

# On a generalized two-fluid Hele-Shaw flow

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Generalized two-phase fluid flows in a Hele-Shaw cell are considered. It is assumed that the flow is driven by the fluid pressure gradient and an external potential field, for example, an electric field. Both the pressure field and the external field may have singularities in the flow domain. Therefore, combined action of these two fields brings into existence some new features, such as non-trivial equilibrium shapes of boundaries between the two fluids, which can be studied analytically. Some examples are presented. It is argued, that the approach and results may find some applications in the theory of fluids flow through porous media and microfluidic devices controlled by electric field.

## 1 Introduction

Analysis of displacement in a Hele-Shaw cell and porous media is a source of a multitude of mathematical problems which provide some insight into general features of nonlinear boundary dynamics [4, 8, 12]. Here, we consider a slightly modified version of the classical problem of flow in a potential external field which displays some new features related to existence of singularities of the external field. The study was prompted by the interest in coupled flow phenomena in saturated porous media in the presence of electric current (“electrokinetic phenomena”). This specific case will be briefly discussed later. However, eventually it became clear that the topic deserves study *per se*. Throughout the text we will use expressions ‘Hele-Shaw flow’ and ‘flow through a porous medium’ interchangeably due to the well known analogy between the Darcy law for a porous medium and the flow rule in a thin gap between two parallel walls.

## 2 The problem statement

We assume that fluid occupies a finite domain  $D(t)$  in the plane  $(x, y)$  of a Hele-Shaw cell of gap thickness  $b$ . The flow velocity  $\mathbf{w}$  is determined by the flow rule

$$\mathbf{w} = -\frac{k}{\eta}\nabla p + \frac{k}{\eta}\mathbf{g}. \quad (2.1)$$

Here,  $p$  is the fluid pressure,  $\eta$  is the fluid viscosity,  $k = b^2/12$  is the gap “permeability”,  $\mathbf{g} = \{g_x, g_y\}$  is the body force due to an external field. We assume the body force to be

potential,

$$\mathbf{g} = -\nabla\Psi(x, y). \quad (2.2)$$

Two familiar examples are the gravity force and the centrifugal force, corresponding to

$$\Psi = \rho gh, \quad \text{and} \quad \Psi = -\frac{1}{2}\rho\omega^2 r^2, \quad (2.3)$$

respectively. Here,  $\rho$  is the fluid density,  $h$  is the height above a datum,  $g$  is the acceleration due to gravity,  $r$  is the distance from the rotation axis normal to the cell plane;  $\omega$  is the rotation rate. These cases allow thorough study; they are considered in particular in Entov *et al.* [4, 3]. Here, we are going to study the case when the external potential  $\Psi$  has singularities within and/or outside the domain  $D(t)$ .

The flow field satisfies the continuity equation

$$\nabla \cdot \mathbf{w} = \sum_{j=1}^N q_j \delta(x - x_j, y - y_j). \quad (2.4)$$

Here,  $q_j$  are strengths (flow rates) of the point sources (sinks) within the flow domain. We assume that at the boundary  $\Gamma = \partial D(t)$  the pressure vanishes,

$$p(x, y) = 0, \quad (x, y) \in \Gamma. \quad (2.5)$$

The boundary dynamics is governed by the relation

$$v_n = w_n = \mathbf{w} \cdot \mathbf{n}, \quad (2.6)$$

$\mathbf{n}$  being the outward normal to  $\Gamma$ , and  $v_n$  the velocity of propagation of the boundary in the normal direction.

We assume now that the external potential field satisfies the equation

$$\Delta G = \sum_{m=1}^M Q_m \delta(x - x'_m, y - y'_m), \quad G = -\frac{k}{\eta} \Psi, \quad (2.7)$$

in the entire plane  $(x, y)$  with boundary condition

$$|\nabla G| \rightarrow 0, \quad (x^2 + y^2) \rightarrow \infty. \quad (2.8)$$

Therefore,

$$G = \operatorname{Re}F(z), \quad F = \sum_{m=1}^M \frac{Q_m}{2\pi} \ln(z - z'_m),$$

$$z = x + iy; \quad z'_m = x'_m + iy'_m. \quad (2.9)$$

Let us introduce the velocity potential

$$\Phi(x, y) = -\frac{k}{\eta}(p + \Psi). \quad (2.10)$$

It satisfies the following problem:

$$\Delta\Phi = \sum_{j=1}^N q_j \delta(x - x_j, y - y_j), \quad (x_j, y_j) \in D(t); \tag{2.11}$$

$$\Phi(x, y) = -\frac{k}{\eta} \Psi(x, y) = G(x, y), \quad (x, y) \in \Gamma(t); \tag{2.12}$$

$$v_n = \frac{\partial\Phi}{\partial n}, \quad (x, y) \in \Gamma(t). \tag{2.13}$$

The last equation serves to describe the moving boundary dynamics.

The only difference with the usual Hele-Shaw problem is that the velocity potential does not vanish at the boundary, but should be equal to a specified function of the boundary point.

Now let  $u(x, y)$  be a harmonic function in a domain  $D^*$  such, that  $D(t)$  remains within  $D^*$ . Then as a straightforward generalization of the Richardson Theorem [10, 11, 4] we find:

$$\frac{d}{dt} \int_{D(t)} u dA = \sum_{j=1}^N q_j u(z_j) + \int_{D(t)} \nabla G \cdot \nabla u dA. \tag{2.14}$$

The following chain of equalities proves statement (2.14):

$$\begin{aligned} \frac{d}{dt} \int_{D(t)} u dA &= \int_{\partial D(t)} u \frac{\partial\Phi}{\partial n} dl \\ &= \int_{\partial D(t)} \Phi \frac{\partial u}{\partial n} dl + \int_{D(t)} (u\Delta\Phi - \Phi\Delta u) dA \\ &= \int_{\partial D(t)} G \frac{\partial u}{\partial n} dl + \sum_{j=1}^N u(z_j)q_j = \int_{D(t)} \nabla \cdot (G\nabla u) dA + \sum_{j=1}^N u(z_j)q_j \\ &= \sum_{j=1}^N q_j u(z_j) + \int_{D(t)} \nabla G \cdot \nabla u dA. \end{aligned}$$

The left-hand side of (2.14) is a time derivative of a harmonic moment of the domain  $D(t)$ .

This equation leads to explicit analytic techniques of predicting domain evolution provided that the operator  $\nabla G \cdot \nabla$  maps harmonic functions to harmonic ones (and the domain initially belongs to a certain class of domains). It can be shown that this is possible only if

$$G = ax + by + c(x^2 + y^2) + d, \tag{2.15}$$

with constant  $a, b, c, d$ . Indeed, assume that the function  $G$  is twice continuously differentiable in a neighborhood of zero, and the operator  $\nabla G \nabla$  maps harmonic functions on this neighborhood to harmonic ones. This operator can be written as  $2(\partial G \bar{\partial} + \bar{\partial} G \partial)$ . Thus, applying this operator to the function  $\bar{z}^2$ , we get that the function  $\partial G \cdot \bar{z}$  is harmonic. Hence, the function  $f(z) = \partial^2 G \cdot \bar{z}$  is holomorphic. But then the function  $f(z)/\bar{z}$  cannot be continuous near zero unless  $f = 0$ . Thus,  $\partial^2 G = 0$ , which implies that  $G = g(\bar{z})z + h(\bar{z})$ , where  $g, h$  are holomorphic. Similarly,  $\bar{\partial}^2 G = 0$ , which implies  $g'' = h'' = 0$ , so  $g$  and  $h$  are linear functions, as desired. Essentially, this is a combination of uniform (“gravity”) and

axisymmetrical (“centrifugal force”) fields treated previously [3, 4]. Of course, for  $c \neq 0$ , it can be reduced to pure rotation about a shifted axis.

Here, we will be interested primarily in equilibrium shapes of the flow domain under combined action of the flow and the external potential. In this case, one can derive effective solutions for a much wider class of the external potentials.

### 3 Steady-state shapes

For the steady state (equilibrium) domain  $D$  we arrive at the following moment problem:

$$\forall u : \Delta u = 0 \text{ in } D, \tag{3.1}$$

$$\int_D \nabla G \cdot \nabla u dA = - \sum_{j=1}^N q_j u(z_j). \tag{3.2}$$

(Of course, equilibrium domain can exist only if the net fluid flux vanishes,  $\sum_{j=1}^N q_j = 0$ .) Now we assume that the external field has the form

$$G(z) = \sum_{m=1}^M \frac{Q_m}{2\pi} \ln |z - z'_m|. \tag{3.3}$$

In other words, it can be considered as an electric field generated by a finite array of point charges in a plane. For brevity’s sake, we will refer to it as an “electric potential” field. Note that  $z'_m$  can be both inside and outside  $D$ . Let us now introduce the corresponding complex potential  $F(z)$  and ‘complex current’  $\omega(z)$ :

$$F(z) = G(z) + i\Xi(z) = \sum_{m=1}^M \frac{Q_m}{2\pi} \ln(z - z'_m); \tag{3.4}$$

$$\omega(z) = F'(z) = \frac{\partial G}{\partial x} - i \frac{\partial G}{\partial y} = \sum_{m=1}^M \frac{Q_m}{2\pi(z - z'_m)}. \tag{3.5}$$

(The complex potential  $F(z)$  is, generally speaking, multivalued, unless all the ‘electric sources’ are outside  $D$ .) Then the moment equation (3.2) can be written as

$$J_D = \int_D \overline{\omega(z)} U'(z) dA = - \sum_{i=1}^N q_i U(z_i), \tag{3.6}$$

for an analytic function  $U$  on a neighborhood of  $D$ , since we have  $\nabla U = (U', iU')$ .

The integral  $J_D$  in the left-hand side of (3.6) converges even if some  $z'_m$  belong to  $D$  as  $\omega(z)$  has simple poles at these points. We write the integral as

$$J_D = \int_D \overline{\omega(z)} U'(z) dx dy = \frac{1}{2i} \oint_{\partial D} \overline{\omega(z)} U(z) d\bar{z} + \sum_{m: z'_m \in D} \frac{Q_m}{2} U(z'_m). \tag{3.7}$$

The last transformation follows from Green’s Theorem; the sum in the right-hand side accounts for the contributions of poles of  $\omega$  in  $D$ .

Let us now choose

$$U(z) = \frac{1}{\pi(w - z)},$$

$w$  being a point outside  $D$ . Then (3.6) becomes

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{\overline{\omega(z)} dz}{w - z} = - \sum_{i=1}^N \frac{q_i}{\pi(w - z_i)} - \sum_{m: z'_m \in D} \frac{Q_m}{2\pi} \frac{1}{w - z'_m}. \tag{3.8}$$

We denote the primitive of the function in the r.h.s. of Eq. (3.8) by  $h(w)$ :

$$h(w) = - \sum_{i=1}^N \frac{q_i}{\pi} \ln(w - z_i) - \sum_{m: z'_m \in D} \frac{Q_m}{2\pi} \ln(w - z'_m). \tag{3.9}$$

The following Theorem due to Richardson [10] plays a crucial role in solving the problem of finding the domain  $D$ :

**Theorem 3.1** *Let  $f : K \rightarrow D$  be a conformal mapping that maps the unit disk  $K$  of the  $\zeta$ -plane onto  $D$ . Then the function*

$$\frac{d}{d\zeta} \left( \overline{F \left( f \left( \frac{1}{\bar{\zeta}} \right) \right)} - h(f(\zeta)) \right), \tag{3.10}$$

*initially defined in a vicinity of the unit circle, extends analytically to a holomorphic function in  $K$ .*

**Proof** On the unit circle  $\zeta = e^{i\varphi}$ ,  $\zeta = 1/\bar{\zeta}$  and thus it suffices to show that the differential  $d(\overline{F(z)} - h(z))$  on  $\partial D$  extends to a holomorphic differential in  $D$ .

So according to the Cauchy Theorem it is necessary to check that

$$\oint_{\partial D} \frac{d(\overline{F(z)} - h(z))}{t - z} = 0, \quad \text{for } t \notin D. \tag{3.11}$$

However, this follows directly from Eqs. (3.8), (3.9) and the fact that  $dh$  is holomorphic outside  $D$ . □

This implies the following important corollary:

**Corollary 1** *The function  $\frac{d}{d\zeta} F(f(\zeta))$  is rational.*

**Proof** For any function  $f(z)$  denote

$$f^*(z) = \overline{f(\bar{z})}.$$

This implies immediately that

$$\overline{F(f(z))} = F^*(f^*(\bar{z})).$$

Then according to the Theorem (3.10) the differential

$$d(F^*(f^*(1/\zeta)) - h(f(\zeta)))$$

is analytic in the unit disc.

Therefore, the differential

$$D(\zeta) := d\left(F^*\left(f^*\left(\frac{1}{\zeta}\right)\right)\right)$$

has the same singularities as  $dh(f(\zeta))$  in the unit disk  $K$ , i.e. simple poles at the points

$$\zeta = f^{-1}(z'_m), \quad z'_m \in D \quad \text{and} \quad \zeta = f^{-1}(z_j), \quad j = 1, \dots, N.$$

Therefore  $dF(f(\zeta))$  has poles at the points

$$1/\overline{f^{-1}(z_j)}, \quad 1/\overline{f^{-1}(z'_m)}, \quad z'_m \in D$$

outside the unit disk. On the other hand,  $dF(f(\zeta))$  has poles at the points

$$\zeta = f^{-1}(z'_m), \quad z'_m \in D$$

within the unit disk.

Therefore,  $D(\zeta)$  has a finite number of poles, and hence it is rational. Then determination of the precise form of  $f(\zeta)$  can be reduced to a set of nonlinear algebraic equations.  $\square$

Note that this analysis can be extended in a standard way to the limiting case of coalescence of hydrodynamic sources and sinks corresponding to multipoles. In such a case, the terms

$$\frac{q_j}{\pi(z - z_j)}$$

have to be replaced with terms

$$\frac{\mu_j}{\pi(z - z_j)^n}, \quad n > 1, \quad \text{etc.}$$

#### 4 Examples of equilibrium domains: harmonic potential

In this section we present several examples of effective construction of the equilibrium domains starting from specified singularities of the hydrodynamic and ‘electric’ fields. We first start from a comparatively simple case when all singularities of the external field are located outside the equilibrium domain, and the field is univalent within the equilibrium domain. In this case, the problem can be reduced to that of equilibrium in a uniform (‘gravity’) field. The second part of the section deals with harmonic external potential having singularities within the flow domain.

##### 4.1 Univalent $F(z)$

Assume that  $F(z)$  is univalent in  $D$ . Then we can solve the problem in a more straightforward way. Let us consider an auxiliary plane of the complex potential  $\tilde{z} = F(z)$  (further referred to as potential plane, or  $F$ -plane), and let a domain  $\tilde{D} = F(D)$  corresponds to  $D$  in this plane. Then (3.6) becomes

$$\int_{\tilde{D}} U'(\tilde{z}) d\tilde{x}d\tilde{y} = - \sum q_j U(F_j); \quad \tilde{z} = F(z); \quad F_j = F(z_j). \tag{4.1}$$

Indeed, Eq. (3.6) is transformed into Eq. (4.1) using

$$U'(z) = U'(\tilde{z}) \frac{d\tilde{z}}{dz} = U'(\tilde{z})\omega(z); \quad |\omega(z)|^2 dx dy = d\tilde{x}d\tilde{y}.$$

However, this is exactly the form of the moment equation that corresponds to a uniform external field with  $G(\tilde{z}) = \tilde{x}$  (the Hele-Shaw flow in the presence of gravity) [4]. If we take  $u(z) = z^n$ , Equation (4.1) becomes

$$\tilde{M}_{n-1} \equiv \int_{\tilde{D}} \tilde{z}^{n-1} d\tilde{x}d\tilde{y} = - \sum q_j F_j^n / n, \tag{4.2}$$

so that all the moments are specified for a given  $F(z)$ .

Letting  $n = 1$  we have

$$\tilde{S} = - \sum q_j F_j > 0. \tag{4.3}$$

This is a necessary condition for the existence of a steady-state solution. In particular, if the flow is generated by a dipole (a source-sink doublet of strength  $\pm q = \pm\mu/(2\epsilon)$  at  $z = \pm\epsilon$ ) then as  $\epsilon \rightarrow 0$  the rhs of Eq. (4.3) tends to  $\mu F'(0)$ , and (4.2) becomes

$$M_0 \equiv \tilde{S} = \int_{\tilde{D}} d\tilde{x}d\tilde{y} = \mu; \quad M_n = 0, \quad n = 1, 2, 3, \dots \tag{4.4}$$

Obviously, in the  $F$ -plane the equilibrium domain is a circle of radius  $R_0 = \sqrt{\mu/\pi}$ .

Thus, there is a fixed value of the area of the equilibrium domain in the potential plane for which such a domain exists. Obviously, the equilibrium in this case is due to a fine balance between hydrodynamic and external forces, and is unstable.

The above elementary example is generic in the sense that for given set of hydrodynamic and electric sources the area of the equilibrium domain, provided such a domain exists, can assume only a discrete set of values, if the electric sources are outside the domain, so that  $F(z)$  is analytic in  $D$ .

It is reasonable to ask about the fate of a domain that evolves under combined action of the balanced hydrodynamic sources and the external field starting from a non-equilibrium shape. While in general case the answer is beyond our capacities, some insight can be derived from the simple case of “gravity”, i.e. uniform potential field,

$$F(z) = -g\rho z. \tag{4.5}$$

(In this case, it is convenient to consider the  $x$ -axis to be directed upward.)

Then the moment dynamics equation (2.14) becomes

$$\frac{d}{dt} \int_{D(t)} u dA = \sum_{i=1}^N q_i u(z_i) - C \int_{D(t)} \frac{\partial u}{\partial x} dA; \quad C = \frac{\rho g k}{\eta}. \tag{4.6}$$

If we now take

$$u(z) = \frac{1}{\pi(w - z)}, \quad w \in \mathbf{Z} \setminus D;$$

and introduce the Cauchy transform of the domain  $D$ :

$$\chi(w) = \int_D \frac{dA}{\pi(w - z)},$$

then Eq. (4.6) can be written as ([4], see also [1])

$$\frac{\partial \chi(z, t)}{\partial t} - C \frac{\partial \chi(z, t)}{\partial x} = \sum_{i=1}^N \frac{q_i}{\pi(z - z_i)}. \tag{4.7}$$

This first order p.d.e. is readily solved explicitly. The solution has the form

$$\chi(z, t) = \chi_0(z + Ct) + \int_0^t \sum_{i=1}^N \frac{q_i d\tau}{\pi(z + C(t - \tau) - z_i)}. \tag{4.8}$$

For the dipole,  $\chi_0(z) = A/z$ , and the integrand becomes  $\mu/[\pi(z + C(t - \tau))^2]$ , and therefore

$$\chi(z, t) = \frac{\mu}{Cz} + \frac{AC - \mu}{C(z + Ct)}.$$

The first term is the Cauchy transform for a circle of area  $A_0 = \mu/C$  centered at the origin; the second term corresponds to the circle of the area  $A - A_0$  centered at  $z = -Ct$ . It means that at large  $t$  the solution is a combination of the stationary circle of area  $A_0$  at the origin corresponding to the limiting steady-state solution, and the circle of the area  $A - A_0$  “sinking” in the gravity field.

As general solution (4.8) is valid for any function  $h$ , it is tempting to state, that an arbitrary initial domain of sufficiently large area under the combined action of gravity and a dipole at the origin ( $z = 0$ ) eventually splits into two parts, namely, a stationary disk of area  $A_0 = \mu/C$  centered at  $z = 0$ , and a “sinking” domain with the Cauchy transform of the form

$$\chi_1 = \chi_0(z + tC) - \mu/[C(z + tC)]. \tag{4.9}$$

At  $t = 0$  the shape of the “sinking” domain is specified by the Cauchy transform

$$h_1(z, 0) = h_0(z) - \mu/(Cz).$$

This can be derived from the initial domain in the absence of gravity by placing a sink at the origin and sucking the amount of fluid corresponding to the equilibrium domain area  $A_0$ . If we now allow this new domain to slide far enough in the gravity field, and then inject back the same amount of fluid at the origin, we will get exactly the Cauchy transform specified by (4.9). Now we see, that this conjectured form of evolution will indeed occur, if the initial domain will evolve smoothly during the initial sucking of fluid. It will be certainly so, if the initial domain itself can be produced from a simply connected domain by injecting the amount of fluid  $A_0$  without violating the simply-connectedness condition.

This argument allows one to develop a number of explicit solutions for domains evolving in gravity field in the presence of a dipole at the origin.

Now we are going to consider some less trivial examples of equilibrium domains. It is worth noting that such domains can exist only for special combinations of hydrodynamic sources and the external potential. Indeed, if in the moment equation (3.6) we let  $U = F(z)$ , it results in

$$\int_D |\omega(z)|^2 dA = - \sum_{j=1}^N q_j F(z_j)$$



and since the l.h.s. of this equation is positive, the (complex) electric potential should satisfy the condition

$$-\sum_{j=1}^N q_j F(z_j) > 0. \tag{4.10}$$

*A priori* the sum in the rhs can be any complex number, but for the equilibrium shape to exist, the r.h.s. should be real and positive. Obviously, this inequality can hold only for special forms of the potential. For example, in the case of a doublet source-sink of equal strength both of them should lie on the same force line of the electric field ( $\text{Im}F(z_1) = \text{Im}F(z_2)$ ).

As we will see later, if some electric field sources (“charges”) are within the flow domain  $D$ , there is a continuous spectrum of the equilibrium domain areas. For example, in the simplest case of absence of hydrodynamic sources,  $q_j = 0$ , there is no flow within the equilibrium domain, the potential  $\Phi = \text{const}$  in  $D$ , and boundary condition (2.12) implies that  $G = \text{const}$  along  $\partial D$ , so the boundary should be a level curve of the electric potential. For example, in the case of a single electric source the equilibrium domains are circles of arbitrary radius centered at the source. Notice that, since the boundary of the equilibrium domain in the absence of hydrodynamic sources is just a level curve of the electric potential the analytic continuation procedure reduces to the reflection principle of electrostatics applied in the  $\zeta$ -plane.

**Example 1** Let us have two hydrodynamic sources of strengths  $q_1 = -q_2 = q$  at  $z_1 = a > 0$ ,  $z_2 = b$  respectively, and an electric “charge”  $Q$  at  $z = 0$ . Then

$$F(z) = \frac{Q}{2\pi} \ln z, \tag{4.11}$$

Inequality (4.10) implies that  $qQ \ln(b/a) > 0$ , so that  $b > a$  for positive  $Q$ .

Using the reduction to the “gravity” case technique, we have in the potential plane  $\tilde{z} = (Q/2\pi) \ln z$  the Cauchy transform for the transformed domain  $\tilde{D} = F(D)$ <sup>1</sup>

$$h_{\tilde{D}} = \frac{q}{\pi} \ln \left( \frac{\tilde{z} - \frac{Q}{2\pi} \ln a}{\tilde{z} - \frac{Q}{2\pi} \ln b} \right). \tag{4.12}$$

Then the conformal map  $\tilde{f}$  of the unit disk  $K$  on  $\tilde{D}$  is given by the expression [3]

$$\tilde{f}(\zeta) = \frac{q}{\pi} \ln \frac{1 + \alpha\zeta}{1 - \alpha\zeta} + \frac{Q}{2\pi} \ln \sqrt{ab}, \tag{4.13}$$

<sup>1</sup> Let in equation (4.1)

$$U(\tilde{z}) = \frac{1}{\pi} \ln(w - \tilde{z});$$

then

$$h_{\tilde{D}}(w) = \int_{\tilde{D}} \frac{d\tilde{A}}{\pi(w - \tilde{z})} = \frac{q}{\pi} [U(w - F(a)) - U(w - F(b))] = \frac{q}{\pi} \ln \left( \frac{w - \frac{Q}{2\pi} \ln a}{w - \frac{Q}{2\pi} \ln b} \right).$$

with  $\alpha$  determined from the equation

$$\frac{q}{\pi} \ln \frac{1 + \alpha^2}{1 - \alpha^2} = \frac{Q}{2\pi} \ln \sqrt{\frac{b}{a}}, \quad 0 < \alpha < 2. \tag{4.14}$$

Therefore,

$$\alpha = \sqrt{\frac{(b/a)^{\lambda/2} - 1}{(b/a)^{\lambda/2} + 1}}; \quad \lambda = \frac{Q}{2q},$$

and

$$f(\zeta) = e^{\frac{2\pi}{Q} \tilde{f}(\zeta)} = \sqrt{ab} \left( \frac{1 + \alpha\zeta}{1 - \alpha\zeta} \right)^{1/\lambda}. \tag{4.15}$$

The solution is a circle for  $\lambda = 1$ , i.e.  $Q = 2q$ . The solution remains physically sensible only for not too large  $b/a$ ; otherwise overlapping of different parts of the predicted domain occurs.

Consider the condition for non-overlapping of the mapping (4.15). The condition of overlapping is

$$f(\zeta) = f(\bar{\zeta}) = f(1/\zeta); \quad \zeta = e^{i\phi}$$

In our case it implies

$$\left( \frac{1 + \alpha\zeta}{1 - \alpha\zeta} \right)^{1/\lambda} = \left( \frac{1 + \alpha\zeta^{-1}}{1 - \alpha\zeta^{-1}} \right)^{1/\lambda}$$

In the plane

$$\zeta_1 = \frac{1 + \alpha\zeta}{1 - \alpha\zeta}$$

the boundary is a circle of radius  $r_1 = 2\alpha/(1 - \alpha^2)$  centered at  $(1 + \alpha^2)/(1 - \alpha^2)$ . As

$$z = C\zeta_1^{1/\lambda},$$

the overlapping occurs at the point of maximum  $\arg(\zeta_1)$ . But

$$\beta = \max(\arg(\zeta_1)) = \sin^{-1}(2\alpha/(1 + \alpha^2)).$$

So overlapping occurs at

$$\beta = \pi\lambda; \quad 2\alpha/(1 + \alpha^2) = \sin \pi\lambda$$

or

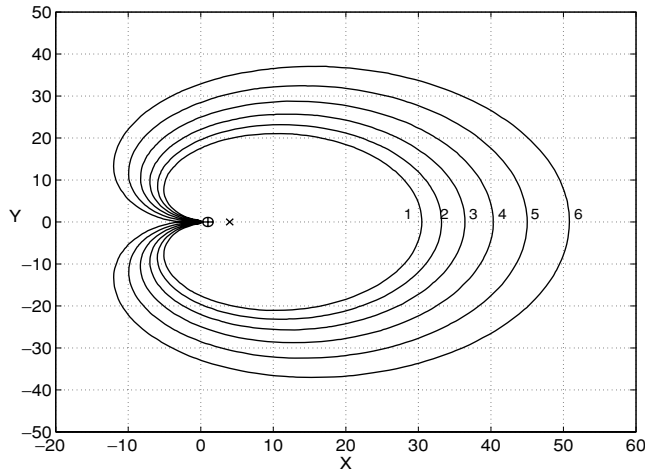
$$\alpha = \tan \frac{\pi Q}{4q}; \quad \frac{(b/a)^{\lambda/2} - 1}{(b/a)^{\lambda/2} + 1} = \tan^2 \frac{\pi Q}{4q}$$

$$(a/b) = \left( \cos \frac{\pi Q}{2q} \right)^{4q/Q}. \tag{4.16}$$

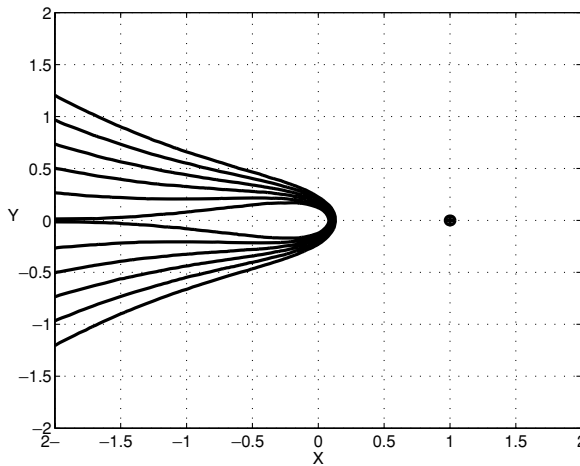
Therefore, non-overlapping equilibrium domain exists in the range of parameters

$$1 \geq a/b \geq (\cos \pi\lambda)^{(2/\lambda)}; \quad \lambda = Q/(2q).$$

For small  $\lambda$  non-overlapping equilibrium domains exist only in a narrow range of  $b/a$  close to unity. If we let the ratio  $b/a$  tend to the critical value  $(\cos \pi\lambda)^{-2/\lambda}$ , the area of



a.



b.

FIGURE 1. Equilibrium domains. Flow is driven by a source at  $z = a$ , and sink at  $z = b$ , an electric charge  $Q$  is located at  $z = 0$ ; plots correspond to  $q = 1$ ;  $a = 1$ ;  $b = 4$ ; and  $Q = 0.2734, 0.2959, 0.3189, 0.3424, 0.3664, 0.3909$ . for curves 1-6 respectively (a). Figure (b) shows blow-up of the upper figure illustrating that the electric charge is outside the equilibrium domain.

the equilibrium domain increases rapidly, and the domain acquires horseshoe shape. For  $\lambda = 1/2$  the equilibrium domain remains simply connected for any  $a/b$ . In the limiting case

$$f(\zeta) = \left( \frac{1 - \zeta}{1 + \zeta} \right)^2$$

the critical domain is the entire  $z$  plane with a cut along the negative real axis.

An example is presented in Fig. 1.

For curve 1 the ratio  $Q/2q$  is quite close to the critical value. Consider now a limiting case when the hydrodynamic source and sink collide, and form a dipole of the moment

$\mu$ . Formally, this corresponds to

$$b/a = e^\epsilon; \quad q = \frac{\mu}{\epsilon a}, \quad \epsilon \rightarrow 0.$$

Then (4.15) becomes

$$f_\epsilon(\zeta) = ae^{\epsilon/2} \left( \frac{1 + \alpha_\epsilon \zeta}{1 - \alpha_\epsilon \zeta} \right)^{1/2\epsilon}, \quad \lambda_\epsilon = \frac{Q}{2q} = \frac{\epsilon a Q}{2\mu}. \tag{4.17}$$

$$\alpha_\epsilon = \left( \frac{e^{\epsilon^2 a Q / 4\mu} - 1}{e^{\epsilon^2 a Q / 4\mu} + 1} \right)^{1/2} \approx \epsilon \sqrt{\frac{aQ}{8\mu}}. \tag{4.18}$$

So we find

$$f_0(\zeta) = \lim_{\epsilon \rightarrow 0} f_\epsilon(\zeta) = \lim_{\epsilon \rightarrow 0} a \left( 1 + \epsilon \sqrt{\frac{aQ}{2\mu}} \zeta \right)^{\frac{2\mu}{aQ}} = a \exp \left( \zeta \sqrt{\frac{2\mu}{aQ}} \right). \tag{4.19}$$

This mapping corresponds to a non-overlapping domain iff

$$\sqrt{2\mu/aQ} \leq \pi; \quad \mu/aQ \leq \pi^2/2. \tag{4.20}$$

At greater values of  $\mu/aQ$ , there is no simply connected equilibrium domain. It can be conjectured that in this case the electric field is too weak to prevent breakthrough caused by the hydrodynamic dipole.

Figure 2 shows shapes of the equilibrium domain for  $\mu = 1, a = 1; Q = 0.2026; 0.2410; 0.2866; 0.3408; 0.4053$ .

### 4.2 Singular points technique

**Example 2** Consider now the equilibrium domains corresponding to a hydrodynamic dipole of the moment  $\mu$  and an electric source of the strength  $Q$  both located at  $z = 0$ . In this case, the differential  $d[F(f(\zeta))]$  has singularities only at 0 and  $\infty$ . The general method described above implies that

$$d[F(f(\zeta))] = \left( \frac{P}{\zeta} + R \right) d\zeta, \tag{4.21}$$

$$F(f(\zeta)) = P \ln \zeta + R\zeta + C, \tag{4.22}$$

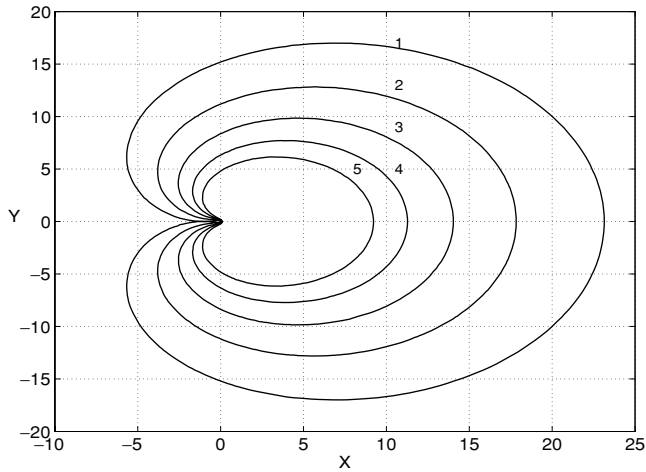
$$f(\zeta) = Ae^{2\pi F/Q} = A\zeta^{2\pi P/Q} e^{(2\pi R/Q)\zeta}. \tag{4.23}$$

Since  $f(\zeta)$  is a conformal map,  $f'(0) \neq 0, \infty$ . Therefore,

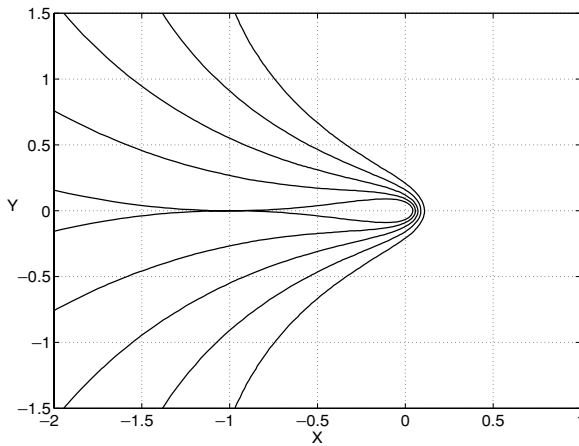
$$P = \frac{Q}{2\pi}; \quad f(\zeta) = A\zeta e^{B\zeta}. \tag{4.24}$$

In order to find  $B$ , we use the moment relation

$$\int_D \overline{\omega(z)} U'(z) dS = \mu U'(0). \tag{4.25}$$



a.



b.

FIGURE 2. Equilibrium domains. Flow is driven by a dipole of moment  $\mu$  at  $z = a$ , an electric charge  $Q$  is located at  $z = 0$ ; plots correspond to  $\mu = 1$ ;  $a = 1$ ; and  $Q = 0.2026; 0.2410; 0.2866; 0.3408; 0.4053$ . for curves 1–5 respectively (a). Figure (b) shows blow-up of the upper figure illustrating that the electric charge is outside the equilibrium domain.

In our case it assumes the form

$$\int_D \frac{Q}{2\pi\bar{z}} U'(z) dS = \mu U'(0). \tag{4.26}$$

Choosing  $U = z$ , we get

$$\int_D \frac{dS}{\bar{z}} = \frac{2\pi\mu}{Q}. \tag{4.27}$$

Substituting

$$z = A\zeta e^{B\zeta},$$

we get

$$\int_K \frac{|A + AB\zeta|^2 |e^{B\zeta}|^2}{\bar{\zeta} A e^{B\bar{\zeta}}} d\sigma = \int_K \frac{(A + AB\zeta)(A + AB\bar{\zeta}) e^{B\zeta}}{A\bar{\zeta}} d\sigma = \frac{2\pi\mu}{Q}. \tag{4.28}$$

Evaluating the integral on the l.h.s. of (4.28) we find

$$B = \frac{2\mu}{QA}; \quad f(\zeta) = A\zeta e^{\frac{2\mu\zeta}{QA}}. \tag{4.29}$$

The parameter  $A$ , still indeterminate, is a size parameter. It should satisfy the condition that  $f(\zeta)$  is single-valued. Therefore the critical values of the parameter correspond to

- (1)  $f'(\zeta) = 0, \quad \zeta \in \partial K$ , or
- (2)  $f(\zeta) = f(\bar{\zeta}) = f(\zeta^{-1}), \quad \zeta \in \partial K$ .

The first condition results in:

$$f'(\zeta) = A(1 + B\zeta)e^{B\zeta} = 0, \quad \zeta \in \partial K, \Rightarrow B = 1, \quad \text{or} \quad \frac{2\mu}{AQ} = 2. \tag{4.30}$$

The second condition implies

$$1 = f(\zeta)/f(\zeta^{-1}) = \zeta^2 e^{\frac{2\mu}{QA}(\zeta - \zeta^{-1})} = e^{2i\phi + 2i\frac{2\mu}{QA} \sin \phi}. \tag{4.31}$$

Then the critical condition becomes

$$L(\phi) = \phi + \frac{2\mu}{QA} \sin \phi = \pi k, \quad 0 < \phi < \pi. \tag{4.32}$$

As  $L(\pi) = \pi$ , a root in the segment  $(0, \pi)$  appears as the derivative

$$L'(\pi) = 1 + \frac{2\mu}{QA} \cos \pi = 1 - \frac{2\mu}{QA}$$

becomes negative, or at

$$\frac{2\mu}{AQ} = 1. \tag{4.33}$$

This condition coincides with (4.30). Hence a simply connected equilibrium domain exists for

$$\frac{2\mu}{AQ} \leq 1, \quad A \geq \frac{2\mu}{Q}. \tag{4.34}$$

Thus there exists a continuous spectrum of sizes of equilibrium domains that is bounded from below.

This result can be interpreted as inability of a given ‘‘charge’’ to prevent breakup due to action of hydrodynamic dipole, if the domain area is too small, or the domain boundary is too close to the dipole.

Figure 3 shows boundaries of the equilibrium domains described by the mapping of the unit disk given by (4.29) for  $Q = 1, \mu = 1, A = 1, \sqrt{2}, 2, 2\sqrt{2},$  and  $4$  for curves 1–5 respectively.

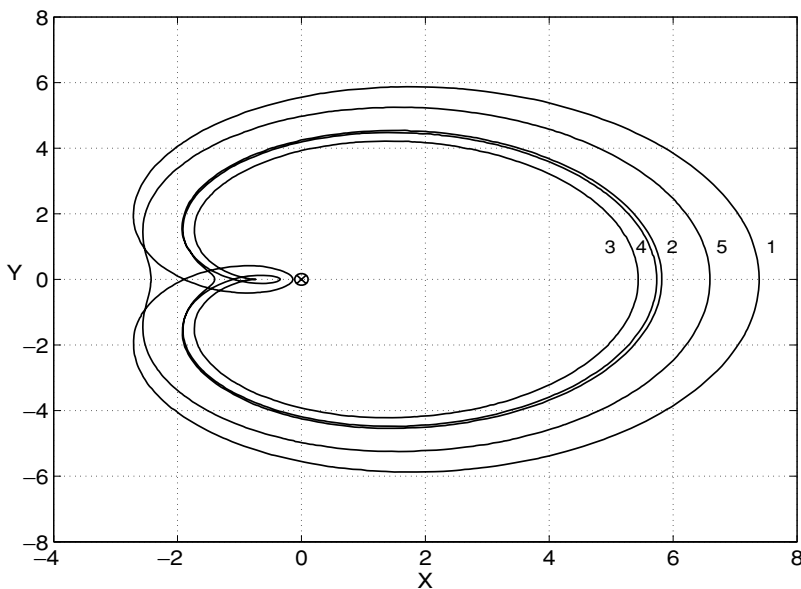


FIGURE 3. Equilibrium domains. Flow is driven by a dipole of moment  $\mu$  and an electric charge  $Q$  located at  $z = 0$ ; plots correspond to  $\mu = 1$ ;  $Q = 1$ ; and  $A = 1, \sqrt{2}, 2, 2\sqrt{2}$ , and 4 for curves 1–5 respectively. Self-intersecting boundaries 1 and 2 correspond to non-physical domains.

**Example 3.** Now we consider interaction of a hydrodynamic quadrupole at the origin with two “charges” of strength  $Q$  at  $z = \pm a$  outside the equilibrium domain  $D$ . We fix the conformal mapping of  $K$  on  $D$  by conditions  $f(0) = 0; f'(0) > 0$ . In this case  $F(f(\zeta))$  has a pole of second order at infinity, and hence

$$F(f(\zeta)) = -\alpha\zeta^2, \tag{4.35}$$

and

$$F(z) = \frac{Q}{2\pi} \ln \left( 1 - \frac{z^2}{a^2} \right). \tag{4.36}$$

Therefore

$$f(\zeta) = z = a\sqrt{1 - e^{-\frac{2\pi\alpha}{Q}\zeta^2}}, \quad \alpha > 0. \tag{4.37}$$

The parameter  $\alpha$  depends on the strength  $\beta$  of the hydrodynamic quadrupole, namely, the pole coefficient of  $F(f(\zeta))$  at  $\infty$  is the same as that of  $h(f(\zeta))$  at  $\zeta = 0$  by the Theorem of §3. As

$$h = \frac{\beta}{2\pi z^2}, \tag{4.38}$$

the requirement implies

$$\frac{\beta}{2\pi a^2 \frac{2\pi\alpha}{Q}} = \alpha; \quad \alpha^2 = \frac{\beta Q}{4\pi^2 a^2}.$$

Therefore the solution exists for  $\beta > 0$ , and

$$\alpha = \frac{\sqrt{\beta Q}}{2\pi a},$$

Thus,

$$f(\zeta) = a\sqrt{1 - \exp\left(-\sqrt{\frac{\beta}{a^2 Q}}\zeta^2\right)} = a\zeta\sqrt{\frac{1 - \exp\left(-\sqrt{\frac{\beta}{a^2 Q}}\zeta^2\right)}{\zeta^2}}. \tag{4.39}$$

This mapping remains univalent while

$$\exp\left(-\sqrt{\frac{\beta}{a^2 Q}}\zeta\right)$$

remains single-valued, so that

$$\frac{1}{a}\sqrt{\frac{\beta}{Q}} \leq \pi; \quad \frac{\beta}{a^2 Q} \leq \pi^2. \tag{4.40}$$

Figure 4 shows mapping of the unit disk given by (4.39) for  $\beta = 1, Q = 1, a = 0.2677, 0.3183 = 1/\pi, 0.3785, 0.4502$  (curves 1–4 respectively).

**5 Non-harmonic external field: reduction to a Riemann-Hilbert problem**

Up to now, we considered equilibrium fluid domains in harmonic external field, and based our analysis on the the general moment dynamics equation (2.14) which becomes (3.2) for equilibrium domains. In other words, the equilibrium is considered as a limiting case of dynamics. However, the requirement of harmonicity of the external field can sometime be lifted, if we consider only equilibrium domains in an external field. Some examples are presented in this section. The problem of finding stationary shapes of the flow domain in an external field can be approached in a different way that allows extension to non-harmonic external field potential of special form.

Let  $D$  be an equilibrium domain for a specified set of hydrodynamic sources and multipoles corresponding to logarithmic singularities and poles of the complex velocity potential  $W(z)$ ,  $\Phi = \text{Re}W$  (cf. (2.10)–(2.12)) in the field of the external force potential  $G(x, y)$ .

Then  $W(z)$  is an analytic (but may be multivalued) function in  $D$  having a prescribed set of singularities; its differential  $dW(z)$  is a meromorphic function in  $D$  and

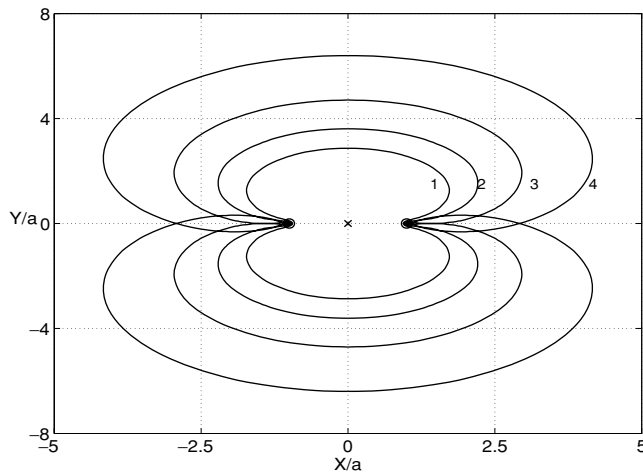
$$W(z)|_{z \in \partial D} = \widehat{G}(z, \bar{z}), \tag{5.1}$$

there,

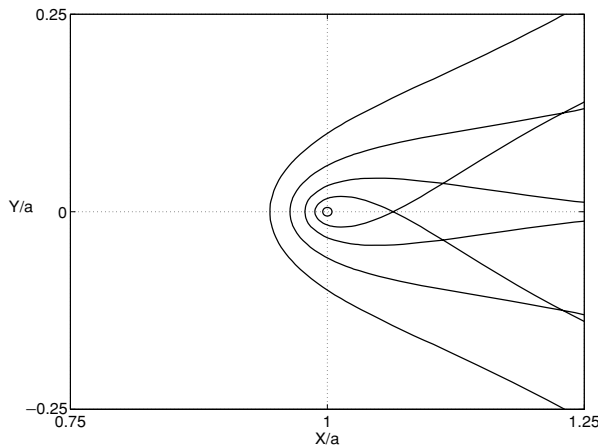
$$\widehat{G}(z, \bar{z}) \equiv G\left(\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})\right). \tag{5.2}$$

We show that in a number of cases due to the special form of the potential  $G(x, y)$  it proves to be possible to find the domain  $D$  explicitly. Consider once more the conformal





a.



b.

FIGURE 4. Equilibrium domains. Flow is driven by a quadrupole of moment  $\beta$  and two electric charges  $Q$  located at  $z = \pm a$ ; plots correspond to  $\beta = 1$ ;  $Q = 1$ ; and  $A = 0.2677$ ;  $0.3183 = 1/\pi$ ;  $0.3785$ ;  $0.4502$  for curves 1–4 respectively (a). Blowup of Figure 4,a (b). The charge remains outside the equilibrium domain; only non-intersecting boundaries correspond to physically admissible equilibrium domains.

map  $f : K \rightarrow D$  and define

$$\Theta(\zeta) = W(f(\zeta)), \quad \zeta \in K. \tag{5.3}$$

This function is analytic up to singularities of the specified type (poles and logarithmic singular points) in  $K$  and assumes real values along the boundary  $\partial K$ . Then it can be analytically continued into the entire complex plane  $\zeta$  using the symmetry principle:

$$\Theta(\zeta) = \overline{\Theta\left(\frac{1}{\bar{\zeta}}\right)}, \quad |\zeta| > 1. \tag{5.4}$$

Then

$$\Theta(\zeta) = \sum_{j=1}^N \frac{q_j}{2\pi} [\ln(\zeta - \zeta_j) + \overline{\ln(\zeta - \bar{\zeta}_j^{-1})}]. \tag{5.5}$$

Hence  $\Theta(\zeta)$  is known in the entire complex plane  $\zeta$  up to locations of the singularities  $\zeta_j$ . At the boundary of the unit disk

$$\Theta(\zeta) = \widehat{G}(z, \bar{z}). \tag{5.6}$$

For given  $\zeta_j$ , this equation for the conformal mapping  $f(\zeta)$  can be written as

$$\widehat{G}(f(\zeta), f^*(1/\zeta)) = \Theta(\zeta). \tag{5.7}$$

In general, it is not clear how to determine the conformal map  $f(\zeta)$  from this equation. However, it proves to be possible under some additional assumptions on  $G$ .

Some of these particular cases are presented below.

### 5.1 Harmonic external potential

Suppose that  $G$  is a harmonic function with maybe a finite set of logarithmic singular points within  $D$ , and let  $F(z)$  be respective complex potential. Then

$$G(z, \bar{z}) = \frac{1}{2}(F(z) + \overline{F(z)}) = \frac{1}{2}(F(z) + F^*(z)). \tag{5.8}$$

Introducing this expression into (5.7), we get

$$F^*(f^*(1/\zeta)) = 2\Theta(\zeta) - F(f(\zeta)). \tag{5.9}$$

Therefore  $F^*(f^*(1/\zeta))$  has finitely many singularities within  $K$ . Those inside  $K$  correspond to singularities of  $\Theta(\zeta)$  and  $F(f(\zeta))$  inside  $K$  [i.e both “hydrodynamic” and “electric” singularities], those outside  $K$  are explicitly determined by the singularities of  $\chi(\zeta) = F(f(\zeta))$ . Therefore the derivative

$$\frac{d}{d\zeta} F(f(\zeta))$$

is meromorphic in the entire  $\zeta$  plane, and hence it is rational with number and order of singular points known beforehand. This allows one to write down its explicit expression up to a number of indetermined coefficients. Then the conformal mapping is expressed as

$$f(\zeta) = F^{-1}(\chi(\zeta)), \tag{5.10}$$

and it remains to write and solve a set of equations for location and strength of singularities. It is the case considered previously.

### 5.2 Unidirectional external field

Suppose now that

$$G = H(x) = H\left(\frac{1}{2}(z + \bar{z})\right), \quad H'(x) > 0. \tag{5.11}$$

This means that the external “force” has only an  $x$ -component, which is independent of  $y$ . Then

$$\Theta(\zeta) = H\left(\frac{1}{2}(f(\zeta) + f^*(1/\zeta))\right), \tag{5.12}$$

$$f(\zeta) + f^*(1/\zeta) = 2H^{-1}(\Theta(\zeta)), \quad \zeta \in \partial K. \tag{5.13}$$

The functions  $f(\zeta)$  and  $f^*(1/\zeta)$  are analytic respectively inside and outside the unit circle. This Riemann-Hilbert problem is solved using the Cauchy-type integral [7, 5]:

$$f(\zeta) = \frac{1}{\pi i} \oint_{\partial K} \frac{H^{-1}(\Theta(u))}{u - \zeta} du - \frac{1}{2\pi i} \oint_{\partial K} \frac{H^{-1}(\Theta(u))}{u} du. \tag{5.14}$$

**Example 5.1** Let  $H(x) = x^2$ , and the flow be generated by a dipole at a location  $z = x_0 > 0$ . Then

$$W(z) \sim \frac{\mu}{z - x_0}, \quad z \rightarrow x_0. \tag{5.15}$$

We assume that  $x_0 = f(0)$ . Then  $\Theta(\zeta)$  should have poles at  $\zeta = 0$  and  $\zeta = \infty$ , and, therefore,

$$\Theta(\zeta) = \alpha \left( \zeta + \frac{1}{\zeta} \right) + \beta. \tag{5.16}$$

Introducing these expressions into (5.14), we get

$$f(\zeta) = \frac{1}{\pi i} \oint_{\partial K} \frac{\sqrt{\alpha(u + u^{-1}) + \beta}}{u - \zeta} du - \frac{1}{2\pi i} \oint_{\partial K} \frac{\sqrt{\alpha(u + u^{-1}) + \beta}}{u} du. \tag{5.17}$$

Then the dipole location is given by the expression

$$x_0 = \frac{1}{2\pi i} \oint_{\partial K} \frac{\sqrt{\alpha(u + u^{-1}) + \beta}}{u} du, \tag{5.18}$$

while for its strength  $\mu$  we find

$$\frac{\mu}{f'(0)} = \alpha; \quad f'(0) = \frac{\mu}{\alpha} = \frac{1}{\pi i} \oint_{\partial K} \frac{\sqrt{\alpha(u + u^{-1}) + \beta}}{u^2} du. \tag{5.19}$$

Equations (5.18) and (5.19) serve to find  $\alpha$  and  $\beta$  for given  $x_0$  and  $\mu$ . They can be reduced to the equations

$$x_0 = \frac{1}{\pi} \int_0^\pi \sqrt{2\alpha \cos \varphi + \beta} d\varphi; \quad \frac{\mu}{\alpha} = \frac{2}{\pi} \int_0^\pi \sqrt{2\alpha \cos \varphi + \beta} \cos \varphi d\varphi. \tag{5.20}$$

Relations (5.20) are shown in Figure 5,a. Using these relations, we can construct explicitly the equilibrium domains predicted by conformal mapping (5.18). Some results are presented in Figure 5,b.

### 5.3 Axially-symmetric external field

We assume now that the external potential has the form

$$G = H(x^2 + y^2) = H(z\bar{z}), \quad H'(r) > 0, r \neq 0 \text{ in } D, \quad r = (z\bar{z})^{1/2}. \tag{5.21}$$

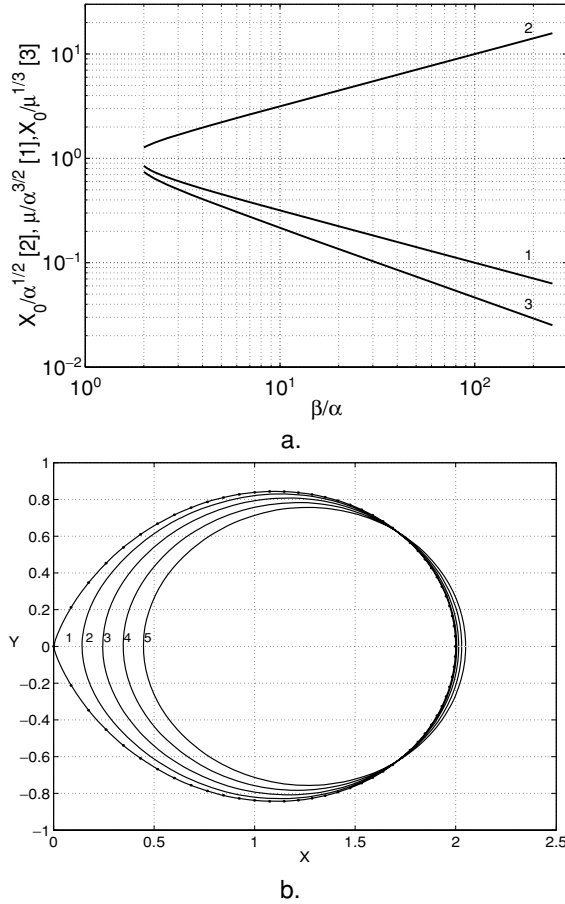


FIGURE 5. **a**: Relation between geometric parameters of the equilibrium domain and relative strength of the dipole; **b**: Shape of equilibrium domains for  $B = 2.00; 2.02; 2.06; 2.12; 2.20$  (curves 1–5, respectively).

that corresponds to a radially-symmetric external field with the symmetry axis outside the equilibrium domain  $D$ . Equation (5.4) implies

$$f(\zeta)f^*(1/\zeta) = H^{-1}(\Theta(\zeta)), \tag{5.22}$$

or

$$\ln f(\zeta) + \ln f^*(1/\zeta) = \ln H^{-1}(\Theta(\zeta)). \tag{5.23}$$

As by assumption  $f(\zeta) \neq 0$ ,  $\zeta \in \bar{D}$ , the logarithms in the l.h.s. of this equation are analytic functions respectively in the unit disk and outside it, and therefore we once more have a Riemann-Hilbert problem. Its solution is

$$f(\zeta) = \exp \left( \frac{1}{2\pi i} \oint_{\partial K} \frac{\ln H^{-1}(\Theta(u))}{u - \zeta} du - \frac{1}{4\pi i} \oint_{\partial K} \frac{\ln H^{-1}(\Theta(u))}{u} du \right). \tag{5.24}$$

These expressions allow us to restore the shape of the equilibrium domain provided the expression for  $\Theta(\zeta)$  can be guessed using properties of the hydrodynamic singularities.

**Example 5.2** Let  $H(x^2 + y^2) = r^2$ , and the flow be generated by a dipole at a location  $z = r_0$ . Then

$$W(z) \sim \frac{\mu}{z - r_0}, \quad z \rightarrow r_0. \tag{5.25}$$

We assume that  $r_0 = f(0)$ . Then repeating the argument of the previous subsection, we find the same expression (5.16) for  $\Theta(\zeta)$ , and keeping in mind that in our case  $H^{-1}(X) = X$ , we have, upon introducing this expression into (5.24),

$$\begin{aligned} f(\zeta) &= \exp\left(\frac{1}{2\pi i} \oint_{\partial K} \frac{\ln(\Theta(u))}{u - \zeta} du - \frac{1}{4\pi i} \oint_{\partial K} \frac{\ln(\Theta(u))}{u} du\right) \\ &= \frac{\exp\left(\frac{1}{2\pi i} \oint_{\partial K} \frac{(\ln[\alpha(u+1/u)+\beta])}{u-\zeta} du\right)}{\exp\left(\frac{1}{4\pi i} \oint_{\partial K} \frac{(\ln[\alpha(u+1/u)+\beta])}{u} du\right)}. \end{aligned} \tag{5.26}$$

Characteristic shapes of the equilibrium domains predicted by the mapping (5.26) are shown in Figure 6.

### 5.4 External field depending on a harmonic function

Let the external potential depend on a function harmonic up to specified logarithmic singularities,

$$\begin{aligned} G &= H(T(x, y)), \\ \Delta T &= \sum_{m=1}^M Q_m \delta(x - x'_m, y - y'_m). \end{aligned} \tag{5.27}$$

Then  $T(x, y)$  is the real part of an analytic function  $\Xi(z)$  having specified logarithmic singularities, and

$$G = H\left(\frac{1}{2}(\Xi(z) + \Xi^*(\bar{z}))\right). \tag{5.28}$$

Let function  $H^{-1}$  be rational and all “hydrodynamic singularities” correspond to multipoles (there are no logarithmic singularities corresponding to sources). Then from (5.28)

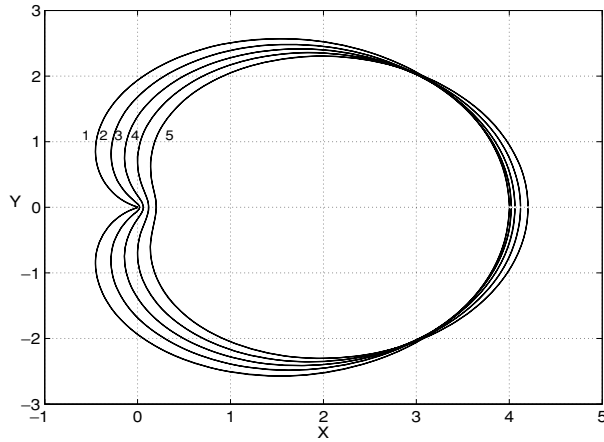
$$\Xi(f(\zeta)) + \Xi^*(f^*(1/\zeta)) = 2H^{-1}(\Theta(\zeta)), \quad \zeta \in \partial K, \tag{5.29}$$

or, denoting

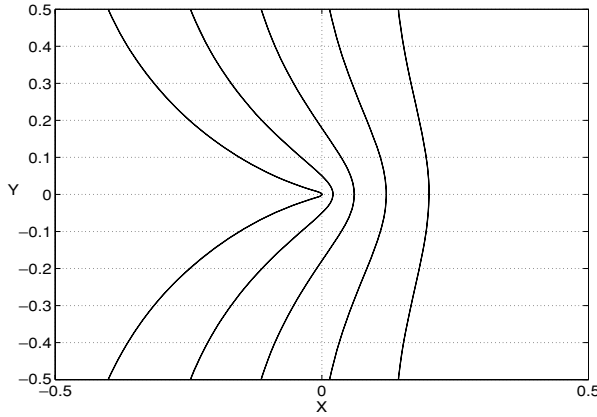
$$\begin{aligned} Z(\zeta) &= \Xi(f(\zeta)), \\ Z(\zeta) + Z^*(1/\zeta) &= H^{-1}(\Theta(\zeta)), \quad \zeta \in \partial K. \end{aligned} \tag{5.30}$$

This is essentially the same equation as (5.13), and it can be solved using the same technique. Then the conformal mapping is given by the expression

$$f(\zeta) = \Xi^{-1}(Z(\zeta)). \tag{5.31}$$



a.



b.

FIGURE 6. Equilibrium domains for flow driven by a dipole in quadratic axisymmetric external potential field for  $B = 2.0; 2.021; 2.061; 2.121; 2.201$  (curves 1–5, respectively), (a); blow-up of Figure 6,a, (b).

**Example 5.3** Let

$$T = x^2 - y^2; \quad H(T) = \sqrt{x^2 - y^2}; \quad \Xi(z) = \frac{1}{2}z^2; \tag{5.32}$$

and let the flow be generated by a single dipole of strength  $\mu$  at  $z = a > 0$ ,

$$W(z) \sim \frac{\mu}{z - a}, \quad z \rightarrow a. \tag{5.33}$$

Then  $\Theta(\zeta)$  is expressed by (5.16), and

$$Z(\zeta) = \frac{1}{\pi i} \oint_{\partial K} \frac{\sqrt{\alpha(u + u^{-1}) + \beta}}{u - \zeta} du - \frac{1}{2\pi i} \oint_{\partial K} \frac{\sqrt{\alpha(u + u^{-1}) + \beta}}{u} du, \tag{5.34}$$

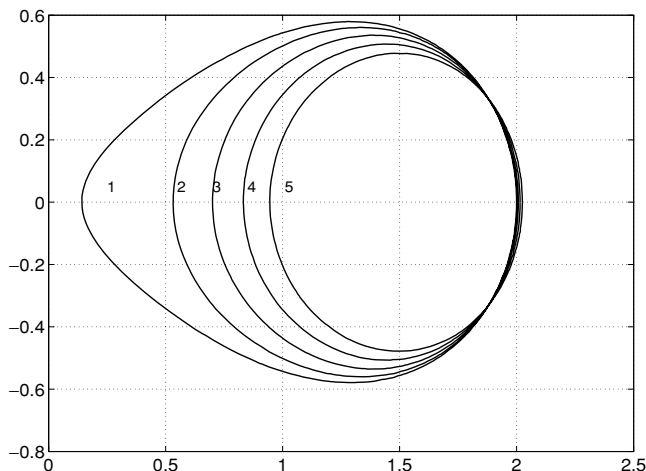


FIGURE 7. Equilibrium domains for flow driven by a dipole in external potential field of the form Eq. (5.32) for  $B = 2.0001; 2.0201; 2.0601; 2.1201; 2.2001$  (curves 1–5, respectively).

$$f(\zeta) = \sqrt{2Z}. \tag{5.35}$$

Then for  $\alpha$  and  $\beta$  we have the equations

$$a = f(0) = \sqrt{\frac{2\sqrt{\alpha}}{\pi} \int_0^\pi \sqrt{2 \cos \varphi + \beta/\alpha} d\varphi}, \tag{5.36}$$

$$\begin{aligned} \frac{\mu}{f'(0)} = \alpha; \quad f'(0) = \frac{\mu}{\alpha} = (2/Z(0))^{1/2} Z'(0) \\ = \frac{4\sqrt{\alpha}}{a\pi} \int_0^\pi \sqrt{2 \cos \varphi + \beta/\alpha} \cos \varphi d\varphi. \end{aligned} \tag{5.37}$$

Shapes of the equilibrium domains for flow driven by a dipole at  $z = 1$  in the external field corresponding to (5.32) are shown in Figure 7.

### 5.5 Non-planar Hele-Shaw cell

Consider a non-planar Hele-Shaw cell in constant gravity field. Let  $(x, y)$  be coordinates in the horizontal plane, and  $h(x, y)$  be the elevation of a cell point over the horizontal plane. Then assuming  $h = h(x)$  it is possible to introduce the conformal coordinate

$$z = s(x) + iy, \quad s(x) = \int_0^x \sqrt{1 + h'(t)^2} dt, \tag{5.38}$$

so that the problem reduces to that of planar Hele-Shaw cell with effective potential

$$H(z) = h(x) = h(s^{-1}(\text{Re}z)). \tag{5.39}$$

Similarly, if

$$h(x, y) = K(r); \quad r = \sqrt{x^2 + y^2}, \tag{5.40}$$

then the conformal coordinate is

$$z = e^{i\varphi} R(r); \quad R(r) = \exp \int_1^r \sqrt{1 + K'(\rho)^2} \frac{d\rho}{\rho}, \quad (5.41)$$

and the effective potential is

$$H(z, \bar{z}) = h(R^{-1}(|z|)). \quad (5.42)$$

## 6 Possible applications

In this section we show that the mathematical model considered above can be used to model electro-osmotically-driven flow in a thin gap between two infinite parallel walls provided that the gap is filled with two immiscible fluids having equal electric conductivities, the viscosity of the fluid in the exterior of the domain  $D(t)$  is negligible, and the electro-osmotic coefficients of the two fluids are different. Therefore, there exists at least one non-trivial physical situation corresponding to the mathematical problem considered in this paper.

### 6.1 Electrokinetic effect: physics, available data

The electrokinetic effect consists in the generation of electric current by fluid flow through porous media or thin gaps between solid walls, and in the reverse effect of inducing flow by application of electric field. It is the last case, usually referred as *electro-osmosis* that serves as the primary motivation of the theory presented here. Electrokinetic phenomena are caused by difference in mobility of ions, some of which are fixed at the surface of the solid skeleton (matrix) of the porous medium, or the solid walls, while dissolved counterions can move with the fluid within the gap or porespace, or force it to move, if an electric field is applied.

Macroscopically, the flow and electric current are governed by the equations

$$\mathbf{u} = -\frac{k}{\eta}(\nabla p - \xi \nabla \psi), \quad (6.1)$$

$$\mathbf{I} = -S(\nabla \psi - C \nabla p), \quad \xi = CS\eta/k. \quad (6.2)$$

Here,  $k$  is the medium permeability,  $\eta$  is the fluid viscosity,  $S$  is the fluid electric conductivity  $C$  is the electrokinetic coupling coefficient (see elsewhere for details [9, 2, 6]).

“Electro-osmotic flow”, the flow driven by electric potential differential, has important applications. It is used in soil remediation and prevention of moisture penetration in underground structures. Recently, electroosmotic flow is also actively studied as an element of microfluidic devices, when flow in narrow gaps or channels is driven by electric potential [13]. Presumably, it is this class of flows, to which the presented above theory can find some applications.

Namely, we consider flow driven both by pressure gradient and external electric field in a narrow plane gap between two solid non-conducting walls. We assume, that due to significant fluid conductivity the flow effect on the electric current is negligible. In this



case, (6.1) and (6.2) become

$$\mathbf{u} = -\frac{k}{\eta}(\nabla p - \zeta \nabla \psi), \tag{6.3}$$

$$\mathbf{I} = -S \nabla \psi. \tag{6.4}$$

here,  $\mathbf{u}(x, y)$  and  $\mathbf{I}(x, y)$  are the flow velocity and electric current averaged over the gap thickness; they satisfy the continuity equations (conservation laws)

$$\nabla \cdot \mathbf{u} = q_u(x, y); \quad \nabla \cdot \mathbf{I} = Q_I(x, y); \tag{6.5}$$

$k = h^2/12$ ; the pressure and the electric potential are functions of the in-plane coordinates  $(x, y)$ .

Now we assume that the gap is filled by two fluids, one of them, within time-dependent plane domain  $D(t)$ , is characterized by the viscosity  $\eta$ , conductivity  $S$  and electrokinetic coupling coefficients  $C$  and  $\zeta$ ; while exterior of  $D(t)$  is filled with another fluid of viscosity  $\eta_1$  and conductivity  $S_1$ , and electrokinetic coupling coefficients  $C_1$  and  $\zeta_1$ . Then at the boundary  $\Gamma(t) = \partial D(t)$  we have

$$p^+ = p^-; \quad \psi^+ = \psi^-; \quad u_n^+ = u_n^-; \quad I_n^+ = I_n^-. \tag{6.6}$$

For given densities of the volume ( $q_u$ ) and electric ( $Q_I$ ) sources (6.3)–(6.6) define a free boundary problem of coupled pressure/electro-osmotically driven flow in the gap.

Now we consider a particular case when both fluids have the same conductivity,  $S_1 = S$ . Then  $\psi(x, y)$  satisfies the equation

$$\Delta \psi(x, y) = \frac{Q_I}{S} \tag{6.7}$$

in the entire plane, so it can be considered as known. Let now the viscosity of the external fluid outside  $D(t)$  be negligible. Then, assuming that there is no net flux to infinity, we have

$$p^+(x, y) = \zeta_1 \psi(x, y, t) + const_1, \quad (x, y) \in \mathbf{Z} \setminus D(t). \tag{6.8}$$

Now we define the “effective pressure” as

$$P(x, y) = p(x, y) - \zeta_1 \psi(x, y) - const_1. \tag{6.9}$$

Then we have

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} = -\frac{k}{\eta}(\nabla P - (\zeta - \zeta_1)\nabla \psi); \quad \mathbf{x} \in D(t); \quad P(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial D(t) \tag{6.10}$$

It is, up to notations, the problem considered in this paper. The above examples show that external electric field can be used to confine the flow to a finite domain  $D$ .

### 7 Conclusion

In this paper, it is shown that the moment approach can be extended on dynamics of free boundaries in Hele-Shaw flows driven by a system of sources and sinks in an external

potential field. For several cases, when the external field is generated by a simple set of singularities, or belongs to a certain particular class, it proves to be possible to develop analytic techniques for finding equilibrium shapes of the fluid domains in the external field. Some simple examples are presented. Combined pressure-driven and electro-osmotic flows are discussed as a possible application of the presented theory.

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