ $(1-|z|^2)^{-2s}\,\mathrm{d}m(z)$ that corresponds to its stationary values, gives rise to the PDE equation

$$(1-(x^2+y^2))\Delta u - 2(1-2s)\left(x\frac{\partial}{\partial x}u + y\frac{\partial}{\partial y}u\right) - (1-2s)^2 u = 0, \tag{3.1}$$

or equivalently to

$$\operatorname{div}\left((\nabla u)(1-x^2-y^2)^{1-2s}\right) - (1-2s)^2(1-x^2-y^2)^{-2s}u = 0. \tag{3.2}$$

The next theorem is proved in [**3**] and establishes that the Dirichlet problem associated with the PDE equation (3.1) has a unique solution. Namely:

Theorem 3.1. *Let $\varphi \in \mathcal{C}(\mathbb{T})$. We then have that the function*

$$u(z) = P_s(\varphi)(z) := \int_{\mathbb{T}} P_s(z,\zeta)\varphi(\zeta)\,\mathrm{d}\sigma(\zeta),$$

where

$$P_s(z,\zeta) = C_s\frac{(1-|z|^2)^{2s}}{|1-z\bar{\zeta}|^{1+2s}}, \quad z \in \mathbb{D}, \zeta \in \mathbb{T},$$

with $C_s = (\Gamma(1+s/2)^2)/\Gamma(2s)$ is the unique solution to the PDE given in (3.1), which is a continuous function on $\overline{\mathbb{D}}$ and extends φ to $\overline{\mathbb{D}}$.

In addition, this solution u can be given in terms of its Fourier expansion. If $a, b, c > 0$, the hypergeometric function is defined by

$$F(a,b;c;x) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n,$$

where $(a)_0 = 1$, $(a)_n = a(a+1)\dots(a+n-1)$ if $n \geqslant 1$.

If $\varphi(\mathrm{e}^{\mathrm{i}x}) = \sum_{k\in\mathbb{Z}}\widehat{\varphi}(k)\,\mathrm{e}^{\mathrm{i}kx}$ is the Fourier expansion of φ, then the function $u = P_s(\varphi)$ can be expressed as

$$u(z) = \sum_{k=0}^{\infty} f_k(r^2)r^k(\hat{\varphi}(k)\,\mathrm{e}^{\mathrm{i}kx} + \hat{\varphi}(-k)\,\mathrm{e}^{-\mathrm{i}kx}), \quad z = r\,\mathrm{e}^{\mathrm{i}x}, \tag{3.3}$$

where $f_k(x) = (F_k(x)/F_k(1))$ and $F_k(x) = F(k-s+(1/2), -s+(1/2); k+1; x)$, with uniform and absolute convergence on compact sets in $\mathbb{D}$. In particular, if $\varphi \equiv 1$, we have that

$$\int_{\mathbb{T}} P_s(z,\zeta)\,\mathrm{d}\sigma(\zeta) = f_0(|z|^2). \tag{3.4}$$

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We will see (see corollary 3.6) that P_s gives an isomorphism between the space $H^s(\mathbb{T})$ and its image in $\mathcal{W}^2_{1,1-2s}$.

Observe that (3.4) gives that, unlike what happens in the classical case for the Poisson kernel ($s = 1/2$), $P_s(1)$ is not constant.

Definition 3.2. Let $0 < s < 1$. If $\varphi \in \mathcal{C}^1(\mathbb{T})$, the fractional derivative of order s is defined by

$$(-\Delta)^s(\varphi)(\zeta) = \frac{2s\Gamma(1/2-s)^2}{\Gamma(1-2s)} P.V. \int_{\mathbb{T}} \frac{\varphi(\zeta)-\varphi(\eta)}{|\zeta-\eta|^{1+2s}} \mathrm{d}\sigma(\eta).$$

As it is usual, if the function φ is regular enough, the principal value of the integral reduces to an ordinary integral of a function. We will see that this operator can be also defined by a Fourier multiplier.

In $\mathbb{R}^n$, in [7] it is proved that an operator analogous to P_s satisfies that

$$\lim_{y\to 0} y^{1-2s} \frac{\partial}{\partial y} P_s(\varphi)(x,y) = -(-\Delta)^s \varphi(x)$$

and that this operator P_s is an isometry. Our next theorems establish a version of these results for the unit circle, which, in particular, permits to study the fractional Laplacian on $\mathbb{T}$ through ordinary derivatives on $\mathcal{W}^2_{1,1-2s}$.

Theorem 3.3. *Let $0 < s < (1/2)$ and let $\varphi \in \mathcal{C}^1(\mathbb{T})$. We then have that*

$$\lim_{r\to 1^-} (1-r^2)^{1-2s} \frac{\partial}{\partial r} P_s(\varphi)(r\zeta) = 2C_s \frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2} \left((-\Delta)^s(\varphi)(\zeta) + \varphi(\zeta)\right).$$

Proof. Let, as before, $f_k(x)$ be the function defined by $f_k(x) = (F_k(x)/F_k(1))$. We will first prove that

$$\lim_{r\to 1^-} (1-r^2)^{1-2s} \frac{\mathrm{d}}{\mathrm{d}r} f_k(r^2) r^k = 2\frac{\Gamma(s+1/2)\Gamma(1-2s)}{\Gamma(2s)\Gamma(1/2-s)} \frac{\Gamma(k+s+1/2)}{\Gamma(k-s+1/2)} \quad k \geqslant 0. \tag{3.5}$$

Indeed, we have that

$$\frac{\mathrm{d}}{\mathrm{d}r} f_k(r^2) r^k = 2f_k'(r^2) r^{k+1} + k r^{k-1} f_k(r^2) = \frac{2F_k'(r^2) r^{k+1} + k r^{k-1} F_k(r^2)}{F_k(1)}.$$

Hence, using that by [13] page 58,

$$F_k'(t) = \frac{(k-s+1/2)(1/2-s)}{k+1} F(k-s+3/2, 3/2-s; k+2; t),$$

we have that

$$\frac{\mathrm{d}}{\mathrm{d}r} f_k(r^2) r^k = \frac{(k-s+1/2)(1/2-s)}{(k+1)F_k(1)} F(k-s+3/2, 3/2-s; k+2; r^2) 2r^{k+1} + \frac{k r^{k-1}}{F_k(1)} F_k(r^2).$$

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Next, it is well known (see, for instance, [13] page 64) that if $|z| < 1$, then

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z).$$

Thus the above equals to

$$\frac{(k-s+1/2)(1/2-s)}{(k+1)F_k(1)}(1-r^2)^{2s-1}F(s+1/2, k+s+1/2; k+2; r^2)2r^{k+1} + \frac{kr^{k-1}}{F_k(1)}F_k(r^2)$$

and

$$\begin{aligned}(1-r^2)^{1-2s}\frac{\mathrm{d}}{\mathrm{d}r}f_k(r^2)r^k &= \frac{(k-s+1/2)(1/2-s)}{(k+1)F_k(1)}F(s+1/2, k+s+1/2; k+2; r^2)2r^{k+1} \\ &\quad + (1-r^2)^{1-2s}\frac{kr^{k-1}}{F_k(1)}F_k(r^2) := A_k(r) + B_k(r). \qquad (3.6)\end{aligned}$$

Since $k+1-(k-s+1/2)-(1/2-s) = 2s > 0$, the series that defines the function $F_k(r^2)$ converges absolutely in $r=1$ ([13] page 57), so we have that $\lim_{r\to 1^-} F_k(r^2) = F_k(1)$. Hence, $\lim_{r\to 1^-} B_k(r) = 0$. Next, by hypothesis, we have that $k+2-(s+1/2)-(k+s+1/2) = 1-2s > 0$, so using the preceding argument, we obtain that

$$\lim_{r\to 1^-} A_k(r) = \frac{2(k-s+1/2)(1/2-s)}{(k+1)}\frac{F(s+1/2, k+s+1/2; k+2; 1)}{F(k-s+(1/2), -s+(1/2); k+1; 1)}.$$

But (see [13] page 61), if $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$ and $\mathrm{Re}\,(c-a-b) > 0$, we have that

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Thus, using that $\Gamma(z+1) = z\Gamma(z)$, we obtain that

$$\begin{aligned}\lim_{r\to 1^-} A_k(r) &= \frac{2(k-s+1/2)(1/2-s)}{(k+1)}\frac{\Gamma(k+2)\Gamma(1-2s)}{\Gamma(k-s+3/2)\Gamma(3/2-s)} \\ &\quad \times \frac{\Gamma(s+1/2)\Gamma(k+s+1/2)}{\Gamma(k+1)\Gamma(2s)} \\ &= \frac{2(k-s+1/2)\Gamma(s+1/2)\Gamma(k+s+1/2)\Gamma(1-2s)}{(k-s+1/2)\Gamma(k-s+1/2)\Gamma(2s)\Gamma(1/2-s)} \qquad (3.7) \\ &= 2\frac{\Gamma(s+1/2)\Gamma(1-2s)}{\Gamma(2s)\Gamma(1/2-s)}\frac{\Gamma(k+s+1/2)}{\Gamma(k-s+1/2)}.\end{aligned}$$

That gives (3.5).

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Next, we finish the proof of the theorem. Using (3.5) for $k=0$, we have that

$$\begin{aligned}\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\int_{\mathbb{T}} P_s(r\zeta,\eta)\,\mathrm{d}\sigma(\eta)\\ =\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\mathrm{d}}{\mathrm{d}r}f_0(r^2)=2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(r\zeta)=\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\int_{\mathbb{T}}P_s(r\zeta,\eta)\varphi(\eta)\,\mathrm{d}\sigma(\eta)\\ =\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\int_{\mathbb{T}}P_s(r\zeta,\eta)\left(\varphi(\eta)-\varphi(\zeta)\right)\,\mathrm{d}\sigma(\eta)+2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\varphi(\zeta).\end{aligned} \tag{3.8}$$

We have that if $\zeta\overline{\eta}=\mathrm{e}^{\mathrm{i}x}$, $|1-\zeta\overline{\eta}|\approx|x|$, and

$$\begin{aligned}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\frac{(1-r^2)^{2s}}{|1-r\zeta\overline{\eta}|^{1+2s}}=(1-r^2)^{1-2s}\frac{\partial}{\partial r}\frac{(1-r^2)^{2s}}{(1+r^2-2r\cos x)^{((1+2s)/2)}}\\ =\frac{-4sr}{(1+r^2-2r\cos x)^{((1+2s)/2)}}-(1+2s)\frac{(1-r^2)(r-\cos x)}{(1+r^2-2r\cos x)^{((1+2s)/2)+1}}.\end{aligned}$$

Consequently, since $\varphi\in\mathcal{C}^1(\mathbb{T})$, $|\varphi(\eta)-\varphi(\zeta)|=\mathrm{O}(|x|)$ if $x\to 0$ and $|1-r\zeta\overline{\eta}|\approx(1-r)+|x|$, we have that

$$(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(r,\eta)\,|\varphi(\eta)-\varphi(\zeta)|\lesssim\frac{1}{|x|^{2s}}\in L^1(\mathbb{T}).$$

Hence, the Dominated Convergence Theorem gives that

$$\begin{aligned}\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\int_{\mathbb{T}}P_s(r,\eta)\left(\varphi(\eta)-\varphi(\zeta)\right)\,\mathrm{d}\sigma(\eta)\\ =-4sC_s\int_{\mathbb{T}}\frac{\varphi(\eta)-\varphi(\zeta)}{|\zeta-\eta|^{1+2s}}\mathrm{d}\sigma(\eta)=2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}(-\Delta)^s(\varphi(\zeta)).\end{aligned} \tag{3.9}$$

This calculation and (3.8) finishes the proof of the theorem. □

THEOREM 3.4. *Let φ be a $\mathcal{C}^\infty$ function on $\mathbb{T}$. We then have that*

$$\begin{aligned}2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left(\int_{\mathbb{T}}\varphi^2+\int_{\mathbb{T}}\varphi(-\Delta)^s\varphi\right)\\ =\int_{\mathbb{D}}|\nabla P_s(\varphi)|^2(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y\\ +(1-2s)^2\int_{\mathbb{D}}|P_s(\varphi)|^2(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

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Proof. Let ω be the form defined on $\mathbb{D}$ by

$$\omega(z) = P_s(\varphi)(z)\left(\frac{\partial P_s(\varphi)}{\partial x}\,\mathrm{d}y - \frac{\partial P_s(\varphi)}{\partial y}\,\mathrm{d}x\right)(1-x^2-y^2)^{1-2s}, \quad z = x+iy.$$

Let $r<1$ be fixed, and let D_r be the disc centred at the origin and of radius $r>0$. Stokes's Theorem and (3.2) give that

$$\int_{\partial D_r}\omega = \int_{D_r}|\nabla P_s(\varphi)|^2(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y + (1-2s)^2\int_{D_r}|P_s(\varphi)|^2(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.10}$$

The Lebesgue's Monotone Convergence Theorem gives that

$$\begin{aligned}\lim_{r\to1^-}&\int_{D_r}|\nabla P_s(\varphi)|^2(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y\\ &+(1-2s)^2\int_{D_r}|P_s(\varphi)|^2(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{\mathbb{D}}|\nabla P_s(\varphi)|^2(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+(1-2s)^2\int_{\mathbb{D}}|P_s(\varphi)|^2(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}\int_{\partial D_r}\omega &= \int_0^{2\pi}P_s(\varphi)_{|\partial D_r}(1-r^2)^{1-2s}\left(\frac{\partial P_s(\varphi)}{\partial x}_{|\partial D_r} r\cos x + \frac{\partial P_s(\varphi)}{\partial y}_{|\partial D_r} r\sin x\right)\mathrm{d}x\\ &=\int_0^{2\pi}(1-r^2)^{1-2s}P_s(\varphi)_{|\partial D_r} r\frac{\partial}{\partial r}P_s(\varphi)_{|\partial D_r}\,\mathrm{d}x.\end{aligned}$$

In order to pass to the limit when $r\to1^-$, we next check that we can apply the Lebesgue's Dominated Convergence Theorem. Since $P_s(\varphi)\in\mathcal{C}(\overline{\mathbb{D}})$, we just have to check that the function $r(1-r^2)^{1-2s}(\partial/\partial r)P_s(\varphi)_{|\partial D_r}$ is uniformly bounded for $r\in[0,1]$ by an integrable function. Using (3.3) and (3.6), we have that

$$\begin{aligned}r(1-r^2)^{1-2s}&\frac{\partial}{\partial r}P_s(\varphi)(r\,\mathrm{e}^{\mathrm{i}x})\\ &=\sum_{k\geqslant0} r\left(A_k(r)+B_k(r)\right)\left(\widehat{\varphi}(k)\,\mathrm{e}^{\mathrm{i}kx}+\widehat{\varphi}(-k)\,\mathrm{e}^{-\mathrm{i}kx}\right).\end{aligned}$$

Since the hypergeometric function is an increasing function on r, we have that

$$\begin{aligned}A_k(r)&\lesssim\frac{F(s+1/2,k+s+1/2;k+2;1)}{F_k(1)}\\ &=\frac{\Gamma(k+2)\Gamma(1-2s)}{\Gamma(k-s+3/2)\Gamma(3/2-s)}\frac{\Gamma(1/2+s)\Gamma(k+s+1/2)}{\Gamma(k+1)\Gamma(2s)}\approx k^{2s}, \quad k\gg1.\end{aligned}$$

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And

$$B_k(r) \lesssim k.$$

Since $\varphi \in \mathcal{C}^\infty(\mathbb{T})$, we deduce that $|\widehat{\varphi}(k)| \lesssim (/1|k|^l)$, for each $l \geqslant 1$. Hence,

$$\sup_{r\leqslant 1} r(1-r^2)^{1-2s}\left|\frac{\partial}{\partial r}P_s(\varphi)(r\,\mathrm{e}^{\mathrm{i}x})\right| < \infty.$$

and consequently applying theorem 3.3,

$$\lim_{r\to 1^-}\int_0^{2\pi} P_s(\varphi)(r\,\mathrm{e}^{\mathrm{i}x})r(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(r\,\mathrm{e}^{\mathrm{i}x})\,\mathrm{d}x$$
$$= 2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left(\int_0^{2\pi}\varphi^2+\int_0^{2\pi}\varphi(-\Delta)^s(\varphi)\right).$$

Thus, using (3.10) and (3.11), we have proved the theorem. □

Corollary 3.5. *Let $\varphi \in \mathcal{C}^\infty(\mathbb{T})$. Let $\psi = ((-\Delta)^s + I)\varphi$. For each $k \in \mathbb{Z}$,*

$$\widehat{\psi}(k) = \frac{\Gamma(1/2-s)\Gamma(|k|+s+1/2)}{\Gamma(1/2+s)\Gamma(|k|-s+1/2)}\widehat{\varphi}(k). \tag{3.12}$$

In particular, $(-\Delta)^s$ can be defined as a Fourier multiplier.

Proof. By (3.6) and (3.7), $\lim_{r\to 1^-}(1-r^2)^{1-2s}(\partial/\partial r)P_s(\varphi)(r\,\mathrm{e}^{\mathrm{i}x})$ has as a sequence of Fourier multipliers

$$\left(\frac{2\Gamma(s+1/2)\Gamma(1-2s)}{\Gamma(2s)\Gamma(1/2-s)}\frac{\Gamma(|k|+s+1/2)}{\Gamma(|k|-s+1/2)}\right)_{k\in\mathbb{Z}}.$$

Since by (3.6) and (3.9),

$$\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(r\,\mathrm{e}^{\mathrm{i}x})$$
$$= 2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left((-\Delta)^s\varphi+\varphi\right)(\mathrm{e}^{\mathrm{i}x}),$$

we obtain (3.12). □

Corollary 3.6. *For any $\varphi \in H^s(\mathbb{T})$,*

$$\|\varphi\|_{H^s(\mathbb{T})} \approx \|P_s(\varphi)\|_{W^2_{1,1-2s}}.$$

Proof. It is a consequence of last theorem, the density of the functions $\mathcal{C}^\infty(\mathbb{T})$ in $H^s(\mathbb{T})$ and lemma 2.2, since by theorem 3.3 and Stirling's formula we have that $\|((-\Delta)^s)^{1/2}(\varphi)\|_{L^2(\mathbb{T})} \approx \|(-\Delta)^{s/2}(\varphi)\|_{L^2(\mathbb{T})}$. □

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Proposition 3.7. *Let $\varphi \in H^s(\mathbb{T})$ and let $\Psi \in \mathcal{W}^2_{1,1-2s}$ and ψ its restriction to $\mathbb{T}$. We then have that*

$$2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left(\int_{\mathbb{T}}\psi\varphi+\int_{\mathbb{T}}\psi(-\Delta)^s\varphi\right)=\int_{\mathbb{D}}\nabla\Psi\nabla P_s(\varphi)(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y$$
$$+(1-2s)^2\int_{\mathbb{D}}\Psi P_s(\varphi)(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. Since $\mathcal{C}^\infty(\overline{\mathbb{D}})$ and $\mathcal{C}^\infty(\mathbb{T})$ are dense in $\mathcal{W}^2_{1,1-2s}$ and in $H^s(\mathbb{T})$, respectively, we may assume, without loss of generality, that $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ and $\Psi \in \mathcal{C}^\infty(\overline{\mathbb{D}})$. Let ω be the form defined on $\mathbb{D}$ by

$$\omega(z)=\Psi(z)\left(\frac{\partial P_s(\varphi)}{\partial x}\,\mathrm{d}y-\frac{\partial P_s(\varphi)}{\partial y}\,\mathrm{d}x\right)(1-x^2-y^2)^{1-2s},\quad z=x+iy.$$

Let $r<1$ be fixed. Stokes's Theorem and (3.2) give that

$$\int_{\partial D_r}\omega=\int_{D_r}\nabla\Psi\nabla P_s(\varphi)(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y$$
$$+(1-2s)^2\int_{D_r}\Psi P_s(\varphi)(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y.$$

We now observe that since both $\nabla\Psi$ and Ψ are bounded on $\overline{\mathbb{D}}$ and by theorem 3.4, $\nabla P_s(\varphi)$ is in the vector-valued space $\mathbf{L}^2((1-|z|^2)^{1-2s}\,\mathrm{d}m(z))$ and $P_s(\varphi)\in L^2((1-|z|^2)^{-2s}\,\mathrm{d}m(z))$, the Lebesgue's Dominated Convergence Theorem gives that

$$\lim_{r\to 1^-}\left(\int_{D_r}\nabla\Psi\nabla P_s(\varphi)(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y\right.$$
$$\left.+(1-2s)^2\int_{D_r}\Psi P_s(\varphi)(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y\right)$$
$$=\int_{\mathbb{D}}\nabla\Psi\nabla P_s(\varphi)(1-x^2-y^2)^{1-2s}\,\mathrm{d}x\,\mathrm{d}y$$
$$+(1-2s)^2\int_{\mathbb{D}}\Psi P_s(\varphi)(1-x^2-y^2)^{-2s}\,\mathrm{d}x\,\mathrm{d}y.$$

A similar argument to the one used in the proof of theorem 3.4, gives that

$$\lim_{r\to 1^-}\int_0^{2\pi}r(1-r^2)^{1-2s}\Psi_{|\partial D_r}\frac{\partial}{\partial r}P_s(\varphi)_{|\partial D_r}\,\mathrm{d}x$$
$$=2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left(\int_0^{2\pi}\psi\varphi+\int_0^{2\pi}\psi(-\Delta)^s(\varphi)\right).\qquad\square$$

Finally, we recall the following estimate that was implicit in [**3**] page 130, and whose proof we include for a sake of completeness.

Let

$$\nabla_{\mathbb{D}}\varphi(z)=\left((1-|z|^2)\mathcal{R}\varphi,(1-|z|^2)\overline{\mathcal{R}}\varphi\right).\tag{3.13}$$

where $\mathcal{R}\varphi(z)=z(\partial\varphi/\partial z)(z)$, $\overline{\mathcal{R}}\varphi(z)=\overline{z}(\partial\varphi/\partial\overline{z})(z)$ are the radial derivatives.

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LEMMA 3.8. *Let $0 < s < 1/2$. Then*

$$\left|\int_{\mathbb{T}} \nabla_{\mathbb{D}} P_s(z,\zeta)\, d\sigma(\zeta)\right| \lesssim (1-|z|^2)^{2s}, \quad z \in \mathbb{D}.$$

Proof. Since (see [3] page 130), $|\int_{\mathbb{D}} \nabla_{\mathbb{T}} P_s(z,\zeta)\, d\sigma(\zeta)| \lesssim (1-|z|^2)|F'(1/2-s, 1/2-s; 1; |z|^2)|$ and, by [13] page 58, we have that $F'(1/2-s, 1/2-s; 1; |z|^2) = (1/2-s)^2 F(3/2-s, 3/2-s; 2; |z|^2) = (1-|z|^2)^{2s-1} F(1/2+s, 1/2+s; 1; |z|^2)$. The assertion is a consequence of the continuity on $\overline{\mathbb{D}}$ of the function $F(1/2+s, 1/2+s; 1; |z|^2)$ (see [13] page 57). □

4. The space $H^s(\mathbb{T})$ and weighted estimates for a Fourier multiplier

4.1. The space $H^s(\mathbb{T})$

DEFINITION 4.1. Let $0 < s < 1$. The Riesz kernel I_s on the unit circle is defined by

$$I_s(\zeta,\eta) = \frac{\Gamma((1+s)/2)^2}{\Gamma(s)} \frac{1}{|1-\zeta\overline{\eta}|^{1-s}}, \quad \zeta,\eta \in \mathbb{T}.$$

If f is an integrable function on $\mathbb{T}$, the Riesz operator is defined by

$$I_s(f)(\zeta) = \int_{\mathbb{T}} I_s(\zeta,\eta) f(\eta)\, d\sigma(\eta).$$

The space $I_s(L^2(\mathbb{T}))$ is the space of functions $\psi = I_s(\varphi)$, $\varphi \in L^2(\mathbb{T})$, normed by $\|\psi\|_{I_s(L^2(\mathbb{T}))} = \|\varphi\|_{L^2(\mathbb{T})}$.

The Fourier coefficients of the Riesz kernel in $\mathbb{T}$ are the following (see for instance, [2]):

LEMMA 4.2. *Let $0 < s < 1$. Then for any $k \in \mathbb{Z}$,*

$$\widehat{I_s}(k) = \frac{\Gamma(|k| + ((1-s)/2))\Gamma((1+s)/2)}{\Gamma((1-s)/2)\Gamma(|k| + ((1+s)/2))}.$$

THEOREM 4.3. *Let $0 < s < 1/2$.*

$$((-\Delta)^s + I)\, I_{2s} = I.$$

Proof. Indeed, by corollary 3.5, we deduce that if $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ has as the sequence of Fourier coefficients $(\widehat{\varphi}(k))_{k\in\mathbb{Z}}$, then the function $((-\Delta)^s + I)\varphi$ has as sequence of Fourier coefficients $([(\Gamma(|k|+s+1/2)\Gamma(1/2-s))/(\Gamma(|k|-s+1/2)\Gamma(1/2+s))]\, \widehat{\varphi}(k))_{k\in\mathbb{Z}}$. The proof of the proposition follows then from the density of $\mathcal{C}^\infty(\mathbb{T})$ in $L^2(\mathbb{T})$ and lemma 4.2. □

COROLLARY 4.4. *Let $0 < s < 1/2$. We then have that $\varphi \in H^s(\mathbb{T})$ (i.e., $(-\Delta)^{s/2}\varphi, \varphi \in L^2(\mathbb{T})$) if and only if $\varphi = I_s(\psi)$, $\psi \in L^2(\mathbb{T})$ and $\|\varphi\|_{L^2(\mathbb{T})} \approx \|\psi\|_{L^2(\mathbb{T})}$.*

COROLLARY 4.5. *If $\varphi(e^{ix}) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)\, e^{ikx}$, then $\|\varphi\|^2_{H^s(\mathbb{T})} \approx \sum_{k\in\mathbb{Z}} |(|k|^s + 1)\, \widehat{\varphi}(k)|^2$.*

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Proof. It is an immediate consequence of lemma 4.2, corollary 4.4 and Stirling's formula. □

From theorem 4.3 we deduce a reformulation of the bilinear problem (1.1). Namely,

PROPOSITION 4.6. *Let $c \in L^2(\mathbb{T})$. We then have that the boundedness of the bilinear form*

$$\left| \int_{\mathbb{T}} \varphi \psi c \, \mathrm{d}\sigma \right| \lesssim \|\varphi\|_{H^s(\mathbb{T})} \|\psi\|_{H^s(\mathbb{T})},$$

is equivalent to the boundedness of the bilinear form

$$|\Lambda_{I_{2s}(c)}(\varphi, \psi)| = \left| \int_{\mathbb{T}} ((-\Delta)^s + I)\, (\varphi\psi) I_{2s}(c) \, \mathrm{d}\sigma \right| \lesssim \|\varphi\|_{H^s(\mathbb{T})} \|\psi\|_{H^s(\mathbb{T})}.$$

Proof. The identity obtained in theorem 4.3, together with the self-adjointness of the operator $(-\Delta)^s + I$, gives that

$$\begin{aligned} \Lambda_{I_{2s}(c)}(\varphi, \psi) &= \int_{\mathbb{T}} \varphi \psi c \, \mathrm{d}\sigma = \int_{\mathbb{T}} \varphi \psi \, ((-\Delta)^s + I)\, (I_{2s})(c) \, \mathrm{d}\sigma \\ &= \int_{\mathbb{T}} ((-\Delta)^s + I)\, (\varphi\psi) I_{2s}(c) \, \mathrm{d}\sigma, \end{aligned}$$

from which we deduce the proposition. □

4.2. Weighted estimates for the operator $I_{2s}^{1/2} I_s^{-1}$

On the real line, the analogous to the operator I_s on $\mathbb{T}$ is the Riesz operator associated with the kernel $1/(|x-y|^{1-s})$. This family of operators on $\mathbb{R}$ is a semigroup and, in particular, $I_s I_s = I_{2s}$. This fact gives that, as multipliers on $L^2(\mathbb{R})$, we have that $I_s = I_{2s}^{1/2}$ or, equivalently, $I_{2s}^{1/2} I_s^{-1} = Id$. Here, in the unit disc, we can also define $I_{2s}^{1/2}$ as a Fourier multiplier, but it does not coincide with I_s. Nevertheless, the asymptotic behaviour is the same and, in particular, $I_{2s}^{1/2} I_s^{-1}$ defines a bounded operator in $L^2(\mathbb{T})$. We will need to check that it also defines a bounded operator on $L^2(\omega)$, for any weight ω the Muckenhoupt class A_2 on $\mathbb{T}$.

We recall that the operator $I_{2s}^{1/2} I_s^{-1}$ is defined as a Fourier multiplier operator, up to a constant, by

$$\widehat{I_{2s}^{1/2} I_s^{-1}}(\varphi)(k) = \Psi_s(k) \widehat{\varphi}(k)$$

$k \in \mathbb{Z}$, where

$$\Psi_s(x) = \left(\frac{\Gamma(|x| + 1/2 - s)}{\Gamma(|x| + 1/2 + s)} \right)^{1/2} \left(\frac{\Gamma(|x| + 1/2 - s/2)}{\Gamma(|x| + 1/2 + s/2)} \right)^{-1}.$$

We observe that Ψ_s is a continuous function on $\mathbb{R}$ and by Stirling's formula, it follows easily that Ψ_s is bounded. The following proposition estimates the derivates of the function $\Psi_s(x)$.

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PROPOSITION 4.7. *Let $0 < s < 1/2$. We then have that $\Psi_s \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$ and satisfies that for any $j \in \mathbb{N}$, there exists $C = C(j, s)$ such that*

$$|\Psi_s^{(j)}(x)| \lesssim \frac{C}{|x|^j}, \quad x \neq 0.$$

Proof. Without loss of generality, we may assume that $x \gg 1$, since Ψ_s has bounded derivatives in a neighbourhood of the origen. We will first obtain estimates of the derivatives of the quotient of Gamma functions involved in the definition of the function Ψ. Let Φ_{2s} be the function defined by $\Phi_{2s}(x) = ((\Gamma(x + 1/2 - s))/(\Gamma(x + 1/2 + s)))$, $x \geqslant 0$. We then have:

(i) For any $j \in \mathbb{N} \cup \{0\}$, there exists $C = C(j, s)$ such that

$$\Phi_{2s}^{(j)}(x) \leqslant C \frac{1}{x^{2s+j}}, \quad x \gg 1.$$

(ii) For any $j \in \mathbb{N} \cup \{0\}$, there exists $C = C(j, s)$ such that

$$(\Phi_s^{-1})^{(j)}(x) \leqslant C \frac{1}{x^{j-s}}, \quad x \gg 1.$$

We begin with the proof of (i). The proof will follow by induction on $j \geqslant 0$. Stirling's formula gives that, provided x is big enough, $(\Gamma(x - s + 1/2))/(\Gamma(x + s + 1/2)) \approx 1/x^{2s}$. Then (i) holds for $j = 0$. Next, if we denote by $\mathcal{P} = (\Gamma'/\Gamma)$, we have that

$$\left(\frac{\Gamma(x - s + 1/2)}{\Gamma(x + s + 1/2)}\right)' = \frac{\Gamma(x - s + 1/2)}{\Gamma(x + s + 1/2)} \left(\mathcal{P}(x - s + 1/2) - \mathcal{P}(x + s + 1/2)\right). \quad (4.1)$$

Let us now estimate the differences $\mathcal{P}(x - s + 1/2) - \mathcal{P}(x + s + 1/2)$. Observe that

$$\mathcal{P}^{(j)}(z) = (-j)_{j-1} \frac{1}{z^{1+j}} + \sum_{n=1}^{\infty} (-j)_{j-1} \frac{1}{(n + z)^{j+1}}.$$

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Hence,

$$|\mathcal{P}(x-s+1/2)-\mathcal{P}(x+s+1/2)| \lesssim \sup_{x-s+1/2<y<x+s+1/2} |\mathcal{P}'(y)|.$$

For x big enough we have that

$$\sum_n \frac{1}{(n+x+t+1/2)^2} \lesssim \frac{1}{x}, \quad -s<t<s.$$

Hence

$$|\mathcal{P}(x-s+1/2)-\mathcal{P}(x+s+1/2)| \lesssim \frac{1}{x}$$

and in general,

$$\left|\mathcal{P}^{(j)}(x-s+1/2)-\mathcal{P}^{(j)}(x+s+1/2)\right| \lesssim \frac{1}{x^{j+1}}.$$

Next, assume that the estimate (i) is true for $l \leqslant j$, and we will check that it also holds for $j+1$. By Leibniz's formula, and the induction hypothesis applied to (4.1),

$$\Phi_{2s}^{(j+1)}(x) = \left(\frac{\Gamma(x-s+1/2)}{\Gamma(x+s+1/2)}\right)^{(j+1)} \lesssim \sum_{i=0}^{j} \frac{1}{x^{2s+i}} \frac{1}{x^{j+1-i}} \approx \frac{1}{x^{2s+j+1}}.$$

A similar argument on induction proves (ii).

Next, Faà di Bruno formula, (see for instance [**11**]) gives that

$$\begin{aligned}\left(\Phi_{2s}^{1/2}\right)^{(j)}(x) = \sum & \frac{j!}{m_1!\cdots m_j!(1!)^{m_1}\cdots(j!)^{m_j}} \\ & \times (1/2-1)\cdots(1/2-(m_1+\cdots+m_j)) \\ & \times \Phi_{2s}(x)^{1/2-(m_1+\cdots+m_j)} \prod_{l=1}^{j} (\Phi_{2s}^{(l)}(x))^{m_l},\end{aligned}$$

where the sum is over all the l-tuples of non-negative integers $(m_1,\ldots,m_l)$ satisfying that $1\cdot m_1+2\cdot m_2+\cdots+j\cdot m_j=j$.

Applying that by (i), $|(\Phi_{2s})^{(l)}(x)| \lesssim (1/(x^{2s+l}))$, for $x \gg 1$, we then have

$$\left|\left(\Phi_{2s}^{1/2}\right)^{(j)}(x)\right| \lesssim \frac{1}{x^{s+j}}, \quad x \gg 1.$$

Finally, Liebniz's rule together with this estimate and (ii) give that

$$|\Psi_s^{(j)}(x)| \lesssim \sum_{i=0}^{j} \frac{1}{x^{s+j-i}} \frac{1}{x^{i-s}} = \frac{1}{x^j}, \quad x \gg 1. \qquad \square$$

We recall the following result that can be found in [**23**], chapter VI, § 4.4, proposition 2.

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PROPOSITION 4.8. *Let m be a bounded function that is in $\mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$ and satisfies that for any $j \geqslant 0$,*

$$|m^{(j)}(x)| \lesssim |x|^{-j}.$$

Let L be the distribution whose Fourier transform is m. We then have that L agrees with a function $L(x)$ away from the origin that is in $\mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$ and satisfies that for any $j \geqslant 0$,

$$|L^{(j)}(x)| \lesssim |x|^{-1-j}.$$

Hence, L defines by convolution a Calderón-Zygmund operator and in particular we have that for any Muckenhoupt weight $\omega \in A_p(\mathbb{R})$, we have that

$$\|L(f)\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}.$$

We will also need the following theorem in [6]:

PROPOSITION 4.9. *Let $1 < p < \infty$ and $\omega \in A_p(\mathbb{T})$ (i.e., a periodic weight in $A_p(\mathbb{R})$). If Ψ is a continuous function on $\mathbb{R}$ which is a Fourier multiplier for $L^p(\mathbb{R},\omega)$, then $\Psi_{|\mathbb{Z}}$ is a Fourier multiplier for $L^p(\mathbb{T},\omega)$.*

THEOREM 4.10. *For any weight ω in the Muckenhoupt class A_p,*

$$\|I_{2s}^{1/2} I_s^{-1}(\varphi)\|_{L^p(\omega)} \lesssim \|\varphi\|_{L^p(\omega)}, \quad \varphi \in L^p(\omega).$$

Proof. Since $I_{2s}^{1/2} I_s^{-1}$ coincides with $(\Psi_s)_{|\mathbb{Z}}$ as multipliers, the proposition follows from propositions 4.7, 4.8 and 4.9. □

REMARK 4.11. By similar arguments to proposition 4.8, the operator $I_{2s}^{1/2}$ can be realized as an operator L satisfying $|L(x)| \lesssim (1/(|x|^{1-s}))$.

5. Weighted estimates for a weighted area function

Let $\mathbf{K} : \mathbb{D}\times\mathbb{T} \to \mathbb{C}\times\mathbb{C}$ be a vector-valued kernel. If φ is a function on $\mathbb{T}$,

$$K(\varphi)(z) = \int_{\mathbb{T}} K(z,\zeta)\varphi(\zeta)\,\mathrm{d}\sigma(\zeta).$$

The area function associated with $\mathbf{K}$ is

$$G_{\mathbf{K}}(\varphi)(\zeta) := \left(\int_{\Gamma_\zeta} |\mathbf{K}(\varphi)(z)|^2 \frac{\mathrm{d}m(z)}{(1-|z|^2)^2}\right)^{1/2}, \tag{5.1}$$

where Γ_ζ, where $\Gamma(\zeta) = \{z \in \mathbb{D};\, |z-\zeta| < \alpha(1-|z|^2)\}$, $\alpha > 1$, is the cone with vertex ζ.

For the proof of our main theorem, we need the estimate

$$\|G_{\mathbf{K}}(\varphi)\|_{L^2(\omega)} \lesssim \|\varphi\|_{L^2(\omega)} \quad \omega \in A_p.$$

The literature on estimates of this type is extense. It started with the area function associated with the Poisson kernel. In the 60's, E.M. Stein introduced

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the so-called Littlewood-Paley function on $\mathbb{R}^n$, $(\int_0^\infty |K_t(f)|^2(\mathrm{d}t/t))^{1/2}$, for kernels $K_t(x)=(1/t)\Phi(x/t)$, where Φ is a function with integral zero. Later, the condition on mean zero on the function φ were replaced by other conditions. Also there are results on kernels that are not of convolution type. We refer, among others, to the paper [**10**] and the references therein.

Since we do not have an explicit reference on the theorem needed for our results (area function and kernel which is not of convolution type), we have opted to include in Appendix a sketch of the proof of this theorem, adapting some of the known results. Specifically, we have adapted to our context the arguments for convolution kernels in [**16**]. Another type of results could have been adapted. For instance, probably it would be possible to obtain the desired weighted estimate from a version of the pointwise estimate in [**12**], lemma 1.6 or adapting for the area function the arguments given for the Littlewood-Paley function in [**10**].

Theorem 5.1. *Let $\mathbf{K}:\mathbb{D}\times\mathbb{T}\to\mathbb{C}\times\mathbb{C}$ be a vector-valued kernel satisfying that there exist constants $C, c>0$ such that:*

(i) $\|G_{\mathbf{K}}(\varphi)\|_{L^2(\mathbb{T})}\lesssim\|\varphi\|_{L^2(\mathbb{T})}$, *for any* $\varphi\in L^2(\mathbb{T})$.

(ii) $|\mathbf{K}(z,\zeta)|\lesssim(((1-|z|^2)^{\varepsilon})/(|1-z\overline{\zeta}|^{1+\varepsilon}))$, *for some* $\varepsilon>0$.

(iii) *For* $\alpha_1,\alpha_2,\zeta\in\mathbb{T}$, $0<r<1$ *such that* $|\alpha_1-\alpha_2|\leqslant c|1-r\zeta\overline{\alpha_1}|$,

$$|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)|\lesssim\frac{(1-r^2)^{\varepsilon}|\alpha_1-\alpha_2|^{\varepsilon}}{|1-r\zeta\overline{\alpha_1}|^{1+2\varepsilon}},$$

Then, for any $\omega\in A_p$, we have,

$$\|G_{\mathbf{K}}(\varphi)\|_{L^p(\omega)}\lesssim\|\varphi\|_{L^p(\omega)}.$$

Our next goal is to check that the (vector-valued) kernel

$$\mathbf{K}(z,\zeta)=(1-|z|^2)^{-s}\int_{\mathbb{T}}\nabla_{\mathbb{D}}P_s(z,\eta)\frac{\mathrm{d}\sigma(\eta)}{|1-\zeta\overline{\eta}|^{1-s}},\tag{5.2}$$

is in the hypothesis of theorem 5.1.

Proposition 5.2. *Let $\mathbf{K}(z,\zeta)$ be the vector-valued kernel defined in (5.2) and $G_{\mathbf{K}}$ as in (5.1). We have that there exist constants $C, c>0$ such that:*

(i') $\|G_{\mathbf{K}}(\varphi)\|_{L^2(\mathbb{T})}\lesssim\|\varphi\|_{L^2(\mathbb{T})}$, *for any* $\varphi\in L^2(\mathbb{T})$.

(ii') $|\mathbf{K}(z,\zeta)|\lesssim((1-|z|^2)^{s})/(|1-z\overline{\zeta}|^{1+s})$.

(iii') *For* $\alpha_1,\alpha_2,\zeta\in\mathbb{T}$, $0<r<1$ *such that* $|\alpha_1-\alpha_2|\leqslant c|1-r\zeta\overline{\alpha_1}|$,

$$|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)|\lesssim\frac{(1-r^2)^{s}|\alpha_1-\alpha_2|^{s}}{|1-r\zeta\overline{\alpha_1}|^{1+2s}}.$$

The proof of (i') is an immediate consequence of Fubini's Theorem, corollaries 3.6 and 4.4.

For the proof of estimate (ii'), we write

$$\mathbf{K}(z,\zeta) = (1-|z|^2)^{-s}\int_{\mathbb{T}} \nabla_{\mathbb{D}} P_s(z,\eta)\left(\frac{1}{|1-\zeta\overline{\eta}|^{1-s}} - \frac{1}{|1-z\overline{\zeta}|^{1-s}}\right) \mathrm{d}\sigma(\eta)$$
$$+ (1-|z|^2)^{-s}\int_{\mathbb{T}} \nabla_{\mathbb{D}} P_s(z,\eta)\frac{\mathrm{d}\sigma(\eta)}{|1-z\overline{\zeta}|^{1-s}} := \mathbf{A}(z,\zeta) + \mathbf{B}(z,\zeta).$$

Let us begin obtaining the desired estimate for $|\mathbf{A}|$. Computing $\nabla_{\mathbb{D}}$ and using that we have that

$$\left|\frac{1}{a^\alpha} - \frac{1}{b^\alpha}\right| \lesssim \frac{|a-b|(a^\alpha + b^\alpha)}{a^\alpha b^\alpha (a+b)}, \quad 0<\alpha<1 \; a,b>0, \tag{5.3}$$

we obtain, considering separately the points $\eta \in \mathbb{T}$ such that $|1-\zeta\overline{\eta}| \leqslant |1-z\overline{\zeta}|$ and $|1-\zeta\overline{\eta}| > |1-z\overline{\zeta}|$ respectively (see (3.13))

$$|\mathbf{A}(z,\zeta)| \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|}\int_{\mathbb{T}} \frac{||1-z\overline{\zeta}|-|1-\zeta\overline{\eta}||}{|1-z\overline{\eta}|^{1+2s}}\frac{1}{|1-\zeta\overline{\eta}|^{1-s}}\,\mathrm{d}\sigma(\eta)$$
$$+ \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1-s}}\int_{\mathbb{T}} \frac{||1-z\overline{\zeta}|-|1-\zeta\overline{\eta}||}{|1-z\overline{\eta}|^{1+2s}|1-\zeta\overline{\eta}|}\mathrm{d}\sigma(\eta) := A_1(z,\zeta) + A_2(z,\zeta).$$
$$A_1(z,\zeta) \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|}\int_{\mathbb{T}} \frac{|1-z\overline{\eta}|}{|1-z\overline{\eta}|^{1+2s}|1-\zeta\overline{\eta}|^{1-s}}\mathrm{d}\sigma(\eta)$$
$$\leqslant \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|}\int_{\mathbb{T}} \frac{\mathrm{d}\sigma(\eta)}{|1-z\overline{\eta}|^{2s}|1-\zeta\overline{\eta}|^{1-s}} \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1+s}}.$$

Next, we bound A_2. Let $0<\delta<2s$ be fixed. We then have:

$$A_2(z,\zeta) \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1-s+\delta}}\int_{\mathbb{T}} \frac{1}{|1-z\overline{\eta}|^{2s}|1-\zeta\overline{\eta}|^{1-\delta}} \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1+s}}.$$

For the estimate of $|\mathbf{B}(z,\zeta)|$, we use that by lemma 3.8, $|\int_{\mathbb{T}} \nabla_{\mathbb{D}} P_s(z,\eta)\,\mathrm{d}\sigma(\eta)| \lesssim (1-|z|^2)^{2s}$. Hence

$$|\mathbf{B}(z,\zeta)| \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1-s}} \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1+s}}.$$

Altogether gives that $\mathbf{K}$ satisfies (ii').

Next we prove the estimate (iii').

Assume first that $|\alpha_1-\alpha_2| \geqslant \delta$, for some $\delta>0$. This case is immediate since if $|\alpha_1-\alpha_2| \leqslant c|1-r\zeta\overline{\alpha_1}|$, with $2c<1$, we have that $|1-r\zeta\overline{\alpha_1}| \approx |1-r\zeta\overline{\alpha_2}|$ and $|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)| \leqslant |\mathbf{K}(r\alpha_1,\zeta)| + |\mathbf{K}(r\alpha_2,\zeta)| \lesssim |\mathbf{K}(r\alpha_1,\zeta)|$. Hence, using estimate (ii') and the condition $|\alpha_1-\alpha_2| \geqslant \delta$, we deduce that

$$|\mathbf{K}(r\alpha_1,\zeta)| \lesssim \frac{(1-r^2)^s}{|1-r\alpha_1\overline{\zeta}|^{1+s}} \lesssim \frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-r\alpha_1\overline{\zeta}|^{1+2s}}.$$

Now assume that $|\alpha_1-\alpha_2|<\delta$, for some $\delta>0$.

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We recall that as before, if $2c < 1$ and $|\alpha_1 - \alpha_2| \leqslant c|1 - r\zeta\overline{\alpha_1}|$, then $|1 - r\zeta\overline{\alpha_1}| \approx |1 - r\zeta\overline{\alpha_2}|$.

$$\begin{aligned}
&|\mathbf{K}(r\alpha_1, \zeta) - \mathbf{K}(r\alpha_2, \zeta)| \\
&\quad = |\mathbf{K}(r\zeta, \alpha_1) - \mathbf{K}(r\zeta, \alpha_2)| \\
&\quad = (1 - r^2)^{-s} \left| \int_{\mathbb{T}} \nabla_{\mathbb{D}} P_s(r\zeta, \eta) \left(\frac{1}{|1 - \eta\overline{\alpha}_1|^{1-s}} - \frac{1}{|1 - \eta\overline{\alpha}_2|^{1-s}} \right) \mathrm{d}\sigma(\eta) \right| \\
&\quad \leqslant (1 - r^2)^{-s} \int_{\mathbb{T}} |\nabla_{\mathbb{D}} P_s(r\zeta, \eta)| \left| \frac{1}{|1 - \eta\overline{\alpha}_1|^{1-s}} - \frac{1}{|1 - \eta\overline{\alpha}_2|^{1-s}} \right. \\
&\qquad \left. - \left(\frac{1}{|1 - r\zeta\overline{\alpha}_1|^{1-s}} - \frac{1}{|1 - r\zeta\overline{\alpha}_2|^{1-s}} \right) \right| \mathrm{d}\sigma(\eta) \\
&\qquad + (1 - r^2)^{-s} \left| \int_{\mathbb{T}} \nabla_{\mathbb{D}} P_s(r\zeta, \eta) \left(\frac{1}{|1 - r\zeta\overline{\alpha}_1|^{1-s}} - \frac{1}{|1 - r\zeta\overline{\alpha}_2|^{1-s}} \right) \mathrm{d}\sigma(\eta) \right| \\
&\quad = D + E.
\end{aligned}$$

This decomposition permits to avoid integrability problems when we introduce the modulus inside the integral.

We first observe that by lemma 3.8, and using again (5.3), we have that since $|1 - r\zeta\overline{\alpha_1}| \approx |1 - r\zeta\overline{\alpha_2}|$,

$$\begin{aligned}
E &\lesssim (1 - r^2)^s \left| \frac{1}{|1 - r\zeta\overline{\alpha_1}|^{1-s}} - \frac{1}{|1 - r\zeta\overline{\alpha_2}|^{1-s}} \right| \lesssim \frac{(1 - r^2)^s |\alpha_1 - \alpha_2|}{|1 - r\zeta\overline{\alpha_1}|^{2-s}} \\
&\lesssim \frac{(1 - r^2)^s |\alpha_1 - \alpha_2|^s}{|1 - r\zeta\overline{\alpha_1}|^{2-s+s-1}} \lesssim \frac{(1 - r^2)^s |\alpha_1 - \alpha_2|^s}{|1 - r\zeta\overline{\alpha_1}|^{1+2s}}.
\end{aligned}$$

In order to obtain the desired estimate for D, we consider separately the integration regions $\Omega_1 := \{\eta \in \mathbb{T},\ |1 - r\zeta\overline{\eta}| \geqslant \varepsilon|1 - r\zeta\overline{\alpha_1}|\}$ and $\Omega_2 := \{\eta \in \mathbb{T},\ |1 - r\zeta\overline{\eta}| \leqslant \varepsilon|1 - r\zeta\overline{\alpha_1}|\}$, where $\varepsilon < 1$ will be fixed later on. We denote the corresponding integrals by D_1 and D_2.

We begin with the estimate of D_1. Since in that case we are assuming that $|1 - r\zeta\overline{\eta}| \geqslant \varepsilon|1 - r\zeta\overline{\alpha_1}|$) these are not integrability problems and we bound separately the two summands. We have that

$$\begin{aligned}
D_1 &\leqslant (1 - r^2)^{-s} \int_{\Omega_1} |\nabla_{\mathbb{D}} P_s(r\zeta, \eta)| \left| \frac{1}{|1 - \eta\overline{\alpha}_1|^{1-s}} - \frac{1}{|1 - \eta\overline{\alpha}_2|^{1-s}} \right| \mathrm{d}\sigma(\eta) \\
&\quad + (1 - r^2)^{-s} \int_{\Omega_1} |\nabla_{\mathbb{D}} P_s(r\zeta, \eta)| \left| \frac{1}{|1 - r\zeta\overline{\alpha}_1|^{1-s}} - \frac{1}{|1 - r\zeta\overline{\alpha}_2|^{1-s}} \right| \mathrm{d}\sigma(\eta) \\
&= D_{11} + D_{12}.
\end{aligned}$$

On D_{11} we assume without loss of generality that $|1 - \eta\overline{\alpha_1}| \leqslant |1 - \eta\overline{\alpha_2}|$. Then, using (5.3) and choosing $0 < \varepsilon < s$ we have that

$$D_{11} \lesssim (1 - r^2)^s \frac{|\alpha_1 - \alpha_2|}{|1 - r\zeta\overline{\alpha_1}|^{1+2s}} \int_{\Omega_1} \frac{\mathrm{d}\sigma(\eta)}{|1 - \eta\overline{\alpha_1}|^{1-s+\varepsilon} |1 - \eta\overline{\alpha_2}|^{1-\varepsilon}},$$

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where ε satisfies that $0 < \varepsilon < s$. Hence,

$$D_{11} \lesssim (1-r^2)^s \frac{|\alpha_1-\alpha_2|}{|1-r\zeta\overline{\alpha_1}|^{1+2s}} \frac{1}{|\alpha_1-\alpha_2|^{1-s}} = (1-r^2)^s \frac{|\alpha_1-\alpha_2|^s}{|1-r\zeta\overline{\alpha_1}|^{1+2s}}.$$

Next, we estimate D_{12}. By (5.3) and the fact that since $2c<1$ for $|\alpha_1-\alpha_2| \leqslant c|1-r\zeta\overline{\alpha_1}|$, we have that $|1-r\zeta\overline{\alpha_1}| \approx |1-r\zeta\overline{\alpha_2}|$, then

$$\begin{aligned} D_{12} &\lesssim \frac{(1-r^2)^s}{|1-r\zeta\overline{\alpha_1}|^{2-s}} \int_{|1-r\zeta\overline{\eta}|\geqslant|1-r\zeta\overline{\alpha_1}|/2} \frac{|\alpha_1-\alpha_2|}{|1-r\zeta\overline{\eta}|^{1+2s}} \,\mathrm{d}\sigma(\eta) \\ &\lesssim \frac{(1-r^2)^s|\alpha_1-\alpha_2|}{|1-r\zeta\overline{\alpha_1}|^{2-s}} \frac{1}{|1-r\zeta\overline{\alpha_1}|^{2s}} \leqslant \frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-r\zeta\overline{\alpha_1}|^{1+2s}}. \end{aligned}$$

We now estimate D_2. In this case, $|1-r\zeta\overline{\eta}| \leqslant \varepsilon|1-r\zeta\overline{\alpha_1}|$ and we will use the parametrization of the unit circle $\mathbb{T}$ by $\mathrm{e}^{\mathrm{i}t}$, $t\in(-\pi,\pi]$. We denote $\zeta=\mathrm{e}^{\mathrm{i}x}$, $\eta=\mathrm{e}^{\mathrm{i}y}$, $\alpha_1=\mathrm{e}^{\mathrm{i}a_1}$, and $\alpha_2=\mathrm{e}^{\mathrm{i}a_2}$, where $x,y,a_1,a_2\in(-\pi,\pi]$. With this notation we have

$$\begin{aligned} D_2 \leqslant \int_{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|\leqslant\varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|} &\frac{(1-r^2)^s}{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|^{1+2s}} \Bigg| \frac{1}{|1-\mathrm{e}^{\mathrm{i}(y-a_1)}|^{1-s}} \\ &- \frac{1}{|1-\mathrm{e}^{\mathrm{i}(y-a_2)}|^{1-s}} - \frac{1}{|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|^{1-s}} + \frac{1}{|1-r\,\mathrm{e}^{\mathrm{i}(x-a_2)}|^{1-s}} \Bigg| \,\mathrm{d}y \end{aligned}$$

Next, for $x,y\in(-\pi,\pi]$, such that $|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}| \leqslant \varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|$, the function

$$\begin{aligned} \Phi(t) &:= \frac{1}{|1-\mathrm{e}^{\mathrm{i}(y-t)}|^{1-s}} - \frac{1}{|1-r\,\mathrm{e}^{\mathrm{i}(x-t)}|^{1-s}} \\ &= \frac{1}{4^{(1-s)/2}\left(\sin^2((y-t)/2)\right)^{((1-s)/2)}} - \frac{1}{\left((1-r)^2+4r\sin^2((x-t)/2)\right)^{(1-s)/2}} \end{aligned}$$

is differentiable for $t\in[a_1,a_2]$. By the Mean-Value Theorem, we deduce that

$$\begin{aligned} D_2 &\lesssim \int_{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|\leqslant\varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a)}|} \frac{(1-r^2)^s}{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|^{1+2s}} \left| \int_{a_1}^{a_2} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)\,\mathrm{d}t \right| \mathrm{d}y \\ &= \int_{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|\leqslant\varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|} \frac{(1-r^2)^s(1-s)}{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|^{1+2s}}(1/2) \\ &\quad\times \Bigg| \int_{a_1}^{a_2} \Bigg(\frac{\sin((y-t)/2)\cos((y-t)/2)}{4^{(1-s)/2}\left(\sin^2((y-t)/2)\right)^{((1-s)/2)+1}} \\ &\quad - \frac{r\sin((x-t)/2)\cos((x-t)/2)}{\left((1-r)^2+4r\sin^2((x-t)/2\right)^{((1-s)/2)+1}} \Bigg) \,\mathrm{d}t \Bigg| \,\mathrm{d}y. \end{aligned}$$

We first observe that if we choose ε small enough, then the condition $|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}| \leqslant \varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|$ gives that $1-r \lesssim |1-\mathrm{e}^{\mathrm{i}(x-a_1)}|$, and consequently, we

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have that $|1-\mathrm{e}^{\mathrm{i}(x-a_1)}| \approx |1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|$. Adding and substracting the intermediate term $[\sin((x-t)/2)\cos((x-t)/2)4^{(1-s)/2}]/[(\sin^2((x-t)/2))^{(1-s)/2)+1}]$, we have that

$$
\begin{aligned}
D_2 \lesssim & \int_{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|\leqslant\varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a)}|} \frac{(1-r^2)^s(1-s)}{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|^{1+2s}} \\
& \times \int_{a_1}^{a_2} \left| \frac{\sin((y-t)/2)\cos((y-t)/2)}{4^{(1-s)/2}\left(\sin^2((y-t)/2)\right)^{(1-s)/2)+1}} \right. \\
& \left. - \frac{\sin((x-t)/2)\cos((x-t)/2)}{4^{(1-s)/2}\left(\sin^2((x-t)/2)\right)^{((1-s)/2)+1}} \right| \mathrm{d}t\,\mathrm{d}y \\
& + \int_{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|\leqslant\varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a)}|} \frac{(1-r^2)^s(1-s)}{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|^{1+2s}} \\
& \times \int_{a_1}^{a_2} \left| \frac{\sin((x-t)/2)\cos((x-t)/2)}{4^{(1-s)/2}\left(\sin^2((x-t)/2)\right)^{((1-s)/2)+1}} \right. \\
& \left. - \frac{r\sin((x-t)/2)\cos((x-t)/2)}{\left((1-r)^2+4r\sin^2((x-t)/2)\right)^{((1-s)/2)+1}} \right| \mathrm{d}t\,\mathrm{d}y := D_{21}+D_{22}.
\end{aligned}
$$

We begin with D_{21}. We apply the Mean Value Theorem, and for each $t\in[a_1,a_2]$, there exists l_t between x and y such that

$$
\begin{aligned}
D_{21} \lesssim & \int_{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|\leqslant\varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|} \frac{(1-r^2)^s|x-y|}{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|^{1+2s}} \int_{a_1}^{a_2} \left\{ \left| \frac{\cos^2((l_t-t)/2)}{|\sin((l_t-t)/2)|^{3-s}} \right| \right. \\
& \left. + \left| \frac{\sin^2((l_t-t)/2)}{|\sin((l_t-t)/2)|^{3-s}} \right| + \left| \frac{\sin^2((l_t-t)/2)\cos^2((l_t-t)/2)}{|\sin((l_t-t)/2)|^{5-s}} \right| \right\} \mathrm{d}t\,\mathrm{d}y \\
\lesssim & |1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|^{1-2s} \frac{(1-r^2)^s|a_1-a_2|}{|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|^{3-s}} \leqslant \frac{(1-r^2)^s|a_1-a_2|^s}{|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|^{1+2s}},
\end{aligned}
$$

where we have used that for any $t\in[a_1,a_2]$, $|\sin((l_t-t)/2)| \approx |1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|$ and $|a_1-a_2| < c|1-r\zeta\overline{a_1}|$.

Finally, for the estimate of D_{22}, we use again the Mean Value Theorem and we obtain that for each $0<r<1$ and each $t\in[a_1,a_2]$, there exists $l\in[r,1)$ such that

$$
\begin{aligned}
D_{22} \lesssim & \int_{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|\leqslant\varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|} \frac{(1-r^2)^s(1-r)}{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|^{1+2s}} \\
& \times \int_{a_1}^{a_2} \left(\frac{|\sin((x-t)/2)\cos((x-t)/2)|}{\left((1-l)^2+4l\sin^2((x-t)/2)\right)^{((1-s)/2)+1}} \right. \\
& \left. + \frac{l|\sin((x-t)/2)\cos((x-t)/2)|(2(1-l)+4\sin^2((x-t)/2)}{\left((1-l)^2+4l\sin^2((x-t)/2)\right)^{((1-s)/2)+2}} \right) \mathrm{d}t\,\mathrm{d}y.
\end{aligned}
$$

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Since $|1 - r\,\mathrm{e}^{\mathrm{i}(x-y)}| \leqslant \varepsilon |1 - r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|$, we have that $|1 - l\,\mathrm{e}^{\mathrm{i}(x-a_1)}| \approx |1 - r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|$ for any $l \in [r, 1)$. Hence the above is bounded by

$$\int_{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|\leqslant\varepsilon|1-r\,\mathrm{e}^{\mathrm{i}(x-a_1)}|} \int_{a_1}^{a_2} \frac{(1-r^2)^s(1-r)}{|1-r\,\mathrm{e}^{\mathrm{i}(x-y)}|^{1+2s}} \times \left(\frac{1}{|1-l\,\mathrm{e}^{\mathrm{i}(x-a_1)}|^{2-s}} + \frac{1}{|1-l\,\mathrm{e}^{\mathrm{i}(x-a_1)}|^{3-s}} \right) \mathrm{d}t\,\mathrm{d}y$$
$$\lesssim \frac{(1-r^2)^s(1-r^2)^{1-2s}|\alpha_1-\alpha_2|}{|1-z\overline{\alpha_1}|^{3-s}} \lesssim \frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-z\overline{\alpha_1}|^{1+2s}},$$

where in the last estimate we have used that $|\alpha_1 - \alpha_2| \lesssim |1 - z\overline{\alpha_1}|$ and $(1-r) \lesssim |1 - z\overline{\alpha_1}|$.

6. Capacities, trace measures for $H^s(\mathbb{T})$ and Carleson measures for $P_s(H^s(\mathbb{T}))$

Definition 6.1. Let $E \subset \mathbb{T}$. The Riesz capacity of E is defined by

$$\mathrm{Cap}_s(E) := \inf\{\|f\|_2^2 : \, I_s(|f|) \geqslant 1 \text{ on } E\}.$$

We list some properties of the equilibrium measure for a compact set in $\mathbb{T}$, which will be used below and that are essentially due to O. Frostman (see [**1**] theorem 2.2.7).

Theorem 6.2. *Given a closed set $E \subset \mathbb{T}$, there exists a positive capacitary measure ν_E on $\mathbb{T}$, such that:*

(i) *ν_E is supported on E and $\nu_E(E) = \mathrm{Cap}_s(E)$.*

(ii) *$q_E := I_s I_s(\nu_E) \geqslant 1$ a.e. on E.*

(iii) *$q_E \in H^s(\mathbb{T})$ and $\|q_E\|^2_{H^s(\mathbb{T})} \lesssim \mathrm{Cap}_s(E)$.*

(iv) *There is a constant $C > 0$ independent of E, such that for any $\zeta \in \mathbb{T}$, $q_E(\zeta) \leqslant C$.*

Remark 6.3. Since $2s < 1$, we have that $I_s * I_s \approx I_{2s}$. This fact and corollary 4.5 give that the function $p_E := I_{2s}(\nu_E)$ satisfies properties (6.2) and (6.2), with property (6.2) replaced by $p_E \gtrsim 1$ a.e. on E. For our purposes this is the function we will use when constructing appropriate test functions.

Let $\varphi : (-\pi, \pi) \to [0, \infty)$ be a $\mathcal{C}^\infty$ function on $(-\pi, \pi]$, non-increasing in $|x|$, with compact support on $(-\pi, \pi)$ and such that $\int_{-\pi}^{\pi} \varphi = 1$. For $\delta > 0$, let $\varphi_\delta(x) = (1/\delta)\varphi(x/\delta)$. We write $\nu_{E,\delta} := \nu_E * \varphi_\delta$, the regularizations of the measure ν_E. We then have that $\nu_{E,\delta}$ are functions in $\mathcal{C}^\infty$ on $\mathbb{T}$ satisfying that $\mathrm{d}\nu_{E,\delta} := \nu_{E,\delta}\,\mathrm{d}x \to \mathrm{d}\nu$ in the sense of distributions and such that $\|\nu_{E,\delta}\|_1 = \mathrm{Cap}_s(E)$.

We denote by $p_{E,\delta} := I_{2s} * \nu_{E,\delta}, \quad \delta > 0$.

Lemma 6.4 [**18**], chapter 2, lemma 3.6. *If $0 < s < 1/2$ and $\beta \in (1, 1/(1-2s))$, then $p^\beta_{E,\delta}$ is in the Muckenhoupt class A_1, with A_1-constant independent of E and δ.*

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THEOREM 6.5. *Let $E \subset \mathbb{T}$ be a closed set and let p_E be the function given in remark 6.3 and $p_{E,\delta}$ the regularization considered before. Let $\alpha > 1/2$. Then,*

(i) $\|p^{\alpha}_{E,\delta}\|^2_{H^s(\mathbb{T})} \lesssim \mathrm{Cap}_s(E)$.

Proof. We define the form ω_δ by

$$\omega_\delta = (P_s(p_{E,\delta}))^{2\alpha-1}(1-r^2)^{1-2s}\left(\frac{\partial}{\partial x}P_s(p_{E,\delta})\,\mathrm{d}y - \frac{\partial}{\partial y}P_s(p_{E,\delta})\,\mathrm{d}x\right).$$

Arguing as in theorem 3.4, using that $p_{E,\delta}$ is bounded we can pass to the limit under the integral sign. Then using theorem 3.3 and proposition 4.3, we have that

$$\begin{aligned}\lim_{r\to 1^-}\int_{\partial D_r}\omega_\delta &= \lim_{r\to 1^-}\int_0^{2\pi}(1-r^2)^{1-2s}(P_s(p_{E,\delta})^{2\alpha-1}_{|\partial D_r} r\frac{\partial}{\partial r}P_s(p_{E,\delta})_{|\partial D_r}\,\mathrm{d}x\\ &= \int_{\mathbb{T}} p^{2\alpha-1}_{E,\delta}(I+(-\Delta)^s)p_{E,\delta}\\ &= \int_{\mathbb{T}} p^{2\alpha-1}_{E,\delta}\,\mathrm{d}\nu_{E,\delta} \lesssim \int_{\mathbb{T}}\mathrm{d}\nu_{E,\delta} = \mathrm{Cap}_s(E),\end{aligned}$$

Next, Stokes's Theorem and the Lebesgue's Monotone Convergence Theorem, give that

$$\begin{aligned}\lim_{r\to 1^-}\int_{\partial D_r}\omega_\delta = \frac{2\alpha-1}{\alpha^2}&\int_{\mathbb{D}}(1-r^2)^{1-2s}|\nabla(P_s(p_{E,\delta}))^{\alpha}|^2\,\mathrm{d}m(z)\\ +(1-2s)^2&\int_{\mathbb{D}}(1-r^2)^{-2s}|(P_s(p_{E,\delta}))^{\alpha}|^2\,\mathrm{d}m(z).\end{aligned}$$

On the other hand, the function $(P_s(p_{E,\delta}))^{\alpha}$ has boundary values $p^{\alpha}_{E,\delta}$. Consequently, by lemma 2.1, we have that

$$\begin{aligned}\|p^{\alpha}_{E,\delta}\|^2_{H^s(\mathbb{T})} &\lesssim \int_{\mathbb{D}}(1-r^2)^{1-2s}|\nabla(P_s(p_{E,\delta}))^{\alpha}|^2\,\mathrm{d}m(z)\\ &\quad+\int_{\mathbb{D}}(1-r^2)^{-2s}|(P_s(p_{E,\delta}))^{\alpha}|^2\,\mathrm{d}m(z)\\ &\approx \lim_{r\to 1^-}\int_{\partial D_r}\omega_\delta \lesssim \mathrm{Cap}_s(E).\end{aligned}$$

□

6.1. Trace measures for $H^s(\mathbb{T})$ and Carleson measures for $P_s(H^s(\mathbb{T}))$

The characterization of the positive trace measures for $H^s(\mathbb{T})$ is well known (see, for instance the book [**17**] or [**1**] theorem 7.2.1 for a proof). Namely

PROPOSITION 6.6. *Let $0 < s < 1/2$ and let μ be a positive Borel measure on $\mathbb{T}$. Then, μ is a trace measure for $H^s(\mathbb{T})$, that is, $\int_{\mathbb{T}}|f|^2\,\mathrm{d}\mu \lesssim \|f\|^2_{H^s(\mathbb{T})}$ for every $f \in H^s(\mathbb{T})$, if and only if there exists $C_\mu > 0$ such that for any compact set $E \subset \mathbb{T}$ $\mu(E) \leqslant C_\mu \mathrm{Cap}_s(E)$.*

DEFINITION 6.7. Let $E \subset \mathbb{T}$. Then the tent over E, $T(E)$ is defined by $T(E) = \mathbb{D} \setminus \bigcup_{\xi \notin E} \Gamma(\xi)$.

The arguments for the proof of the next elementary lemma can be found, for instance, [9] lemma 3.25.

LEMMA 6.8. *Let $0 < s < 1/2$, and let $E \subset \mathbb{T}$. Let f be a non-negative measurable function on $\mathbb{T}$ such that $f \geqslant 1$ a.e. on E. Then $P_s(f) \gtrsim 1$ on $T(E)$.*

Our next result gives a characterization of the Carleson measures for the space $P_s(H^s(\mathbb{T}))$. The proof heavily relies on Hanson's strong capacitary estimate (see, for instance theorem 7.1.1 in [1] for a proof) and in lemma 6.8 (see [9] theorem 3.26 for the details of the arguments of the proof).

THEOREM 6.9. *Let $0 < s < 1/2$ and let μ be a positive Borel measure on $\mathbb{D}$. Then, μ is a Carleson measure for $P_s(H^s(\mathbb{T}))$, that is, $\int_{\mathbb{D}} P_s(\varphi)^2 \, \mathrm{d}\mu \lesssim \|\varphi\|^2_{H^s(\mathbb{T})}$ for every $\varphi \in H^s(\mathbb{T})$ if and only if, there exists $C_\mu > 0$ such that for any compact set $E \subset \mathbb{R}$, $\mu(T(E)) \leqslant C_\mu \mathrm{Cap}_s(E)$.*

We finish the section with two results that will give equivalent reformulations to (iv) and (v) in theorem 1.1 and that will be used when needed in the proof of this Theorem.

LEMMA 6.10. *Assume that the measure $|\nabla P_s(b)|^2(1-|z|^2)^{1-2s} \, \mathrm{d}m(z)$ is a Carleson measure for $P_s(H^s(\mathbb{T}))$, then the measure $|P_s(b)|^2(1-|z|^2)^{-2s} \, \mathrm{d}m(z)$ is also a Carleson measure for $P_s(H^s(\mathbb{T}))$.*

In particular, $|\nabla P_s(b)|^2(1-|z|^2)^{1-2s} \, \mathrm{d}m(z)$ is a Carleson measure for $P_s(H^s(\mathbb{T}))$ if and only if $(|\nabla P_s(b)|^2(1-|z|^2)^{1-2s} + |P_s(b)|^2(1-|z|^2)^{-2s}) \, \mathrm{d}m(z)$ is a Carleson measure for $P_s(H^s(\mathbb{T}))$.

Proof. Let $\varphi \in H^s(\mathbb{T})$. Applying lemma 2.2, we deduce that

$$\begin{aligned}\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{-2s} \, \mathrm{d}m(z) &\lesssim \int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{1-2s} \, \mathrm{d}m(z) \\ &+ \int_{\mathbb{D}} |P_s(\varphi)||\nabla P_s(\varphi)|P_s(b)|^2 (1-|z|^2)^{1-2s} \, \mathrm{d}m(z) \\ &+ \int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)||\nabla P_s(b)|(1-|z|^2)^{1-2s} \, \mathrm{d}m(z) = I + II + III.\end{aligned}$$

Now, we use the pointwise estimate for extensions of L^2 functions in $\mathbb{T}$ given in [3] lemma 2.8, which gives in particular that $|P_s(b)(z)| \lesssim (1/((1-|z|^2)^{1/2}))$, together with corollary 3.6 to obtain

$$I \lesssim \int_{\mathbb{D}} |P_s(\varphi)|^2 (1-|z|^2)^{-2s} \, \mathrm{d}m(z) \lesssim \|P_s(\varphi)\|^2_{W^2_{1,1-2s}} \approx \|\varphi\|^2_{H^s(\mathbb{T})}.$$

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Next, Hölder's inequality and the same pointwise estimate $|P_s(b)(z)| \lesssim ((1/((1-|z|^2)^{1/2}))$, give that

$$\begin{aligned} II &\leqslant \left(\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{-2s}\, \mathrm{d}m(z)\right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{D}} |\nabla P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{2-2s}\, \mathrm{d}m(z)\right)^{1/2} \\ &\leqslant \frac{\varepsilon}{2}\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{-2s}\, \mathrm{d}m(z) + \frac{1}{2\varepsilon}\|P_s(\varphi)\|^2_{W^2_{1,1-2s}}, \end{aligned}$$

where $\varepsilon < 1$. Hence, $II \lesssim \|P_s(\varphi)\|^2_{W^2_{1,1-2s}} \approx \|\varphi\|^2_{H^s(\mathbb{T})}$.

Finally, Hölder's inequality, the hypothesis and the pointwise estimate $|P_s(b)(z)| \lesssim ((1/((1-|z|^2)^{1/2}))$ give

$$\begin{aligned} III &\leqslant \left(\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{1-2s}\, \mathrm{d}m(z)\right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{D}} |P_s(\varphi)|^2 |\nabla P_s(b)|^2 (1-|z|^2)^{1-2s}\, \mathrm{d}m(z)\right)^{1/2} \\ &\lesssim \left(\int_{\mathbb{D}} |P_s(\varphi)|^2 (1-|z|^2)^{-2s}\, \mathrm{d}m(z)\right)^{1/2} \|P_s(\varphi)\|_{W^2_{1,1-2s}} \\ &\lesssim \|P_s(\varphi)\|^2_{W^2_{1,1-2s}} \approx \|\varphi\|^2_{H^s(\mathbb{T})}. \end{aligned}$$

Altogether gives finally that

$$\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{-2s}\, \mathrm{d}m(z) \lesssim \|P_s(\varphi)\|^2_{W^2_{1,1-2s}} \approx \|\varphi\|^2_{H^s(\mathbb{T})}.$$

□

LEMMA 6.11. *The following assertions are equivalent:*

(i) *The measure* $\mathrm{d}\nu := \left|(-\Delta)^{s/2}(b)\right|^2 \mathrm{d}\sigma$ *is a trace measure for* $H^s(\mathbb{T})$.

(ii) *The measure* $\mathrm{d}\widetilde{\nu} := \left|((-\Delta)^s + I)^{1/2}(b)\right|^2 \mathrm{d}\sigma$ *is a trace measure for* $H^s(\mathbb{T})$.

Proof. The proof is based in the following result by V. Maz'ya and I.E. Verbitsky (see [**19**]):

PROPOSITION 6.12. *Let* g *be an integrable function on* $\mathbb{T}$ *such that* $|g|^p\,\mathrm{d}\sigma$ *is a trace measure for* $I_s[L^p]$. *Let* h *be a measurable function on* $\mathbb{T}$ *satisfying that for any weight* w *in* A_1,

$$\int_{\mathbb{T}} |h|^p w \lesssim \int_{\mathbb{T}} |g|^p w.$$

We then have that $|h|^p\,\mathrm{d}\sigma$ *is a trace measure for* $I_s[L^p]$.

Assume that (6.11) holds, that is $\left|(-\Delta)^{s/2}(b)\right|^2 \,\mathrm{d}\sigma$ is a trace measure for $H^s(\mathbb{T})$. Let $h = (-\Delta)^{s/2}(b)$ and $g = ((-\Delta)^s + I)^{1/2}(b)$.

Then $g = ((-\Delta)^s + I)^{1/2}(-\Delta)^{-s/2}(-\Delta)^{s/2}b = Th$, where $T = ((-\Delta)^s + I)^{1/2}(-\Delta)^{-s/2}$. Applying corollary 3.5 and lemma 4.2 and using an argument similar to the one used in proposition 4.7, we deduce that T is an operator of Calderón–Zygmund type.

Hence, applying Theorem 4.10, we have that for any $\omega \in A_1 \subset A_2$,

$$\int_{\mathbb{T}} |g|^2 \,\mathrm{d}\omega = \int_T |Th|^2 \,\mathrm{d}\omega \lesssim \int_{\mathbb{T}} |h|^2 \,\mathrm{d}\omega.$$

Now, proposition 6.12 gives that $\mathrm{d}\widetilde{\nu}$ is a trace measure for $H^s(\mathbb{T})$, which is (6.11).

The implication in the other sense is proved in an analogous way. □

7. Proof of the main result (theorem 1.1)

7.1. Proof of (i) ⇔ (ii) ⇔ (iii)

If $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{T})$, then $P_s(\varphi)$, $P_s(\varphi)$ and $P_s(\varphi\psi)$ are in $\mathcal{W}^2_{1,1-2s} \cap L^\infty$. Hence $P_s(\varphi)P_s(\varphi) \in \mathcal{W}^2_{1,1-2s}$ with the same boundary values, $\varphi\psi$, as the function $P_s(\varphi\psi)$. Consequently, the equivalences between (i), (ii) and (iii) follow from proposition 3.7.

7.2. Proof of (v) ⇒ (iii)

We first observe that if

$$|\nabla P_s(b)|^2(1-|z|^2)^{1-2s} \,\mathrm{d}m(z)$$

is a Carleson measure for $P_s(H^s(\mathbb{T}))$, lemma 6.10 gives that the measure

$$\left(|\nabla P_s(b)|^2(1-|z|^2)^{1-2s} + |P_s(b)|^2(1-|z|^2)^{-2s}\right) \,\mathrm{d}m(z)$$

is also a Carleson measure for $P_s(H^s(\mathbb{T}))$.

Next, in order to prove (iii), it is enough to consider the case $\varphi = \psi$. Then Hölder's inequality and the above observation gives that

$$\begin{aligned}
&\left|\int_{\mathbb{D}} \nabla(P_s(\varphi)^2)(z)\nabla P_s(b)(z)(1-|z|^2)^{1-2s} \,\mathrm{d}m(z)\right| \\
&\lesssim \int_{\mathbb{D}} |P_s(\varphi)(z)||\nabla P_s(\varphi)(z)||\nabla P_s(b)(z)|(1-|z|^2)^{1-2s} \,\mathrm{d}m(z) \\
&\leqslant \left(\int_{\mathbb{D}} |P_s(\varphi)(z)|^2|\nabla P_s(b)|^2(1-|z|^2)^{1-2s} \,\mathrm{d}m(z)\right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{D}} |\nabla P_s(\varphi)(z)|^2(1-|z|^2)^{1-2s} \,\mathrm{d}m(z)\right)^{1/2} \lesssim \|\varphi\|^2_{H^s(\mathbb{T})}.
\end{aligned}$$

Similarly,

$$\left|\int_{\mathbb{D}} (P_s(\varphi))^2(z)P_s(b)(z)(1-|z|^2)^{-2s} \,\mathrm{d}m(z)\right| \lesssim \|\varphi\|^2_{H^s(\mathbb{T})},$$

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7.3. Proof of (iv) $\Rightarrow$ (v)

By lemmas 6.10 and 6.11 it is enough to show that if $|((-\Delta)^s+I)^{1/2}(b)|^2\,\mathrm{d}\sigma$ is a trace measure for H^s, then $|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z)$ is a Carleson measure for $P_s(H^s(\mathbb{T})(\mathbb{T}))$. Using theorem 6.9, we must show that for any closed set $E\subset\mathbb{T}$, $\int_{T(E)}|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z)\lesssim \mathrm{Cap}_s(E)$.

Let $E\subset\mathbb{T}$ be closed and let p_E be the potential of the extremal measure for the set E. For $z\in\mathbb{D}$, let $I_z=\{\zeta\in\mathbb{T};\, z\in\Gamma(\zeta)\}$. We have that if $z\in T(E)$, then $I_z\subset E$ and $|I_z|\approx(1-|z|^2)$. Let $\alpha\in(1/2,1/(2(1-2s))]$. Then, lemma 6.4, gives that $p_E^{2\alpha}\in A_1$, and, in particular, $p_E^{2\alpha}\in A_2$.

Since $p_E\gtrsim 1$ a.e. on E, Fubini's theorem gives,

$$\begin{aligned}
&\int_{T(E)}|\nabla P_s(b)(z)|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z)\\
&\lesssim \int_{\mathbb{D}}|\nabla P_s(b)(z)|^2(1-|z|^2)^{1-2s}\frac{1}{(1-|z|^2)}\int_{I_z}p_E^{2\alpha}(\zeta)\,\mathrm{d}\sigma(\zeta)\,\mathrm{d}m(z)\\
&\lesssim \int_{\mathbb{T}}\int_{\Gamma(\zeta)}|\nabla P_s(b)(z)|^2(1-|z|^2)^{-2s}p_E^{2\alpha}(\zeta)\,\mathrm{d}m(z)\,\mathrm{d}\sigma(\zeta)\\
&= \int_{\mathbb{T}}\int_{\Gamma(\zeta)}|(1-|z|^2)^{-s}\nabla P_s(I_s(I_s^{-1}(b)))(z)|^2p_E^{2\alpha}(\zeta)\,\mathrm{d}m(z)\,\mathrm{d}\sigma(\zeta)\\
&= \|G_{\mathbf{K}}(I_s^{-1}(b))\|^2_{L^2(p_E^{2\alpha})}.
\end{aligned}$$

Since $p_E^{2\alpha}\in A_2$, proposition 5.2 and theorem 5.1 give that the above is bounded by

$$\left(\int_{\mathbb{T}}|I_s^{-1}(b)(\zeta)|^2p_E^{2\alpha}(\zeta)\,\mathrm{d}\sigma(\zeta)\right)^2=\|I_s^{-1}(b)\|^2_{L^2(p_E^{2\alpha})}.$$

Since $I_s^{-1}=I_{2s}^{1/2}I_s^{-1}I_{2s}^{-1/2}$, theorem 4.10 gives

$$\begin{aligned}
\|I_s^{-1}(b)\|^2_{L^2(p_E^{2\alpha})}&=\|I_{2s}^{1/2}I_s^{-1}I_{2s}^{-1/2}(b)\|^2_{L^2(p_E^{2\alpha})}\lesssim\|I_{2s}^{-1/2}(b)\|^2_{L^2(p_E^{2\alpha})}\\
&\leqslant \liminf_{\delta\to 0^+}\int_{\mathbb{T}}p_{E,\delta}^{2\alpha}(\zeta)|I_{2s}^{-1/2}(b)(\zeta)|^2\,\mathrm{d}\sigma(\zeta)
\end{aligned}$$

But $I_{2s}^{-1/2}=((-\Delta)^s+I)^{1/2}$ and $|I_{2s}^{-1/2}(b)(\zeta)|^2\,\mathrm{d}\sigma$ is by hypothesis a trace measure for $H^s(\mathbb{T})$. Then we have that the above is bounded by $\liminf_{\delta\to 0}\|p_{E,\delta}^{\alpha}\|^2_{H^s(\mathbb{T})}$, which by theorem 6.5, is in turn bounded by $\mathrm{Cap}_s(E)$.

7.4. Proof of (i) $\Rightarrow$ (iv)

By lemma 6.11, we have that proving condition (iv) is equivalent to proving that $\mathrm{d}\mu(\zeta)=|((-\Delta)^s+I)^{1/2}(b)(\zeta)|^2\,\mathrm{d}\sigma(\zeta)=|I_{2s}^{-1/2}(b)(\zeta)|^2\,\mathrm{d}\sigma(\zeta)$ is a trace measure for H^s. This will be checked by proving that it satisfies the capacitary characterization given in proposition 6.6, that is, we will show that for each compact

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set $E \subset \mathbb{T}$,

$$\int_E |I_{2s}^{-1/2}(b)(\zeta)|^2 \,\mathrm{d}\sigma(\zeta) \lesssim \mathrm{Cap}_s(E).$$

Let $E \subset \mathbb{T}$ be a closed subset of $\mathbb{T}$ and let $p_{E,\delta} = I_{2s} * \nu_{E,\delta}$, $\delta > 0$, where $\nu_{E,\delta}$ is a regularization of the extremal capacitary measure of E. Let $\alpha \in (1/2, 1/(2(1-2s)))$ be fixed. We consider the test functions

$$\varphi_\delta := \frac{I_{2s}^{1/2}(\chi_E I_{2s}^{-1/2}(b))}{p_{E,\delta}^\alpha}, \quad \psi_\delta := p_{E,\delta}^\alpha.$$

We write $g_E = \chi_E(I_{2s}^{-1/2}(b))$. Applying the hypothesis (i), we have that

$$\int_E |I_{2s}^{-1/2}(b)|^2 \,\mathrm{d}\sigma = \int_{\mathbb{T}} I_{2s}^{-1/2}(\varphi_\delta \psi_\delta) I_{2s}^{-1/2}(b) \,\mathrm{d}\sigma \lesssim \|\varphi_\delta\|_{H^s(\mathbb{T})} \|\psi_\delta\|_{H^s(\mathbb{T})}. \tag{7.1}$$

We next estimate each of these last norms. First, we have that by theorem 6.5, $\|\psi_\delta\|^2_{H^s(\mathbb{T})} = \|p^\alpha_{E,\delta}\|^2_{H^s(\mathbb{T})} \lesssim \mathrm{Cap}_s(E)$. Our next objective is to prove that

$$\lim_{\delta \to 0} \|\varphi_\delta\|^2_{H^s(\mathbb{T})} \lesssim \int_{\mathbb{T}} |g_E|^2 \,\mathrm{d}\sigma. \tag{7.2}$$

If this estimate holds, we will have by (7.1) that $\int_E |I_{2s}^{-1/2}(b)|^2 \,\mathrm{d}\sigma \lesssim \mathrm{Cap}_s(E)$, which is the estimate we wanted to prove.

Using lemma 2.1,

$$\begin{aligned}
\|\varphi_\delta\|^2_{H^s(\mathbb{T})} &\lesssim \int_{\mathbb{D}} \left| \nabla \left(\frac{P_s\left(I_{2s}^{1/2}(g_E)\right)}{(P_s(p_{E,\delta}))^\alpha} \right) \right|^2 (1-|z|^2)^{1-2s} \,\mathrm{d}m(z) \\
&\quad + \int_{\mathbb{D}} \left| \frac{P_s\left(I_{2s}^{1/2}(g_E)\right)}{(P_s(p_{E,\delta}))^\alpha} \right|^2 (1-|z|^2)^{1-2s} \,\mathrm{d}m(z) \\
&\lesssim \int_{\mathbb{D}} \left| \frac{\nabla\left(P_s\left(I_{2s}^{1/2}(g_E)\right)\right)}{(P_s(p_{E,\delta}))^\alpha} \right|^2 (1-|z|^2)^{1-2s} \,\mathrm{d}m(z) \\
&\quad + \int_{\mathbb{D}} \frac{\left| P_s\left(I_{2s}^{1/2}(g_E)\right) \nabla\left(P_s(p_{E,\delta})\right)\right|^2}{(P_s(p_{E,\delta}))^{2\alpha+2}} (1-|z|^2)^{1-2s} \,\mathrm{d}m(z) \\
&\quad + \int_{\mathbb{D}} \left| \frac{P_s\left(I_{2s}^{1/2}(g_E)\right)}{(P_s(p_{E,\delta}))^\alpha} \right|^2 (1-|z|^2)^{1-2s} \,\mathrm{d}m(z) \\
&= I + II + III.
\end{aligned}$$

We begin with the estimate of I. Let $z \in \mathbb{D}$. We have that

$$P_s(p_{E,\delta})(z) \gtrsim \frac{1}{|I_z|} \int_{I_z} p_{E,\delta}.$$

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Using this estimate and Hölder's inequality twice, we obtain that

$$I \lesssim \int_{\mathbb{D}} |\nabla \left(P_s(I_{2s}^{1/2}(g_E))\right)|^2(1-|z|^2)^{1-2s}\left(\frac{1}{|I_z|}\int_{I_z} p_{E,\delta}(\eta)\,\mathrm{d}\sigma(\eta)\right)^{-2\alpha}\mathrm{d}m(z)$$

$$\lesssim \int_{\mathbb{D}} |\nabla \left(P_s(I_{2s}^{1/2}(g_E))\right)|^2(1-|z|^2)^{1-2s}\left(\frac{1}{|I_z|}\int_{I_z} p^{-1}_{E,\delta}(\eta)\,\mathrm{d}\sigma(\eta)\right)^{2\alpha}\mathrm{d}m(z)$$

$$\lesssim \int_{\mathbb{T}}\int_{\Gamma(\zeta)} |\nabla \left(P_s(I_sI_s^{-1}I_{2s}^{1/2}(g_E))\right)|^2(1-|z|^2)^{-2s}\,\mathrm{d}m(z)\frac{1}{p^{2\alpha}_{E,\delta}}(\eta)\,\mathrm{d}\sigma(\eta).$$

Since by lemma 6.4, $p^{2\alpha}_{E,\delta}\in A_2$ with constants independent of E and δ (and hence also $p^{-2\alpha}_{E,\delta}\in A_2$), proposition 5.2 and theorem 5.1 give that the above is bounded, up to a constant, by

$$\int_{\mathbb{T}} |I_s^{-1}I_{2s}^{1/2}(g_E)|^2\frac{1}{p^{2\alpha}_{E,\delta}}(\eta)\,\mathrm{d}\sigma(\eta) \lesssim \int_{\mathbb{T}} g_E^2(\eta)\frac{1}{p^{2\alpha}_{E,\delta}}(\eta)\,\mathrm{d}\sigma(\eta),$$

where in the last estimate we have used theorem 4.10, since $p^{-2\alpha}_{E,\delta}\in A_2$. Altogether we deduce that

$$I \lesssim \int_{\mathbb{T}} g_E^2(\eta)\frac{1}{p^{2\alpha}_{E,\delta}}(\eta)\,\mathrm{d}\sigma(\eta).$$

Now we proceed to estimate II. We consider the form given by

$$\omega_\delta(z) = \frac{(P_s(I_{2s}^{1/2}g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha+1}}(1-|z|^2)^{1-2s}\left(\frac{\partial P_s(p_{E,\delta})}{\partial x}\,\mathrm{d}y - \frac{\partial P_s(p_{E,\delta})}{\partial y}\,\mathrm{d}x\right).$$

Integrating on the circle of radius $r<1$, taking polar coordinates and letting $r\to 1^-$, we have (see theorem 3.3) that

$$\lim_{r\to 1^-}\int_{\partial D_r}\omega_\delta = \lim_{r\to 1^-}\int_{\partial D_r}(1-r^2)^{1-2s}r\frac{(P_s(I_{2s}^{1/2}g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha+1}}\frac{\partial}{\partial r}P_s(p_{E,\delta})$$

$$= \int_{\mathbb{T}} \frac{(I_{2s}^{1/2}g_{E_\delta})^2}{p^{2\alpha+1}_{E,\delta}}((-\Delta)^s+I)p_{E,\delta} = \int_{\mathbb{T}}\frac{(I_{2s}^{1/2}g_{E_\delta})^2}{p^{2\alpha+1}_{E,\delta}}\,\mathrm{d}\nu_\delta \geqslant 0.$$

Applying Stokes's Theorem on D_r, and letting $r\to 1^-$ as in theorem 3.4, we have that

$$\int_{\mathbb{T}}\omega_\delta = \int_{\mathbb{T}}\frac{I_{2s}^{1/2}g^2_{E_\delta}}{p^{2\alpha+1}_{E,\delta}}\,\mathrm{d}\nu_\delta$$

$$= -(2\alpha+1)\int_{\mathbb{D}}\frac{(P_s(I_s^{1/2}g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha+2}}|\nabla P_s(p_{E,\delta})|^2(1-|z|^2)^{1-2s}\,\mathrm{d}m(z)$$

$$+ 2\int_{\mathbb{D}}\frac{P_s(I_{2s}^{1/2}g_E)\nabla P_s(I_{2s}^{1/2}g_E)\nabla P_s(p_{E,\delta})}{P_s(p_{E,\delta})^{2\alpha+1}}(1-|z|^2)^{1-2s}\,\mathrm{d}m(z)$$

$$+ (1-2s)^2\int_{\mathbb{D}}\frac{(P_s(I_s^{1/2}g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha+1}}\int_{\mathbb{T}}\frac{p_{E,\delta}(\zeta)}{|z-\zeta|^{1+2s}}\mathrm{d}\sigma(\zeta)\,\mathrm{d}m(z).$$

Since we have shown that $\int_{\mathbb{T}} \omega_\delta \geqslant 0$, we deduce that

$$II \lesssim \int_{\mathbb{D}} \frac{P_s(I_{2s}^{1/2} g_E) \nabla P_s(I_{2s}^{1/2} g_E) \nabla P_s(p_{E,\delta})}{P_s(p_{E,\delta})^{2\alpha+1}} (1-|z|^2)^{1-2s} \,\mathrm{d}m(z)$$
$$+ \int_{\mathbb{D}} \frac{(P_s(I_s^{1/2} g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha}} (1-|z|^2)^{-2s} \,\mathrm{d}m(z).$$

Next, we proceed to estimate the first term on the right. Hölder's inequality gives that

$$\int_{\mathbb{D}} \frac{P_s(I_{2s}^{1/2} g_E) \nabla P_s(I_{2s}^{1/2} g_E) \nabla P_s(p_{E,\delta})}{P_s(p_{E,\delta})^{2\alpha+1}} (1-|z|^2)^{1-2s} \,\mathrm{d}m(z)$$
$$\lesssim \left(\int_{\mathbb{D}} \frac{|P_s\left(I_{2s}^{1/2}(g_E)\right)|^2 |\nabla\left(P_s(p_{E,\delta})\right)|^2}{|P_s(p_{E,\delta})|^{2\alpha+2}} (1-|z|^2)^{1-2s} \,\mathrm{d}m(z) \right)^{1/2}$$
$$\times \left(\int_{\mathbb{D}} \frac{|\nabla P_s\left(I_{2s}^{1/2}(g_E)\right)|^2}{P_s(p_{E,\delta})^{2\alpha}} (1-|z|^2)^{1-2s} \,\mathrm{d}m(z) \right)^{1/2}$$
$$= II^{1/2} I^{1/2} \lesssim (1/\varepsilon) I + \varepsilon II.$$

In addition, (2.1) gives that

$$\int_{\mathbb{D}} \frac{(P_s(I_s^{1/2} g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha}} (1-|z|^2)^{-2s} \,\mathrm{d}m(z) \lesssim III + \varepsilon(I + II).$$

Consequently, we have shown that

$$II \lesssim I + III. \tag{7.3}$$

Next, if we now choose $0 < \varepsilon' < 1$, we have

$$III \lesssim$$
$$= \int_{1-|z|^2<\varepsilon'} \frac{P_s(I_{2s}^{1/2} g_E)^2}{P_s(p_{E,\delta})^{2\alpha}} (1-|z|^2)^{1-2s} \,\mathrm{d}m(z)$$
$$+ \int_{1-|z|^2\geqslant\varepsilon'} \frac{P_s(I_{2s}^{1/2} g_E)^2}{P_s(p_{E,\delta})^{2\alpha}} (1-|z|^2)^{1-2s} \,\mathrm{d}m(z).$$

Since in the first integral $1-|z|^2 < \varepsilon'$, using (2.1) and (7.3), it is bounded by $\varepsilon'(I + II + III)$. We pass that to the left-hand side and obtain that

$$I + II + III \lesssim I + \int_{1-|z|^2\geqslant\varepsilon'} \frac{P_s(I_{2s}^{1/2} g_E)^2}{P_s(p_{E,\delta})^{2\alpha}} (1-|z|^2)^{1-2s} \,\mathrm{d}m(z). \tag{7.4}$$

But, when $1-|z|^2 \geqslant \varepsilon'$, we have that $P_s(I_{2s}^{1/2}(g_E)) \approx \int_{\mathbb{T}} I_{2s}^{1/2}(g_E)$ and $P_s(p_{E,\delta}) \approx \int_{\mathbb{T}} p_{E,\delta}$.

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Hence, using Hölder's inequality and that by lemma 6.4, $p_{E,\delta}^{2\alpha} \in A_2$, with constants independent of E and δ we have that

$$\int_{1-|z|^2\geqslant\varepsilon'} \frac{P_s(I_{2s}^{1/2} g_E)^2}{P_s(p_{E,\delta})^{2\alpha}}(1-|z|^2)^{1-2s}\,\mathrm{d}m(z)$$
$$\lesssim \frac{\left(\int_{\mathbb{T}} I_{2s}^{1/2}(g_E)\right)^2}{\left(\int_{\mathbb{T}} p_{E,\delta}\right)^{2\alpha}} \lesssim \frac{\left(\int_{\mathbb{T}} I_{2s}^{1/2}(g_E)\right)^2}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}}. \tag{7.5}$$

Fubini's Theorem and the fact that the operator $I_{2s}^{1/2}$ can be represented as a convolution by a kernel $T_{\mathbf{K}}(\zeta,\eta)$ satisfying that $T_{\mathbf{K}}(\zeta,\eta) \lesssim (1/(|\zeta-\eta|^{1-s}))$ (see remark 4.11), give that

$$\left|\int_{\mathbb{T}} I_{2s}^{1/2}(g_E)(\zeta)\,\mathrm{d}\sigma(\zeta)\right| \lesssim \left|\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{1}{|\zeta-\eta|^{1-s}}\mathrm{d}\sigma(\eta) g_E(\zeta)\,\mathrm{d}\sigma(\zeta)\right| \lesssim \int_{\mathbb{T}} |g_E(\zeta)|\,\mathrm{d}\sigma(\zeta).$$

Plugging this estimate in (7.5), and using that g_E is supported on E, we have that Hölder's inequality gives that

$$\frac{\left(\displaystyle\int_{\mathbb{T}} I_{2s}^{1/2}(g_E)\,\mathrm{d}\sigma\right)^2}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}\,\mathrm{d}\sigma} \lesssim \frac{\left(\int_{\mathbb{T}} |g_E(\zeta)|\,\mathrm{d}\sigma(\zeta)\right)^2}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}\,\mathrm{d}\sigma} \lesssim \frac{m(E)\int_{\mathbb{T}} g_E^2(\zeta)\,\mathrm{d}\sigma(\zeta)}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}\,\mathrm{d}\sigma}.$$

So, we have just proved that, using (7.1) and (7.4), that

$$\|\varphi_\delta\|_{H^s(\mathbb{T})}^2 \lesssim I + II + III \lesssim \int_{\mathbb{T}} g_E^2(\eta)\frac{1}{p_{E,\delta}^{2\alpha}}(\eta)\,\mathrm{d}\sigma(\eta) + \frac{m(E)\int_{\mathbb{T}} g_E^2(\zeta)\,\mathrm{d}\sigma(\zeta)}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}\,\mathrm{d}\sigma}.$$

We next have that $p_{E,\delta}^{2\alpha}$ is bounded above, and also bounded below (with constant depending on E) since,

$$p_{E,\delta}(\zeta) = \int_{\mathbb{T}} \frac{\mathrm{d}\nu_\delta(\eta)}{|1-\zeta\bar{\eta}|^{1-2s}} \gtrsim \int_{\mathbb{T}} \mathrm{d}\nu_\delta(\eta) = \nu_s(\mathbb{T}) = \nu(\mathbb{T}).$$

Hence, we can apply the Lebesgue's Dominated Convergence Theorem and deduce that

$$\lim_{\delta\to 0} \|\varphi_\delta\|_{H^s(\mathbb{T})}^2 \lesssim \int_{\mathbb{T}} g_E^2(\eta)\frac{1}{p_E^{2\alpha}}(\eta)\,\mathrm{d}\sigma(\eta) + \frac{m(E)\int_{\mathbb{T}} g_E^2(\zeta)\,\mathrm{d}\sigma(\zeta)}{\int_{\mathbb{T}} p_E^{2\alpha}\,\mathrm{d}\sigma}.$$

Next, since g_E is supported on E and $p_E \gtrsim 1$ on E, we deduce that

$$\lim_{\delta\to 0} \|\varphi_\delta\|_{H^s(\mathbb{T})}^2 \lesssim \int_{\mathbb{T}} g_E^2(\eta)\,\mathrm{d}\sigma(\eta),$$

which prove (7.2) and, as it was pointed out, finishes the proof of the theorem.

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8. Appendix: Proof of theorem 5.1

The proof of theorem 5.1 follows the scheme given in [16]. In consequence, we will just sketch the specific parts of the proof for our situation and remit to this paper to find the proofs of the remaining parts used here. We recall some definitions.

If f is a measurable function on $\mathbb{T}$ and Q is an interval on $\mathbb{T}$, the local mean oscillation of f on Q is given by

$$\omega_\lambda(f;Q) = \inf_{c\in\mathbb{R}}((f-c)\chi_Q)^*(\lambda|Q|), \quad 0<\lambda<1,$$

where $((f-c)\chi_Q)^*$ is the non-increasing rearrangement of $(f-c)\chi_Q$.

Let $m(f,Q)$ be the median value of f over Q, as a (possibly non-unique) real number such that

$$\max\left(|\{\zeta\in Q\,;\, f(\zeta)>m(f,Q)\}|, |\{\zeta\in Q\,;\, f(\zeta)<m_f(Q)\}|\right) \leqslant |Q|/2.$$

Next, given an interval Q_0, let us denote $\mathcal{D}(Q_0)$ the dyadic intervals with respect to Q_0. The dyadic local sharp maximal function $m^{\#,d}_{\lambda;Q_0}f$ is defined by

$$m^{\#}_{\lambda;Q_0}f(\zeta) = \sup_{\zeta\in Q'\in\mathcal{D}(Q_0)} \omega_\lambda(f;Q').$$

One key ingredient in the proof of the theorem is the decomposition of A.K. Lerner in terms of the local mean oscillation. In [4], it is proved the following version of Lerner's estimate for homogeneous spaces:

THEOREM 8.1. *Let f a measurable function on $\mathbb{T}$, $\mathcal{D}$ a dyadic decomposition of intervals of $\mathbb{T}$. Let $Q_0\in\mathcal{D}$. Then there exists $\varepsilon>0$ and a (possibly empty) sparse family $\mathcal{S}(Q_0)$ of intervals in $\mathcal{D}$ included in Q_0 such that for a.e. $\zeta\in Q_0$,*

$$|f(\zeta)-m(f,Q_0)| \leqslant m^{\#}_{\varepsilon,Q_0}(f)(\zeta) + \sum_{Q\in\mathcal{S}(Q_0)} \omega_\varepsilon(f,Q)\chi_Q(\zeta).$$

We would like to apply this theorem to the function $f = G_{\mathbf{K}}(\varphi)^2$, and we will need to obtain estimates for $m^{\#}_{\varepsilon,Q_0}(G_{\mathbf{K}}(\varphi)^2)$ and $\omega_\varepsilon(G_{\mathbf{K}}(\varphi)^2,Q)$.

The following lemma follows from well-known techniques of splitting functions in 'good' and 'bad' parts, which come from a method stated by A.P. Calderón and A. Zygmund (see [16]).

LEMMA 8.2. *There exists $C>0$ such that for any $\lambda>0$, $f\in L^1(\mathbb{T})$,*

$$|\{\eta;\, G_{\mathbf{K}}(\varphi)(\eta)>\lambda\}| \lesssim \frac{\|\varphi\|_{L^1(\,\mathrm{d}\sigma)}}{\lambda}.$$

We now can prove the following version of Lerner's estimate:

LEMMA 8.3. *Let $0<\lambda<1$. Then, for any cube $Q\in\mathcal{D}_j$,*

$$\omega_\lambda(G_{\mathbf{K}}(\varphi)^2;Q) \lesssim \sum_{k\geqslant 0}\frac{1}{2^{ks}}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|\varphi|\right)^2.$$

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Proof. Let $Q \in \mathcal{D}_j$. We decompose $G_{\mathbf{K}}(\varphi)^2(\eta)$ in two terms given by

$$\begin{aligned} G_{\mathbf{K}}(\varphi)^2(\eta) &= \int_{T(2Q)} \chi_{\Gamma_\eta}(z) \left| \int_{\mathbb{T}} \mathbf{K}(z,\zeta)\varphi(\zeta)\,\mathrm{d}\sigma(\zeta) \right|^2 \frac{\mathrm{d}m(z)}{(1-|z|^2)^2} \\ &\quad + \int_{\mathbb{D}\setminus T(2Q)} \chi_{\Gamma_\eta}(z) \left| \int_{\mathbb{T}} \mathbf{K}(z,\zeta)\varphi(\zeta)\,\mathrm{d}\sigma(\zeta) \right|^2 \frac{\mathrm{d}m(z)}{(1-|z|^2)^2} \\ &= I_1(\varphi)(\eta) + I_2(\varphi)(\eta). \end{aligned}$$

We will then have that if ζ_1 is an arbitrary point in Q,

$$\begin{aligned} &\omega_\lambda(G_{\mathbf{K}}(\varphi)^2; Q) \\ &\quad \leqslant \left((G_{\mathbf{K}}(\varphi)^2 - I_2(\varphi)(\zeta_1))\chi_Q\right)^* (\lambda|Q|) \\ &\quad \lesssim (I_1(\varphi)\chi_Q)^* ((\lambda|Q|/2) + ((I_2(\varphi) - I_2(\varphi)(\zeta_1))\chi_Q)^* ((\lambda|Q|/2)) \\ &\quad \lesssim (I_1(\varphi)\chi_Q)^* ((\lambda|Q|)/2) + \|I_2(\varphi) - I_2(\varphi)(\zeta_1)\|_{L^\infty(Q)}. \end{aligned} \tag{8.1}$$

We will first show that

$$(I_1(\varphi)\chi_Q)^* (\lambda|Q|/2) \lesssim \sum_{2^k l_Q \leqslant 1} \frac{1}{2^k} \left(\frac{1}{|2^k Q|} \int_{2^k Q} |\varphi|\,\mathrm{d}\sigma \right)^2.$$

Since $(x+y)^2 \leqslant 2(x^2+y^2)$, we have that for any $\eta \in Q$,

$$I_1(\varphi)(\eta) \leqslant 2\left(I_1(\varphi\chi_{4Q})(\eta) + I_1(\varphi\chi_{\mathbb{T}\setminus 4Q})(\eta)\right),$$

and consequently,

$$(I_1(\varphi)\chi_Q)^* (\lambda|Q|/2) \lesssim (I_1(\varphi\chi_{4Q}))^* (\lambda|Q|/4) + \left(I_1(\varphi\chi_{\mathbb{T}\setminus 4Q})\right)^* (\lambda|Q|/4).$$

By lemma 8.2 we have that

$$(I_1(\varphi\chi_{4Q}))^* (\lambda|Q|/4) \leqslant \left((G_{\mathbf{K}}(\varphi\chi_{4Q}))^2\right)^* (\lambda|Q|/2) \lesssim \left(\frac{1}{|4Q|} \int_{4Q} |\varphi|\,\mathrm{d}\sigma \right)^2.$$

Consider now the term $(I_1(\varphi\chi_{\mathbb{T}\setminus 4Q}))^*(\lambda|Q|/2)$. It will be enough to obtain, for $z \in T(2Q)$, the following pointwise estimate:

$$|\mathbf{K}(\varphi\chi_{\mathbb{T}\setminus 4Q})(z)| \lesssim \left(\frac{1-|z|^2}{l_Q} \right)^s \sum_{k\geqslant 1;\, 2^k l_Q \lesssim 1} \frac{1}{2^{ks}} \frac{1}{2^k l_Q} \int_{2^k Q} |\varphi|\,\mathrm{d}\sigma. \tag{8.2}$$

Indeed, if (8.2) holds, then we will have that by Chebyshev's inequality

$$\begin{aligned} &\left(I_1(\varphi\chi_{\mathbb{D}\setminus 4Q})\chi_Q\right)^* (\lambda|Q|/4) \lesssim \frac{\|I_1(\varphi\chi_{\mathbb{T}\setminus 4B(Q)})\chi_Q\|_{L^1}}{(\lambda|Q|)/4} \\ &\lesssim \frac{4}{\lambda|Q|} \int_{\mathbb{T}} \int_{\mathbb{D}} \chi_{\Gamma_\eta}(z) \left| \int_{\mathbb{T}} \mathbf{K}(z,\zeta)\varphi(\zeta)\chi_{\mathbb{D}\setminus 4Q}(\zeta)\,\mathrm{d}\sigma(\zeta) \right|^2 (1-|z|^2)^{-2}\mathrm{d}V(z)\,\mathrm{d}\sigma(\eta) \\ &\lesssim \left(\sum_{k\geqslant 2,\, 2^k l_Q \leqslant 1} \frac{1}{2^{ks}} \frac{1}{|2^k Q|} \int_{2^k Q} |\varphi|\,\mathrm{d}\sigma \right)^2. \end{aligned}$$

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Consequently, applying Schwartz's inequality,

$$\left(I_1(f\chi_{\mathbb{T}\setminus 4Q})\chi_Q\right)^*(\lambda|Q|/4) \lesssim \sum_{k\geqslant 2,\, 2^k l_Q\leqslant 1} \frac{1}{2^{ks}}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|\varphi(\eta)|\,\mathrm{d}\sigma(\eta)\right)^2.$$

Let us prove (8.2). If $z\in T(2Q)$, we then have that

$$\begin{aligned}|\mathbf{K}(\varphi\chi_{\mathbb{T}\setminus 4Q})(z)| &= |\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)\chi_{\mathbb{T}\setminus 4Q}(\zeta)\,\mathrm{d}\sigma(\zeta)|\\ &\lesssim \left(\frac{(1-|z|^2)}{l_Q}\right)^s \sum_{k\geqslant 2,\, 2^k l_Q\leqslant 1}\frac{1}{2^{ks}}\frac{1}{2^k l_Q}\int_{2^kQ}|\varphi|\end{aligned}$$

So, in order to finish the proof of (8.1), we are left to estimate $\|I_2(\varphi)-I_2(\varphi)(\zeta_1)\|_{L^\infty(Q)}$. Let $\omega_1,\omega_2\in Q$. Then,

$$\begin{aligned}|I_2(\varphi)(\omega_1)-I_2(\varphi)(\omega_2)| \leqslant \sum_{k\geqslant 1,\, 2^k l_Q\leqslant 1}\int_{T(2^{k+1}Q)\setminus T(2^kQ)}\left|\chi_{\Gamma_{\omega_1}}(z)-\chi_{\Gamma_{\omega_2}}(z)\right|\\ \times\left|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)\,\mathrm{d}\sigma(\zeta)\right|^2\frac{\mathrm{d}V(z)}{(1-|z|^2)^2}.\end{aligned} \tag{8.3}$$

We split the points $z\in T(2^{k+1}Q)\setminus T(2^kQ)$ such that $\chi_{\Gamma_{\omega_1}}(z)-\chi_{\Gamma_{\omega_2}}(z)\neq 0$ in two connected sets, Ω_1^k,Ω_2^k. We will obtain estimates for the integrals over one of them, say Ω_1^k, being the estimates over Ω_2^k analogous. Then,

$$\begin{aligned}|I_2(\varphi)(\omega_1)-I_2(\varphi)(\omega_2)| &\lesssim \sum_{k\geqslant 1,\, 2^k l_Q\leqslant 1}\int_{\Omega_1^k}\left|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)\,\mathrm{d}\sigma(\zeta)\right|^2\frac{\mathrm{d}V(z)}{(1-|z|^2)^2}\\ &\lesssim \sum_{k\geqslant 1,\, 2^k l_Q\leqslant 1}\int_{\Omega_1^k}\left|\int_{2^kQ}\mathbf{K}(z,\zeta)\varphi(\zeta)\,\mathrm{d}\sigma(\zeta)\right.\\ &\quad\left.+\sum_{j>k,\, 2^j l_Q\leqslant 1}\int_{2^jQ\setminus 2^{j-1}Q}\mathbf{K}(z,\zeta)\varphi(\zeta)\,\mathrm{d}\sigma(\zeta)\right|^2\frac{\mathrm{d}V(z)}{(1-|z|^2)^2}.\end{aligned}$$

Next, observe that if $\zeta\in 2^kQ$ and $z\in\Omega_1^k$, we have that $(1-|z|^2)\approx 2^k l_Q$ and $|1-z\overline{\zeta}|\approx 2^k l_Q$. On the other hand, if $\zeta\in 2^jQ\setminus 2^{j-1}Q$, $j>k$, we have that $|1-z\overline{\zeta}|\gtrsim 2^j l_Q$. Altogether gives, integrating in polar coordinates on Ω_1^k and using the fact that the angle width is of order l_Q, whereas the line integral on r is of order $2^k l_Q$, that the above is bounded, up to constant, by

$$\frac{(2^k l_Q)^{2s}}{(2^k l_Q)^{2(1+s)}}\frac{l_Q}{(2^k l_Q)}\left(\int_{2^k l_Q}|\varphi|\,\mathrm{d}\sigma\right)^2+\frac{l_Q}{(2^k l_Q)}\left(\sum_{j>k,\, 2^j l_Q\leqslant 1}\frac{(2^k l_Q)^s\int_{2^jQ}|\varphi|\,\mathrm{d}\sigma}{(2^j l_Q)^{1+s}}\right)^2.$$

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Hence, adding up in k, we will have that (8.3) is bounded, up to a constant, by

$$\sum_{k\geqslant 1,\, 2^k l_Q\leqslant 1}\frac{1}{2^k}\left(\frac{1}{2^k l_Q}\int_{2^k l_Q}|\varphi|\,\mathrm{d}\sigma\right)^2$$
$$+\sum_{k\geqslant 1,\, 2^k l_Q\leqslant 1}\frac{1}{2^{k(1-2s)}}\left(\sum_{j>k,\, 2^j l_Q\leqslant 1}\frac{1}{2^{js}}\frac{1}{2^j l_Q}\int_{2^j Q}|\varphi|\right)^2$$

By Hölder's inequality, the above is bounded by

$$\sum_{k\geqslant 1,\, 2^k l_Q\leqslant 1}\frac{1}{2^{ks}}\left(\frac{1}{2^k l_Q}\int_{2^k l_Q}|\varphi)|\right)^2.$$

We now sketch how to finish the proof of theorem 5.1. First, lemma 8.3 gives that a.e. $\zeta\in Q$, $m^{\#}_{\lambda,Q}G(\psi)^2(\zeta)\lesssim M(\psi)(\zeta)^2$, where $M(\psi)$ denotes the Hardy-Littlewood maximal function. Next, we have that for any $Q\in\mathcal{D}^i$, there exists a sparse family $\mathcal{S}(Q)=(Q^k_j)$, $Q^k_j\in\mathcal{D}^i$ so that if we denote by

$$\mathcal{T}^{\mathcal{S}}_l(\psi)(\zeta)=\left(\sum_{Q^k_j\in\mathcal{S}(Q)}(\psi_{2^l B(Q^k_j)})^2\chi_{Q^k_j}(\zeta)\right)^{1/2},$$

then by theorem 8.1, we have that if

$$\mathcal{T}^{\mathcal{S}}(\psi)(\zeta)=\sum_{l\geqslant 0}\frac{1}{2^{l/4}}\mathcal{T}^{\mathcal{S}}_l(\psi)(\zeta),$$

then for a.e $\zeta\in Q$,

$$|G(\psi)(\zeta)^2-m_Q(G(\psi)^2)|\lesssim\left(M(\psi)(\zeta)^2+\sum_{l\geqslant 0}\frac{1}{2^{l/2}}\left(\mathcal{T}^{\mathcal{S}}_l(\psi)\right)^2\right).$$

Hence

$$|G(\psi)(\zeta)^2-m_Q(G(\psi)^2)|^{1/2}\lesssim M(\psi)(\zeta)+\mathcal{T}^{\mathcal{S}}(\psi)(\zeta),$$

where M is the Hardy-Littlewood maximal function.

It is proved in [**16**] that for any $\omega\in A_3$,

$$\|\mathcal{T}^{\mathcal{S}}(\psi)\|_{L^3(\omega)}\lesssim\|\psi\|_{L^3(\omega)}.$$

Observe that here we are not interested in obtaining sharpest estimates and in consequence, we could have chosen other index $p_0>2$ instead of $p_0=3$.

On the other hand, the Hardy-Littlewood maximal function maps $L^3(\omega)$ to $L^3(\omega)$, so, $\|M(\psi)\|_{L^3(\omega)} \lesssim \|\psi\|_{L^3(\omega)}$. Altogether gives that

$$\| \left(G(\psi)^2 - m_Q(G(\psi)^2)\right)^{1/2} \|_{L^3(\omega)} \lesssim \|\psi\|_{L^3(\omega)}.$$

Hence

$$\begin{aligned} \|G(\psi)\|_{L^3(\omega)} &= \|G(\psi)^2\|^{1/2}_{L^{3/2}(\omega)} \\ &\lesssim \|G(\psi)^2 - m_Q(G(\psi)^2)\|^{1/2}_{L^{3/2}(\omega)} + \|m_Q(G(\psi)^2)\|^{1/2}_{L^{3/2}(\omega)} \\ &\lesssim \|\psi\|_{L^3(\omega)} + \|m_Q(G(\psi)^2)\|^{1/2}_{L^{3/2}(\omega)}. \end{aligned}$$

Finally, it is proved in [**9**] that

$$\|m_Q(G(\psi)^2)\|^{1/2}_{L^{3/2}(\omega)} \lesssim \|\psi\|_{L^3(\omega)}.$$

Rubio de Francia's extrapolation theorem gives then that $\|G(\psi)\|_{L^p(\omega)} \lesssim \|\psi\|_{L^p(\omega)}$. □

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References

1 D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*. Grundlehren der Mathematischen Wissenschaften, vol. 314 (Berlin: Springer-Verlag, 1999).

2 P. Ahern and C. Cascante. Exceptional sets for Poisson integrals of potentials on the unit sphere in C^n, $p \leqslant 1$. *Pacific J. Math.* **153** (1992), 1–13.

3 P. Ahern, J. Bruna and C. Cascante. H^p-theory for generalized M-harmonic functions in the unit ball. *Indiana Univ. Math. J.* **45** (1996), 103–135.

4 T. C. Anderson and A. Vagharshakyan. A simple proof of the sharp weighted estimate for Calderon-Zygmund operators on homogeneous spaces. *J. Geom. Anal.* **24** (2014), 1276–1297.

5 N. Arcozzi, R. Rochberg, E. Sawyer and B. D. Wick. Bilinear forms on the Dirichlet space. *Anal. & PDE* **3** (2010), 21–47.

6 E. Berkson and T. A. Gillespie. On restrictions of multipliers in weighted spaces. *Indiana Univ. Math. J.* **52** (2003), 927–961.

7 L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Commun. Partial Diff. Equ.* **32** (2007), 1245–1260.

8 C. Cascante and J. M. Ortega. On a characterization of bilinear forms on the Dirichlet space. *Proc. Am. Math. Soc.* **140** (2012), 2429–2440.

https://doi.org/10.1017/prm.2019.16 Published online by Cambridge University Press

9 C. Cascante, J. Fàbrega and J. M. Ortega. Sharp norm estimates for the Bergman operator from weighted mixed-norm spaces to weighted Hardy space. *Canad. J. Math.* **68** (2016), 1257–1284.

10 L. Chaffee, J. Hart and L. Oliveira. Weighted multilinear square functions bounds. *Michigan Math. J.* **63** (2014), 371–400.

11 A. D. D. Craik. Prehistory of Faà di Bruno's Formula. *Am. Math. Mon. (Math. Assoc. Am.)* **112** (2005), 217–234.

12 D. Cruz-Uribe and C. Pérez. On two-weight problem for singular integral operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **I** (2002), 821–849.

13 A. Erdelyi. *Higher transcendental functions*, vol. 1 (New York: McGraw-Hill, 1953).

14 D. Kim. Trace theorems for Sobolev–Slobodeckij spaces with or without weights. *J. Funct. Spaces Appl.* **5** (2007), 243–268.

15 P. G. Lemarié-Rieusset and S. Gala. Multipliers between Sobolev spaces and fractional differentiation. *J. Math. Anal. Appl.* **322** (2006), 1030–1054.

16 A. K. Lerner. On sharp aperture-weighted estimates for square functions. *J. Fourier Anal. Appl.* **20** (2014), 784–800.

17 V. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations.* Grundlehren der Mathematischen Wissenschaften, vol. 342 (Berlin: Springer-Verlag, 2011).

18 V. Maz'ya and T.O. Shaposhnikova. *Theory of multipliers in spaces of differentiable functions.* Monographs and Studies in Mathematics, vol. 23 (Boston: Pitman, 1985).

19 V. G. Maz'ya and I. E. Verbitsky. Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. *Ark. Mat.* **33** (1995), 81–115.

20 V. Maz'ya and I. E. Verbitsky. The Schrödinger operator on the energy space: boundedness and compactness criteria. *Acta Math.* **188** (2002), 263–302.

21 V. Maz'ya and I. E. Verbitsky. The form boundedness criterion for the relativistic Schrödinger operator. *Ann. de l'Inst. Fourier* **54** (2004), 317–339.

22 J. Nečas. Sur un méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnell. *An. Scuona Norm. Sup. Pisa* **16** (1962), 305–326.

23 E.M. Stein. *Elias M. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton Mathematical Series, vol. 43 (Princeton, NJ: Princeton University Press, 1993).

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