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A_∞-structures, Brill–Noether loci and the Fourier–Mukai transform

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Abstract

We use the A_∞-formalism to study variation in cohomology spaces under formal deformations of coherent sheaves on projective varieties. As an application we describe formal neighborhoods of some singular points in twisted Brill–Noether loci in the moduli spaces of vector bundles on a curve. Another application is a computation of the Fourier–Mukai transform of some natural line bundles on symmetric powers of a curve.

0. Introduction

The goal of this paper is to show how the techniques of A_∞-categories can be applied to the study of variation of cohomology spaces of coherent sheaves under deformations.

Our starting point is the fact that the derived category $D^b(X)$ of coherent sheaves on a k-scheme X, where k is a field, can be equipped with a natural A_∞-category structure (canonical up to a homotopy). The notion of an A_∞-category generalizes the concept of an A_∞-algebra due to Stasheff [Sta63]. It was introduced by Fukaya in [Fuk93] in connection with Floer homology and then used by Kontsevich in his homological formulation of mirror symmetry (see [Kon95]). The A_∞-structure on the derived category $D^b(X)$ can be defined naturally using a dg-category producing $D^b(X)$ by passing to cohomology. This construction was first introduced by Kadeishvili [Kad82] in the setting of A_∞-algebras. The idea of considering additional structures on derived categories coming from dg-categories goes back to [BK90].

Let us assume that X is projective over k. The observation we make is that the A_∞-structure on $D^b(X)$ can be used to describe the variation of cohomology spaces under formal deformations of coherent sheaves on X. Our main result, Theorem 2.2, gives an explicit description in terms of the A_∞-structure of a complex that governs such a variation over a formal neighborhood of a given coherent sheaf in its moduli space. This theorem was inspired by the study by Green and Lazarsfeld of the variation of cohomology spaces under deformations of topologically trivial holomorphic line bundles on a Kähler manifold (see [GL91, Theorem 3.2]). Note that A_∞-structures do not appear in [GL91] since for topologically trivial line bundles on a Kähler manifold all higher products are homotopic to zero.

The main idea is that an A_∞-structure on $D^b(X)$ gives rise to a canonical formal deformation of every coherent sheaf on X. For sheaves with unobstructed deformations the obtained families are universal. In general we conjecture that they are miniversal. A homotopy between A_∞-structures leads to formal changes of variables in the corresponding formal coordinate systems on the moduli spaces. Thus, a choice of an A_∞-structure can be considered as an algebraic analog of choosing

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hermitian metrics on all vector bundles, so that the above construction is an algebraic analog of the Kuranishi construction (see [Fuk01] for more on this analogy). On the other hand, using an A_∞-structure one can control the variation of cohomology spaces in the above formal universal families. The crucial notion that helps organize these deformed spaces is that of an A_∞-functor. Namely, we show that for every object of an A_∞-category there is a canonical deformation of the corresponding representable A_∞-functor. In the case of a coherent sheaf on X this A_∞-functor corresponds to the canonical formal deformation mentioned earlier. Looking at the variation of values of this A_∞-functor on specific objects of $D^b(X)$ one can obtain information about the formal neighborhoods of the loci where the dimensions of cohomology jump.

As an application of our techniques we calculate formal neighborhoods of 'sufficiently nice' points in Brill–Noether loci parametrizing special vector bundles on curves. Recall that the classical Brill–Noether loci for a smooth projective curve C parametrize line bundles of given degree on C with a given number of linearly independent global sections. More precisely, for every $d \geqslant 0$, $r \geqslant 0$, one has a subscheme W_d^r in the Jacobian J^d of degree d line bundles on C, supported on the set of line bundles L (of degree d) with $h^0(L) > r$ (for a precise definition see [ACGH84]). Perhaps, the most important example of a Brill–Noether locus is the theta divisor, $\Theta = W_{g-1}^0 \subset J^{g-1}$, where g is the genus of C, consisting of line bundles L with $h^0(L) > 0$. Riemann's theorem asserts that the multiplicity of Θ at a point L is equal to $h^0(L)$. In [Kem73], Kempf generalized this theorem by describing tangent cones to points of $W_d := W_d^0$. The same techniques can be used to calculate tangent cones to some points of W_d^r for $r > 0$ (see [ACGH84]).

Similar Brill–Noether loci can be defined in the moduli spaces of stable (or semistable) vector bundles of higher rank on C. More generally, one can consider *twisted* Brill–Noether loci $W_{n,d}^r(E)$ parametrizing stable vector bundles V of rank n and degree d such that $h^0(V \otimes E) > r$, where E is a fixed vector bundle on C (see [Tex95]). Kempf's results admit partial generalization to these loci (see [Li93] for the case $E = \mathcal{O}_C$, [Tex98] for the case $n = 1$, $r = 0$). Using A_∞-techniques we will prove the following theorem complementing these results.

Theorem 0.1. *Let E be a vector bundle on C, V be a stable vector bundle on C of rank n and degree d, such that the natural map*

$$\mu_{V,E} : H^0(C, V \otimes E) \otimes H^0(C, V^\vee \otimes E^\vee \otimes \omega) \to H^0(V \otimes V^\vee \otimes \omega)$$

is injective. We think about $\mu_{V,E}$ as a matrix of linear forms on $T = \mathrm{Ext}^1(V, V) \simeq H^0(V \otimes V^\vee \otimes \omega)^*$. Then the formal neighborhood of $W_{n,d}^r(E)$ at V, where $r < h := h^0(V \otimes E)$, is isomorphic over k to the formal neighborhood of zero in the subscheme of T defined by the $(h-r) \times (h-r)$ minors of $\mu_{V,E}$.

Note that the case when $n = 1$, $E = \mathcal{O}_C$ and k is algebraically closed follows essentially from the definition of Brill–Noether loci (see § 3.1). However, the case $n = 1$, $\mathrm{rk}\, E > 1$ already seems to be non-trivial. The map $\mu_{V,E}$ is called the (generalized) *Gieseker–Petri map*. Its injectivity is equivalent to the condition that the smallest Brill–Noether locus associated with E containing V, namely, $W_{n,d}^{h-1}(E)$, where $h = h^0(V \otimes E)$, is smooth of expected dimension at V. In this situation, Theorem 0.1 describes the formal neighborhoods of all larger Brill–Noether loci $W_{n,d}^r(E)$, $r < h$, at V. Note that Kempf's theorem and its generalizations state that the tangent cone to $W_{n,d}^r(\mathcal{O})$ at V is isomorphic to the subscheme of T considered in Theorem 0.1 under a weaker assumption on V (see [Li93]).

Theorem 0.1 follows from a stronger result, Theorem 3.1, asserting that certain higher products associated with V and E are homotopic to zero. We expect that this statement should play a role in a non-commutative version of our results formulated in terms of canonical non-commutative thickenings of the moduli space of vector bundles on C (see [Kap98]).

We also apply our techniques to the study of the Fourier transform of certain line bundles on symmetric powers $\mathrm{Sym}^d C$ of a curve C. Namely, for a line bundle L on C let us denote by $L^{(d)}$ the dth symmetric power of L which is a line bundle on $\mathrm{Sym}^d C$. Let us denote by $F_d(L)$ the derived push-forward of $L^{(d)}$ under the natural morphism $\sigma^d : \mathrm{Sym}^d C \to J^d$. It is not difficult to show that if $\deg(L) \geqslant -1$ then $F_d(L)$ is actually a sheaf concentrated in degree 0 (see Lemma 3.2, part ii). We fix a point $p \in C$ and identify J^d with J by $L \mapsto L(-dp)$. Recall that for every abelian variety A the Fourier–Mukai transform is an equivalence $\mathcal{S} : D^b(A) \to D^b(\hat{A})$, where $\hat{A}$ is the dual abelian variety (see [Muk81]). Using the self-duality of J we can consider the Fourier–Mukai transform as an autoequivalence $\mathcal{S} : D^b(J) \to D^b(J)$. In § 3.2 we will prove the following theorem.

THEOREM 0.2. *Assume that $1 \leqslant d \leqslant g-1$. Then one has the following isomorphisms in $D^b(J)$:*

$$[-1]_J^* \mathcal{S}(F_d(\mathcal{O}_C((g-d)p))) \simeq F_{g-d}(\mathcal{O}_C(-p))(\Theta)[-d] \simeq R\underline{\mathrm{Hom}}(F_{g-d}(\mathcal{O}_C(dp)), \mathcal{O}_J),$$

where $[-1]_J : J \to J$ is the inversion map, $\Theta = W_{g-1} \subset J$ is the theta divisor.

This theorem provides a new collection of coherent sheaves on Jacobians for which W.I.T. holds (see [Muk81] for terminology and for other examples). Note that the case $d = 1$ was considered in [BP01] in connection with Torelli's theorem.

1. A_∞-structures

In this section we present some A_∞-formalism that, for the most part, is well known.

1.1 A_∞-categories and functors

For more details concerning most of the following definitions the reader can consult [Kel01].

Let k be a field. All the categories (and A_∞-categories) considered below are going to be k-linear. This means that all morphism spaces are k-vector spaces and all operations are k-linear. By the Koszul sign rule we mean the appearance of $(-1)^{\widetilde{a}\widetilde{b}}$ when switching graded symbols a and b, where we use the notation $\widetilde{a} = \deg(a)$.

DEFINITION 1.1.

i) An A_∞-*category* $\mathcal{C}$ consists of a class of objects and a collection of graded morphism spaces $\mathrm{Hom}^*(O_1, O_2) = \mathrm{Hom}^*_{\mathcal{C}}(O_1, O_2)$ for every pair of objects O_1, O_2 equipped with the operations

$$m_n : \mathrm{Hom}^*(O_2, O_1) \otimes_k \mathrm{Hom}^*(O_3, O_2) \otimes_k \cdots \otimes_k \mathrm{Hom}^*(O_{n+1}, O_n) \to \mathrm{Hom}^*(O_{n+1}, O_1),$$

where $n = 1, 2, 3, \ldots$, homogeneous of degree $2 - n$. These operations satisfy the following A_∞-*constraint*:

$$\sum_{k+l=n+1} \sum_{j=1}^{k} (-1)^{j+l(k-j)+\epsilon} m_k(a_1, \ldots, a_{j-1}, m_l(a_j, \ldots, a_{j+l-1}), a_{j+l}, \ldots, a_n) = 0,$$

where $n = 1, 2, 3 \ldots$, $(-1)^\epsilon$ comes from the Koszul sign rule (m_l is exchanged with $a_1, \ldots, a_{j-1}$).

ii) An A_∞-category $\mathcal{C}$ is called *minimal* if $m_1 = 0$.

In the following by a *non-unital* category we mean a version of the notion of a category in which the existence of identity morphisms is not required.

Example 1.1. A (non-unital) *dg-category* is an A_∞-category with $m_n = 0$ for $n > 2$. It can be considered as a non-unital category, such that all morphism spaces are equipped with the structure of complexes and the composition satisfies the Leibnitz rule. On the other hand, every minimal

A_∞-category can be considered as a non-unital category with an additional structure given by higher products.

The most important example of a dg-category is the dg-category of complexes $\mathrm{Com}(\mathcal{A})$ over some k-linear category $\mathcal{A}$. It is defined as follows (see [BK90]). For every pair of complexes $K^\bullet$, $L^\bullet$ we set

$$\mathrm{Hom}^n(K^\bullet, L^\bullet) = \prod_{j-i=n} \mathrm{Hom}_{\mathcal{A}}(K^i, L^j).$$

The differential is given by $m_1(f) = d \circ f - (-1)^{\tilde{f}} f \circ d$ and the composition by $m_2(f, g) = f \circ g$.

Example 1.2. Let A be a dg-algebra (respectively dg-coalgebra). Then we can consider left dg-modules (respectively dg-comodules) over A as objects of a dg-category. Namely, for a pair of dg-modules (respectively dg-comodules) M, M' we set $\mathrm{Hom}^n(M, M')$ to be the space of maps $f : M \to M'$ such that $f(M_i) \subset M_{i+n}$ and f commutes with the A-action (respectively coaction) in the graded sense. The differential m_1 on these spaces and the composition m_2 are defined by the same formulae as in Example 1.1. We will denote the dg-category of dg-modules (respectively dg-comodules) over A by $A - dg - \mathrm{mod}$ (respectively $A - dg - \mathrm{comod}$). Similarly, one can define the dg-category of right dg-modules $dg - \mathrm{mod} - A$ (respectively $dg - \mathrm{comod} - A$).

Definition 1.2. Let $\mathcal{C}$ be an A_∞-category. The opposite A_∞-category $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, the morphism spaces $\mathrm{Hom}^*_{\mathcal{C}^{\mathrm{op}}}(O_1, O_2) = \mathrm{Hom}^*_{\mathcal{C}}(O_2, O_1)$, and the operations

$$m_n^{\mathrm{op}}(a_1, \dots, a_n) = (-1)^{\binom{n+1}{2}+1+\epsilon} m_n(a_n, \dots, a_1),$$

where ϵ is determined by the Koszul sign rule, i.e. $\epsilon = \sum_{i<j} \tilde{a}_i \tilde{a}_j$.

It is easy to check that the A_∞-constraint is indeed satisfied for (m_n^{op}). The linear and the constant term in the quadratic function defining the sign in m_n^{op} are chosen in such a way that $m_1^{\mathrm{op}} = m_1$ and the sign m_2^{op} comes only from the Koszul sign rule.

Definition 1.3. An *A_∞-functor* $F : \mathcal{C} \to \mathcal{C}'$ between A_∞-categories associates to every object O of $\mathcal{C}$ an object $F(O)$ of $\mathcal{C}'$ and to every collection of objects $O_1, \dots, O_{n+1}$ in $\mathcal{C}$, where $n \geqslant 1$, a k-linear map

$$F_n : \mathrm{Hom}^*_{\mathcal{C}}(O_2, O_1) \otimes_k \mathrm{Hom}^*_{\mathcal{C}}(O_3, O_2) \otimes_k \cdots \otimes_k \mathrm{Hom}^*_{\mathcal{C}}(O_{n+1}, O_n) \to \mathrm{Hom}^*_{\mathcal{C}'}(F(O_{n+1}), F(O_1))$$

of degree $1 - n$. These maps are compatible with the operations in $\mathcal{C}$ and $\mathcal{C}'$ in the following way:

$$\sum (-1)^{\epsilon + d(k_\bullet)} m_i(F_{k_1}(a_1, \dots, a_{k_1}), F_{k_2-k_1}(a_{k_1+1}, \dots, a_{k_2}), \dots, F_{n-k_{i-1}}(a_{k_{i-1}+1}, \dots, a_n))$$
$$= \sum_{k+l=n+1} \sum_{j=1}^{k} (-1)^{\epsilon' + j - 1 + l(k-j)} F_k(a_1, \dots, a_{j-1}, m_l(a_j, \dots, a_{j+l-1}), a_{j+l}, \dots, a_n),$$

where in the left-hand side (LHS) the summation is taken over all sequences $0 = k_0 < k_1 < k_2 < \cdots < k_{i-1} < n$, $d(k_\bullet) = \sum_{j=1}^{i-1} (i-j)(k_j - k_{j-1} - 1)$, ϵ and ϵ' come from the Koszul sign rule.

Definition 1.4. Every A_∞-functor $F : \mathcal{C} \to \mathcal{C}'$ defines the *opposite* A_∞-functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}')^{\mathrm{op}}$ between the opposite A_∞-categories by the rule

$$F_n^{\mathrm{op}}(a_1, \dots, a_n) = (-1)^{\binom{n+1}{2}+1+\epsilon} F_n(a_n, \dots, a_1),$$

where ϵ comes from the Koszul sign rule.

One can define a *contravariant* A_∞-functor from $\mathcal{C}$ to $\mathcal{C}'$ as an A_∞-functor from $\mathcal{C}$ to $(\mathcal{C}')^{\mathrm{op}}$. This is equivalent to giving an A_∞-functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$. To avoid any confusion in the signs we will consider only covariant A_∞-functors, replacing the target by the opposite A_∞-category when necessary.

Example 1.3. Every object O of an A_∞-category $\mathcal{C}$ defines the *representable A_∞-functor* $h_O : \mathcal{C} \to \mathrm{Com}(k - \mathrm{mod})$, where $\mathrm{Com}(k - \mathrm{mod})$ is the dg-category of complexes of k-vector spaces. Namely, $h_O(O') = \mathrm{Hom}^*(O, O')$ with the differential m_1 and

$$h_{O,n}(a_1, \dots, a_n)(a) = (-1)^{(n+1)} m_{n+1}(a_1, \dots, a_n, a).$$

Similarly, we have the representable A_∞-functor $h'_A : \mathcal{C} \to \mathrm{Com}(k - \mathrm{mod})^{\mathrm{op}}$ defined by $h'_O(O') = (\mathrm{Hom}^*(O', O), m_1)$,

$$h'_{A,n}(a_1, \dots, a_n)(a) = (-1)^{\widetilde{a}(\widetilde{a_1} + \dots + \widetilde{a_n})} m_{n+1}(a, a_1, \dots, a_n).$$

One can define the composition of A_∞-functors (see [Kel01, 3.4], or § 1.3 below) and the notion of a homotopy between two A_∞-functors with the same source and target (for A_∞-algebras this reduces to the notion of a homotopy between A_∞-morphisms defined in [Kel01, 3.7]; see also § 1.3). Using this one can define the notion of A_∞-equivalence between A_∞-categories. It is known that an A_∞-functor $F : \mathcal{C} \to \mathcal{C}'$ is an A_∞-equivalence if and only if H^*F is an equivalence (see [Kad87], [Pro84] for the case of A_∞-algebras).

DEFINITION 1.5.

i) Let $\mathcal{C}$ be an A_∞-category. Then we define the graded non-unital category $H^*\mathcal{C}$ and the non-unital category $H^0\mathcal{C}$ having the same objects as $\mathcal{C}$ by setting

$$\mathrm{Hom}_{H^*\mathcal{C}}(O, O') = H^*(\mathrm{Hom}_{\mathcal{C}}(O, O'), m_1), \quad \mathrm{Hom}_{H^0\mathcal{C}}(O, O') = H^0(\mathrm{Hom}_{\mathcal{C}}(O, O'), m_1)$$

and by considering the composition law induced by m_2 for these spaces.

ii) The component F_1 of an A_∞-functor $F : \mathcal{C} \to \mathcal{C}'$ between A_∞-categories induces the graded non-unital functor $H^*F : H^*\mathcal{C} \to H^*\mathcal{C}'$ and the non-unital functor $H^0 : H^0\mathcal{C} \to H^0\mathcal{C}'$.

Theorem 1.1 below is essentially due to Kadeishvili (in [Kad82] only the case of A_∞-algebras is considered, however, the generalization to A_∞-categories is straightforward). It was rediscovered several times in different contexts, see [Kel01] and references therein.

THEOREM 1.1. *Let $\mathcal{C}$ be an A_∞-category. Then there exists an extension of the structure of graded non-unital category on $H^*\mathcal{C}$ to that of A_∞-category, such that $H^*\mathcal{C}$ with this structure is A_∞-equivalent to $\mathcal{C}$. More precisely, there exists an A_∞-functor $F : \mathcal{C} \to H^*\mathcal{C}$ such that H^*F is the identity functor on $H^*\mathcal{C}$.*

Note that an A_∞-structure on $H^*\mathcal{C}$ constructed in Theorem 1.1 is not unique. However, all these structures are homotopic in the sense of the following definition.

DEFINITION 1.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be two minimal A_∞-categories. An A_∞-functor $F : \mathcal{C} \to \mathcal{C}'$ is called a *homotopy* if the functor H^*F is the identity.

In other words, if there is a homotopy $F : \mathcal{C} \to \mathcal{C}'$ then $\mathcal{C}$ and $\mathcal{C}'$ should have the same objects and the same spaces of morphisms but possibly different sets of operations $m = (m_n)$ and $m' = (m'_n)$. The fact that F_1 is the identity together with the minimality assumption implies that $m_2 = m'_2$, i.e. $\mathcal{C}$ and $\mathcal{C}'$ coincide as usual (i.e. non-unital) categories. By changing the point of view, we can consider m and m' as two minimal A_∞-structures on a non-unital category $\mathcal{C}$ and say that F is a homotopy from the A_∞-structure m to m'. In fact, it is easy to see that for every A_∞-category $\mathcal{C}$ and every collection of morphisms

$$F_n : \mathrm{Hom}^*_{\mathcal{C}}(O_2, O_1) \otimes_k \mathrm{Hom}^*_{\mathcal{C}}(O_3, O_2) \otimes_k \dots \otimes_k \mathrm{Hom}^*_{\mathcal{C}}(O_{n+1}, O_n) \to \mathrm{Hom}^*_{\mathcal{C}}(F(O_{n+1}), F(O_1))$$

of degree $1 - n$, where $n = 2, 3, \dots$, there exists a unique A_∞-structure $m' = (m'_n)$ on $\mathcal{C}$, such that F defines a homotopy from m to m' (see [Pol00]). We will use the notation $m' = m + \delta(F)$ for this new A_∞-structure.

Note that composition of two homotopies is again a homotopy. It is easy to see that with respect to this composition the set of all homotopies for a given non-unital category $\mathcal{C}$ forms a group acting on the set of all minimal A_∞-structures on $\mathcal{C}$ with m_2 given by the composition in $\mathcal{C}$.

1.2 A_∞-structures on derived categories

Let $\mathcal{A}$ be a k-linear abelian category with enough injective objects. Then one can equip the derived category $D^+(\mathcal{A})$ of bounded below complexes with a canonical structure (up to a homotopy) of a minimal A_∞-category such that m_2 is the standard composition in $D^+(\mathcal{A})$. More precisely, one first has to make a graded category out of $D^+(\mathcal{A})$ by taking

$$\operatorname{Hom}^*(K, K') = \oplus_{i\in\mathbb{Z}} \operatorname{Hom}^i_{D(\mathcal{A})}(K, K')$$

as morphism spaces, where $\operatorname{Hom}^i_{D(\mathcal{A})}(K, K') = \operatorname{Hom}_{D(\mathcal{A})}(K, K'[i])$. Let us denote this graded category by $D^+_{\mathbb{Z}}(\mathcal{A})$. Then there is a canonical homotopy class of minimal A_∞-structures on $D^+_{\mathbb{Z}}(\mathcal{A})$ with m_2 equal to the standard composition. It is constructed as follows. As is well known (see [GM96, III, 5.20]), the category $D^+(\mathcal{A})$ is equivalent to the homotopy category $H^0 \operatorname{Com}^+(\mathcal{I})$ of complexes of injective objects in $\mathcal{A}$ bounded below (here $\mathcal{I}$ denotes the subcategory of injective objects in $\mathcal{A}$). This category is obtained by taking H^0 from the dg-category $\operatorname{Com}^+(\mathcal{I})$ of complexes. Similarly, the $\mathbb{Z}$-graded category $D^+_{\mathbb{Z}}(\mathcal{A})$ is equivalent to $H^* \operatorname{Com}^+(\mathcal{I})$. Now applying Theorem 1.1 we get a canonical homotopy class of minimal A_∞-structures on $H^* \operatorname{Com}^+(\mathcal{I})$.

We can restrict this A_∞-structure on $D^+_{\mathbb{Z}}(\mathcal{A})$ to the subcategory $D^b_{\mathbb{Z}}(\mathcal{A})$.

It is natural to ask whether the canonical homotopy class of A_∞-structures on $D^b_{\mathbb{Z}}(\mathcal{A})$ constructed above, contains the trivial one (with $m_n = 0$ for $n > 2$). The examples of computations of Massey products (see [Pol03]) show that for derived categories of coherent sheaves on projective curves this is not the case – otherwise all these Massey products would vanish. This implies that similar nontriviality holds for an arbitrary projective variety of dimension $\geqslant 1$. However, in Theorem 1.1 we assert that in some cases the part of the A_∞-structure on $D^b_{\mathbb{Z}}(C)$ (where C is a curve) responsible for the variation of cohomology near given stable vector bundle, is homotopically trivial in the above sense.

1.3 Bar construction and A_∞-modules

Bar construction is a convenient tool to record the A_∞-data. In particular, it explains the signs arising in A_∞-definitions and allows us to define A_∞-morphisms and homotopies between them in a concise way.

Definition 1.7. Let A be an A_∞-algebra over k. Its *bar construction* is the space

$$\operatorname{Bar}(A) = T(A[1]) = \oplus_{n\geqslant 0}(A[1])^{\otimes n}$$

considered as a cofree (coassociative) coalgebra (with counit) with the coderivation $b_A : \operatorname{Bar}(A) \to \operatorname{Bar}(A)$ of degree 1 whose components $b_n : (A[n])^{\otimes n} \to A[1]$, $n \geqslant 1$, are defined by the products m_n via the following commutative diagram:

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{\ m_n\ } & A \\ \Big\downarrow{\scriptstyle s^{\otimes n}} & & \Big\downarrow{\scriptstyle s} \\ (A[1])^{\otimes n} & \xrightarrow{\ b_n\ } & A[1] \end{array} \tag{1.1}$$

where $s : A \to A[1]$ is the canonical map of degree -1.

Note that when the map b_n applied to elements of $(A[n])^{\otimes n}$ is expressed in terms of m_n, some signs will arise because of the Koszul sign rule, i.e.

$$b_n(s(a_1)\otimes\cdots\otimes s(a_n)) = (-1)^{(n-1)\widetilde{a_1}+(n-2)\widetilde{a_2}+\cdots+\widetilde{a_{n-1}}}s(m_n(a_1,\ldots,a_n)).$$

The A_∞-constraint is equivalent to the statement that $b_A^2 = 0$ (see [Sta63]) and thus we can consider $(\mathrm{Bar}(A), b_A)$ as a dg-coalgebra.

The importance of the bar construction is due to the fact that an A_∞-morphism between A_∞-algebras $f : A \to A'$ is the same as a morphism of dg-coalgebras $F : \mathrm{Bar}(A) \to \mathrm{Bar}(A')$: one should just take the components $f_n : A^{\otimes n} \to A'$ and make the map $\mathrm{Bar}(A) \to A'[1]$ out of them as above. This interpretation leads to a natural definition of the composition of A_∞-morphisms between A_∞-algebras.

DEFINITION 1.8. A homotopy between a pair of A_∞-morphisms $f, g : A \to A'$ of A_∞-algebras is the morphism $H : \mathrm{Bar}(A) \to \mathrm{Bar}(A')$ of degree -1 such that

$$\Delta\circ H = (F\otimes H + H\otimes G)\circ\Delta,$$

$$F - G = b\circ H + H\circ b,$$

where $F, G : \mathrm{Bar}(A) \to \mathrm{Bar}(A')$ are morphisms of coalgebras corresponding to f and g.

The first condition in this definition allows us to recover a homotopy H from its component $\mathrm{Bar}(A) \to A'$, so H corresponds to a collection of maps $h_n : A^{\otimes n} \to A'$ of degree $-n$, $n \geqslant 1$, satisfying some equations. It turns out that for A_∞-algebras over a field k the homotopy between A_∞-morphisms is an equivalence relation (see [Pro84]). One can also consider the corresponding notion of homotopy equivalence between A_∞-algebras. The following important theorem was proven by Kadeishvili (see [Kad85, Kad87]) and independently by Prouté (see [Pro84]). An A_∞-morphism $f = (f_n) : A \to B$ between A_∞-algebras is called a *quasi-isomorphism* if the corresponding map $H^*f_1 : H^*A \to H^*B$ is an isomorphism.

THEOREM 1.2. *Every quasi-isomorphism of A_∞-algebras is a homotopy equivalence.*

We leave the reader to define the bar construction of an A_∞-category and the notion of homotopy between A_∞-functors imitating the above definitions for A_∞-categories. The analog of Theorem 1.2 also holds for A_∞-categories.

DEFINITION 1.9. A (left) A_∞-module M over an A_∞-algebra A is a graded vector space over k equipped with k-linear operations

$$m_n^M : A^{\otimes(n-1)}\otimes_k M \to M$$

of degree $2-n$, where $n \geqslant 1$, satisfying the A_∞-constraint. Equivalently, one can say that an A_∞-module M over A is the same as an A_∞-category $\mathcal{C}_M$ with two objects X, Y such that $\mathrm{Hom}^*(Y,Y) = A$, $\mathrm{Hom}(X,Y) = M$, $\mathrm{Hom}(Y,X) = 0$, $\mathrm{Hom}(X,X) = 0$.

It is easy to see that the structure of an A_∞-module over A on a graded k-vector space M is equivalent to the datum of a differential b_M of degree 1 on a cofree $\mathrm{Bar}(A)$-comodule

$$\mathrm{Bar}(M) := \mathrm{Bar}(A)\otimes_k M[1]$$

such that $(\mathrm{Bar}(M), b_M)$ is a dg-comodule over $(\mathrm{Bar}(A), b_A)$, i.e. $b_M^2 = 0$ and the pair (b_M, b_A) satisfies the co-Leibnitz rule. The explicit formula for the cogenerating components of $b = b_M$ is

$$b_n(s(a_1)\otimes\cdots\otimes s(a_{n-1})\otimes s(x)) = (-1)^{(n-1)\widetilde{a_1}+(n-2)\widetilde{a_2}+\cdots+\widetilde{a_{n-1}}}s(m_n(a_1,\ldots,a_{n-1},x)),$$

where $x\in M$, $a_1,\ldots,a_{n-1}\in A$.

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Definition 1.10. A *closed morphism* of A_∞-modules $f : M \to M'$ over an A_∞-algebra A is defined as a sequence of maps $f_n : A^{\otimes(n-1)} \otimes_k M \to M'$ of degree $1-n$ (where $n \geqslant 1$), such that the data $(f : M \to M', \mathrm{id} : A \to A)$ defines an A_∞-functor $\mathcal{C}_M \to \mathcal{C}_{M'}$ that is identical on objects.

Again we can interpret this notion in terms of the bar constructions: an A_∞-morphism $M \to M'$ is the same as a closed morphism of dg-comodules $\mathrm{Bar}(M) \to \mathrm{Bar}(M')$ over $\mathrm{Bar}(A)$. Here we equip dg-comodules over $\mathrm{Bar}(A)$ with the structure of a dg-category as in § 1.1. More generally, using the correspondence between A_∞-modules over A and dg-comodules over $\mathrm{Bar}(A)$, we will consider A_∞-modules over A as objects of a dg-category denoted by $A - \mathrm{mod}_\infty$. By definition, the map $M \mapsto \mathrm{Bar}(M)$ defines a dg-functor $A - \mathrm{mod}_\infty \to \mathrm{Bar}(A) - dg - \mathrm{comod}$.

Finally, let us quote the following important theorem (see [Kel01, § 4.2]).

Theorem 1.3. *Let $f : M \to M'$ be a closed A_∞-morphism of A_∞-modules. If f is a quasi-isomorphism (i.e. induces an isomorphism $H^*(M, m_1) \to H^*(M', m_1)$), then f is a homotopic equivalence.*

2. A_∞-structures and formal deformations

2.1 Completed cobar construction and the canonical deformation of an A_∞-module

Let B be a dg-coalgebra over k, $d : B \to B$ be the corresponding differential of degree 1. Then there is a natural structure of dg-algebra on the dual graded vector space B^*. Namely, we define the differential $d : B^* \to B^*$ such that for an element $b^* \in B^*$ one has $(db^*)(b) = (-1)^{\tilde{b}} b^*(db)$. The multiplication on B^* is given by the following composition:

$$B^* \otimes B^* \to (B \otimes B)^* \xrightarrow{\Delta^*} B^*,$$

where $\Delta : B \to B \otimes B$ is the comultiplication. Thus, for $b_1^*, b_2^* \in B^*$ one has

$$(b_1^* b_2^*)(b) = \sum_i (-1)^{\tilde{b_i}\tilde{b'_i}} b_1^*(b_i) b_2^*(b'_i),$$

where $\Delta(b) = \sum_i b_i \otimes b'_i$. When case B has a counit $\epsilon : B \to k$, the dual map $\epsilon^* : k \to B^*$ will be a unit for B^*.

Let A be an A_∞-algebra. Applying the above construction to the dg-coalgebra structure on $\mathrm{Bar}(A)$ we obtain a dg-algebra structure on the dual space $C(A) := \mathrm{Bar}(A)^*$ with the differential c_A induced by b_A as above. We will call $C(A)$ the *completed cobar construction* of A. In particular, we obtain the associative algebra structure (with a unit) on

$$A^! = H^0(C(A), c_A) \simeq H^0(\mathrm{Bar}(A), b_A)^*.$$

Our notation is motivated by the non-homogeneous quadratic duality: if $A = k \oplus A_+$ is a quadratic dg-algebra (i.e. A is generated by A_1 as a k-algebra and the defining relations are quadratic), then $(A_+)^!$ is a completion of the quadratic-linear algebra dual to A (see [PP]).

Proposition 2.1. *An A_∞-morphism $f : A \to B$ between A_∞-algebras induces a homomorphism $f^! : B^! \to A^!$ of associative algebras. This correspondence extends to a contravariant functor from the homotopy category of A_∞-algebras to the category of associative algebras.*

Proof. By definition, homotopy classes of A_∞-maps $f : A \to B$ are in bijection with homotopy classes of homomorphisms of dg-coalgebras $\mathrm{Bar}(A) \to \mathrm{Bar}(B)$. It remains to use the natural functor $C \mapsto H^0(C)^*$ from the homotopy category of coalgebras to the category of associative algebras. □

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For a pair of graded k-vector spaces V and W we set

$$\mathrm{Hom}_{gr}(V, W) = \oplus_{n\in\mathbb{Z}} \mathrm{Hom}(V, W)_n$$

where $\mathrm{Hom}(V, W)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}(V_i, V_{i+n})$.

Let M be an A_∞-module over A. Then we can define a canonical differential c_M on

$$C(M) = \mathrm{Hom}_{gr}(\mathrm{Bar}(A), M)$$

which makes $(C(M), c_M)$ into a right dg-module over the dg-algebra $C(A)$ via some natural right action of $C(A)$ on $C(M)$. Let us state this construction in slightly more general terms.

PROPOSITION 2.2. *Let (B, d) be a dg-coalgebra with a counit and M be a vector space. Consider the dual dg-algebra (B^*, d). Then every dg-comodule differential $d_M : B \otimes M \to B \otimes M$ on a free B-comodule $B\otimes M$ naturally induces a dg-module differential $d_M^\vee : \mathrm{Hom}_{gr}(B, M) \to \mathrm{Hom}_{gr}(B, M)$, where $\mathrm{Hom}_{gr}(B, M)$ is considered to be a right B^*-module via the action*

$$(\phi \cdot b^*) = (\phi \otimes b^*) \circ \Delta,$$

where $\phi \in \mathrm{Hom}_{gr}(B, M)$, $b^* \in B^*$, $\phi \otimes b^*$ is considered to be a map from $B \otimes B$ to $M \otimes k = M$. This construction extends to a dg-functor from the dg-category of dg-comodules over B that are free as B-comodules to $dg - \mathrm{mod} - B^*$.

Proof. The differential d_M is uniquely determined by its component $\overline{d}_M : B \otimes M \to M$. Namely, we have

$$d_M(b \otimes m) = (\mathrm{id}_B \otimes \overline{d}_M)(\Delta(b) \otimes m) + db \otimes m.$$

The condition $d_M^2 = 0$ is equivalent to $\overline{d} \circ d_M = 0$, i.e.

$$\sum_i (-1)^{\widetilde{b_i}} \overline{d}_M(b_i \otimes \overline{d}_M(b_i' \otimes m)) + \overline{d}_M(db \otimes m), \tag{2.1}$$

where $\Delta(b) = \sum_i b_i \otimes b_i'$, $m \in M$. Now we define $d_M^\vee$ by

$$d_M^\vee(\phi)(b) = \sum_i (-1)^{\widetilde{b_i}\widetilde{\phi}} \overline{d}_M(b_i \otimes \phi(b_i')) - (-1)^{\widetilde{\phi}} \phi(db),$$

where $\phi \in \mathrm{Hom}_{gr}(B, M)$ is a homogeneous element. It is easy to see that c is a derivation of $\mathrm{Hom}_{gr}(B, M)$ as a right dg-module over $C(B)$. The condition $(d_M^\vee)^2 = 0$ follows easily from Equation (2.1) using the coassociativity of Δ.

Every morphism of B-comodules $f : B \otimes M \to B \otimes M'$ is uniquely determined by its component $\overline{f} : B \otimes M \to M'$. We define the corresponding morphism of right B^*-modules $f^\vee : \mathrm{Hom}_{gr}(B, M) \to \mathrm{Hom}_{gr}(B, M')$ by

$$f^\vee(\phi)(b) = \sum_i (-1)^{\widetilde{\beta_i}\widetilde{\phi}} \overline{f}(b_i \otimes \phi(b_i')).$$

This gives the required dg-functor. □

We apply this construction to $B = \mathrm{Bar}(A)$ and the differential $d_M = b_M$ on $B \otimes M$ corresponding to the A_∞-module structure on M, and call the obtained right dg-module $(C(M), c_M)$ over the dg-algebra $C(A)$ the *completed cobar construction* of M.

Now let us assume that an A_∞-algebra A is concentrated in positive degrees, i.e. $A = \oplus_{n\geqslant 1} A_n$. Then $\mathrm{Bar}(A)$ is concentrated in non-negative degrees, while $C(A)$ is concentrated in non-positive degrees. Hence, in this case we have a surjective homomorphism of algebras

$$C(A) \to H^0(C(A)) = A^!.$$

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In fact, in this case the algebra $C(A)_0$

$$C(A)_0 = \prod_{n\geqslant 0} (A_1^{\otimes n})^*$$

is a completion of the tensor algebra and $A^!$ is the quotient of $C(A)_0$ by the two-sided ideal generated by the image of the map

$$A_2^* \to \prod_{n\geqslant 0} (A_1^{\otimes n})^*$$

with components dual to the maps

$$(-1)^{\binom{n}{2}} m_n : A_1^{\otimes n} \to A_2, \quad n \geqslant 1.$$

Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be an A_∞-module over A such that all the spaces M_n are finite-dimensional (in this case we say that M is *locally finite-dimensional*). We are going to define a natural $A^!$-linear differential of degree 1 on the free right $A^!$-module $M \otimes A^!$. In the case when M is finite-dimensional this differential can be immediately obtained from the completed cobar construction of M. Namely, in this case $C(M) \simeq M \otimes C(A)$, so we get a dg-module structure on the free right $C(A)$-module $M \otimes C(A)$. Tensoring with $A^! = H_0(C(A))$ over $C(A)$ we obtain an $A^!$-linear differential on $M \otimes A^!$. In the general case when only graded components of M_n are finite-dimensional we have to take the dual route. First, we claim that the embedding $H^0 \operatorname{Bar}(A) \subset \operatorname{Bar}(A)$ is a morphism of dg-coalgebras. Let us set $B = \operatorname{Bar}(A)$ for brevity. Then we have $H^0 B = \ker(b : B_0 \to B_1)$, hence $\Delta(H^0 B)$ is contained in

$$\ker(B_0 \otimes B_0 \xrightarrow{(b\otimes \mathrm{id}, \mathrm{id}\otimes b)} B_1 \otimes B_0 \oplus B_0 \otimes B_1) = H^0 B \otimes H^0 B$$

which proves our claim. This implies that the subspace $H^0 B \otimes M \subset B \otimes M$ is preserved by the differential b_M. Applying the construction of Proposition 2.2 to the coalgebra $H^0 B$ (with the zero differential) we obtain the $A^!$-linear differential on

$$\operatorname{Hom}_{gr}(H^0 B, M) = \oplus_{n\in\mathbb{Z}} \operatorname{Hom}(H^0 B, M_n) = M \otimes A^!$$

(this last equality follows from the fact that M_n are finite-dimensional). We will denote this differential by c_M and the complex of $A^!$-modules $(M \otimes A^!, c_M)$ by $M_{A^!}$. Note that $A^!$ has a natural augmentation $A^! \to k$ and tensoring $M_{A^!}$ with k over $A^!$ we obtain the complex (M, m_1). So the differential c_M can be considered as a deformation of the differential m_1 on M.

Let us write the explicit formula for the differential c_M assuming for simplicity that A_1 has countable dimension. Let $(e_1, e_2, \dots)$ be a basis of A_1, $(e_1^*, e_2^*, \dots)$ be the dual vectors in A_1^*, so that elements of A_1^* are infinite series $\sum_{n=1}^\infty c_n e_n^*$. Then we have

$$c_M(x \otimes r) = \sum_{n\geqslant 0; i_1,\dots,i_n} (-1)^{\binom{n+1}{2}} m_{n+1}(e_{i_1}, \dots, e_{i_n}, x) \otimes e_{i_1}^* \cdots e_{i_n}^* \cdot r \tag{2.2}$$

where $x \in M$, $r \in A^!$. This infinite series makes sense as an element of $M \otimes A^!$ since M is locally finite-dimensional. Note that using this notation we can write the defining relation in $A^!$ as follows:

$$\sum_{n\geqslant 1; i_1,\dots,i_n} (-1)^{\binom{n}{2}} m_n(e_{i_1}, \dots, e_{i_n}) \otimes e_{i_1}^* \cdots e_{i_n}^* = 0 \tag{2.3}$$

in $A_2 \otimes A^!$.

Proposition 2.3. *The map $M \mapsto M_{A^!}$ extends to a dg-functor $A - \operatorname{mod}_\infty^{lf} \to \operatorname{Com}(\operatorname{mod} - A^!)$, where $A - \operatorname{mod}_\infty^{lf}$ is the category of locally finite-dimensional A_∞-modules.*

Proof. Let M and M' be A_∞-modules over A. A degree n morphism of $\operatorname{Bar}(A)$-comodules $f : \operatorname{Bar}(A) \otimes M \to \operatorname{Bar}(A) \otimes M'$ is determined by its component $\overline{f} : \operatorname{Bar}(A) \otimes M \to M'$. It induces a

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degree n map

$$f^\vee : M \to \mathrm{Hom}_{gr}(H^0\,\mathrm{Bar}(A), M') \simeq M' \otimes A^!$$

such that $f^\vee(m)(b) = \overline{f}(b \otimes m)$ for $m \in M$, $b \in H^0\,\mathrm{Bar}(A)$. We define the value of our functor on f to be the corresponding morphism of the free $A^!$-modules $M \otimes A^! \to M' \otimes A^!$. It is easy to check that this is indeed a dg-functor. □

DEFINITION 2.1. Let (A, M) and (A', M') be two pairs each consisting of an A_∞-algebra and an A_∞-module over it. We define an A_∞-map $f : (A, M) \to (A', M')$ as an A_∞-functor between the corresponding A_∞-categories with two objects. A homotopy between two such A_∞-maps is defined as a homotopy between the corresponding A_∞-functors.

Thus, we can consider the homotopy category of pairs (A, M). Let us also consider pairs of the form (C, K) where C is an associative algebra, K is a complex of right C-modules. We define a morphism $(C, K) \to (C', K')$ between such pairs as a pair (α, β), where $\alpha : C \to C'$ is a homomorphism of algebras, $\beta : K' \to K \otimes_C C'$ is a morphism in the homotopy category of complexes of C'-modules. Proposition 2.1 can then be extended to pairs (A, M) as follows.

PROPOSITION 2.4. *The map $(A, M) \mapsto (A^!, M_{A^!})$ extends to a contravariant functor from the homotopy category of pairs (A, M), such that A is a positively graded A_∞-algebra and M is a locally finite-dimensional A_∞-module over it, to the homotopy category of pairs consisting of an associative algebra and a complex of right modules over it.*

Proof. A homotopy class of maps $(A, M) \to (A', M')$ defines a coalgebra homomorphism $H^0\,\mathrm{Bar}(A) \to H^0\,\mathrm{Bar}(A')$ and a homotopy class of compatible comodule morphisms

$$H^0\,\mathrm{Bar}(A) \otimes M \to H^0\,\mathrm{Bar}(B) \otimes M'.$$

The former map induces a homomorphism of algebras $(A')^! \to A^!$. We can use the component $H^0\,\mathrm{Bar}(A) \otimes M \to M'$ of the latter map to define a map $M \to M' \otimes A^!$ as in the proof of Proposition 2.3. The corresponding map of free $A^!$-modules $M \otimes A^! \to M' \otimes A^!$ is the required chain map $M_{A^!} \to M'_{(A')^!} \otimes_{(A')^!} A^!$. □

2.2 The canonical deformation of a representable A_∞-functor

PROPOSITION 2.5. *Let $\mathcal{C}$ be an A_∞-category, O be an object of $\mathcal{C}$. Then $A = \mathrm{Hom}^*_{\mathcal{C}}(O, O)$ has a natural structure of A_∞-algebra and the representable A_∞-functor $h'_O : \mathcal{C} \to \mathrm{Com}(k - \mathrm{mod})^{\mathrm{op}}$ factors as the composition of an A_∞-functor $H'_O : \mathcal{C} \to A - \mathrm{mod}^{\mathrm{op}}_\infty$ with the forgetting dg-functor $A - \mathrm{mod}^{\mathrm{op}}_\infty \to \mathrm{Com}(k - \mathrm{mod})^{\mathrm{op}}$.*

Proof. The structure of an A_∞-algebra on A and of an A_∞-module on $\mathrm{Hom}^*_{\mathcal{C}}(X, O)$ is simply given by the operations in $\mathcal{C}$. To define an A_∞-functor H'_O extending h'_O we have to define for every sequence $(x_1 : X_1 \to X_0, \ldots, x_p : X_p \to X_{p-1})$ of morphisms in $\mathcal{C}$ a morphism

$$H'_{O,p}(x_1, \ldots, x_p) : \mathrm{Hom}^*_{\mathcal{C}}(X_0, O) \to \mathrm{Hom}^*_{\mathcal{C}}(X_p, O)$$

in the dg-category of A_∞-modules over A. Such a morphism corresponds to a morphism of $\mathrm{Bar}(A)$-comodules

$$\mathrm{Bar}(A) \otimes \mathrm{Hom}^*_{\mathcal{C}}(X_0, O)[1] \to \mathrm{Bar}(A) \otimes \mathrm{Hom}^*_{\mathcal{C}}(X_p, O)[1].$$

By the definition, this morphism is obtained by substituting $x_1, \ldots, x_p$ in the component

$$\mathrm{Bar}(A) \otimes \mathrm{Hom}^*_{\mathcal{C}}(X_0, O)[1] \otimes \mathrm{Hom}^*_{\mathcal{C}}(X_1, X_0)[1] \otimes \cdots \otimes \mathrm{Hom}^*_{\mathcal{C}}(X_p, X_{p-1})[1] \to \mathrm{Bar}(A) \otimes \mathrm{Hom}^*_{\mathcal{C}}(X_p, O)[1]$$

of the differential in the bar construction of $\mathcal{C}$. Thus, the components of $H'_{O,p}(x_1, \ldots, x_p)$ have form

$$H'_{O,p}(x_1, \ldots, x_p)_n(a_1, \ldots, a_{n-1}, x) = \pm m_{n+p}(a_1, \ldots, a_{n-1}, x, x_1, \ldots, x_p),$$

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where $x \in \mathrm{Hom}^*_{\mathcal{C}}(X_0, O)$, $a_1, \ldots, a_{n-1} \in A = \mathrm{Hom}^*_{\mathcal{C}}(O, O)$. It is not difficult to check that the axioms of an A_∞-functor are satisfied. □

Let O be an object of an A_∞-category $\mathcal{C}$.

Definition 2.2. Let us define the associative k-algebra $R(O)$ (with a unit) by setting $R(O) = (\mathrm{Hom}^*(O, O)_+)^!$, where $\mathrm{Hom}^*(O, O)_+ = \oplus_{n\geqslant 1} \mathrm{Hom}^n(O, O)$.

As we have seen already, $R(O)$ actually depends just on $\mathrm{Hom}^1(O, O)$, $\mathrm{Hom}^2(O, O)$ and the operations $m_n : T^n \mathrm{Hom}^1(O, O) \to \mathrm{Hom}^2(O, O)$.

Composing the representable A_∞-functor $h'_O : \mathcal{C} \to \mathit{Hom}^*(O, O)_+ - \mathrm{mod}^{\mathrm{op}}_\infty$ with the dg-functor $M \mapsto (M \otimes_k R(O), c_M)$ defined in Proposition 2.3, we obtain the A_∞-functor

$$F_O : \mathcal{C} \to \mathrm{Com}(\mathrm{mod} - R(O))^{\mathrm{op}}.$$

By the definition we have

$$F_O(O') = \mathrm{Hom}^*(O', O) \otimes R(O)$$

with the differential given by Equation (2.2). The formula for the structure of an A_∞-functor is of the form

$$F_{O,p}(x_1, \ldots, x_p)(x \otimes r) = \sum_{n\geqslant 0; i_1, \ldots, i_n} \pm m_{p+n+1}(e_{i_1}, \ldots, e_{i_n}, x, x_1, \ldots, x_p) \otimes e^*_{i_1}, \ldots, e^*_{i_n} \cdot r.$$

Note that the composition of F_O with the dg-functor $\mathrm{Com}(\mathrm{mod} - R(O))^{\mathrm{op}} \to \mathrm{Com}(\mathrm{mod} - k)^{\mathrm{op}}$ given by $M \mapsto M \otimes_{R(O)} k$ is exactly the representable A_∞-functor h'_O. In addition, the terms of all the complexes $F_O(O')$ are free $R(O)$-modules. Thus, we can consider F_O to be a formal deformation of the functor h'_O.

In particular, for every object O' such that $\mathrm{Hom}^*(O', O')$ and $\mathrm{Hom}^*(O', O)$ are concentrated in degree 0, we get a formal deformation of the structure of right $\mathrm{Hom}^0(O', O')$-module on $\mathrm{Hom}^0(O', O) \otimes_k R(O)$: the deformed action of $a \in \mathrm{Hom}^0(O', O')$ is given by $F_{O,1}(a)$.

Proposition 2.1 easily implies that under a homotopy of the A_∞-structure the algebra $R(O)$ gets replaced by an isomorphic one. Moreover, one can check that under this isomorphism the functor F_O gets replaced by a homotopic one.

Example 2.1. Let A be a complete local Noetherian commutative k-algebra with a residue field k. Consider the derived category $D^b_{\mathbb{Z}}(R - \mathrm{mod})$ and equip it with an A_∞-structure as in § 1.2. Let ${}_Ak$ denote k which is considered to be an A-module. Then there is an isomorphism of k-algebras $R({}_Ak) \simeq A$.

2.3 Computation for derived categories

Now we consider the derived category $\mathcal{C} = D^+_{\mathbb{Z}}(\mathcal{A})$, where $\mathcal{A}$ is an abelian category with enough injectives, so that $\mathcal{C}$ has a minimal A_∞-structure as introduced in § 1.2. Let O be an object of $D^+(\mathcal{A})$. We want to compute the value of the corresponding functor F_O on an object $Q \in D^b(\mathcal{A})$ using a certain adapted resolution. Namely, let us assume that there exists a bounded above complex

$$P^\bullet :\to \cdots P^{n-1} \to P^n,$$

which is quasi-isomorphic to Q, such that for every $i \in \mathbb{Z}$ the space $\mathrm{Hom}^*_{D(\mathcal{A})}(P^i, O)$ is concentrated in degree 0 (e.g. in the following we will consider the situation where O is a coherent sheaf on a projective scheme and P^i are sufficiently negative vector bundles). Then $F_O(P^i) = \mathrm{Hom}^0_{D(\mathcal{A})}(P^i, O) \otimes R(O)$

and we have a complex of $R(O)$-modules

$$F_O(P^n) \to F_O(P^{n-1}) \to \cdots \tag{2.4}$$

with the differentials induced by the morphisms $P^i \to P^{i+1}$.

THEOREM 2.1. *In the above situation the complex of $R(O)$-modules $F_O(Q)$ is quasi-isomorphic to the complex (2.4).*

Proof. Let $O \to I^\bullet$, $P^\bullet \to J^\bullet$ be quasi-isomorphisms with the complexes of injective objects bounded below. We have a homotopic equivalence of pairs

$$(\mathrm{Hom}^*_{D(\mathcal{A})}(O,O), \mathrm{Hom}^*_{D(\mathcal{A})}(Q,O)) \simeq (A, \widetilde{M})$$

with $A := \mathrm{tot}\,\mathrm{Hom}_{\mathcal{A}}(I^\bullet, I^\bullet)$ and $\widetilde{M} := \mathrm{tot}\,\mathrm{Hom}_{\mathcal{A}}(J^\bullet, I^\bullet)$, where 'tot' denotes the convolution of a bicomplex. Therefore, by Proposition 2.4 we get an isomorphism $R(O) \simeq A^!$ and the homotopy equivalence of complexes $F_O(Q) \simeq \widetilde{M}_{A^!}$. The morphism of the complexes $P^\bullet \to J^\bullet$ induces a quasi-isomorphism of dg-modules over A

$$\widetilde{M} \to M := \mathrm{tot}\,\mathrm{Hom}_{\mathcal{A}}(P^\bullet, I^\bullet).$$

It follows that $\widetilde{M}_{A^!}$ and $M_{A^!}$ are homotopically equivalent as A_∞-modules over A (see Theorem 1.3).

Now we observe that $M_{A^!}$ is the total complex associated with the bicomplex of $A^!$-modules

$$(\mathcal{B} := \oplus_{i,j} M^{i,j} \otimes A^!, \partial_1, \partial_2),$$

where $M^{i,j} = \mathrm{Hom}_{\mathcal{A}}(J^{-i}, I^j)$, the differential ∂_1 is the $A^!$-linear map induced by the standard map $M^{i,j} \to M^{i+1,j}$, while the differential ∂_2 is induced by the structures of dg-modules over A on the complexes $M^{i,\bullet}$. Thus, the rows of this bicomplex are exactly the complexes $M^{i,\bullet}_{A^!}$. Our assumption that $\mathrm{Hom}^*_{D(\mathcal{A})}(P^{-i}, O) = \mathrm{Hom}^0_{D(\mathcal{A})}(P^{-i}, O)$ implies that the cohomology of $M^{i,\bullet}_{A^!}$ is concentrated in degree 0. Therefore, the embedding of the complexes of $A^!$-modules

$$(H^0(\mathcal{B}, \partial_2), \partial_1) \to \mathrm{tot}\,\mathcal{B} = (\mathcal{B}, \partial_1 + \partial_2)$$

is a quasi-isomorphism. Finally, we note that for every i there is an isomorphism $H^0(M^{i,\bullet}_{A^!}) \simeq \mathrm{Hom}_{D(\mathcal{A})}(P^{-i}, O) \otimes R(O) = F(P^{-i})$ and the morphisms $H^0(M^{i,\bullet}_{A^!}) \to H^0(M^{i+1,\bullet}_{A^!})$ induced by ∂_1 are identified with the differentials in the complex (2.4). $\square$

2.4 The deformation of a coherent sheaf

Now let us specialize to the case when $\mathcal{C}$ is the derived category $D^b_{\mathbb{Z}}(X)$ of coherent sheaves on a projective scheme X over k. Let $\mathcal{F}$ be a coherent sheaf on X, and let $\mathcal{O}_X(1)$ be an ample line bundle on X. We can consider the associative algebra $R(\mathcal{F})$ and the A_∞-functor $F_{\mathcal{F}} : D^b_{\mathbb{Z}}(X) \to \mathrm{Com}(\mathrm{mod}-R(\mathcal{F}))^{\mathrm{op}}$ constructed above. Note that $R(\mathcal{F})$ is a quotient of the completed tensor algebra of the space $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})^*$. Assume that for $n > 0$ the cohomology spaces $H^*(X, \mathcal{O}_X(n))$ and $H^*(X, \mathcal{F}(n))$ are concentrated in degree zero. Then the formula

$$(s \otimes r) * a = F_{\mathcal{F},1}(a)(s \otimes r) = \sum_{n \geqslant 0; i_1, \ldots, i_n} (-1)^{\binom{n}{2}} m_{n+2}(e_{i_1}, \ldots, e_{i_n}, s, a) \otimes e^*_{i_1}, \ldots, e^*_{i_n} \cdot r,$$

where $a \in H^0(X, \mathcal{O}_X(m))$, $m > 0$, $s \in H^0(X, \mathcal{F}(m'))$, $m' > 0$, $r \in R(\mathcal{F})$, defines the structure of a graded $\oplus_{n>0} H^0(X, \mathcal{O}_X(n))$-module on

$$M_{\mathcal{F}} = \oplus_{n>0} H^0(X, \mathcal{F}(n)) \otimes_k R(\mathcal{F}).$$

Clearly, this structure commutes with the $R(\mathcal{F})$-module structure on $M_{\mathcal{F}}$.

If we replace $R(\mathcal{F})$ by its abelianization, $R = R(\mathcal{F})^{ab}$, the above construction still works, and thus we get a structure of a graded $\oplus_{n>0} H^0(X, \mathcal{O}_X(n)) \otimes R$-module on $M = \oplus_{n>0} H^0(X, \mathcal{F}(n)) \otimes R$.

Then the localization $\widetilde{M}$ of M will be a coherent sheaf $\mathcal{F}_R$ on $X \times \mathrm{Spec}(R)$ that is flat over R. Let us consider the natural homomorphism $R \to k$ given by the augmentation of R. The natural isomorphism

$$M \otimes_R k \simeq H^0(X, \mathcal{F}(n)),$$

compatible with the $\oplus_{n>0} H^0(X, \mathcal{O}_X(n))$-action, induces an isomorphism of $\mathcal{F}_R|_{X \times \mathrm{Spec}(k)}$ with $\mathcal{F}$. Thus, the family $\mathcal{F}_R$ is a deformation of $\mathcal{F}$.

Theorem 2.2. *Let $\mathcal{G} \in D^b(X)$ be an object. Then for every $n > 0$ one has an isomorphism*

$$F_{\mathcal{F}}(\mathcal{G}) \simeq Rp_{2*} R\underline{\mathrm{Hom}}(p_1^* \mathcal{G}, \mathcal{F}_R) \tag{2.5}$$

in the derived category of complexes of R-modules, where p_1 and p_2 are the projections of the product $X \times \mathrm{Spec}(R)$ to its factors.

Proof. This follows easily from Theorem 2.1. Indeed, let N be an integer such that $H^i(X, \mathcal{F}(n)) = 0$ for $i > 0$, $n > N$. We can choose a quasi-isomorphism $P^\bullet \to \mathcal{G}$, where $P^\bullet$ is a bounded above complex, such that each P^i is a direct sum of the line bundles $\mathcal{O}_X(-n)$ with $n > N$. Then $Rp_{2*} R\underline{\mathrm{Hom}}(p_1^* \mathcal{G}, \mathcal{F}_R)$ is represented by the complex of R-modules

$$\cdots \to H^0(X \times \mathrm{Spec}(R), (P^n)^\vee \otimes \mathcal{F}_R) \to H^0(X \times \mathrm{Spec}(R), (P^{n-1})^\vee \otimes \mathcal{F}_R) \to \cdots$$

But this complex coincides with the complex $\cdots \to F_{\mathcal{F}}(P^n) \to F_{\mathcal{F}}(P^{n-1}) \to \cdots$ □

Remark 2.1. Theorem 2.2 is a generalization of the formal analog of Theorem 3.2 in [GL91]. The latter theorem states that in the context of Kähler geometry the variation of cohomology groups in a family of topologically trivial line bundles can be described locally by a complex similar to $F_{\mathcal{F}}(\mathcal{O})$ but with the differential depending only on m_2. The reason for the absence of higher corrections to this differential is that in this situation there is a natural choice of the A_∞-structure for which the relevant higher products vanish (this A_∞-structure is constructed using Dolbeault complexes and harmonic projectors, see [Pol01]).

Corollary 2.1. *For every $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ consider the family $\mathcal{F}_\xi$ over $X \times \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$ induced by $\mathcal{F}_R$ via the natural homomorphism of k-algebras $\pi_\xi : R \to k[\epsilon]/(\epsilon^2)$ defined by $\pi_\xi(e) = e(\xi) \cdot \epsilon$ for $e \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})^*$. If $\xi \neq 0$ then the family $\mathcal{F}_\xi$ is non-constant.*

Proof. Let p_1, p_2 be the projections of the product $X \times \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$ to its factors. Theorem 2.2 implies that the object

$$Rp_{2*} R\underline{\mathrm{Hom}}(p_1^* \mathcal{F}, \mathcal{F}_\xi) \in D^+(k[\epsilon]/(\epsilon^2) - \mathrm{mod})$$

is represented by the complex $\mathrm{Hom}^*(\mathcal{F}, \mathcal{F}) \otimes_k k[\epsilon]/(\epsilon^2)$ with the differential

$$d(a \otimes r) = m_2(a, \xi) \otimes r\epsilon,$$

where $a \in \mathrm{Ext}^i(\mathcal{F}, \mathcal{F})$, $r \in k[\epsilon]/(\epsilon^2)$. Thus, $d(\mathrm{id}_{\mathcal{F}}) = \xi \otimes \epsilon \neq 0$, hence the zeroth cohomology of this complex is a proper subspace of $\mathrm{Hom}(\mathcal{F}, \mathcal{F}) \otimes_k k[\epsilon]/(\epsilon^2)$. It follows that the dimension of $R^0 p_{2*} R\underline{\mathrm{Hom}}(p_1^* \mathcal{F}, \mathcal{F}_\xi)$ over k is less than $2 \dim_k \mathrm{Hom}(\mathcal{F}, \mathcal{F})$. On the other hand, for the constant family $p_1^* \mathcal{F}$ we have

$$R^0 p_{2*} R\underline{\mathrm{Hom}}(p_1^* \mathcal{F}, p_1^* \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{F}) \otimes_k k[\epsilon]/(\epsilon^2),$$

which has dimension $2 \dim_k \mathrm{Hom}(\mathcal{F}, \mathcal{F})$. Hence $\mathcal{F}_\xi$ cannot be isomorphic to $p_1^* \mathcal{F}$. □

Remark 2.2. It seems plausible that in the above situation the family $\mathcal{F}_R$ is the miniversal formal (commutative) deformation of $\mathcal{F}$. In the case of module deformation a similar statement follows from the work of Laudal [Lau83]. However, it seems that A_∞-techniques allows us to simplify the calculations in [Lau83]. We plan to return to this question and its non-commutative analog in a future paper.

3. Applications

3.1 Brill–Noether loci

Now let C be a projective curve over a field k. Below we assume that C is smooth although most probably this condition can be relaxed. Let $U(n, d)$ be the moduli space of stable bundles of rank n and degree d on a curve C. Then for every vector bundle E on C and every $i \geqslant 0$ one can define a subscheme $W^r_{n,d}(E) \subset U(n, d)$ corresponding to stable bundles V such that $\dim H^0(V \otimes E) > r$.

Consider first the case $n = 1$. Then $U(1, d) = J^d$ is the Jacobian of line bundles of degree d on C. Let $\mathcal{P}$ be the universal family on $C \times J^d$ and let p_1, p_2 be the projections of the product $C \times J^d$ to its factors. Then the derived push-forward $Rp_{2*}(p_1^* E \otimes \mathcal{P})$ can be represented by a complex $V_0 \xrightarrow{\delta} V_1$ of vector bundles on J^d. By definition, the ideal sheaf of the subscheme $W^r_{1,d}(E) \subset J^d$ is generated locally by the $(v_0 - r) \times (v_0 - r)$ minors of the matrix representing δ in some local bases of V_0 and V_1 (here $v_0 = \mathrm{rk}\, V_0$).

In the case $n > 1$ the definition is similar. The situation is slightly complicated by the fact that in general there is no universal family on $C \times U(n, d)$ (even locally in Zariski topology on $U(n, d)$). However, one can get around this difficulty by working with stacks of vector bundles (essentially this boils down to considering the universal family over the relevant Quot-scheme). The reader can consult [Lau91] and [Mer] for details.

Proof of Theorem 0.1. Applying the construction of § 2.4 to $\mathcal{F} = V$ (and using the A_∞-structure on $D^b_{\mathbb{Z}}(C)$), we obtain the family V_R on $C \times \mathrm{Spec}(R)$, where $R = \hat{S}(\mathrm{Ext}^1(V, V)^*)$ is the completed symmetric algebra of the space $\mathrm{Ext}^1(V, V)^*$. Let $\iota : \mathrm{Spec}(R) \to U(n, d)$ be the corresponding morphism to the moduli space. Then according to Corollary 2.1, the tangent map to ι at the closed point of $\mathrm{Spec}(R)$ is an isomorphism. Therefore, ι induces an isomorphism of $\mathrm{Spec}(R)$ with the formal neighborhood of V in $U(n, d)$.

Now applying Theorem 2.2 to $\mathcal{F} = V$ and $\mathcal{G} = E^\vee$, we find that the object $Rp_{2*}(p_1^* E \otimes V_R) \in D(R - \mathrm{mod})$ is represented by the complex

$$F_V(E^\vee) : H^0(C, V \otimes E) \otimes_k R \xrightarrow{d} H^1(C, V \otimes E) \otimes_k R,$$

where the differential d is given by (2.2). Let us choose some bases in $H^0(C, V \otimes E)$ and $H^1(C, V \otimes E)$ and view d as an R-valued matrix. Note that the condition of injectivity of $\mu_{V,E}$ is equivalent to surjectivity of the map

$$\mathrm{Ext}^1(V, V) \to \mathrm{Hom}(H^0(C, V \otimes E), H^1(C, V \otimes E))$$

obtained from $\mu_{V,E}$ via Serre duality. It follows that the leading terms of the entries of d are linearly independent elements of $\mathrm{Ext}^1(V, V)^*$. Therefore, we can choose a formal coordinate system on $U(n, d)$ at V, such that the entries of d will be coordinate functions. This immediately implies the result. □

Remark 3.1. In the case when $n = 1$, $E = \mathcal{O}_C$ and k is algebraically closed, the assertion of Theorem 0.1 follows easily from the fact that for every special line bundle L on C there exists an effective divisor D such that the natural map $H^0(C, L) \to H^0(C, L(D))$ is an isomorphism and $h^1(L(D)) = 0$ (this trick is considered in details in [Kem83]). Indeed, one just have to use the resolution $\mathcal{L}(D) \to \mathcal{L}(D)|_D$ for line bundles $\mathcal{L}$ in a neighborhood of L and apply the definition of Brill–Noether loci to the corresponding complex $H^0(\mathcal{L}(D)) \to H^0(\mathcal{L}(D)|_D)$.

Remark 3.2. It seems that the condition of injectivity of $\mu_{V,E}$ in Theorem 0.1 can be relaxed. For example, we checked that the conclusion of the theorem is satisfied for double points of theta divisors in hyperelliptic curves, even though the corresponding Petri map has a one-dimensional kernel (the details will appear elsewhere).

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The moduli spaces of stable vector bundles admit canonical non-commutative thickenings (see [Kap98]). Note that in § 2.4 we obtained naturally non-commutative deformations of coherent sheaves and then passed to abelianization. In particular, an A_∞-structure gives rise to formal coordinates on the above non-commutative thickenings of the moduli spaces of vector bundles. We believe that there is a way to define naturally some non-commutative thickenings of the Brill–Noether loci, so that the analog of Theorem 0.1 still holds for them. This should be a consequence of the following result.

Theorem 3.1. *Let V and E be vector bundles on C, such that V is stable and the Gieseker–Petri map $\mu_{V,E}$ is injective. Then one can choose an A_∞-structure on $D^b_{\mathbb{Z}}(C)$ from the canonical homotopy class in such a way that all the products*

$$m_n : \operatorname{Ext}^1(V,V)^{\otimes(n-1)} \otimes H^0(C, V\otimes E) \to H^1(C, V\otimes E)$$

vanish for $n > 2$.

Proof. Let us start with an A_∞-structure on $D^b(C)$ from the canonical homotopy class. We want to change it to a homotopic one, so that for the new structure the products in question vanish. We construct the required homotopy as the infinite composition of homotopies $(f^{(n)})$, $n = 2, 3, \dots$, where the only non-zero component of $f^{(n)}$ is

$$f^{(n)}_n : \operatorname{Ext}^1(V,V)^{\otimes n} \to \operatorname{Ext}^1(V,V).$$

It is easy to see that such an infinite composition necessarily converges. We want to choose the first map $f^{(2)}_2$ in such a way that the following diagram would be commutative:

$$\begin{array}{ccc}
\operatorname{Ext}^1(V,V)\otimes\operatorname{Ext}^1(V,V) & & \\
\Big\downarrow f^{(2)}_2 & \searrow & \\
\operatorname{Ext}^1(V,V) & \longrightarrow & \operatorname{Hom}(H^0(V\otimes E), H^1(V\otimes E))
\end{array} \qquad (3.1)$$

where the horizontal and the diagonal arrows are partial dualizations of the maps given by m_2 and m_3, respectively. In fact, by Serre duality the bottom arrow can be identified with the dual of $\mu_{V,W}$. Hence, it is surjective, so there exists a map $f^{(2)}_2$ making the above diagram commutative. Let us replace the A_∞-structure on $D^b(C)$ by the homotopic one: $m \mapsto m + \delta(-f^{(2)})$. For this new structure the map

$$m_3 : \operatorname{Ext}^1(V,V)\otimes\operatorname{Ext}^1(V,V)\otimes H^0(V\otimes E) \to H^1(V\otimes E)$$

will be zero. Now we can choose a map $f^{(3)}_3$ which makes the following diagram commutative:

$$\begin{array}{ccc}
\operatorname{Ext}^1(V,V)^{\otimes 3} & & \\
\Big\downarrow f^{(3)}_3 & \searrow & \\
\operatorname{Ext}^1(V,V) & \longrightarrow & \operatorname{Hom}(H^0(V\otimes E), H^1(V\otimes E))
\end{array} \qquad (3.2)$$

where the horizontal arrow is the same as before, while the diagonal arrow is given by the partial dualization of m_4. Then we again replace the A_∞-structure by the homotopic one: $m \mapsto m + \delta(f^{(3)})$. For this new A_∞-structure the maps

$$m_n : \operatorname{Ext}^1(V,V)^{\otimes(n-1)} \otimes H^0(V\otimes E) \to H^1(V\otimes E)$$

will be zero for $n = 3, 4$. Continuing in this way we will eventually kill all of these maps for $n \geqslant 3$. □

3.2 Computation of the Fourier–Mukai transform

Let C be a smooth projective curve, J^d the Jacobian of line bundles of degree d, $\sigma^d : \mathrm{Sym}^d C \to J^d$ the natural morphism sending $p_1 + \cdots + p_d$ to $\mathcal{O}_C(p_1 + \cdots + p_d)$. For a line bundle L of degree n we set $F_d(L) = R\sigma^d_* L^{(d)}$, where $L^{(d)}$ is the dth symmetric power of L, which is a line bundle on $\mathrm{Sym}^d C$. Below we identify J^d with J as before using the fixed point p. In particular, we consider the Brill–Noether loci W^r_d as subschemes of J.

We need to recall some facts about the Picard group of $\mathrm{Sym}^d C$. For every $d \geqslant 2$ there is an exact sequence

$$0 \to \mathrm{Pic}(J) \to \mathrm{Pic}(\mathrm{Sym}^d C) \stackrel{\deg}{\to} \mathbb{Z} \to 0,$$

where the embedding of $\mathrm{Pic}(J)$ is given by the pull-back with respect to σ^d, while the map deg to $\mathbb{Z}$ is normalized by the condition $\deg(L^{(d)}) = \deg(L)$ for $L \in \mathrm{Pic}(C)$ (see [Col75, BP01]). In particular, $\mathrm{Pic}^0(\mathrm{Sym}^d C)$ is naturally identified with $\mathrm{Pic}^0(J)$. In addition, for every line bundle M on $\mathrm{Pic}(\mathrm{Sym}^d C)$ and for every linear system $\mathbb{P} \subset \mathrm{Sym}^d C$ of positive dimension, the degree of M is equal to the usual degree of $M|_{\mathbb{P}}$.

LEMMA 3.1. *Let E be a vector bundle on a projective space $\mathbb{P}^n$ such that there is an exact sequence*

$$0 \to V \otimes \mathcal{O}(-1) \to W \otimes \mathcal{O} \to E \to 0.$$

Then $H^i(\mathbb{P}^n, S^j E(m)) = 0$ for $i > 0$, $j \geqslant 0$, $m \geqslant -i$.

Proof. We have the following Koszul resolution for $S^j E$:

$$0 \to \bigwedge^j V \otimes \mathcal{O}(-j) \to \bigwedge^{j-1} V \otimes W \otimes \mathcal{O}(-j+1) \to \cdots \to V \otimes S^{j-1} W \otimes \mathcal{O}(-1) \to S^j W \otimes \mathcal{O} \to S^j E \to 0.$$

Using this resolution to compute the cohomology of $S^j E(m)$ we immediately derive the result from the vanishing of $H^i(\mathbb{P}^n, \mathcal{O}(m))$ for $i > 0$, $m \geqslant -i$. $\square$

LEMMA 3.2.

i) *For every line bundle ξ of degree zero on C one has*

$$F_d(L \otimes \xi) \simeq F_d(L) \otimes \mathcal{P}_\xi^{-1},$$

where $\mathcal{P}_\xi$ is the line bundle on J corresponding to $\xi \in J$ via the self-duality of J.

ii) *If $\deg(L) \geqslant -1$ then $F_d(L)$ is a sheaf on J^d, i.e. $R^i \sigma^d_* L^{(d)} = 0$ for $i > 0$.*

iii) *Assume that $d \leqslant g + 1$. Let $j : J \setminus W^1_d \hookrightarrow J$ be the open embedding. If $-1 \leqslant \deg(L) \leqslant g - d$ then $F_d(L) \simeq j_* j^* F_d(L)$. If C is not hyperelliptic then this is true for every L such that $\deg(L) \geqslant -1$.*

Proof. i) It suffices to prove that

$$(L \otimes \xi)^{(d)} \simeq L^{(d)} \otimes (\sigma^d)^* \mathcal{P}_\xi^{-1}.$$

Since the line bundle $(L \otimes \xi)^{(d)} \otimes (L^{(d)})^{-1}$ on $\mathrm{Sym}^d C$ is algebraically equivalent to the trivial bundle, it has form $(\sigma^d)^* \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Pic}^0(J)$. Let us embed C into $\mathrm{Sym}^d C$ by the map $x \mapsto x + (d-1)p$. Restricting our line bundles to C we obtain an isomorphism $\xi \simeq (\sigma^d)^* \mathcal{L}$, which precisely means that $\mathcal{L} \simeq \mathcal{P}_\xi^{-1}$. Note that the sign appears here because the pull-back map $\mathrm{Pic}^0(J) \to \mathrm{Pic}^0(C) \simeq J(k)$ differs from the standard isomorphism of $\hat{J}$ with J by $[-1]_J$.

ii) Recall that the fibers of σ^d are projective spaces corresponding to complete linear systems of degree d. Note that the restriction of $L^{(d)}$ to every fiber $\mathbb{P} = (\sigma^d)^{-1}(\xi)$, where $\xi \in J$, has degree $\deg(L)$. Also, it is well known that the normal bundle N to the embedding $\mathbb{P} = (\sigma^d)^{-1}(\xi) \subset \mathrm{Sym}^d C$

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fits into the exact sequence

$$0 \to N \to H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_{\mathbb{P}} \to H^1(C, \xi(dp)) \otimes \mathcal{O}_{\mathbb{P}}(1) \to 0.$$

Using Lemma 3.1 we deduce that higher cohomology of the bundle $L^{(d)}|_{\mathbb{P}} \otimes S^j(N^\vee)$ on the projective space $\mathbb{P}$ vanish. By the formal functions theorem this implies that $R^i\sigma^d_* L^{(d)}|_\xi = 0$ for $i > 0$.

iii) Let $Q_d \subset \mathrm{Sym}^d$ be the preimage of W^1_d under the morphism σ^d. Let us denote by j' the open embedding $\mathrm{Sym}^d C \setminus Q_d \hookrightarrow \mathrm{Sym}^d C$. Then it suffices to prove that the natural map

$$\sigma^d_* L^{(d)} \to \sigma^d_* j'_* j'^* L^{(d)}$$

is an isomorphism. If C is non-hyperelliptic then Q_d has codimension $\geqslant 2$ in $\mathrm{Sym}^d C$ (this follows from Martens' theorem, see [ACGH84]), so in this case $L^{(d)} \simeq j'_* j'^* L^{(d)}$. Now assume that C is hyperelliptic. Then Q_d is an irreducible divisor in $\mathrm{Sym}^d C$. It is easy to check that $\deg(Q_d) = d-g-1$ (see the proof of Lemma 2.5 in [BP01]). Therefore, for every $n > 0$ we have $\deg(L^{(d)}(nQ_d)) < 0$. Let us consider the open subset $U = \mathrm{Sym}^d C \setminus Q'_d \subset \mathrm{Sym}^d C$, where $Q'_d = (\sigma^d)^{-1}(W^2_d)$. Since the morphism $\pi : Q_d \cap U \to W^1_d \setminus W^2_d$ induced by σ^d is flat, the base change theorem implies that

$$\pi_*(L^{(d)}(nQ_d)|_{Q_d \cap U}) = 0$$

for every $n > 0$. Hence, the natural map

$$\sigma^d_{U*}(L^{(d)}|_U) \to \sigma^d_{U*}(L^{(d)}(*Q_d)|_U) \tag{3.3}$$

is an isomorphism, where $\sigma^d_U : U \to W_d \setminus W^2_d$ is the map induced by σ^d, $L^{(d)}(*Q_d) = \cup_n L^{(d)}(nQ_d)$. Let $j_U : U \hookrightarrow \mathrm{Sym}^d C$ and $j'' : \mathrm{Sym}^d C \setminus Q_d \to U$ be the open embeddings, so that $j' = j_U \circ j''$. Since the codimension of Q'_d in $\mathrm{Sym}^d C$ is $\geqslant 2$, we have $L^{(d)} \simeq j_{U*}(L^{(d)}|_U)$ (respectively $L^{(d)}(*Q_d) \simeq j_{U*}(L^{(d)}(*Q_d)|_U)$). Therefore, applying the push-forward with respect to the embedding $W_d \setminus W^2_d \hookrightarrow W_d$ to the isomorphism (3.3) we obtain the isomorphism

$$\sigma^d_* L^{(d)} \widetilde{\to} \sigma^d_* L^{(d)}(*Q_d). \qquad \square$$

Note that the morphism σ^d induces an isomorphism

$$H^1(J, \mathcal{O}_J) \simeq H^1(\mathrm{Sym}^d C, \mathcal{O}_{\mathrm{Sym}^d C}).$$

Hence, both spaces are naturally isomorphic to $H^1(C, \mathcal{O}_C)$. This isomorphism is used in the following result.

Proposition 3.1. *Let L be a line bundle on C. For every $d \geqslant 1$, $i \geqslant 0$, there is a canonical isomorphism*

$$H^i(\mathrm{Sym}^d C, L^{(d)}) \simeq TS^{d-i} H^0(C, L) \otimes \bigwedge^i H^1(C, L),$$

where for every vector space V we denote by $TS^n V \subset V^{\otimes n}$ the space of symmetric n-tensors. Under these isomorphisms the cup-product maps

$$H^1(\mathrm{Sym}^d C, \mathcal{O}_{\mathrm{Sym}^d C}) \otimes H^i(\mathrm{Sym}^d C, L^{(d)}) \to H^{i+1}(\mathrm{Sym}^d C, L^{(d)})$$

are identified with the natural maps

$$H^1(\mathcal{O}_C) \otimes TS^{d-i} H^0(L) \otimes \bigwedge^i H^1(L) \to TS^{d-i-1} H^0(L) \otimes \bigwedge^{i+1} H^1(L)$$

induced by the cup-product map $H^1(\mathcal{O}_C) \otimes H^0(L) \to H^1(L)$.

Proof. The first assertion is a consequence of the *symmetric Künneth isomorphism* constructed by Deligne in [AGV73, XVII, (5.5.17.2), (5.5.32.1)]. This is a canonical isomorphism of graded vector

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spaces

$$TS^d R\Gamma(C, L) \to R\Gamma(\mathrm{Sym}^d C, L^{(d)}) \tag{3.4}$$

where in the LHS we take the dth symmetric power of the graded vector space $R\Gamma(C, L) = H^0(C, L) \oplus H^1(C, L)[-1]$, so that

$$TS^d R\Gamma(C, L) \simeq \oplus_i TS^{d-i} H^0(C, L) \otimes \bigwedge^i H^1(C, L).$$

The compatibility with cup-products can be easily checked by considering pull-backs to the dth cartesian power of C. □

COROLLARY 3.1. *Assume that* $\deg(L) \geqslant g - d$ *and* $h^0(L) = 0$*. Then* $H^i(\mathrm{Sym}^d C, L^{(d)}) = 0$ *for all* i*.*

Let $\Theta = W_{g-1} \subset J$ be the theta divisor. The following lemma is probably well known, however, we could not find the reference in the literature (the case $d = 1$ is the classical Riemann's theorem on intersection of C with Θ).

LEMMA 3.3. *For every* $d \geqslant 1$ *one has an isomorphism*

$$\omega_{\mathrm{Sym}^d C} \simeq (\sigma^d)^*(\mathcal{O}_J(\Theta))((g - d - 1)R_p^d),$$

where $R_p^d \subset \mathrm{Sym}^d C$ *is the image of the natural embedding* $\mathrm{Sym}^{d-1} C \to \mathrm{Sym}^d C : D \mapsto p + D$*.*

Proof. Assume first that $d > 2g - 2$. Then σ^d identifies $\mathrm{Sym}^d C$ with the projective bundle $\mathbb{P}(E_d)$ of lines in a vector bundle E_d over J in such a way that the line bundle $\mathcal{O}_{\mathrm{Sym}^d C}(R_p^d)$ corresponds to $\mathcal{O}(1)$. Furthermore, it is known that $\det E_d \simeq \mathcal{O}_J(-\Theta)$ (see [Mat61]). This easily implies the result in this case. To deduce it for all $d \geqslant 1$ one can use the descending induction together with the fact that the normal bundle to the embedding $\mathrm{Sym}^d C \to \mathrm{Sym}^{d+1} C : D \mapsto p + D$ is isomorphic to $\mathcal{O}_C(R_p^d)$. □

LEMMA 3.4. *Let* A *be a local regular ring,* $P_0 \to P_1 \to \cdots \to P_n$ *be a complex of free* A*-modules of finite rank. Assume that all modules* $H^i(P_\bullet)$ *have support of codimension* $\geqslant n$*. Then* $H^i(P_\bullet) = 0$ *for* $i < n$*.*

Proof. Let g be the dimension of A. First of all, note that the cases $n = 1$ and $n > g$ are trivial. On the other hand, the case $n = g$ is equivalent to the lemma in [Mum74, III.13]. To deduce the general case we will use the induction in g. We can assume that $n < g$. Let us choose an element x in the maximal ideal $\mathfrak{m} \subset A$, such that x does not belong to $\mathfrak{m}^2$ and to all associate primes of height n of the modules $(H^i(P_\bullet))$ (such an element exists since $n < g$). Then the ring $\overline{A} = A/xA$ is regular of dimension $g - 1$ and we claim that the assumptions of the lemma are satisfied for the complex $\overline{P}_\bullet = P_\bullet/xP_\bullet$ of free $\overline{A}$-modules. Indeed, from the long exact sequence

$$\cdots \to H^i(P_\bullet) \xrightarrow{x} H^i(P_\bullet) \to H^i(\overline{P}_\bullet) \to H^{i+1}(P_\bullet) \xrightarrow{x} H^{i+1}(P_\bullet) \to \cdots$$

we immediately obtain that the modules $H^i(\overline{P}_\bullet)$ have codimension $\geqslant n + 1$ in $\mathrm{Spec}(A)$, or equivalently, codimension $\geqslant n$ in $\mathrm{Spec}(\overline{A})$. By the induction assumption $H^i(\overline{P}_\bullet) = 0$ for $i < n$. Now the above exact sequence implies that the endomorphism of multiplication by x on $H^i(P_\bullet)$ is surjective for $i < n$. By Nakayama lemma this implies that $H^i(P_\bullet)$ vanishes for $i < n$. □

Proof of Theorem 0.2. The fact that $F_d(\mathcal{O}_C((g-d)p))$ is concentrated in degree 0 was already proven in Lemma 3.2, part ii. Let us denote for brevity the functor $[-1]^* \circ \mathcal{S}$ by $\mathcal{S}^-$. Note that the transform $\mathcal{S}^-(F_d(\mathcal{O}_C((g-d)p)))$ is isomorphic to

$$Rp_2^*(p_1^*(\mathcal{O}_C((g-d)p))^{(d)} \otimes (\sigma^d \times \mathrm{id})^* \mathcal{P}^{-1}),$$

where p_1 and p_2 are projections of the product $\mathrm{Sym}^d C \times J$ to its factors, $\mathcal{P}$ is the Poincaré line bundle on $J \times J$. It follows that $\mathcal{S}^-(F_d(\mathcal{O}_C((g-d)p)))$ can be represented locally on J by a complex of vector bundles

$$V_0 \to V_1 \to \cdots \to V_d. \tag{3.5}$$

Now Lemma 3.2(i) and Corollary 3.1 imply that the cohomology sheaves of $\mathcal{S}^-(F_d(\mathcal{O}_C((g-d)p)))$ are supported on $W_{g-d} = \sigma^{g-d}(\mathrm{Sym}^d C) \subset J$. Applying Lemma 3.4 we derive that the cohomology of the complex $V_\bullet$ is concentrated in degree d. Thus,

$$S_d := \mathcal{S}^-(F_d(\mathcal{O}_C((g-d)p)))[d]$$

is actually a coherent sheaf (placed in degree zero).

We claim that the sheaf S_d is obtained as the (non-derived) push-forward of its restriction to the open subset $J \setminus W^1_{g-d}$ (since $W^1_1 = \emptyset$, we can assume below that $g - d \geqslant 2$). Indeed, since locally we have a resolution of S_d by vector bundles (3.5) of length d, it follows that $\mathcal{H}^i_Y(S_d) = 0$ for every closed subset $Y \subset J$ of codimension greater than $i + d$ (where $\mathcal{H}^i_Y$ denotes local cohomology with support at Y). Since $\dim W^1_{g-d} \leqslant g - d - 2$, we find that $\mathcal{H}^i_{W^1_{g-d}}(S_d) = 0$ for $i = 0, 1$. This implies our claim that

$$S_d \simeq j_* j^* S_d, \tag{3.6}$$

where $j : J \setminus W^1_{g-d} \to J$ is the natural embedding.

On the other hand, we can use Theorem 2.2 and Lemma 3.1 to study the sheaf S_d near a point $L(-(g-d)p)$ of $W_{g-d} \setminus W^1_{g-d} \subset J$, where $\deg(L) = g - d$ (so that $h^0(L) = 1$ and $h^1(L) = d$). More precisely, choosing a non-zero section s of L, we find that in a formal neighborhood of this point S_d is isomorphic to the dth cohomology of the complex

$$R \to H^1(L) \otimes R \to \cdots \to \bigwedge^{d-1} H^1(L) \otimes R \to \bigwedge^{d} H^1(L) \otimes R,$$

where $R = \hat{S}(H^1(\mathcal{O})^*)$. The differential is the sum of the usual Koszul differential associated with the surjection $H^1(\mathcal{O}_C) \to H^1(L)$ (given by the cup-product with s) and some terms that vanish modulo $\mathfrak{m}^2$, where $\mathfrak{m} \subset R$ is a maximal ideal. Therefore, its dth cohomology is isomorphic to $R/(f_1, \ldots, f_d)$ with $f_i \in \mathfrak{m}$, where $(f_1, \ldots, f_d) \bmod \mathfrak{m}^2$ are components of the above map $H^1(\mathcal{O}) \otimes H^1(L)$ with respect to some basis of $H^1(L)$. It follows that the ideal $(f_1, \ldots, f_d)$ is prime and has height d. Since S_d is supported on W_{g-d} which is smooth at $L(-(g-d)p)$ and also has codimension d, the ideal $(f_1, \ldots, f_d) \subset R$ coincides with the ideal defining W_{g-d} near $L(-(g-d)p)$. Therefore, $j^* S_d$ is actually a line bundle on $W_{g-d} \setminus W^1_{g-d}$.

Next, we want to study the derived pull-back $L(\sigma^{g-d})^* S_d$. First, we need to calculate the pull-back of the Poincaré line bundle $\mathcal{P}$ on $J \times J$ under the morphism

$$\sigma^d \times \sigma^{g-d} : \mathrm{Sym}^d C \times \mathrm{Sym}^{g-d} C \to J \times J.$$

For this purpose it is convenient to use the Deligne symbol of a pair of line bundles on a relative curve (see [Del87]). Namely, it is well known that

$$\mathcal{P}^{-1} \simeq \langle p_{13}^* \mathcal{P}_C, p_{23}^* \mathcal{P}_C \rangle$$

where p_{ij} are projections from the product $C \times J \times J$, $\mathcal{P}_C$ is the Poincaré line bundle on $C \times J$

(which we always take to be normalized at p). Therefore,

$$(\sigma^d\times\sigma^{g-d})^*\mathcal{P}^{-1}\simeq\langle\mathcal{O}(\mathcal{D}_{12}-d[p\times\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C]),\mathcal{O}(\mathcal{D}_{13}-(g-d)[p\times\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C])\rangle,$$

where we consider $C\times\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C$ as a relative curve over $\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C$,

$$\mathcal{D}_{12}=\{(x,D,D')\in C\times\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C : x\in D\},$$
$$\mathcal{D}_{13}=\{(x,D,D')\in C\times\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C : x\in D'\}.$$

Note that the intersection $\mathcal{D}_{12}\cap\mathcal{D}_{13}$ is irreducible and projects birationally to the divisor $\mathcal{D}\subset\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C$ supported on the set of (D,D') such that $D\cap D'\neq\emptyset$. It follows that

$$\langle\mathcal{O}(\mathcal{D}_{12}),\mathcal{O}(\mathcal{D}_{13})\rangle\simeq\mathcal{O}_{\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C}(\mathcal{D}).$$

Similarly, we derive that

$$\langle\mathcal{O}(\mathcal{D}_{12}),\mathcal{O}(p\times\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C)\rangle\simeq\mathcal{O}_{\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C}(R_p^d\times\operatorname{Sym}^{g-d} C),$$
$$\langle\mathcal{O}(\mathcal{D}_{13}),\mathcal{O}(p\times\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C)\rangle\simeq\mathcal{O}_{\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C}(\operatorname{Sym}^d C\times R_p^{g-d}),$$

where we use the notation of Lemma 3.3. Thus, we obtain

$$(\sigma^d\times\sigma^{g-d})^*\mathcal{P}^{-1}\simeq\mathcal{O}_{\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C}(\mathcal{D}-(g-d)[R_p^d\times\operatorname{Sym}^{g-d} C]-d[\operatorname{Sym}^d C\times R_p^{g-d}]).$$

Note that $\mathcal{O}_{\operatorname{Sym}^i C}(R_p^i)$ is isomorphic to the ith symmetric power of $\mathcal{O}_C(p)$. Therefore, we derive that

$$L(\sigma^{g-d})^*\mathcal{S}^-(F_d(\mathcal{O}_C((g-d)p)))\simeq Rp_{2*}((\sigma^d\times\sigma^{g-d})^*\mathcal{P}^{-1}((g-d)[R_p^d\times\operatorname{Sym}^{g-d} C]))$$
$$\simeq Rp_{2*}(\mathcal{O}(\mathcal{D}))(-dR_p^{g-d}),$$

where p_1,p_2 are the projections of the product $\operatorname{Sym}^d C\times\operatorname{Sym}^{g-d} C$ onto its factors. It is easy to see that for every $D\in\operatorname{Sym}^{g-d} C$, the restriction of the divisor $\mathcal{D}$ to $\operatorname{Sym}^d C\times\{D\}$ is the divisor $R_D^d\subset\operatorname{Sym}^d C$ such that $\mathcal{O}_{\operatorname{Sym}^d C}(R_D^p)\simeq(\mathcal{O}_C(D))^{(d)}$. It follows that over the complement to $Q^{g-d}\subset\operatorname{Sym}^{g-d}(C)$ the natural map $\mathcal{O}_{\operatorname{Sym}^{g-d} C}\to Rp_{2*}(\mathcal{O}(\mathcal{D}))$ is an isomorphism (by the base change theorem). Therefore, we obtain

$$L^d i^* j^* S_d\simeq j^*\sigma_*^{g-d}(\mathcal{O}_{\operatorname{Sym}^{g-d} C}(-dR_p^{g-d})),$$

where $i: W_{g-d}^{ns}=W_{g-d}\setminus W_{g-d}^1\to J\setminus W_{g-d}^1$ is the closed embedding, $j: J\setminus W_{g-d}^1\to J$ is the open embedding. On the other hand, applying duality theory we get

$$L^d i^* j^* S_d\simeq j^* S_d\otimes\omega_{W_{g-d}^{ns}}^{-1}.$$

Hence, the above isomorphism can be rewritten as

$$j^* S_d\simeq j^*\sigma_*^{g-d}(\omega_{\operatorname{Sym}^{g-d} C}(-dR_p^{g-d})).$$

Now Lemma 3.2, part iii together with (3.6) imply that

$$S_d\simeq\sigma_*^{g-d}(\omega_{\operatorname{Sym}^{g-d} C})(-dR_p^{g-d}).$$

Using Lemma 3.3 we obtain that

$$S_d\simeq\sigma_*^{g-d}(\mathcal{O}_{\operatorname{Sym}^{g-d} C})(-\Theta)$$

which proves the first isomorphism of the theorem. Finally, the duality for the morphism σ^{g-d} implies that

$$R\sigma_*^{g-d}(\omega_{\operatorname{Sym}^{g-d} C}(-dR_p^{g-d}))[-d]\simeq R\underline{\operatorname{Hom}}(R\sigma_*^{g-d}\mathcal{O}_{\operatorname{Sym}^{g-d} C}(dR_p^{g-d}),\mathcal{O}_J)$$

which gives the second isomorphism of the theorem. □

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