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REACH OF REPULSION FOR DETERMINANTAL POINT PROCESSES IN HIGH DIMENSIONS

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Abstract

Goldman (2010) proved that the distribution of a stationary determinantal point process (DPP) Φ can be coupled with its reduced Palm version $\Phi^{0,!}$ such that there exists a point process η where $\Phi = \Phi^{0,!} \cup \eta$ in distribution and $\Phi^{0,!} \cap \eta = \varnothing$. The points of η characterize the repulsive nature of a typical point of Φ. In this paper we use the first-moment measure of η to study the repulsive behavior of DPPs in high dimensions. We show that many families of DPPs have the property that the total number of points in η converges in probability to 0 as the space dimension $n \to \infty$. We also prove that for some DPPs, there exists an R^* such that the decay of the first-moment measure of η is slowest in a small annulus around the sphere of radius $\sqrt{n}R^*$. This R^* can be interpreted as the asymptotic reach of repulsion of the DPP. Examples of classes of DPP models exhibiting this behavior are presented and an application to high-dimensional Boolean models is given.

Keywords: Laguerre–Gaussian model; normal-variance mixture model; Bessel-type model; high-dimensional geometry; Palm calculus; pair correlation function; stochastic ordering; Boolean model; information theory; error exponent; log-concave density; large deviation;

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1. Introduction

Determinantal point processes (DPPs) are useful models for point patterns where the points exhibit some repulsion from each other, resulting in a more regularly spaced pattern than a Poisson point process. These models originally appeared in random matrix theory and the formalism was introduced by Macchi [17], who was motivated by modeling Fermionic particles in quantum mechanics. They have since been used in many applications; for example, telecommunication networks, machine learning, and ecology; see [10], [12], [15], [16], and the references therein. In this paper we describe the repulsive behavior of stationary and isotropic DPPs as the space dimension goes to ∞.

In the following, a ball with center at the origin and radius r in $\mathbb{R}^n$ is denoted by $B_n(r)$. The ℓ^2 vector norm is denoted by $|\cdot|$ and the L^2-norm on the space $L^2(\mathbb{R}^n)$ by $\|\cdot\|_2$. Now, consider a sequence of point processes Φ_n indexed by dimension, each with constant intensity ρ_n.

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If $\rho_n = \mathrm{e}^{n\rho}$ and $R_n = \sqrt{n}R$, with $\rho \in \mathbb{R}$ and $R > 0$, then Stirling's formula yields

$$\operatorname{vol}(B_n(R_n)) \sim \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi\mathrm{e}}{n}\right)^{n/2} R_n^n \quad \text{as } n \to \infty.$$

This implies there exists a threshold $R^* = 1/\sqrt{2\pi\mathrm{e}}\mathrm{e}^{\rho}$ such that, as $n \to \infty$,

$$\mathbb{E}[\Phi_n(B_n(R_n))] \sim \exp\big(n\big(\rho + \tfrac{1}{2}\log 2\pi\mathrm{e} + \log R\big) + o(n)\big) \to \begin{cases} 0, & R < R^*, \\ \infty, & R > R^*. \end{cases} \tag{1.1}$$

This justifies the interest in considering this regime where the intensities grow exponentially with the dimension and distances grow with the square root of the dimension. This regime also naturally arises in information theory, and following [1] we call it the Shannon regime. In this paper we study the effect of repulsion in this regime and quantify the range and strength at which DPPs asymptotically exhibit repulsion between points.

These issues were discussed in [24], where the authors characterized a certain class of DPPs by an effective 'hard-core' diameter D that grows as $\sqrt{n}$, aligning with our observations. They observed that for $r < D$, the number of points in a ball of radius r around a typical point will be 0 with probability approaching 1, and for $r > D$, the number of points in a ball of radius r around a typical point is 0 with probability approaching 0 as dimension $n \to \infty$. The behavior for $r < D$ is a result of the natural separation due to dimensionality as exhibited in (1.1). However, the observation that D is the maximal such separation is due to the ν-weakly sub-Poisson property of DPPs as defined in [3], and is a feature of all DPPs, not just those studied in [24]. This behavior is the same as a sequence of Poisson point processes in the same regime and, thus, this separation of points in high dimensions is due to dimensionality and not the repulsion of the DPP model. In this paper we provide a more precise description of the repulsive behavior in high dimensions that is specific to the associated kernel of the DPP.

The measure of repulsiveness used in this paper is a refinement of the global measure of repulsiveness for stationary DPPs described in the supplementary material to [15]; see [14]. In that work, the authors considered the measure

$$\gamma := \rho \int (1 - g(x))\,\mathrm{d}x, \tag{1.2}$$

where ρ is the intensity, and $(x, y) \mapsto g(x - y)$ is the pair correlation function of the point process. A point process is considered more repulsive the farther g is away from 1; $g \equiv 1$ corresponds to a Poisson point process. As observed in [13], this measure has the upper bound $\gamma \le 1$ for all stationary point processes.

This measure can be refined in order to examine the repulsive effect of a point of the point process across some finite distance. Goldman [7] proved that for a stationary DPP Φ satisfying certain conditions, there exists a point process η such that

$$\Phi \overset{\mathrm{D}}{=} \Phi^{0,!} \cup \eta \quad \text{and} \quad \Phi^{0,!} \cap \eta = \varnothing,$$

where $\Phi^{0,!}$ denotes a point process with the reduced Palm distribution of Φ and '$\overset{\mathrm{D}}{=}$' denotes equality in distribution. Thus, η is the set of points that have to be removed from Φ due to repulsion when a point is 'placed at' the origin. In the following, the first-moment measure of η will be used as a measure of the repulsiveness of a DPP Φ, and the repulsive effect of a typical point over a finite distance R is quantified by $\mathbb{E}[\eta(B_n(R))]$. Note also that

$$\mathbb{E}[\eta(B_n(R))] = \rho\operatorname{vol}(B_n(R)) - \mathbb{E}[\Phi^{0,!}(B_n(R))] = \rho[K_{\mathrm{Poi}}(R) - K_{\mathrm{DPP}}(R)],$$

where K_{Poi} and K_{DPP} are Ripley's K functions [20] for a Poisson point process and Φ, respectively. Finally, note that the measure of global repulsiveness (1.2) corresponds to η in the sense that $\gamma = \mathbb{E}[\eta(\mathbb{R}^n)]$.

Our main results describe the behavior of the first-moment measure of η in the Shannon regime. Consider a sequence of stationary DPPs $\{\Phi_n\}$ such that Φ_n lies in $\mathbb{R}^n$. For each n, let η_n be the point process such that $\Phi_n \stackrel{\text{D}}{=} \Phi_n^{0,!} \cup \eta_n$ and $\Phi_n^{0,!} \cap \eta_n = \varnothing$. We consider the quantity $\mathbb{E}[\eta_n(\mathbb{R}^n)]$ and the probability measure $\mathbb{E}[\eta_n(\cdot)]/\mathbb{E}[\eta_n(\mathbb{R}^n)]$ on $\mathbb{R}^n$ that is defined to estimate the strength and reach of the repulsiveness of a DPP in any dimension.

It is often the case that $\mathbb{E}[\eta_n(\mathbb{R}^n)] \to 0$ as $n \to \infty$. In this case, Markov's inequality and the coupling inequality imply that, in high dimensions, the total variation distance is small between Φ_n and $\Phi_n^{0,!}$. Indeed,

$$\|\Phi_n - \Phi_n^{0,!}\|_{\text{TV}} \leq \mathbb{P}(\eta_n(\mathbb{R}^n) > 0) \leq \mathbb{E}[\eta_n(\mathbb{R}^n)]. \tag{1.3}$$

Since Φ_n and $\Phi_n^{0,!}$ have the same distribution if and only if Φ_n is Poisson by Slivnyak's theorem (see [4]), we see that such DPPs look increasingly like Poisson point processes as the space dimension increases.

However, the effect of the repulsion can still be observed by examining the probability measure $\mathbb{E}[\eta_n(\cdot)]/\mathbb{E}[\eta_n(\mathbb{R}^n)]$ on $\mathbb{R}^n$ as seen in Propositions 3.1–3.3. Letting X_n be a random vector in $\mathbb{R}^n$ with this probability distribution, we show that if $|X_n|/\sqrt{n} \xrightarrow{\mathbb{P}} R^* \in (0, \infty)$ then

$$\lim_{n\to\infty} \frac{\mathbb{E}[\eta_n(B_n(R\sqrt{n}))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \begin{cases} 0, & R < R^*, \\ 1, & R > R^*, \end{cases}$$

where '$\xrightarrow{\mathbb{P}}$' denotes convergence in probability. Here, R^* is interpreted as the *asymptotic reach of repulsion* in the Shannon regime for these DPPs. This result implies that in high dimensions a typical point has its strongest repulsive effect on points that are at a distance $\sqrt{n}R^*$ away.

The parametric families of DPP kernels presented in [2] and [15] provide examples of DPPs exhibiting a reach of repulsion R^* and counterexamples where no finite R^* exists, as well as computational results on the rates of convergence when a threshold does occur. In Section 4 we study four classes of DPPs: Laguerre–Gaussian DPPs, power exponential DPPs, Bessel-type DPPs, and normal-variance mixture DPPs. For Laguerre–Gaussian DPPs, the sequence $|X_n|/\sqrt{n}$ satisfies a large deviation principle; see Lemma 4.1. As a consequence, the reach of repulsion R^* becomes a phase transition for the exponential rate at which $\mathbb{E}[\eta_n(B_n(R\sqrt{n}))] \to 0$ as $n \to \infty$; see Proposition 4.1. Power exponential DPPs are shown to have a finite reach of repulsion in the Shannon regime for certain parameters; see Proposition 4.2. Bessel-type DPPs are a more repulsive family that does not exhibit an R^*; see Proposition 4.3. Finally, normal-variance mixture DPPs provide additional examples of DPPs that exhibit an R^*, including the Whittle–Matérn and Cauchy models; see Propositions 4.4 and 4.5.

An application of these results is presented in Section 5. We show that some threshold results of [1] for Poisson–Boolean models can be extended to generalized Laguerre–Gaussian DPP Boolean models in the Shannon regime using the rates of convergence computed for these DPPs. Finally, concluding remarks and open questions are stated in Section 6.

2. Preliminaries

DPPs are characterized by an integral operator $\mathcal{K}$ with kernel K, and can be defined in terms of their joint intensities, also known as correlation functions; see [10] and [15].

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Definition 2.1. A simple, locally finite, spatial point process Φ on $\mathbb{R}^n$ is a DPP with kernel $K: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ ($\Phi \sim \mathrm{DPP}(K)$) if its joint intensities exist for all order k and satisfy

$$\rho^{(k)}(x_1, \dots, x_k) = \det(K(x_i, x_j))_{1 \le i, j \le k}, \qquad k = 1, 2, \dots .$$

Note that the intensity function of Φ is given by $\rho(x) = K(x, x)$. The degenerate case where $K(x, y) = \delta_{\{x=y\}}$ coincides with a Poisson point process with unit intensity.

The following conditions on K are imposed to ensure that $\Phi \sim \mathrm{DPP}(K)$ is well defined. Let $K: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous kernel and assume that K is symmetric, i.e. $K(x, y) = K(y, x)$. The kernel K then defines a self-adjoint integral operator $\mathcal{K}$ on $L^2(\mathbb{R}^n)$ given by $\mathcal{K} f(x) = \int K(x, y) f(y)\,\mathrm{d}y$. For any compact set $S \subset \mathbb{R}^n$, the restricted operator $\mathcal{K}_S$ given by

$$\mathcal{K}_S f(x) = \int_S K(x, y) f(y)\,\mathrm{d}y, \qquad x \in S,$$

is a compact operator. By the spectral theory for self-adjoint compact operators, the spectrum of $\mathcal{K}_S$ consists solely of countably many eigenvalues $\{\lambda_k^S\}_{k \in \mathbb{N}}$ with an accumulation point possible only at 0. See [21] for more on compact operators. These conditions imply that for any compact $S \subset \mathbb{R}^n$, the kernel K restricted to $S \times S$ has a spectral representation

$$K(x, y) = \sum_{k=1}^{\infty} \lambda_k^S \phi_k^S(x) \overline{\phi_k^S(y)}, \qquad (x, y) \in S \times S,$$

where $\{\phi_k^S\}_{k \in \mathbb{N}}$ are the eigenvectors of $\mathcal{K}_S$, and form an orthonormal basis of $L^2(S)$.

Theorem 2.1. (Macchi [17].) *Under the above conditions, a kernel K defines a determinantal process on $\mathbb{R}^n$ if and only if the spectrum of $\mathcal{K}$ is contained in* $[0, 1]$.

If $K(x, y) = K_0(x - y)$ then $\Phi \sim \mathrm{DPP}(K)$ is stationary. In this case, the operator $\mathcal{K}$ is the convolution operator $\mathcal{K}(f) = K_0 \star f$ on $L^2(\mathbb{R}^n)$. The intensity function $\rho(x)$ is then constant and satisfies $\rho = K_0(0)$. For these stationary DPPs, there is a simple spectral condition for existence.

Theorem 2.2. (Lavancier *et al.* [15, Theorem 2.3].) *Assume that K_0 is a symmetric continuous real-valued function in $L^2(\mathbb{R}^n)$. Let $K(x, y) = K_0(x - y)$. Then* $\mathrm{DPP}(K)$ *exists if and only if* $0 \le \hat{K}_0 \le 1$, *where $\hat{K}_0$ denotes the Fourier transform of K_0.*

For the rest of this paper, when we state that $\Phi \sim \mathrm{DPP}(K)$ is stationary, we assume that $K(x, y) = K_0(x - y)$ for a real-valued $K_0 \in L^2(\mathbb{R}^n)$, and K is used to mean K_0. There exist stationary DPPs with kernels that are not of this form (see [10, Equation (4.3.7)]), but they are complex-valued and not considered here. In addition, when we stated that Φ is isotropic, we mean that $K_0(x) = R_0(|x|)$ and the distribution of Φ is thus invariant under rotations about the origin in $\mathbb{R}^n$.

The reduced Palm distribution of a stationary point process Φ can be interpreted as the distribution of Φ conditioned on there being a point at the origin with the point at the origin removed (see [4, Chapter 4]) and is denoted by $\mathbb{P}^{0,!}$. A point process with the Palm distribution $\mathbb{P}^{0,!}$ of Φ is denoted $\Phi^{0,!}$. The following theorem is a special case of a useful result about the Palm distribution of DPPs.

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Theorem 2.3. (Shirai and Takahashi [23, Theorem 6.5].) *Let* $\Phi \sim \mathrm{DPP}(K)$ *in* $\mathbb{R}^n$ *be stationary with intensity* $\rho = K(0) > 0$. *Then* $\Phi^{0,!}$ *is a DPP with associated kernel*

$$K_0^!(x, y) = \frac{1}{K(0)} \det \begin{pmatrix} K(x-y) & K(x) \\ K(y) & K(0) \end{pmatrix} = K(x-y) - \frac{1}{\rho} K(x) K(y).$$

The nearest-neighbor function of a stationary point process Φ in $\mathbb{R}^n$ is defined as

$$D(r) := \mathbb{P}^{0,!}(\Phi(B_n(r)) > 0). \tag{2.1}$$

If Φ is Poisson, Slivnyak's theorem yields $D(r) = 1 - \mathrm{e}^{-\mathbb{E}[\Phi(B_n(r))]}$. For $\Phi \sim \mathrm{DPP}(K)$, Theorem 2.3 implies that $D(r) = \mathbb{P}(\Phi^{0,!}(B_n(r)) > 0)$ with $\Phi^{0,!} \sim \mathrm{DPP}(K_0^!)$.

As mentioned in the introduction, Goldman [7] proved the following result.

Theorem 2.4. (Goldman [7, Theorem 7].) *Let* $\Phi \sim \mathrm{DPP}(K)$, *where* K *is continuous, and the spectrum of the integral operator* $\mathcal{K}$ *with kernel* K *is contained in* $[0, 1)$. *Then, there exists a point process* η *such that*

$$\Phi \stackrel{\mathrm{D}}{=} \Phi^{0,!} \cup \eta \quad \textit{and} \quad \Phi^{0,!} \cap \eta = \varnothing.$$

From this theorem we see that a point process with the distribution of $\Phi^{0,!}$ can be obtained from Φ by removing a subset of points η. This is a striking result since the procedure does not include shifting any of the remaining points. The points in η characterize the repulsive nature of the DPP Φ, since these are the points that are 'pushed out' by the point at zero under the reduced Palm distribution. It also makes sense to compare the repulsiveness of DPPs using η. For two stationary DPPs Φ_1 and Φ_2 with the same intensity, Φ_1 is defined to be more repulsive than Φ_2 if $\mathbb{E}[\eta_1(\mathbb{R}^n)] > \mathbb{E}[\eta_2(\mathbb{R}^n)]$. This corresponds to the definition in [15] using the measure γ defined in (1.2). Note that the assumptions for Theorem 2.4 excludes the interesting case of $\mathcal{K}$ with an eigenvalue of 1, corresponding to when $\hat{K}(x)$ attains a value of 1 for some x.

3. Main results

When considering the reach of repulsion of a DPP, it is natural to first consider the nearest-neighbor function (2.1). The following threshold behavior was observed for stationary DPPs in [24]. It is stated here for a sequence of DPPs in the Shannon regime. For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ in $\mathbb{R}^n$ be stationary with intensity $K_n(0) = \mathrm{e}^{n\rho}$ for some $\rho \in \mathbb{R}$. Then, for $\tilde{R} := (2\pi \mathrm{e})^{-1/2} \mathrm{e}^{-\rho}$,

$$\lim_{n\to\infty} \mathbb{P}(\Phi_n^{0,!}(B_n(\sqrt{n}R)) > 0) = \begin{cases} 0, & R < \tilde{R}, \\ 1, & R > \tilde{R}. \end{cases} \tag{3.1}$$

A proof of this fact can be found in Appendix A.

From this result we see that there is a separation of points as the dimension $n \to \infty$ for any stationary DPP. However, the same threshold behavior occurs if the elements of the sequence $\{\Phi_n\}$ are stationary Poisson point processes, as a consequence of (1.1). Using this observation we see that this separation is due purely to dimensionality and is not a result of the repulsiveness of DPPs.

The point process η_n as defined in Theorem 2.4 can be used to provide an alternative characterization of the repulsiveness of a DPP and can be used to measure the consequence of repulsiveness in high dimensions that depends on the determinantal structure.

Lemma 3.1. *Let $\Phi_n \sim \mathrm{DPP}(K_n)$ in $\mathbb{R}^n$ be stationary and assume that $0 \le \hat{K}_n < 1$. Let η_n be the point process given in Theorem 2.4 and define the random vector X_n in $\mathbb{R}^n$ with probability density $K_n(x)^2/\|K_n\|_2^2$. Then*

$$\mathbb{P}(X_n \in B) = \frac{\mathbb{E}[\eta_n(B)]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]}, \qquad B \in \mathcal{B}(\mathbb{R}^n).$$

In the following result we show that under certain limit conditions on the kernels of a sequence of DPPs, the repulsiveness measured by the first-moment measure of η_n is concentrated at a distance $\sqrt{n}R^*$ for some $R^* \in (0, \infty)$ as $n \to \infty$.

Proposition 3.1. *For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ be a stationary and isotropic DPP in $\mathbb{R}^n$, and assume that $0 \le \hat{K}_n < 1$. Let X_n be a random vector in $\mathbb{R}^n$ with probability density $K_n(x)^2/\|K_n\|_2^2$. Assume that, as $n \to \infty$,*

$$\frac{|X_n|}{\sqrt{n}} \xrightarrow{\mathbb{P}} R^*. \tag{3.2}$$

Then

$$\lim_{n\to\infty} \frac{\mathbb{E}[\eta_n(B(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \begin{cases} 0, & R < R^*, \\ 1, & R > R^*. \end{cases} \tag{3.3}$$

Remark 3.1. One way to prove (3.2) is to show that

$$\lim_{n\to\infty} \frac{\mathrm{var}(|X_n|^2)}{n^2} = 0 \quad \text{and} \quad \lim_{n\to\infty} \left(\frac{\mathbb{E}[|X_n|^2]}{n}\right)^{1/2} = R^* \in (0, \infty),$$

and then apply Chebychev's inequality.

Remark 3.2. For general vectors X_n in $\mathbb{R}^n$, the concentration of $|X_n|$ for large n has been well studied; see [6], [9], and [11]. Indeed, Fradelizi *et al.* [6, Proposition 3] proved that X_n is concentrated in a 'thin shell', i.e. there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and for each n,

$$\mathbb{P}\left(\left|\frac{|X_n|}{\mathbb{E}[|X_n|^2]^{1/2}} - 1\right| \ge \varepsilon_n\right) \le \varepsilon_n$$

if and only if $|X_n|$ has a finite rth moment for $r > 2$, and for some $2 < p < r$,

$$\left|\frac{\mathbb{E}[|X_n|^p]^{1/p}}{\mathbb{E}[|X_n|^2]^{1/2}} - 1\right| \to 0 \quad \text{as } n \to \infty.$$

For random vectors with log-concave distributions, the deviation estimate can be improved from the estimate obtained through Chebychev's inequality; see Remark 3.1. The best known estimate is given by the following theorem from [9].

Theorem 3.1. (Guédon and Milman [9, Theorem 1.1].) *Let X denote a random vector in $\mathbb{R}^n$ such that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X \otimes X] = I_n$. Assume X has a log-concave density. Then, for some $C > 0$ and $c > 0$,*

$$\mathbb{P}\left(\left|\frac{|X|}{\sqrt{n}} - 1\right| \ge t\right) \le C\mathrm{e}^{-c\sqrt{n}\min(t^3, t)}.$$

This leads to the following result.

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Proposition 3.2. *For each n, let* $\Phi_n \sim \mathrm{DPP}(K_n)$ *be a stationary and isotropic DPP in* $\mathbb{R}^n$*, and assume that* $0 \le \hat{K}_n < 1$*. Let* X_n *be a random vector with density* $K_n(x)^2/\|K_n\|_2^2$ *and let* $\sigma_n^2 = \mathbb{E}[|X_n|^2]$*. If* K_n^2 *is log-concave for all n then there exist positive constants C and c such that for all* $\delta \in (0, 1)$*,*

$$\frac{\mathbb{E}[\eta_n(B_n(\sigma_n(1-\delta)))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \le C\mathrm{e}^{-c\sqrt{n}\delta^3},$$

and for all $\delta > 0$*,*

$$\frac{\mathbb{E}[\eta_n(\mathbb{R}^n \setminus B_n(\sigma_n(1+\delta)))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \le C\mathrm{e}^{-c\sqrt{n}\min(\delta^3,\delta)}.$$

If, in addition,

$$\lim_{n\to\infty} \frac{\sigma_n}{\sqrt{n}} = R^* \in (0, \infty), \tag{3.4}$$

then for this R^**, the threshold (3.3) occurs, and for all* $R < R^*$*, there exists a constant* $C(R) > 0$ *such that*

$$\liminf_{n\to\infty} -\frac{1}{\sqrt{n}} \ln \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \ge C(R).$$

Remark 3.3. The final conclusion of Proposition 3.2 concerning the rate also holds for $R > R^*$ if $B_n(\sqrt{n}R)$ is replaced by $\mathbb{R}^n \setminus B_n(\sqrt{n}R)$.

The assumption of a large deviation principle (LDP) concentration leads to an estimate of the exponential rate of convergence with speed n and an exact computation of the reach of repulsion R^*.

Proposition 3.3. *For each n, let* $\Phi_n \sim \mathrm{DPP}(K_n)$ *be a stationary and isotropic DPP in* $\mathbb{R}^n$*, and assume that* $0 \le \hat{K}_n < 1$*. Let* X_n *be a random vector with density* $K_n(x)^2/\|K_n\|_2^2$ *and suppose that* $|X_n|/\sqrt{n}$ *satisfies an LDP with strictly convex rate function* I*. Then, for* R^* *such that* $I(R^*) = 0$*, the threshold (3.3) occurs. Also, for* $R < R^*$*,*

$$-\inf_{r<R} I(r) \le \liminf_{n\to\infty} \frac{1}{n} \ln \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \le \limsup_{n\to\infty} \frac{1}{n} \ln \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \le -\inf_{r\le R} I(r),$$

and if the rate function I *is continuous at* R*,*

$$\lim_{n\to\infty} -\frac{1}{n} \ln \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = I(R).$$

Remark 3.4. The second conclusion of Proposition 3.3 concerning the rate also holds for $R > R^*$ if $B_n(\sqrt{n}R)$ is replaced by $\mathbb{R}^n \setminus B_n(\sqrt{n}R)$.

If a sequence of DPPs in increasing dimensions exhibits a reach of repulsion R^*, this means that the points of η_n are most likely to be near distance $\sqrt{n}R^*$ away from the origin in high dimensions. If R^* is less than $\tilde{R}$ from (3.1), points are most likely to be removed at a distance where points of Φ_n appear with probability decreasing to 0 as n increases, due to dimensionality. If R^* can reach past $\tilde{R}$, the points 'pushed out' by repulsion are most likely to lie at a distance where points of Φ_n appear with high probability. Thus, it is of interest to check whether there exist DPP models such that R^* is greater than or equal to $\tilde{R}$, i.e. if $\mathbb{P}(\Phi_n^{0,!}(B_n(\sqrt{n}R^*)) = 0) \to 0$ as $n \to \infty$. In Sections 4.1 and 4.2 we provide examples of DPP models with this reach.

The above results have strong assumptions, and open up additional questions. The first question is whether the points of η_n tend to lie at distances scaling with $\sqrt{n}$, i.e. is the Shannon

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regime the correct one to examine the repulsiveness between points of a family of DPPs in high dimensions? By the radial symmetry of the density of each X_n, the coordinates $\{X_{n,k}\}_{k=1}^n$ are identically distributed, and the sequence $|X_n|^2$ is the sequence of row sums of a triangular array of random variables with identically distributed rows. If the coordinate distributions depend on the dimension in such a way that $\mathbb{E}[|X_n|^2] \neq O(n)$ then a different scaling is needed.

4. Examples

In the following we examine specific families presented in [2] and [15] that illustrate both examples of DPP models satisfying the above results, as well as examples that do not. These examples provide a window into the wide scope of repulsive behavior that can be described using this framework.

The first task is to determine the behavior of $\mathbb{E}[\eta_n(\mathbb{R}^n)]$ as n increases. For each of the examples provided in this section, $\lim_{n\to\infty} \mathbb{E}[\eta_n(\mathbb{R}^n)] = 0$, but each class exhibits this convergence at different speeds. Then the goal is to determine if the DPP models satisfy the conditions of Propositions 3.1, 3.2, or 3.3.

4.1. Laguerre–Gaussian models

For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ in $\mathbb{R}^n$ be a Laguerre–Gaussian DPP as described in [2] with intensity $K_n(0) = \mathrm{e}^{n\rho}$, i.e. for some $m \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, let

$$K_n(x) = \frac{\mathrm{e}^{n\rho}}{\binom{m-1+n/2}{m-1}} L_{m-1}^{n/2}\left(\frac{1}{m}\left|\frac{x}{\alpha}\right|^2\right)\mathrm{e}^{-|x/\alpha|^2/m}, \qquad x \in \mathbb{R}^n, \tag{4.1}$$

where $L_m^\beta(r) = \sum_{k=0}^m \binom{m+\beta}{m-k}(-r)^k/k!$ for all $r \in \mathbb{R}$ denote the Laguerre polynomials. From [2], the condition $0 \le K_n < 1$ translates to a bound on α_n, i.e.

$$\alpha < \frac{1}{\mathrm{e}^\rho (m\pi)^{1/2}} \binom{m-1+n/2}{m-1}^{1/n}. \tag{4.2}$$

Direct calculations yield that the global measure of repulsiveness is

$$\begin{aligned}\mathbb{E}[\eta_n(\mathbb{R}^n)] = {} & \frac{\mathrm{e}^{n\rho}\alpha_n^n}{\binom{m-1+n/2}{m-1}^2}\left(\frac{m\pi}{2}\right)^{n/2} \sum_{k,j=0}^{m-1} \binom{m-1+n/2}{m-1-k}\binom{m-1+n/2}{m-1-j} \\ & \times \frac{(-1)^{k+j}}{k!\,j!}\frac{\Gamma(n/2+k+j)}{2^{k+j}\Gamma(n/2)}.\end{aligned} \tag{4.3}$$

From (4.2), $\mathbb{E}[\eta(\mathbb{R}^n)] < 2^{-n/2} f(n,m)$, where

$$f(n,m) = \sum_{k,j=0}^{m-1} \frac{\binom{m-1+n/2}{m-1-k}\binom{m-1+n/2}{m-1-j}}{\binom{m-1+n/2}{m-1}} \frac{(-1)^{k+j}}{k!\,j!}\frac{\Gamma(n/2+k+j)}{2^{k+j}\Gamma(n/2)} = O(n^{m-1}).$$

It follows from [2, Equation (5.7)] that for fixed n, $\lim_{m\to\infty} 2^{-n/2} f(n,m) = 1$, and as $\alpha \to 0$, K_n approaches the Poisson kernel. Thus, this class of DPPs covers a wide range of repulsiveness for fixed dimension n. However, for any fixed m, the dominant behavior as n increases is $2^{-n/2}$.

Since $\binom{m-1+n/2}{m-1}^{1/n}$ decreases to 1 as $n \to \infty$, a sufficient condition for (4.2) to hold for all n is $0 < \alpha < \mathrm{e}^{-\rho}(m\pi)^{-1/2}$. Note that this scaling for the intensity is the correct one for

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observing interactions between the parameters of the model since it provides a trade-off between how large the parameter α can be and the magnitude of ρ. If the intensity does not grow as quickly with the dimension, the upper bound on α depends less and less on changes in ρ as the dimension increases, and if the intensity grows more quickly, the upper bound for α would tend to 0 as $n \to \infty$.

Proposition 3.3 holds for this sequence of DPPs. Indeed, in the next lemma we show that the sequence of $\mathbb{R}^+$-valued random variables $|X_n|/\sqrt{n}$ satisfies an LDP.

Lemma 4.1. *Fix $m \in \mathbb{N}$, $\rho \in \mathbb{R}$, and let $\alpha \in (0, \mathrm{e}^{-\rho}(m\pi)^{-1/2})$. For each n, let X_n be a random vector in $\mathbb{R}^n$ with probability density $K_n(x)^2/\|K_n\|_2^2$, where K_n is given by (4.1). Then the sequence $\{|X_n|/\sqrt{n}\}_n$ satisfies an LDP with rate function*

$$\Lambda^*(x) = \frac{2x^2}{\alpha^2 m} - \frac{1}{2} + \frac{1}{2}\log\left(\frac{\alpha^2 m}{4x^2}\right).$$

Using this lemma, Proposition 3.3 implies that an R^* exists, and the exponential rates can be determined. In addition, using (4.3), the exponential rate of decay of $\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]$ can be computed.

Proposition 4.1. *Fix $m \in \mathbb{N}$, $\rho \in \mathbb{R}$, and let $\alpha \in (0, \mathrm{e}^{-\rho}(m\pi)^{-1/2})$. For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$, where K_n is given by (4.1). Then, for $R^* := \frac{1}{2}\alpha\sqrt{m}$,*

$$\lim_{n\to\infty} -\frac{1}{n}\log\mathbb{E}[\eta_n(B_n(\sqrt{n}R))] = \begin{cases} -\rho - \dfrac{1}{2}\log 2\pi\mathrm{e} + \dfrac{2R^2}{\alpha^2 m} - \log R, & 0 < R < R^*, \\ -\rho - \log\alpha - \dfrac{1}{2}\log\dfrac{m\pi}{2}, & R > R^*. \end{cases}$$

The rate decays as R increases to $R^* := \frac{1}{2}\alpha\sqrt{m}$, and then for $R > R^*$, the rate no longer depends on R. This coincides with our interpretation of R^* as the asymptotic reach of repulsion of the sequence of DPPs.

For a fixed α, a larger m will result in a farther reach, and for a fixed m, a larger α will provide a farther reach. However, using the bound $\alpha < 1/\mathrm{e}^{\rho}(m\pi)^{1/2}$, the following upper bound on the reach holds uniformly for all m:

$$R^* := \frac{\alpha}{2}\sqrt{m} < \frac{1}{2\mathrm{e}^{\rho}\pi^{1/2}}.$$

Note that the larger ρ is, the smaller the upper bound on R^* can be. This follows from the relationship between α and ρ: the higher the intensity, the smaller α must be for the DPP to exist. Since a larger α implies a larger value of $\mathbb{E}[\eta_n(\mathbb{R}^n)]$, the parameter α is associated with the strength of the repulsiveness. The relationship with ρ showcases the following trade-off observed in [15]: the higher the intensity of the DPP, the less repulsive it can be.

As mentioned in the previous section, it is of interest to know whether there is a range of parameters such that R^* is greater than $\tilde{R}$, the threshold for the convergence of the nearest-neighbor function of Φ; see (3.1). For Laguerre–Gaussian models, $R^* := \frac{1}{2}\sqrt{m}\alpha$ is larger than $\tilde{R}$ and α satisfies the condition of Lemma 4.1 if

$$\left(\frac{2}{\mathrm{e}}\right)^{1/2} < \mathrm{e}^{\rho}\sqrt{m\pi}\alpha < 1.$$

Since the lower bound is strictly less than 1, there is a nonempty range for α such that the reach of repulsion reaches past $\tilde{R}$.

4.2. Power exponential spectral models

The power exponential spectral models, introduced in [15], are defined through the Fourier transform of the kernel. For almost all these models, there is no closed form for the kernel K. Using properties of the Fourier transform, a similar analysis of the repulsive behavior can still be performed.

For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ be a power exponential DPP with intensity $K_n(0) = \mathrm{e}^{n\rho}$ and parameters $\nu > 0$ and $\alpha_n > 0$, i.e. let

$$\hat{K}_n(x) = \mathrm{e}^{n\rho} \frac{\Gamma(n/2+1)\alpha_n^n}{\pi^{n/2}\Gamma(n/\nu+1)} \mathrm{e}^{-|\alpha_n x|^\nu}, \qquad x \in \mathbb{R}^n. \tag{4.4}$$

When $\nu = 2$, a closed-form expression for K_n exists and is called the Gaussian kernel. The condition $0 \le \hat{K}_n < 1$ implies the following upper bound on α_n:

$$\alpha_n < \frac{\Gamma(n/\nu+1)^{1/n}\pi^{1/2}}{\mathrm{e}^{\rho}\Gamma(n/2+1)^{1/n}}, \tag{4.5}$$

and the asymptotic expansion for the upper bound on α_n as $n \to \infty$ is

$$\begin{aligned}\left(\frac{\Gamma(n/\nu+1)\pi^{n/2}}{\mathrm{e}^{n\rho}\Gamma(n/2+1)}\right)^{1/n} &\sim \left(\frac{\sqrt{2\pi n/\nu}(n/\nu\mathrm{e})^{n/\nu}\pi^{n/2}}{\mathrm{e}^{n\rho}\sqrt{2\pi n/2}(n/2\mathrm{e})^{n/2}}\right)^{1/n} \\ &\sim \mathrm{e}^{-\rho} n^{1/\nu-1/2} \frac{(2\pi\mathrm{e})^{1/2}}{(\nu\mathrm{e})^{1/\nu}} \\ &= O(n^{1/\nu-1/2}).\end{aligned}$$

By Parseval's theorem and a change of variables,

$$\begin{aligned}\mathbb{E}[\eta_n(\mathbb{R}^n)] &= \frac{1}{\mathrm{e}^{n\rho}}\|K_n\|_2^2 \\ &= \frac{1}{\mathrm{e}^{n\rho}}\|\hat{K}_n\|_2^2 \\ &= \frac{1}{\mathrm{e}^{n\rho}}\left(\mathrm{e}^{n\rho}\frac{\Gamma(n/2+1)\alpha_n^n}{\pi^{n/2}\Gamma(n/\nu+1)}\right)^2 \int_{\mathbb{R}^n} \mathrm{e}^{-2|\alpha_n x|^\nu}\,\mathrm{d}x \\ &= \mathrm{e}^{n\rho}\left(\frac{\Gamma(n/2+1)\alpha_n^n}{\pi^{n/2}\Gamma(n/\nu+1)}\right)^2 \frac{n\pi^{n/2}}{\Gamma(n/2+1)} \int_0^\infty r^{n-1}\mathrm{e}^{-2(\alpha r)^\nu}\,\mathrm{d}r \\ &= \mathrm{e}^{n\rho}\frac{\Gamma(n/2+1)\alpha_n^{2n}}{\pi^{n/2}\Gamma(n/\nu+1)^2}\frac{n}{2^{n/\nu}\nu\alpha_n^n}\int_0^\infty t^{n/\nu-1}\mathrm{e}^{-t}\,\mathrm{d}t \\ &= 2^{-n/\nu}\alpha_n^n\frac{\mathrm{e}^{n\rho}\Gamma(n/2+1)}{\pi^{n/2}\Gamma(n/\nu+1)}.\end{aligned} \tag{4.6}$$

Using the bound on α_n (4.5),

$$\mathbb{E}[\eta_n(\mathbb{R}^n)] < 2^{-n/\nu}.$$

For fixed dimension n, the global measure of repulsion approaches its upper bound of 1 for large ν. Thus, this class covers a wide range of repulsiveness similar to the Laguerre–Gaussian DPPs. However, for fixed ν, the measure decays exponentially as $n \to \infty$. Note that for $\nu > 2$, the rate is smaller than for the Laguerre–Gaussian models, i.e. the decay is slower.

In the following results we show that if the parameters α_n grow appropriately with n, this sequence satisfies the assumptions of Proposition 3.1.

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Lemma 4.2. *For each n, let X_n be a vector in $\mathbb{R}^n$ with density $K_n^2/\|K_n\|_2^2$ such that $\hat{K}_n$ is given by (4.4). Assume that $\alpha_n \sim \alpha n^{1/\nu-1/2}$ as $n \to \infty$ for $\alpha \in (0,\infty)$, and*

$$\alpha_n < \left(\frac{\Gamma(n/\nu+1)\pi^{n/2}}{\mathrm{e}^{n\rho}\Gamma(n/2+1)}\right)^{1/n} \quad \text{for all } n.$$

Then, as $n \to \infty$,

$$\frac{|X_n|}{\sqrt{n}} \xrightarrow{\mathbb{P}} \alpha\frac{(2\nu)^{1/\nu}}{4\pi}.$$

Now, applying Proposition 3.1, the following result holds for a sequence of power exponential DPPs in the Shannon regime.

Proposition 4.2. *For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$, where $\hat{K}_n$ satisfies the assumptions in Lemma 4.2. Then, for $R^* := \alpha((2\nu)^{1/\nu}/4\pi)$,*

$$\lim_{n\to\infty}\frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \begin{cases} 0, & R < R^*, \\ 1, & R > R^*. \end{cases}$$

For $\nu > 1$, the reach of repulsion R^* for the power exponential models can also reach past the nearest-neighbor threshold $\tilde{R}$. Indeed, for $\alpha_n \sim \alpha n^{1/\nu-1/2}$, $R^* := \alpha((2\nu)^{1/\nu}/4\pi)$ satisfies $\mathbb{P}(\Phi_n(B_n(0,\sqrt{n}R^*)) = 0) \to 0$ as $n \to \infty$ if

$$\alpha\frac{(2\nu)^{1/\nu}}{4\pi} > \frac{1}{\sqrt{2\pi\mathrm{e}}\mathrm{e}^{\rho}}.$$

By asymptotic equation (4.5) for the upper bound of α_n, $\alpha < \sqrt{2\pi\mathrm{e}}/\mathrm{e}^{\rho}(\nu\mathrm{e})^{1/\nu}$. Thus, R^* reaches past $\tilde{R}$ when $\alpha_n \sim \alpha n^{1/\nu-1/2}$ and

$$\frac{4\pi}{(2\nu)^{1/\nu}\mathrm{e}^{\rho}\sqrt{2\pi\mathrm{e}}} < \alpha < \frac{\sqrt{2\pi\mathrm{e}}}{\mathrm{e}^{\rho}(\nu\mathrm{e})^{1/\nu}}.$$

The interval is nonempty since the upper bound is strictly greater than the lower bound for $\nu > 1$.

4.3. Bessel-type models

Another class of DPP models presented in [2] is the Bessel type. This class is more repulsive than the previous two families of models. We show that while the Shannon regime is the correct scaling to examine the repulsiveness of this class in high dimensions, a sequence of these DPPs does not satisfy the conditions of Proposition 3.1.

For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ be a Bessel-type DPP with parameters $\sigma \geq 0$ and $\alpha > 0$, and intensity $K_n(0) = \mathrm{e}^{n\rho}$ for $\rho \in \mathbb{R}$. That is, let

$$K_n(x) = \mathrm{e}^{n\rho}2^{(\sigma+n)/2}\Gamma\left(\frac{\sigma+n+2}{2}\right)\frac{J_{(\sigma+n)/2}(2|x/\alpha|\sqrt{(\sigma+n)/2})}{(2|x/\alpha|\sqrt{(\sigma+n)/2})^{(\sigma+n)/2}}. \tag{4.7}$$

From [2], the bound $0 \leq \hat{K}_n < 1$ implies that

$$\alpha_n^n < \frac{(\sigma+n)^{n/2}\Gamma(\sigma/2+1)}{\mathrm{e}^{n\rho}(2\pi)^{n/2}\Gamma((\sigma+n)/2+1)}. \tag{4.8}$$

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Similar to the previous examples, this family contains DPPs covering a wide range of repulsiveness measured by η_n, and as $n \to \infty$ they are more repulsive in the sense that $\mathbb{E}[\eta_n(\mathbb{R}^n)]$ decays more slowly. Indeed,

$$\begin{aligned}\mathbb{E}[\eta_n(\mathbb{R}^n)] &= \frac{1}{\mathrm{e}^{n\rho}} \int_{\mathbb{R}^n} K_n(x)^2 \, \mathrm{d}x \\ &= \mathrm{e}^{n\rho} \frac{(2\pi)^{n/2}\alpha^n}{(\sigma+n)^{n/2}\Gamma(n/2)} \frac{\Gamma((\sigma+n+2)/2)^2\Gamma(n/2)\Gamma(\sigma+1)}{\Gamma(\sigma/2+1)^2\Gamma(\sigma+n/2+1)} \\ &= \mathrm{e}^{n\rho} \frac{(2\pi)^{n/2}\alpha^n}{(\sigma+n)^{n/2}} \frac{\Gamma(\sigma+1)\Gamma(\sigma/2+n/2+1)^2}{\Gamma(\sigma/2+1)^2\Gamma(\sigma+n/2+1)},\end{aligned}$$

and from the upper bound (4.8),

$$\mathbb{E}[\eta_n(\mathbb{R}^n)] < \frac{\Gamma(\sigma+1)\Gamma(\sigma/2+n/2+1)}{\Gamma(\sigma/2+1)\Gamma(\sigma+n/2+1)}.$$

By Stirling's formula,

$$\frac{\Gamma(\sigma+1)\Gamma(\sigma/2+n/2+1)}{\Gamma(\sigma/2+1)\Gamma(\sigma+n/2+1)} = O(n^{-\sigma/2}) \quad \text{as } n \to \infty.$$

These DPPs do not satisfy the conditions of Proposition 3.1, and so the concentration of the first-moment measure does not occur, contrary to the first two families presented. However, the repulsive measure does not reach past the $\sqrt{n}$ scale in the sense of the following proposition.

Proposition 4.3. *Let $\rho \in \mathbb{R}$, $\alpha > 0$, and $\sigma > 0$. For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ in $\mathbb{R}^n$ with K_n given by (4.7). Then, for any $\beta > \frac{1}{2}$ and $R > 0$,*

$$\lim_{n\to\infty} \frac{\mathbb{E}[\eta_n(\mathbb{R}^n \setminus B_n(Rn^\beta))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = 0.$$

4.4. Normal-variance mixture models

Another class of DPPs described in [15] are those with normal-variance mixture kernels. Let $\Phi_n \sim \mathrm{DPP}(K_n)$ be a normal-variance mixture DPP in $\mathbb{R}^n$ with intensity $\mathrm{e}^{n\rho}$ for $\rho \in \mathbb{R}$, i.e. let

$$K_n(x) = \mathrm{e}^{n\rho} \frac{\mathbb{E}[W^{-n/2}\mathrm{e}^{-|x|^2/2W}]}{\mathbb{E}[W^{-n/2}]}, \qquad x \in \mathbb{R}^n,$$

for some nonnegative real-valued random variable W such that $\mathbb{E}[W^{-n/2}] < \infty$. From [15], the bound $0 \le \hat{K} < 1$ translates to the following bound on the intensity:

$$\mathrm{e}^{n\rho} < \frac{\mathbb{E}[W^{-n/2}]}{(2\pi)^{n/2}}. \tag{4.9}$$

If $\sqrt{2W} = \alpha$, this is known as the Gaussian DPP model. If $W \sim \mathrm{gamma}(\nu + \frac{1}{2}n, 2\alpha^2)$, this is called the Whittle–Matérn model. This is the Cauchy model when $1/W \sim \mathrm{gamma}(\nu, 2\alpha^{-2})$. In both cases, $\nu > 0$ and $\alpha > 0$ are parameters.

This family of DPPs does not cover a wide range of repulsiveness as in the previous families. Indeed, for any random variable W in $\mathbb{R}^+$ such that $\mathbb{E}[W^{-n/2}] < \infty$, Parseval's theorem,

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Jensen's inequality, (4.9), and Fubini's theorem imply that

$$\begin{aligned}
\mathbb{E}[\eta_n(\mathbb{R}^n)] &= \frac{1}{e^{n\rho}} \int_{\mathbb{R}^n} \hat{K}_n(x)^2 \, dx \\
&= \frac{1}{e^{n\rho}} \int_{\mathbb{R}^n} \bigg(e^{n\rho} \frac{(2\pi)^{n/2}}{\mathbb{E}[W^{-n/2}]} \mathbb{E}[e^{-2\pi^2|x|^2 W}] \bigg)^2 \, dx \\
&\le \frac{(2\pi)^{n/2}}{\mathbb{E}[W^{-n/2}]} \int_{\mathbb{R}^n} \mathbb{E}[e^{-4\pi^2|x|^2 W}] \, dx \\
&= \frac{(2\pi)^{n/2}}{\mathbb{E}[W^{-n/2}]} \mathbb{E}\bigg[(4\pi W)^{-n/2} \mathbb{E}\bigg[(4\pi W)^{n/2} \int_{\mathbb{R}^n} e^{-4\pi^2|x|^2 W} \, dx \;\bigg|\; W \bigg]\bigg] \\
&= 2^{-n/2}.
\end{aligned}$$

It is difficult to make further general statements about this class because the behavior of the sequence $|X_n|/\sqrt{n}$ depends heavily on the distribution of the $\mathbb{R}^+$-valued random variable W. In the rest of the section we will describe results for specific models in this class.

Consider a sequence of normal-variance mixture DPPs all associated with the same random variable W. If W is a constant α, the random variables X_n become multivariate Gaussian vectors with mean zero and variance depending on α. The scaled norms of these vectors are well known to satisfy an LDP since the coordinates are independent. This also corresponds to a Laguerre–Gaussian DPP with parameter $m = 2$.

There is also a subclass of the normal-variance mixture models that satisfy Proposition 3.2. In [25], it was proved that if W has a log-concave density then the normal-variance mixture distribution is log-concave. This implies that K_n^2 is log-concave and, thus, if condition (3.4) holds, the conclusion of Proposition 3.2 holds. Since the gamma distribution for a shape parameter ν greater than 1 is log-concave and $\nu + \frac{1}{2}n \ge 1$ for large n, Whittle–Matérn DPPs are an example from this subclass and exhibit an R^* as we demonstrate in the following proposition.

Proposition 4.4. *For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ be a Whittle–Matérn model in $\mathbb{R}^n$ with intensity $e^{n\rho}$ and parameters $\nu > 0$ and $\alpha > 0$, i.e. let*

$$K_n(x) = e^{n\rho} \frac{2^{1-\nu}}{\Gamma(\nu)} \frac{|x|^\nu}{\alpha^\nu} \mathbb{K}_\nu\bigg(\frac{|x|}{\alpha}\bigg), \qquad x \in \mathbb{R}^n, \tag{4.10}$$

where

$$\alpha < \frac{\Gamma(\nu)^{1/n}}{\Gamma(\nu + n/2)^{1/n} 2\sqrt{\pi} e^{\rho}}$$

and $\mathbb{K}_\nu$ is the modified Bessel kernel of the second kind. Then, for $R^* := \frac{1}{2}\alpha$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \begin{cases} 0, & R < R^*, \\ 1, & R > R^*. \end{cases}$$

Remark 4.1. The upper bound on α needed for existence implies that for all ν,

$$R^* := \frac{\alpha}{2} < \frac{\Gamma(\nu)^{1/n}}{\Gamma(\nu + n/2)^{1/n} 4\sqrt{\pi} e^{\rho}} < \frac{1}{\sqrt{2\pi e} e^{\rho}} := \tilde{R},$$

since $(\Gamma(\nu)/\Gamma(\nu + \frac{1}{2}n))^{1/n} \le 1$ and $4 > \sqrt{2e}$. Thus, for these models, R^* never reaches past the nearest-neighbor threshold $\tilde{R}$.

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Finally, in the following proposition we show that the Cauchy models satisfy the conditions of Proposition 3.1 if the α parameter grows appropriately with n.

Proposition 4.5. *For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ be a Cauchy model in $\mathbb{R}^n$ with intensity $\mathrm{e}^{n\rho}$ and parameters $\nu > 0$ and $\alpha_n > 0$, i.e. let*

$$K_n(x) = \frac{\mathrm{e}^{n\rho}}{(1+|x/\alpha_n|^2)^{\nu+n/2}}, \qquad x \in \mathbb{R}^n.$$

Assume $\alpha_n \sim \alpha n^{1/2}$ as $n \to \infty$ for some $\alpha > 0$ such that $\alpha_n < \Gamma(\nu + \frac{1}{2}n)^{1/n}/\sqrt{\pi}\mathrm{e}^{\rho}\Gamma(\nu)^{1/n}$ for each n. Then, for $R^* := \alpha$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \begin{cases} 0, & R < R^*, \\ 1, & R > R^*. \end{cases}$$

Remark 4.2. The upper bound on α_n has the following asymptotic expansion as $n \to \infty$:

$$\alpha_n < \frac{\Gamma(\nu+n/2)^{1/n}}{\sqrt{\pi}\mathrm{e}^{\rho}\Gamma(\nu)^{1/n}} \sim \frac{n^{1/2}}{\sqrt{2\pi \mathrm{e}}\mathrm{e}^{\rho}}.$$

Thus, if $\alpha_n \sim \alpha n^{1/2}$, the reach of repulsion has the upper bound

$$R^* := \alpha < \frac{1}{\sqrt{2\pi \mathrm{e}}\mathrm{e}^{\rho}}.$$

This upper bound is precisely the threshold $\tilde{R}$ for the nearest-neighbor function, and so unlike in the case of Laguerre–Gaussian DPPs and power exponential DPPs, the reach of repulsion R^* for a sequence of Cauchy models with fixed parameter ν cannot reach past $\tilde{R}$.

5. Application to determinantal Boolean models in the Shannon regime

Poisson–Boolean models in the Shannon regime were studied in [1], and the degree threshold results can be extended to Laguerre–Gaussian DPPs using Proposition 4.1.

The setting is as follows. Consider a sequence of stationary DPPs Φ_n, indexed by dimension, where $\Phi_n \sim \mathrm{DPP}(K_n)$ in $\mathbb{R}^n$. Assume that for each n, K_n is continuous, symmetric, and $0 \le \hat{K}_n < 1$. Let the intensity of Φ_n be $K_n(0) = \mathrm{e}^{n\rho}$. Let $\Phi_n = \sum_k \delta_{T_n^{(k)}}$ and $R > 0$. Then consider the sequence of particle processes (see [22]), called determinantal Boolean models,

$$\mathcal{C}_n = \bigcup_k B_n\big(T_n^{(k)}, \tfrac{1}{2}\sqrt{n}R\big).$$

The degree of each model is the expected number of balls that intersect the ball centered at 0 under the reduced Palm distribution, i.e. $\mathbb{E}[\Phi_n^{0,!}(B_n(\sqrt{n}R))]$. In the case when Φ_n is Poisson, $\mathbb{E}^{0,!}[\Phi_n(B(\sqrt{n}R))] = \mathbb{E}[\Phi_n(B(\sqrt{n}R))]$ by Slivnyak's theorem, and

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}^{0,!}[\Phi_n(B_n(\sqrt{n}R))] = \rho + \frac{1}{2}\log 2\pi\mathrm{e} + \log R.$$

To extend this result to DPPs, we need, as $n \to \infty$,

$$\mathbb{E}[\Phi_n^{0,!}(B_n(\sqrt{n}R))] \sim \mathbb{E}[\Phi_n(B_n(\sqrt{n}R))].$$

Note that this would be impossible for a repulsive point process such as the Matérn hardcore process, since $\mathbb{E}[\Phi_n^{0,!}(B_n(R_n))] = 0$ for all R_n less than the hardcore radius.

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However, for DPPs, note that

$$\frac{\mathbb{E}[\Phi_n^{0,!}(B_n(\sqrt{n}R))]}{\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))]} = 1 - \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))]}.$$

Thus, if

$$\frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))]} \to 0 \quad \text{as } n \to \infty,$$

then the degree of the determinantal Boolean model has the same asymptotic behavior as the Poisson–Boolean model.

In the case of Laguerre–Gaussian kernels, this is the case, and the earlier results even provide the rate at which the quantity goes to 0, which exhibits a threshold at R^* as expected.

Proposition 5.1. *Let $m \in \mathbb{N}$ and $\rho \in \mathbb{R}$. For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ in $\mathbb{R}^n$, where*

$$K_n(x) = \frac{\mathrm{e}^{n\rho}}{\binom{m-1+n/2}{m-1}} L_{m-1}^{n/2}\left(\frac{1}{m}\left|\frac{x}{\alpha}\right|^2\right)\mathrm{e}^{-|x/\alpha|^2/m},$$

and α is a parameter such that $0 < \alpha < 1/\sqrt{m\pi}\mathrm{e}^{\rho}$. Then

$$\lim_{n\to\infty} -\frac{1}{n}\ln\frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))]} = \begin{cases} \dfrac{2R^2}{\alpha^2 m}, & 0 < R < \frac{1}{2}\alpha\sqrt{m}, \\ \frac{1}{2} + \log 2 - \log\alpha - \frac{1}{2}\log m + \log R, & R > \frac{1}{2}\alpha\sqrt{m}. \end{cases}$$

6. Conclusion

By examining a measure of repulsiveness of DPPs, we have provided insight into the high-dimensional behavior of different families of DPP models. Most of the families of DPPs presented in this paper have a global measure of repulsion decreasing to 0 as the dimension increases, indicating that they become more and more similar to Poisson point processes in high dimensions as a result of (1.3). However, the reach of the small repulsive effect can still be quantified. By making a connection between the kernel of the DPP and the concentration in high dimensions of the norm of a random vector, we have shown under certain conditions that there exists a distance on the $\sqrt{n}$ scale at which the repulsive effect of a point of the DPP model is strongest as $n \to \infty$. We have illustrated that some families of DPPs exhibit this reach of repulsion and some do not. The results are summarized in Table 1.

Many questions remain concerning the range of possible repulsive behavior of DPPs in high dimensions. First, the results can be extended to scalings other than the Shannon regime in the following way. Assumption (3.2) in Proposition 3.1 can be generalized to the assumption that for some sequence b_n, $|X_n|/b_n \to R^*$ as $n \to \infty$. If $b_n \neq O(n^{1/2})$, the result holds for a different scaling than the Shannon regime, and the repulsiveness is strongest near $R^* b_n$ in high dimensions. While this is precisely what is shown not to happen for the Bessel-type DPPs if $\sigma > 0$, examples of this generalization for $b_n = o(n)$ can be obtained from the power exponential DPPs when $\alpha_n = o(n^{1/\nu - 1/2})$. However, as noted in the introduction, any distance scaling smaller than $\sqrt{n}$ will not reach the regime where the expected number of points goes to ∞ as the dimension grows. Thus, this scaling appears less interesting. It would be interesting to find a family of DPPs that exhibits the concentration for $b_n \gg \sqrt{n}$.

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TABLE 1: Summary of results.

DPP class	$\mathbb{E}[\eta_n(\mathbb{R}^n)]$	R^*	Rate type	$R^* > \tilde{R}$
Laguerre–Gaussian	$< 2^{-n/2}O(n^{m-1})$	$\frac{1}{2}\alpha\sqrt{m}$	LDP	$\left(\frac{2}{\mathrm{e}}\right)^{1/2} < \mathrm{e}^{\rho}\sqrt{m\pi}\alpha < 1$
Power exponential	$< 2^{-n/\nu}$	$\alpha\dfrac{(2\nu)^{1/\nu}}{4\pi}$	Chebychev	$\dfrac{2}{2^{1/\nu}\mathrm{e}} < \dfrac{\mathrm{e}^{\rho}\nu^{1/\nu}}{\sqrt{2\pi\mathrm{e}}}\alpha < \dfrac{1}{\mathrm{e}^{1/\nu}}$
Bessel-type	$< O(n^{-\sigma/2})$	—	—	—
Whittle–Matérn	$< 2^{-n/2}$	$\frac{1}{2}\alpha$	Log-concave	—
Cauchy	$< 2^{-n/2}$	α	Chebychev	—

For all the DPPs studied in this paper, $\mathbb{E}[\eta_n(\mathbb{R}^n)] \to 0$ as $n \to \infty$. This is not always the case. For instance, there exists a class of DPPs such that for $c \in (0, 1)$, $\mathbb{E}[\eta_n(\mathbb{R}^n)] = c$ for all n. Indeed, let $K_n \in L^2(\mathbb{R}^n)$ be such that its Fourier transform is

$$\hat{K}_n(\xi) = \sqrt{c}\,\mathbf{1}_{B_n(r_n)}(\xi), \qquad \xi \in \mathbb{R}^n, \tag{6.1}$$

where $r_n \in \mathbb{R}^+$ is such that $\mathrm{vol}(B_n(r_n)) = K_n(0)$ and $\mathbf{1}.$ is the indicator function. Then

$$\mathbb{E}[\eta_n(\mathbb{R}^n)] = \frac{1}{K_n(0)}\int_{\mathbb{R}^n} K_n(x)^2\,\mathrm{d}x = \frac{1}{K_n(0)}\int_{\mathbb{R}^n} \hat{K}_n(\xi)^2\,\mathrm{d}\xi = \frac{c}{K_n(0)}\mathrm{vol}(B_n(r_n)) = c.$$

It would be useful to find a necessary and sufficient condition for $\mathbb{E}[\eta_n(\mathbb{R}^n)]$ to converge to 0.

There is an important class of stationary and isotropic DPPs that should be mentioned. Recall that to ensure η is well defined, it is assumed that the kernel K associated with Φ satisfies $0 \le \hat{K} < 1$. However, Φ still exists when $\hat{K}$ is allowed to attain the maximum value of 1. For the models studied in this paper, it is the case when the parameter achieves its upper bound. In this case, we can still define the measure of repulsiveness (1.2) even though it may not be interpretable as the intensity measure of a point process η. Replacing $\mathbb{E}[\eta(B)]$ with $\int_B (1 - g(x))\,\mathrm{d}x$ for $B \in \mathcal{B}(\mathbb{R}^n)$, the main results (Propositions 3.1–3.3) can be restated with the condition that $0 \le \hat{K} \le 1$. In this case, the reach of repulsion R^* is interpreted as the distance on the $\sqrt{n}$ scale at which the measure of repulsion is strongest.

A particularly interesting subclass of the DPPs described in the previous paragraph are the most repulsive stationary DPPs, introduced in the supplementary material to [15]; see [14]. These DPPs maximize the measure of repulsiveness γ, and have a kernel K such that $\hat{K}$ is defined as in (6.1) but with $c = 1$. For the most repulsive DPPs, $\gamma = 1$ in any dimension. In addition, for a sequence of DPPs $\{\Phi_n\}_{n\in\mathbb{N}}$, where Φ_n is the most repulsive DPP in $\mathbb{R}^n$ with intensity $\mathrm{e}^{n\rho}$, X_n as defined in Proposition 3.1 satisfies

$$\begin{aligned}
\mathbb{E}[|X_n|^2] &= \int_{\mathbb{R}^n} |x|^2 \frac{K_n(x)^2}{\|K_n\|_2^2}\,\mathrm{d}x \\
&= \frac{\Gamma(n/2+1)}{\pi^{n/2}}\int_{\mathbb{R}^n} |x|^2 \frac{J_{n/2}^2(2\sqrt{\pi}\,\Gamma(n/2+1)^{1/n}\mathrm{e}^{\rho}|x|)}{|x|^n}\,\mathrm{d}x \\
&= n\int_0^\infty r J_{n/2}^2\big(2\sqrt{\pi}\,\Gamma\big(\tfrac{1}{2}n+1\big)^{1/n}\mathrm{e}^{\rho} r\big)\,\mathrm{d}r,
\end{aligned}$$

where J_ν is the Bessel function of the first kind of order ν; see [2]. By [19, Equation (1.17.13)], this integral does not converge, i.e. $|X_n|$ does not have a finite second moment.

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In the recent work of Møller and O'Reilly [18], it was proved that there exists a coupling $(\Phi, \Phi^{0,!})$ such that $\eta := \Phi \setminus \Phi^{0,!}$ contains at most one point. In this case, $\mathbb{E}[\eta(\mathbb{R}^n)] = \mathbb{P}(\eta(\mathbb{R}^n) > 0)$, and the random vector X_n with probability measure $\mathbb{E}[\eta(\cdot)]/\mathbb{E}[\eta(\mathbb{R}^n)]$ has the distribution of the point of η conditioned on $\eta \neq \varnothing$.

Appendix A

Proof of Equation (3.1). For each n, let $\Phi_n \sim \mathrm{DPP}(K_n)$ in $\mathbb{R}^n$ be stationary with intensity $K_n(0) = \mathrm{e}^{n\rho}$. From (1.1), there exists $\tilde{R} := 1/\sqrt{2\pi \mathrm{e}}\mathrm{e}^{\rho}$ such that

$$\lim_{n\to\infty} \mathbb{E}[\Phi_n(B_n(\sqrt{n}R))] = \begin{cases} 0, & R < \tilde{R}, \\ \infty, & R > \tilde{R}. \end{cases}$$

From Theorem 2.3,

$$\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))] - \mathbb{E}[\Phi_n^{0,!}(B_n(\sqrt{n}R))] = \frac{1}{\mathrm{e}^{n\rho}} \int_{B_n(\sqrt{n}R)} K_n(x)^2 \,\mathrm{d}x.$$

Then, using Parseval's theorem and Theorem 2.2,

$$\frac{1}{\mathrm{e}^{n\rho}} \int_{B_n(\sqrt{n}R)} K_n(x)^2 \,\mathrm{d}x \leq \frac{1}{\mathrm{e}^{n\rho}} \int_{\mathbb{R}^n} \hat{K}_n(\xi)^2 \,\mathrm{d}\xi \leq \frac{1}{\mathrm{e}^{n\rho}} \int_{\mathbb{R}^n} \hat{K}_n(\xi) \,\mathrm{d}\xi = 1.$$

Also, since $(1/\mathrm{e}^{n\rho})\int_{B_n(\sqrt{n}R)} K_n(x)^2 \,\mathrm{d}x \geq 0$, the following bounds hold:

$$\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))] - 1 \leq \mathbb{E}[\Phi_n^{0,!}(B_n(\sqrt{n}R))] \leq \mathbb{E}[\Phi_n(B_n(\sqrt{n}R))].$$

Thus, the threshold remains the same for the reduced Palm expectation, i.e.

$$\lim_{n\to\infty} \mathbb{E}[\Phi_n^{0,!}(B_n(\sqrt{n}R))] = \begin{cases} 0, & R < \tilde{R}, \\ \infty, & R > \tilde{R}. \end{cases}$$

By the first-moment inequality and [3, Proposition 5.1], we have the following bounds:

$$1 - \mathbb{E}[\Phi_n^{0,!}(B(\sqrt{n}R))] \leq \mathbb{P}(\Phi_n^{0,!}(B_n(\sqrt{n}R)) = 0) \leq \exp(-\mathbb{E}[\Phi_n^{0,!}(B(\sqrt{n}R))]).$$

Thus,

$$\lim_{n\to\infty} \mathbb{P}(\Phi_n^{0,!}(B_n(\sqrt{n}R)) > 0) = \begin{cases} 0, & R < \tilde{R}, \\ 1, & R > \tilde{R}. \end{cases}$$ □

Appendix B

Proof of Lemma 3.1. From Theorem 2.3, for any $B \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathbb{E}[\eta_n(B)] = \mathbb{E}[\Phi_n(B)] - \mathbb{E}[\Phi_n^{0,!}(B)] = \frac{1}{K_n(0)} \int_B K_n(x)^2 \,\mathrm{d}x,$$

i.e. the first-moment measure of η_n has a density with respect to the Lebesgue measure equal to $(1/K_n(0))K_n(x)^2$. Then, by the monotone convergence theorem,

$$\mathbb{E}[\eta_n(\mathbb{R}^n)] = \lim_{R\to\infty} \mathbb{E}[\eta_n(B_n(R))] = \frac{1}{K_n(0)} \int_{\mathbb{R}^n} K_n(x)^2 \,\mathrm{d}x = \frac{\|K_n\|_2^2}{K_n(0)}.$$

Thus, for all $B \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathbb{P}(X_n \in B) = \int_B \frac{K_n(x)^2}{\|K_n\|_2^2} \,\mathrm{d}x = \frac{\mathbb{E}[\eta_n(B)]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]}.$$ □

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Appendix C

We now present the proofs of our main results.

Proof of Proposition 3.1. The assumption $|X_n|/\sqrt{n} \xrightarrow{\mathbb{P}} R^*$ means that for all $\varepsilon > 0$,

$$\mathbb{P}\bigg(\bigg|\frac{|X_n|}{\sqrt{n}} - R^*\bigg| > \varepsilon\bigg) \to 0 \quad \text{as } n \to \infty.$$

First, assume that $R < R^*$. Then there exists $\varepsilon > 0$ such that $R = R^* - \varepsilon$. Thus,

$$\mathbb{P}(|X_n| \le \sqrt{n}R) = \mathbb{P}\bigg(\frac{|X_n|}{\sqrt{n}} \le R^* - \varepsilon\bigg) \le \mathbb{P}\bigg(\bigg|\frac{|X_n|}{\sqrt{n}} - R^*\bigg| > \varepsilon\bigg) \to 0 \quad \text{as } n \to \infty.$$

Second, assume that $R > R^*$. Then there exists $\varepsilon > 0$ such that $R = R^* + \varepsilon$, and

$$\mathbb{P}(|X_n| \le \sqrt{n}R) = 1 - \mathbb{P}\bigg(\frac{|X_n|}{\sqrt{n}} > R^* + \varepsilon\bigg) \ge 1 - \mathbb{P}\bigg(\bigg|\frac{|X_n|}{\sqrt{n}} - R^*\bigg| > \varepsilon\bigg) \to 1.$$

Finally, by Lemma 3.1, as $n \to \infty$,

$$\frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \mathbb{P}(|X_n| \le \sqrt{n}R) \to \begin{cases} 0, & R < R^*, \\ 1, & R > R^*, \end{cases}$$

completing the proof. □

Proof of Proposition 3.2. Since, for all n, Φ_n is isotropic, X_n as defined in Proposition 3.1 has a radially symmetric density. Thus, X_n has the same distribution as the product $R_n U_n$, where R_n is equal in distribution to $|X_n|$, U_n is uniformly distributed on $\mathbb{S}^{n-1}$, and R_n and U_n are independent. Letting $\sigma_n^2 = \mathbb{E}[|X_n|^2]$ for each n, $(\sqrt{n}/\sigma_n)X_n$ then satisfies the conditions of Theorem 3.1 for each n. Then, by Theorem 3.1, for any $\delta > 0$, there exist absolute constants $C, c > 0$ such that

$$\mathbb{P}\bigg(\bigg|\frac{|X_n|}{\sigma_n} - 1\bigg| \ge \delta\bigg) \le C\mathrm{e}^{-c\sqrt{n}\min(\delta,\delta^3)}.$$

Now, let $\delta \in (0, 1)$. By Lemma 3.1,

$$\frac{\mathbb{E}[\eta_n(B_n(\sigma_n(1-\delta)))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \mathbb{P}\bigg(\frac{|X_n|}{\sigma_n} \le 1 - \delta\bigg) \le C\mathrm{e}^{-c\sqrt{n}\delta^3},$$

since $\min(\delta^3, \delta) = \delta^3$ for $\delta \in (0, 1)$. Similarly, for any $\delta > 0$,

$$\frac{\mathbb{E}[\eta_n(\mathbb{R}^n \setminus B_n(\sigma_n(1+\delta)))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \mathbb{P}\bigg(\frac{|X_n|}{\sigma_n} \ge 1 + \delta\bigg) \le C\mathrm{e}^{-c\sqrt{n}\min(\delta^3,\delta)}.$$

Now, assume that $\sigma_n/\sqrt{n} \to R^* \in (0, \infty)$ as $n \to \infty$. For $R < R^*$, there exists $\varepsilon \in (0, 1)$ such that $R = R^*(1 - \varepsilon)$. Then, for all large enough n, $\sqrt{n}R^*/\sigma_n < (1 - \frac{1}{2}\varepsilon)/(1 - \varepsilon)$ and

$$\begin{aligned}\frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} &= \mathbb{P}(|X_n| \le \sqrt{n}R) \\ &= \mathbb{P}\bigg(\frac{|X_n|}{\sigma_n} \le \frac{\sqrt{n}R}{\sigma_n}\bigg)\end{aligned}$$

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$$
\begin{aligned}
&= \mathbb{P}\bigg(\frac{|X_n|}{\sigma_n} \le \frac{\sqrt{n}R^*(1-\varepsilon)}{\sigma_n}\bigg) \\
&\le \mathbb{P}\bigg(\frac{|X_n|}{\sigma_n} \le 1 - \frac{\varepsilon}{2}\bigg) \\
&\le \mathbb{P}\bigg(\bigg|\frac{|X_n|}{\sigma_n} - 1\bigg| \ge \frac{\varepsilon}{2}\bigg) \\
&\le C\mathrm{e}^{-c\sqrt{n}(\varepsilon/2)^3}.
\end{aligned}
$$

Thus, for all $R < R^*$, there exists a constant $C(\varepsilon(R)) = c\varepsilon^3/2^3$ such that

$$\liminf_{n\to\infty} -\frac{1}{\sqrt{n}} \ln \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \ge C(\varepsilon(R)).$$

A similar argument can be used to show that for all $R > R^*$, there exists $C(\varepsilon(R))$ such that

$$\liminf_{n\to\infty} -\frac{1}{\sqrt{n}} \ln \frac{\mathbb{E}[\eta_n(\mathbb{R}^n \setminus B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \ge C(\varepsilon(R)).$$

This implies the threshold (3.3). □

Proof of Proposition 3.3. If $|X_n|/\sqrt{n}$ satisfies an LDP with convex rate function I then, by definition,

$$-\inf_{r<R} I(r) \le \liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\bigg(\frac{|X_n|}{\sqrt{n}} \le R\bigg) \le \limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\bigg(\frac{|X_n|}{\sqrt{n}} \le R\bigg) \le -\inf_{r\le R} I(r).$$

Thus, using Lemma 3.1,

$$-\inf_{r<R} I(r) \le \liminf_{n\to\infty} \frac{1}{n} \ln \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \le \limsup_{n\to\infty} \frac{1}{n} \ln \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} \le -\inf_{r\le R} I(r).$$

By the assumption that the rate function I is strictly convex, there exists a unique R^* such that $I(R^*) = 0$. Note that $\inf_{r\le R} I(r)$ is then 0 for $R > R^*$. Thus,

$$\lim_{n\to\infty} \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \begin{cases} 0, & R < R^*, \\ 1, & R > R^*. \end{cases}$$

Let $R < R^*$. If the rate function I is continuous at R then the above inequalities become equalities and

$$\lim_{n\to\infty} -\frac{1}{n} \ln \frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = I(R).$$ □

Appendix D

Proof of Lemma 4.1. We show that the sequence of random variables satisfies the conditions of the Gärtner–Ellis theorem; see [5]. First,

$$\mathbb{E}[\mathrm{e}^{s|X_n|^2}] = \frac{\mathrm{e}^{2n\rho}}{\binom{m-1+n/2}{m-1}^2 \|K_n\|_2^2} \underbrace{\int_{\mathbb{R}^n} \exp\bigg(-\bigg(\frac{2}{\alpha^2 m} - s\bigg)|x|^2\bigg)\bigg(L_{m-1}^{n/2}\bigg(\frac{1}{m}\bigg|\frac{x}{\alpha}\bigg|^2\bigg)\bigg)^2 \,\mathrm{d}x}_{I(s)}.$$

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Writing out the polynomial, the integral above becomes

$$I(s) = \sum_{k,j=0}^{m-1} \binom{m-1+n/2}{m-1-k}\binom{m-1+n/2}{m-1-j} \frac{(-1)^{k+j}}{k!\,j!(m\alpha^2)^{k+j}} \times \int_{\mathbb{R}^n} \exp\biggl(-\biggl(\frac{2}{\alpha^2 m} - s\biggr)|x|^2\biggr)|x|^{2k+2j}\,\mathrm{d}x.$$

A quick calculation shows that for $a > 0$,

$$\int_{\mathbb{R}^n} \mathrm{e}^{-a|x|^2}|x|^b\,\mathrm{d}x = \frac{\pi^{n/2}}{a^{(n+b)/2}}\frac{\Gamma(n/2+b/2)}{\Gamma(n/2)}. \tag{D.1}$$

Then

$$I(s) = \begin{cases} \dfrac{\pi^{n/2}}{(2/\alpha^2 m - s)^{n/2}\Gamma(n/2)} \displaystyle\sum_{k,j=0}^{m-1} \binom{m-1+n/2}{m-1-k}\binom{m-1+n/2}{m-1-j} \\ \qquad \times \dfrac{(-1)^{k+j}\Gamma(n/2+k+j)}{k!\,j!(2-sm\alpha^2)^{k+j}} & \text{if } s < 2/\alpha^2 m, \\ \infty & \text{otherwise.} \end{cases}$$

For each $k, j \in \mathbb{N}$,

$$\binom{m-1+n/2}{m-1-k}\binom{m-1+n/2}{m-1-j}\Gamma\Bigl(\frac{n}{2}+k+j\Bigr) \sim \frac{1}{(m-1-k)!\,(m-1-j)!}\Bigl(\frac{n}{2}\Bigr)^{2m-2}\Gamma\Bigl(\frac{n}{2}\Bigr) \quad \text{as } n \to \infty. \tag{D.2}$$

So $I(s)$ has the following asymptotic expansion for $s < 2/\alpha^2 m$ as $n \to \infty$:

$$I(s) \sim \frac{\pi^{n/2}}{(2/\alpha^2 m - s)^{n/2}}\Bigl(\frac{n}{2}\Bigr)^{2m-2} \sum_{k=0}^{m-1}\sum_{j=0}^{m-1} \frac{(-1)^{k+j}}{k!\,j!\,(m-1-k)!\,(m-1-j)!}\frac{1}{(2-sm\alpha^2)^{k+j}}.$$

From (4.3) and (D.2),

$$\frac{1}{\mathrm{e}^{2n\rho}}\|K_n\|_2^2 \sim \frac{\alpha^n}{\binom{m-1+n/2}{m-1}^2}\Bigl(\frac{m\pi}{2}\Bigr)^{n/2}\Bigl(\frac{n}{2}\Bigr)^{2m-2} \sum_{k,j=0}^{m-1} \frac{(-1)^{k+j}}{k!\,j!\,(m-1-k)!\,(m-1-j)!}\frac{1}{2^{k+j}}, \tag{D.3}$$

and, hence,

$$\mathbb{E}[\mathrm{e}^{s|X_n|^2}] \sim \Bigl(1 - \frac{s\alpha^2 m}{2}\Bigr)^{-n/2}\Biggl(\sum_{k,j=0}^{m-1} \frac{(-1)^{k+j}}{k!\,j!\,(m-1-k)!\,(m-1-j)!}\frac{1}{(2-sm\alpha^2)^{k+j}} \times \Biggl[\sum_{k,j=0}^{m-1} \frac{(-1)^{k+j}}{k!\,j!\,(m-1-k)!\,(m-1-j)!}\frac{1}{2^{k+j}}\Biggr]^{-1}\Biggr) \quad \text{as } n \to \infty.$$

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Thus,

$$\Lambda(s) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[\mathrm{e}^{s|X_n|^2}] = \begin{cases} -\frac{1}{2}\log\left(1 - \frac{s\alpha^2 m}{2}\right) & \text{if } s < 2/\alpha^2 m, \\ \infty & \text{otherwise.} \end{cases}$$

It is clear that $0 \in (D(\Lambda))^\circ$, where $D(\Lambda) = \{s \in \mathbb{R} \colon \Lambda(s) < \infty\}$. Thus, the Gärtner–Ellis conditions are satisfied. The rate function for the LDP can be computed with the optimization

$$\Lambda^*(x) = \sup_{\lambda\in\mathbb{R}}[x\lambda - \Lambda(\lambda)] = \sup_{\lambda\in\mathbb{R}}\big[x\lambda + \tfrac{1}{2}\log\big(1 - \tfrac{1}{2}\lambda\alpha^2 m\big)\big].$$

Then, since

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[x\lambda + \frac{1}{2}\log\left(1 - \frac{\lambda\alpha^2 m}{2}\right)\right] = x - \frac{\alpha^2 m}{4 - 2\alpha^2 m\lambda} \quad \Longleftrightarrow \quad \lambda = \frac{2}{\alpha^2 m} - \frac{1}{2x},$$

the rate function is

$$\Lambda^*(x) = x\left(\frac{2}{\alpha^2 m} - \frac{1}{2x}\right) + \frac{1}{2}\log\left(\left(1 - \frac{1}{2}\left(\frac{2}{\alpha^2 m} - \frac{1}{2x}\right)\alpha^2 m\right)\right) = \frac{2x}{\alpha^2 m} - \frac{1}{2} + \frac{1}{2}\log\left(\frac{\alpha^2 m}{4x}\right).$$

Then, by the contraction principle (see [5]), the sequence $|X_n|/\sqrt{n}$ satisfies an LDP with rate function

$$\Lambda^*(x) = \frac{2x^2}{\alpha^2 m} - \frac{1}{2} + \frac{1}{2}\log\left(\frac{\alpha^2 m}{4x^2}\right).$$

Note that $\Lambda^*(x) = 0$ if and only if $x = \frac{1}{2}\alpha\sqrt{m}$, implying $|X_n|/\sqrt{n} \xrightarrow{\mathbb{P}} \frac{1}{2}\alpha\sqrt{m}$. $\square$

Appendix E

Proof of Proposition 4.1. For each n, let X_n be a random vector in $\mathbb{R}^n$ with density $K_n^2/\|K_n\|_2^2$. From Lemma 4.1, for $R < \frac{1}{2}\alpha\sqrt{m}$,

$$\lim_{n\to\infty} -\frac{1}{n}\log\mathbb{P}\left(\frac{|X_n|}{\sqrt{n}} \le R\right) = \frac{2R^2}{\alpha^2 m} - \frac{1}{2} + \frac{1}{2}\log\left(\frac{\alpha^2 m}{4R^2}\right).$$

Then, using (D.3), as $n \to \infty$,

$$\mathbb{E}[\eta_n(\mathbb{R}^n)] = \frac{1}{\mathrm{e}^{n\rho}}\|K_n\|_2^2 \sim \left(\frac{\mathrm{e}^{2\rho}\alpha^2 m\pi}{2}\right)^{n/2} \sum_{k,j=0}^{m-1} \frac{(-1)^{k+j}}{k!\, j!\, (m-1-k)!\, (m-1-j)!} \frac{1}{2^{k+j}},$$

Thus, from Lemma 3.1,

$$\begin{aligned} &\lim_{n\to\infty} -\frac{1}{n}\log\mathbb{E}[\eta_n(B_n(\sqrt{n}R))] \\ &\quad = \lim_{n\to\infty} -\frac{1}{n}\log\mathbb{E}[\eta_n(\mathbb{R}^n)] + \lim_{n\to\infty} -\frac{1}{n}\log\mathbb{P}\left(\frac{|X_n|}{\sqrt{n}} \le R\right) \end{aligned}$$

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$$
= \begin{cases} -\rho - \log\alpha - \dfrac{1}{2}\log\left(\dfrac{m\pi}{2}\right) + \left(\dfrac{2R^2}{\alpha^2 m} - \dfrac{1}{2} + \dfrac{1}{2}\log\left(\dfrac{\alpha^2 m}{4R^2}\right)\right), & 0 < R < \frac{1}{2}\alpha\sqrt{m}, \\ -\rho - \log\alpha - \dfrac{1}{2}\log\left(\dfrac{m\pi}{2}\right), & R > \frac{1}{2}\alpha\sqrt{m} \end{cases}
$$

$$
= \begin{cases} -\rho - \dfrac{1}{2}\log 2\pi\mathrm{e} + \dfrac{2R^2}{\alpha^2 m} - \log R, & 0 < R < \frac{1}{2}\alpha\sqrt{m}, \\ -\rho - \log\alpha - \dfrac{1}{2}\log\left(\dfrac{m\pi}{2}\right), & R > \frac{1}{2}\alpha\sqrt{m}, \end{cases}
$$

and the proof is complete. □

Appendix F

Proof of Lemma 4.2. Since, for all n, $\hat{K}_n \in C^2(\mathbb{R}^n)$, Parseval's theorem implies that

$$
\mathbb{E}[|X_n|^2] = \frac{1}{\|K_n\|_2^2}\int_{\mathbb{R}^n} |x|^2 K_n(x)^2\,\mathrm{d}x = \frac{1}{\|\hat{K}_n\|_2^2}\int_{\mathbb{R}^n} -\frac{\triangle \hat{K}_n(\xi)}{(2\pi)^2}\hat{K}_n(\xi)\,\mathrm{d}\xi. \tag{F.1}
$$

To compute the Laplacian of $\hat{K}$, we first note that for each i,

$$
\begin{aligned}
\frac{\partial^2}{\partial x_i^2}\mathrm{e}^{-|\alpha x|^\nu} &= \frac{\partial}{\partial x_i}(-\nu\alpha^\nu x_i|x|^{\nu-2}\mathrm{e}^{-|\alpha x|^\nu}) \\
&= -\nu\alpha^\nu|x|^{\nu-2}\mathrm{e}^{-|\alpha x|^\nu} - \nu\alpha^\nu x_i\left(\frac{\partial}{\partial x_i}|x|^{\nu-2}\right)\mathrm{e}^{-|\alpha x|^\nu} + (\nu\alpha^\nu x_i|x|^{\nu-2})^2\mathrm{e}^{-|\alpha x|^\nu} \\
&= \mathrm{e}^{-|\alpha x|^\nu}(-\nu\alpha^\nu|x|^{\nu-2} - \nu(\nu-2)\alpha^\nu x_i^2|x|^{\nu-4} + \nu^2\alpha^{2\nu}x_i^2|x|^{2\nu-4}) \\
&= \mathrm{e}^{-|\alpha x|^\nu}(x_i^2(\nu^2\alpha^{2\nu}|x|^{2\nu-4} - \nu(\nu-2)\alpha^\nu|x|^{\nu-4}) - \nu\alpha^\nu|x|^{\nu-2}).
\end{aligned}
$$

Then

$$
\begin{aligned}
\triangle\mathrm{e}^{-|\alpha x|^\nu} &= \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\mathrm{e}^{-|\alpha x|^\nu} \\
&= \sum_{i=1}^n \mathrm{e}^{-|\alpha x|^\nu}(x_i^2(\nu^2\alpha^{2\nu}|x|^{2\nu-4} - \nu(\nu-2)\alpha^\nu|x|^{\nu-4}) - \nu\alpha^\nu|x|^{\nu-2}) \\
&= \mathrm{e}^{-|\alpha x|^\nu}(|x|^2(\nu^2\alpha^{2\nu}|x|^{2\nu-4} - \nu(\nu-2)\alpha^\nu|x|^{\nu-4}) - n\nu\alpha^\nu|x|^{\nu-2}) \\
&= \mathrm{e}^{-|\alpha x|^\nu}(\nu^2\alpha^{2\nu}|x|^{2\nu-2} - (\nu(\nu-2)\alpha^\nu + n\nu\alpha^\nu)|x|^{\nu-2}).
\end{aligned}
$$

Thus, from (4.6) and (F.1),

$$
\begin{aligned}
\mathbb{E}[|X_n|^2] &= \frac{\Gamma(n/2+1)\alpha_n^n 2^{n/\nu}}{4\pi^2\pi^{n/2}\Gamma(n/\nu+1)} \\
&\quad\times \int_{\mathbb{R}^n} \mathrm{e}^{-2|\alpha_n x|^\nu}((\nu(\nu-2)\alpha_n^\nu + n\nu\alpha_n^\nu)|x|^{\nu-2} - \nu^2\alpha_n^{2\nu}|x|^{2\nu-2})\,\mathrm{d}x \\
&= \frac{\Gamma(n/2+1)\alpha_n^{n+\nu}2^{n/\nu}\nu}{4\pi^2\pi^{n/2}\Gamma(n/\nu+1)} \\
&\quad\times \left[(\nu-2+n)\int_{\mathbb{R}^n}|x|^{\nu-2}\mathrm{e}^{-2|\alpha_n x|^\nu}\,\mathrm{d}x - \nu\alpha_n^\nu\int_{\mathbb{R}^n}\mathrm{e}^{-2|\alpha_n x|^\nu}|x|^{2\nu-2}\,\mathrm{d}x\right].
\end{aligned}
$$

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Then, using (D.1),

$$\begin{aligned}\mathbb{E}[|X_n|^2] &= n\frac{\alpha_n^{n+\nu}2^{n/\nu}\nu}{4\pi^2\Gamma(n/\nu+1)}\bigg[-\frac{\nu\alpha_n^\nu\Gamma((n+2\nu-2)/\nu)}{\nu 2^{(n+2\nu-2)/\nu}\alpha_n^{n+2\nu-2}}+\frac{(\nu-2+n)\Gamma((n+\nu-2)/\nu)}{\nu 2^{(n+\nu-2)/\nu}\alpha_n^{n+\nu-2}}\bigg]\\ &= n\frac{2^{2/\nu}\alpha_n^2}{4\pi^2\Gamma(n/\nu+1)}\bigg[\frac{(\nu-2+n)}{2}\Gamma\bigg(\frac{n-2}{\nu}+1\bigg)-\frac{\nu}{4}\Gamma\bigg(\frac{n-2}{\nu}+2\bigg)\bigg]\\ &= n\frac{2^{2/\nu}\alpha_n^2\Gamma((n-2)/\nu+1)}{4\pi^2\Gamma(n/\nu+1)}\bigg[\frac{n}{4}+\frac{\nu}{4}-\frac{1}{2}\bigg].\end{aligned}$$

From the asymptotic formula for the gamma function, as $n\to\infty$,

$$\begin{aligned}\mathbb{E}[|X_n|^2] &\sim n\frac{\alpha_n^2 2^{2/\nu}}{4\pi^2}\bigg(\sqrt{\frac{\nu}{2\pi n}}\bigg(\frac{\nu \mathrm{e}}{n}\bigg)^{n/\nu}\bigg)\bigg(\sqrt{\frac{2\pi(n-2)}{\nu}}\bigg(\frac{n-2}{\nu\mathrm{e}}\bigg)^{(n-2)/\nu}\bigg)\bigg[\frac{n}{4}+\frac{\nu}{4}-\frac{1}{2}\bigg]\\ &= n\frac{\alpha_n^2 2^{2/\nu}}{4\pi^2}\frac{\sqrt{n-2}}{\sqrt{n}}\bigg(1-\frac{2}{n}\bigg)^{n/\nu}\bigg(\frac{n-2}{\nu\mathrm{e}}\bigg)^{-2/\nu}\bigg[\frac{n}{4}+\frac{\nu}{4}-\frac{1}{2}\bigg]\\ &\sim n^{2-2/\nu}\alpha_n^2\frac{(2\nu)^{2/\nu}}{16\pi^2}.\end{aligned}$$

By assumption, $\alpha_n\sim\alpha n^{1/\nu-1/2}$ for some constant $\alpha\in(0,\infty)$. Thus,

$$\lim_{n\to\infty}\frac{\mathbb{E}[|X_n|^2]}{n}=\alpha^2\frac{(2\nu)^{2/\nu}}{16\pi^2}.$$

For the second moment of $|X_n|^2$, Parseval's theorem is applied again and we obtain

$$\mathbb{E}[(|X_n|^2)^2]=\frac{1}{\|K_n\|_2^2}\int_{\mathbb{R}^n}(|x|^2K_n(x))^2\,\mathrm{d}x=\frac{1}{\|K_n\|_2^2}\int_{\mathbb{R}^n}\frac{(\triangle\hat{K}_n(\xi))^2}{(2\pi)^4}\,\mathrm{d}\xi.$$

Then, by the above computation of the Laplacian of $\hat{K}$, (4.6), and (D.1),

$$\begin{aligned}\mathbb{E}[(|X_n|^2)^2] &= \frac{\Gamma(n/2+1)\alpha_n^n2^{n/\nu}\nu^2\alpha_n^{2\nu}}{(2\pi)^4\pi^{n/2}\Gamma(n/\nu+1)}\\ &\quad\times\int_{\mathbb{R}^n}\mathrm{e}^{-2|\alpha_nx|^\nu}(\nu\alpha_n^\nu|x|^{2\nu-2}-(\nu-2+n)|x|^{\nu-2})^2\,\mathrm{d}x\\ &= \frac{\Gamma(n/2+1)\alpha_n^n2^{n/\nu}\nu^2\alpha_n^{2\nu}}{(2\pi)^4\pi^{n/2}\Gamma(n/\nu+1)}\\ &\quad\times\bigg[(\nu\alpha_n^\nu)^2\int_{\mathbb{R}^n}\mathrm{e}^{-2|\alpha_nx|^\nu}|x|^{4\nu-4}\,\mathrm{d}x\\ &\qquad-2\nu\alpha_n^\nu(\nu-2+n)\int_{\mathbb{R}^n}\mathrm{e}^{-2|\alpha_nx|^\nu}|x|^{3\nu-4}\,\mathrm{d}x\\ &\qquad+(\nu-2+n)^2\int_{\mathbb{R}^n}\mathrm{e}^{-2|\alpha_nx|^\nu}|x|^{2\nu-4}\,\mathrm{d}x\bigg]\end{aligned}$$

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$$
\begin{aligned}
&= n\frac{\alpha_n^n 2^{n/\nu}\nu^2\alpha_n^{2\nu}}{(2\pi)^4\Gamma(n/\nu+1)}\bigg[\frac{(\nu\alpha_n^\nu)^2\Gamma((n+4\nu-4)/\nu)}{\nu 2^{(n+4\nu-4)/\nu}\alpha_n^{n+4\nu-4}} \\
&\qquad - \frac{2\nu\alpha_n^\nu(\nu-2+n)\Gamma((n+3\nu-4)/\nu)}{\nu 2^{(n+3\nu-4)/\nu}\alpha_n^{n+3\nu-4}} \\
&\qquad + \frac{(\nu-2+n)^2\Gamma((n+2\nu-4)/\nu)}{\nu 2^{(n+2\nu-4)/\nu}\alpha_n^{n+2\nu-4}}\bigg] \\
&= \frac{n2^{4/\nu}\nu^2\alpha_n^4}{(2\pi)^4\Gamma(n/\nu+1)}\bigg[\frac{\nu\Gamma((n-4)/\nu+4)}{2^4} - \frac{2(\nu-2+n)\Gamma((n-4)/\nu+3)}{2^3} \\
&\qquad + \frac{(\nu-2+n)^2\Gamma((n-4)/\nu+2)}{\nu 2^2}\bigg] \\
&= n\frac{2^{4/\nu}\alpha_n^4\Gamma((n-4)/\nu+1)}{(2\pi)^4\Gamma(n/\nu+1)}\bigg[\frac{\nu^3}{2^4}\Big(\frac{n-4}{\nu}+3\Big)\Big(\frac{n-4}{\nu}+2\Big)\Big(\frac{n-4}{\nu}+1\Big) \\
&\qquad - \frac{\nu^2(n+\nu-2)}{2^2}\Big(\frac{n-4}{\nu}+2\Big)\Big(\frac{n-4}{\nu}+1\Big) \\
&\qquad + \frac{\nu(n+\nu-2)^2}{2^2}\Big(\frac{n-4}{\nu}+1\Big)\bigg] \\
&= n\frac{2^{4/\nu}\alpha_n^4}{(2\pi)^4}\frac{\Gamma((n-4)/\nu+1)}{\Gamma(n/\nu+1)}\Big(\frac{n^3}{2^4} - \frac{n^3}{2^2} + \frac{n^3}{2^2} + o(n^3)\Big) \\
&= n^4\frac{2^{4/\nu}\alpha_n^4}{(2\pi)^4}\frac{\Gamma((n-4)/\nu+1)}{\Gamma(n/\nu+1)}\Big(\frac{1}{16}+o(1)\Big) \\
&\sim n^4\frac{2^{4/\nu}\alpha_n^4}{16(2\pi)^4}\sqrt{\frac{\nu}{2\pi n}}\Big(\frac{\nu e}{n}\Big)^{n/\nu}\sqrt{\frac{2\pi(n-4)}{\nu}}\Big(\frac{n-4}{\nu e}\Big)^{(n-4)/\nu} \\
&= n^4\sqrt{\frac{n-4}{n}}\Big(1-\frac{4}{n}\Big)^{n/\nu}\Big(\frac{n-4}{\nu e}\Big)^{-4/\nu}\frac{\alpha_n^4 2^{4/\nu}}{16(2\pi)^4} \\
&\sim n^4(n-4)^{-4/\nu}\frac{\alpha_n^4(2\nu)^{4/\nu}}{16(2\pi)^4}.
\end{aligned}
$$

Again, since $\alpha_n \sim \alpha n^{1/\nu-1/2}$,

$$
\mathbb{E}[(|X_n|^2)^2] = O(n^2) \quad \text{and} \quad \lim_{n\to\infty}\frac{\mathbb{E}[(|X_n|^2)^2]}{n^2} = \alpha^4\frac{(2\nu)^{4/\nu}}{16(2\pi)^4}.
$$

Note that this limit is exactly the square of the limit of the expectation of $|X_n|^2/n$, implying that

$$
\operatorname{var}\Big(\frac{|X_n|^2}{n^2}\Big) = \frac{\mathbb{E}[(|X_n|^2)^2]}{n^2} - \Big(\frac{\mathbb{E}[|X_n|^2]}{n}\Big)^2 \to 0 \quad \text{as } n\to\infty.
$$

Thus, by Chebychev's inequality,

$$
\frac{|X_n|}{\sqrt{n}} \xrightarrow{\mathbb{P}} \alpha\frac{(2\nu)^{1/\nu}}{4\pi}. \qquad \square
$$

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Appendix G

Proof of Proposition 4.3. First, for $k \geq 0$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x|^k K(x)^2 \, \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} |x|^k \left(\mathrm{e}^{n\rho} 2^{(\sigma+n)/2} \Gamma\left(\frac{\sigma+n+2}{2}\right) \frac{J_{(\sigma+n)/2}(2|x/\alpha|\sqrt{(\sigma+n)/2})}{(2|x/\alpha|\sqrt{(\sigma+n)/2})^{(\sigma+n)/2}} \right)^2 \mathrm{d}x \\
&\quad = \mathrm{e}^{2n\rho} 2^{(\sigma+n)} \Gamma\left(\frac{\sigma+n+2}{2}\right)^2 \int_{\mathbb{R}^n} |x|^k \frac{J_{(\sigma+n)/2}(2|x/\alpha|\sqrt{(\sigma+n)/2})^2}{(2|x/\alpha|\sqrt{(\sigma+n)/2})^{(\sigma+n)}} \, \mathrm{d}x \\
&\quad = \mathrm{e}^{2n\rho} 2^{(\sigma+n)} \Gamma\left(\frac{\sigma+n+2}{2}\right)^2 \frac{2\pi^{n/2}}{\Gamma(n/2)} \\
&\qquad \times \int_0^\infty r^{n-1} r^k \frac{J_{(\sigma+n)/2}(2(r/\alpha)\sqrt{(\sigma+n)/2})^2}{(2(r/\alpha)\sqrt{(\sigma+n)/2})^{(\sigma+n)}} \, \mathrm{d}x,
\end{aligned}$$

and by the change of variables $y = ((2/\alpha)\sqrt{(\sigma+n)/2})r$, we have

$$\begin{aligned}
&\mathrm{e}^{2n\rho} 2^{\sigma+n} \frac{2\pi^{n/2} \Gamma((\sigma+n+2)/2)^2}{\Gamma(n/2)} \\
&\quad \times \int_0^\infty \left(\frac{2}{\alpha}\sqrt{\frac{\sigma+n}{2}}\right)^{-k-n+1} \frac{J_{(\sigma+n)/2}(y)^2}{y^{\sigma+1-k}} \left(\frac{2}{\alpha}\sqrt{\frac{\sigma+n}{2}}\right)^{-1} \mathrm{d}y \\
&\qquad = \mathrm{e}^{2n\rho} 2^{\sigma+n} \frac{2\pi^{n/2} \Gamma((\sigma+n+2)/2)^2 \alpha^{k+n}}{\Gamma(n/2)(2(\sigma+n))^{(k+n)/2}} \int_0^\infty \frac{J_{(\sigma+n)/2}(y)^2}{y^{\sigma+1-k}} \, \mathrm{d}y.
\end{aligned}$$

For $\sigma + 1 - k > 0$, from [19, Equation (10.22.57)],

$$\int_0^\infty \frac{J_{(\sigma+n)/2}(y)^2}{y^{\sigma+1-k}} \, \mathrm{d}y = \frac{\Gamma(n/2+k/2)\Gamma(\sigma+1-k)}{2^{\sigma-k+1}\Gamma((\sigma-k)/2+1)^2\Gamma(\sigma-k/2+n/2+1)},$$

and, thus,

$$\begin{aligned}
\int_{\mathbb{R}^n} |x|^k K(x)^2 \, \mathrm{d}x &= \mathrm{e}^{2n\rho} 2^{\sigma+n} \frac{2\pi^{n/2} \Gamma((\sigma+n+2)/2)^2 \alpha^{k+n}}{\Gamma(n/2)(2(\sigma+n))^{(k+n)/2}} \\
&\qquad \times \frac{\Gamma(n/2+k/2)\Gamma(\sigma+1-k)}{2^{\sigma-k+1}\Gamma((\sigma-k/2+1)^2\Gamma(\sigma-k/2+n/2+1)} \\
&= \mathrm{e}^{2n\rho} \frac{(2\pi)^{n/2} \alpha^{k+n} 2^{k/2} \Gamma((\sigma+n+2)/2)^2}{(\sigma+n)^{(k+n)/2}\Gamma(n/2)} \\
&\qquad \times \frac{\Gamma(n/2+k/2)\Gamma(\sigma+1-k)}{\Gamma((\sigma-k)/2+1)^2\Gamma(\sigma-k/2+n/2+1)}.
\end{aligned}$$

Then, for $\sigma > 0$,

$$\begin{aligned}
\mathbb{E}[|X_n|] &= \frac{1}{\|K_n\|_2^2} \int_{\mathbb{R}^n} |x| K_n(x)^2 \, \mathrm{d}x \\
&= \frac{(2\pi)^{n/2} \alpha^{1+n} 2^{1/2} \Gamma((\sigma+n+2)/2)^2 \Gamma(n/2+1/2)\Gamma(\sigma)}{(\sigma+n)^{(1+n)/2}\Gamma(n/2)\Gamma((\sigma+1)/2)^2\Gamma(\sigma-1/2+n/2+1)} \\
&\quad \times \frac{(\sigma+n)^{n/2}\Gamma(\sigma/2+1)^2\Gamma(\sigma+n/2+1)}{(2\pi)^{n/2}\alpha^n \Gamma(\sigma+1)\Gamma(\sigma/2+n/2+1)^2}
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{\alpha 2^{1/2}}{(\sigma+n)^{1/2}\Gamma(n/2)} \frac{\Gamma(n/2+1/2)\Gamma(\sigma)\Gamma(\sigma/2+1)^2\Gamma(\sigma+n/2+1)}{\Gamma(\sigma/2+1/2)^2\Gamma(\sigma+n/2+1/2)\Gamma(\sigma+1)} \\
&\sim \frac{\alpha 2^{1/2}}{(\sigma+n)^{1/2}\Gamma(n/2)} \frac{\Gamma(n/2)(n/2)^{1/2}\Gamma(\sigma)\Gamma(\sigma/2+1)^2\Gamma(n/2)(n/2)^{\sigma+1}}{\Gamma((\sigma+1)/2)^2\Gamma(n/2)(n/2)^{\sigma+1/2}\Gamma(\sigma+1)} \\
&= \frac{\alpha 2^{1/2}}{(\sigma+n)^{1/2}} \frac{(n/2)\Gamma(\sigma)\Gamma(\sigma/2+1)^2}{\Gamma((\sigma+1)/2)^2\Gamma(\sigma+1)} \\
&\sim n^{1/2}\frac{\alpha}{2^{1/2}} \frac{\Gamma(\sigma)\Gamma(\sigma/2+1)^2}{\Gamma((\sigma+1)/2)^2\Gamma(\sigma+1)} \\
&= O(n^{1/2}).
\end{aligned}
$$

Now, let $\beta > \frac{1}{2}$. By Markov's inequality,

$$
\lim_{n\to\infty} \frac{\mathbb{E}[\eta_n(B_n(Rn^\beta)^c)]}{\mathbb{E}[\eta_n(\mathbb{R}^n)]} = \lim_{n\to\infty} \mathbb{P}(|X_n| \geq Rn^\beta) \leq \lim_{n\to\infty} \frac{\mathbb{E}[|X_n|]}{Rn^\beta} = 0. \qquad \square
$$

Appendix H

Proof of Proposition 4.4. First, from [8, Equation (6.576.3)], we have, for all $\nu > 0$ and $k > 2\nu - 1$,

$$
\int_0^\infty r^k \mathbb{K}_\nu\left(\frac{r}{\alpha}\right)^2 \mathrm{d}r = \frac{2^{-2+k}\alpha^{k+1}}{\Gamma(1+k)} \Gamma\left(\frac{1+k}{2}+\nu\right)\Gamma\left(\frac{k+1}{2}\right)^2\Gamma\left(\frac{1+k}{2}-\nu\right), \tag{H.1}
$$

where $\mathbb{K}_\nu$ is the modified Bessel function of the second kind.

For the Whittle–Matérn kernel (4.10),

$$
\begin{aligned}
\int_{\mathbb{R}^n} K_n(x)^2 \,\mathrm{d}x &= \int_{\mathbb{R}^n} \mathrm{e}^{2n\rho} \frac{2^{2-2\nu}}{\Gamma(\nu)^2} \frac{|x|^{2\nu}}{\alpha^{2\nu}} \mathbb{K}_\nu\left(\frac{|x|}{\alpha}\right)^2 \mathrm{d}x \\
&= \frac{2\pi^{n/2}}{\Gamma(n/2)} \mathrm{e}^{2n\rho} \frac{2^{2-2\nu}}{\Gamma(\nu)^2\alpha^{2\nu}} \int_0^\infty r^{n-1} r^{2\nu} \mathbb{K}_\nu\left(\frac{r}{\alpha}\right)^2 \mathrm{d}r \\
&= \mathrm{e}^{2n\rho} \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{2^{2-2\nu}}{\Gamma(\nu)^2\alpha^{2\nu}} \int_0^\infty r^{n-1+2\nu} \mathbb{K}_\nu\left(\frac{r}{\alpha}\right)^2 \mathrm{d}r.
\end{aligned}
$$

Then, from (H.1),

$$
\begin{aligned}
\int_0^\infty r^{n-1+2\nu} \mathbb{K}_\nu\left(\frac{r}{\alpha}\right)^2 \mathrm{d}r &= \frac{2^{-3+n+2\nu}\alpha^{n+2\nu}}{\Gamma(n+2\nu)} \Gamma\left(\frac{n+2\nu}{2}+\nu\right)\Gamma\left(\frac{n+2\nu}{2}\right)^2\Gamma\left(\frac{n+2\nu}{2}-\nu\right) \\
&= \frac{2^{-3+n+2\nu}\alpha^{n+2\nu}}{\Gamma(n+2\nu)} \Gamma\left(\frac{n}{2}+2\nu\right)\Gamma\left(\frac{n}{2}+\nu\right)^2\Gamma\left(\frac{n}{2}\right).
\end{aligned}
$$

Similarly,

$$
\int_{\mathbb{R}^n} |x|^2 K_n(x)^2 \,\mathrm{d}x = \mathrm{e}^{2n\rho} \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{2^{2-2\nu}}{\Gamma(\nu)^2\alpha^{2\nu}} \int_0^\infty r^{n+1+2\nu} \mathbb{K}_\nu\left(\frac{r}{\alpha}\right)^2 \mathrm{d}r.
$$

and also from (H.1),

$$
\int_0^\infty r^{n+1+2\nu} \mathbb{K}_n\left(\frac{r}{\alpha}\right)^2 \mathrm{d}r = \frac{2^{-1+n+2\nu}\alpha^{n+2+2\nu}}{\Gamma(n+2+2\nu)} \Gamma\left(\frac{n}{2}+2\nu+1\right)\Gamma\left(\frac{n}{2}+\nu+1\right)^2\Gamma\left(\frac{n}{2}+1\right)
$$

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Then

$$\begin{aligned}
\mathbb{E}[|X_n|^2] &= \frac{\int_{\mathbb{R}^n} |x|^2 K_n(x)^2 \, \mathrm{d}x}{\int_{\mathbb{R}^n} K_n(x)^2 \, \mathrm{d}x} \\
&= \frac{(2\alpha)^2 \Gamma(n+2\nu)\Gamma(n/2+2\nu+1)\Gamma(n/2+\nu+1)^2\Gamma(n/2+1)}{\Gamma(n+2+2\nu)\Gamma(n/2+2\nu)\Gamma(n/2+\nu)^2\Gamma(n/2)} \\
&= \frac{(2\alpha)^2(n/2+2\nu)(n/2+\nu)^2(n/2)}{(n+1+2\nu)(n+2\nu)} \\
&\sim \left(\frac{\alpha}{2}\right)^2 n \quad \text{as } n \to \infty,
\end{aligned}$$

and this implies

$$\frac{\mathbb{E}[|X_n|^2]^{1/2}}{\sqrt{n}} \to \frac{\alpha}{2} \quad \text{as } n \to \infty.$$

Thus, since the Whittle–Matérn kernel is log-concave, the conclusion holds by Theorem 3.2 and the proof is complete. □

Appendix I

Proof of Proposition 4.5. First, recall that the beta function satisfies

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} \, \mathrm{d}t = \int_0^\infty t^{x-1}(1+t)^{-(x+y)} \, \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Then, for any $k \geq 0$,

$$\begin{aligned}
\int_{\mathbb{R}^n} |x|^k K_n(x)^2 \, \mathrm{d}x &= \int_{\mathbb{R}^n} |x|^k \frac{\mathrm{e}^{2n\rho}}{(1+|x/\alpha_n|^2)^{2\nu+n}} \, \mathrm{d}x \\
&= \mathrm{e}^{2n\rho} \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty r^{n-1+k} \left(1+\frac{r^2}{\alpha_n^2}\right)^{-2\nu-n} \mathrm{d}r \\
&= \mathrm{e}^{2n\rho} \frac{\pi^{n/2}}{\Gamma(n/2)} \alpha_n^{n+k} \int_0^\infty t^{n/2-1+k/2}(1+t)^{-(2\nu+n)} \, \mathrm{d}t \\
&= \mathrm{e}^{2n\rho} \frac{\pi^{n/2}}{\Gamma(n/2)} \alpha_n^{n+k} B\left(\frac{n}{2}+\frac{k}{2}, 2\nu+\frac{n}{2}-\frac{k}{2}\right).
\end{aligned}$$

Thus, the expectation of $|X_n|^2$ is

$$\begin{aligned}
\mathbb{E}[|X_n|^2] &= \frac{1}{\|K_n\|_2^2} \int_{\mathbb{R}^n} |x|^2 K_n(x)^2 \, \mathrm{d}x \\
&= \alpha_n^2 \frac{B(n/2+1, 2\nu+n/2-1)}{B(n/2, 2\nu+n/2)} \\
&= \alpha_n^2 \frac{\Gamma(n/2+1)\Gamma(2\nu+n/2-1)\Gamma(n+2\nu)}{\Gamma(n+2\nu)\Gamma(n/2)\Gamma(2\nu+n/2)} \\
&= \alpha_n^2 \frac{n}{2(n/2+2\nu-1)} \\
&= \alpha_n^2 \frac{n}{n+4\nu-2},
\end{aligned}$$

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and

$$\begin{aligned}\mathbb{E}[|X_n|^4] &= \alpha_n^4 \frac{B(n/2+2, 2\nu+n/2-2)}{B(n/2, 2\nu+n/2)} \\ &= \alpha_n^4 \frac{\Gamma(n/2+2)\Gamma(2\nu+n/2-2)\Gamma(n+2\nu)}{\Gamma(n+2\nu)\Gamma(n/2)\Gamma(2\nu+n/2)} \\ &= \alpha_n^4 \frac{(n/2+1)n/2}{(2\nu+n/2-2)(2\nu+n/2-1)} \\ &= \alpha_n^4 \frac{n(n+2)}{(n+4\nu-4)(n+4\nu-2)}.\end{aligned}$$

Thus, by the assumption that $\alpha_n \sim \alpha n^{1/2}$ as $n \to \infty$ for some $\alpha > 0$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[|X_n|^2]}{n} = \alpha^2 \quad \text{and} \quad \lim_{n\to\infty} \frac{\operatorname{var}(|X_n|^2)}{n^2} = 0.$$

Thus, by Chebychev's inequality, $|X_n|/\sqrt{n} \xrightarrow{\mathbb{P}} \alpha$. □

Appendix J

Proof of Proposition 5.1. By Proposition 4.1,

$$\lim_{n\to\infty} -\frac{1}{n} \ln \mathbb{E}[\eta_n(B_n(\sqrt{n}R))] = \begin{cases} -\rho - \dfrac{1}{2}\log 2\pi\mathrm{e} + \dfrac{2R^2}{\alpha^2 m} - \log R, & 0 < R < \frac{1}{2}\alpha\sqrt{m}, \\ -\rho - \log\alpha - \dfrac{1}{2}\log\dfrac{m\pi}{2}, & R > \frac{1}{2}\alpha\sqrt{m}. \end{cases}$$

Recall that $\lim_{n\to\infty}(1/n)\ln\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))] = \rho + \frac{1}{2}\log 2\pi\mathrm{e} + \log R$. Thus,

$$\begin{aligned}&\lim_{n\to\infty} -\frac{1}{n}\ln\frac{\mathbb{E}[\eta_n(B_n(\sqrt{n}R))]}{\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))]} \\ &\quad = \lim_{n\to\infty} -\frac{1}{n}\ln\mathbb{E}[\eta_n(B_n(\sqrt{n}R))] + \frac{1}{n}\ln\mathbb{E}[\Phi_n(B_n(\sqrt{n}R))] \\ &\quad = \begin{cases} -\rho - \dfrac{1}{2}\log 2\pi\mathrm{e} + \dfrac{2R^2}{\alpha^2 m} - \log R + \rho + \dfrac{1}{2}\log 2\pi\mathrm{e} + \log R, & 0 < R < \frac{1}{2}\alpha\sqrt{m}, \\ -\rho - \log\alpha - \dfrac{1}{2}\log\dfrac{m\pi}{2} + \rho + \dfrac{1}{2}\log 2\pi\mathrm{e} + \log R, & R > \frac{1}{2}\alpha\sqrt{m}, \end{cases} \\ &\quad = \begin{cases} \dfrac{2R^2}{\alpha^2 m}, & 0 < R < \frac{1}{2}\alpha\sqrt{m}, \\ \frac{1}{2} + \log 2 - \log\alpha - \frac{1}{2}\log m + \log R, & R > \frac{1}{2}\alpha\sqrt{m}, \end{cases}\end{aligned}$$

and the proof is complete. □

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