J. Inst. Math. Jussieu (2017) **16**(5), 899–945
doi:10.1017/S1474748015000109

IMPROPER INTERSECTIONS OF KUDLA–RAPOPORT DIVISORS AND EISENSTEIN SERIES

SIDDARTH SANKARAN

Department of Mathematics, McGill University, Montreal, Canada
(siddarth.sankaran@mcgill.ca)

(Received 2 July 2014; accepted 13 March 2015; first published online 23 April 2015)

Abstract We consider a certain family of Kudla–Rapoport cycles on an integral model of a Shimura variety attached to a unitary group of signature (1, 1), and prove that the arithmetic degrees of these cycles are Fourier coefficients of the central derivative of an Eisenstein series of genus 2. The integral model in question parameterizes abelian surfaces equipped with a non-principal polarization and an action of an imaginary quadratic number ring, and in this setting the cycles are degenerate: they may contain components of positive dimension. This result can be viewed as confirmation, in the degenerate setting and for dimension 2, of conjectures of Kudla and Kudla–Rapoport that predict relations between the intersection numbers of special cycles and the Fourier coefficients of automorphic forms.

Keywords: Shimura varieties; arithmetic cycles; Eisenstein series; Kudla programme

2010 *Mathematics subject classification*: Primary 14G35
Secondary 11G18; 11F27

1. Introduction

In their article [8], Kudla and Rapoport investigate integral models of Shimura varieties attached to unitary groups of signature $(n-1, 1)$. These models are defined as moduli spaces of abelian varieties equipped with an action of the maximal order o_k in a fixed imaginary quadratic field, together with a compatible principal polarization. Kudla and Rapoport go on to construct a family of 'special' cycles, and prove that when such a cycle is zero dimensional and is supported in the fibre of an unramified prime, its degree can be identified with a Fourier coefficient of the derivative of an incoherent Eisenstein series for $U(n, n)$ at its centre of symmetry. This result is in line with a deep conjectural programme, initiated by Kudla and supported by his collaborators, that aims to establish systematic relations between arithmetic cycles on Shimura varieties and Fourier coefficients of automorphic forms; see the survey article [5].

In this paper, we study an extension of the problem of Kudla and Rapoport in the case when $n = 2$, where we allow the polarizations to be non-principal in a controlled way. Though the cycles in this setting might not be zero dimensional, our main result asserts that their degrees, suitably defined, are again identified with the Fourier coefficients of the central derivative of a (non-standard) Eisenstein series for $U(2,2)$.

We now give a more precise account of this result. Let k be an imaginary quadratic field, with ring of integers o_k, and fix an odd squarefree integer $d \in \mathbb{Z}_{>0}$, all of whose factors are inert primes in k. We define $\mathrm{M}^d_{(1,1)}$ to be the moduli stack of triples $\underline{A} = (A, i, \lambda)$, where A is an abelian surface equipped with an action

$$i : o_k \to \mathrm{End}(A)$$

that satisfies a *signature* $(1, 1)$ *condition*, see Definition 4.1 below, and λ is a polarization such that

(i) the corresponding Rosati involution induces Galois conjugation on the image $i(o_k)$; and

(ii) $\ker(\lambda) \subset A[d]$ is contained in the d-torsion of A with $|\ker(\lambda)| = d^2$.

This moduli problem is representable by a Deligne–Mumford (DM) stack that is flat over $\mathrm{Spec}(o_k)$.

Next, we let $\mathcal{E}$ be the DM stack parameterizing triples $\underline{E} = (E, i_E, \lambda_E)$ consisting of an elliptic curve E with an o_k-action $i_E : o_k \to \mathrm{End}(E)$ satisfying the *signature* $(1, 0)$ condition, and a principal polarization λ_E whose Rosati involution again induces Galois conjugation on the image $i_E(o_k)$.

Following [8], we define the *Kudla–Rapoport cycles* on the product $\mathcal{M} := \mathcal{E} \times_{o_k} \mathrm{M}^d_{(1,1)}$ as follows. Suppose that we are given points $\underline{E} \in \mathcal{E}(S)$ and $\underline{A} \in \mathrm{M}^d_{(1,1)}(S)$ valued in some connected base scheme S over $\mathrm{Spec}(o_k)$. Then the space

$$\mathrm{Hom}_{S,o_k}(E, A)$$

of o_k-linear morphisms admits a positive-definite o_k-Hermitian form defined by the formula

$$(x, y) := \lambda_E^{-1} \circ y^\vee \circ \lambda_A \circ x \in \mathrm{End}_{o_k}(E) \simeq o_k.$$

Given an integer $m \in \mathbb{Z}$, we define $\mathfrak{Z}(m)$ to be the moduli space of tuples

$$\mathfrak{Z}(m)(S) = \{(\underline{E}, \underline{A}, x) \mid (\underline{E}, \underline{A}) \in \mathcal{M}(S), x \in \mathrm{Hom}_{S,o_k}(E, A) \text{ with } (x, x) = m\}.$$

This moduli problem is representable by a DM stack, and the natural forgetful map $\mathfrak{Z}(m) \to \mathcal{M}$ is finite and unramified. We thereby obtain a cycle on $\mathcal{M}$, which, abusing notation, we denote by the same symbol $\mathfrak{Z}(m)$.

Similarly, for any matrix $T \in \mathrm{Herm}_2(o_k)$, we define $\mathfrak{Z}(T)$ to be the moduli space of tuples

$$\mathfrak{Z}(T)(S) = \{(\underline{E}, \underline{A}, \mathbf{x})\},$$

where $(\underline{E}, \underline{A}) \in \mathcal{M}(S)$ as before, and $\mathbf{x} = [x_1, x_2] \in \mathrm{Hom}_{S,o_k}(E, A)^2$ is a pair of maps such that

$$(\mathbf{x}, \mathbf{x}) := \begin{pmatrix} (x_1, x_1) & (x_1, x_2) \\ (x_2, x_1) & (x_2, x_2) \end{pmatrix} = T.$$

As before, the forgetful map $\mathfrak{Z}(T) \to \mathcal{M}$ defines a cycle, denoted by the same symbol.

Let $T = \begin{pmatrix} t_1 & * \\ * & t_2 \end{pmatrix}$. Then there is a decomposition

$$\mathfrak{Z}(t_1) \times_{\mathcal{M}} \mathfrak{Z}(t_2) = \coprod_{T' = \begin{pmatrix} t_1 & * \\ * & t_2 \end{pmatrix}} \mathfrak{Z}(T')$$

over all cycles $\mathfrak{Z}(T')$ corresponding to matrices T' with the same diagonal entries as T.

When T is non-singular, it turns out that the generic fibre $\mathfrak{Z}(T)_k$ is empty, and so the support of $\mathfrak{Z}(T)$ is concentrated in finitely many fibres of non-zero characteristic. In this case, we define

$$\widehat{\deg}\, \mathfrak{Z}(T) := \sum_{\mathfrak{p} \subset o_k} \chi(\mathfrak{Z}(T)_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{Z}(t_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathfrak{Z}(t_2)}) \log(N(\mathfrak{p}))$$

as the sum of the contributions to the Serre intersection multiplicity of $\mathfrak{Z}(t_1)$ and $\mathfrak{Z}(t_2)$ that appear within the support of $\mathfrak{Z}(T)$, weighted by the factors $\log N(\mathfrak{p})$.

Our main result relates this degree to the Tth Fourier coefficient of the derivative at the centre of symmetry $(s = 0)$ of an Eisenstein series $\mathcal{E}(z, s)$, constructed in § 4.3, on the Hermitian upper half-space $\mathfrak{H}_2$ of genus 2; here, the derivative

$$\mathcal{E}'(z, s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(z, s)$$

is taken with respect to the variable $s \in \mathbb{C}$.

Main Theorem. *Suppose that* $T \in \mathrm{Herm}_2(o_k)$ *is positive definite, and define*

$$\mathrm{Diff}(T) := \{\ell \;\, inert, \ell \nmid d, \mathrm{ord}_\ell \det T \;\; odd\} \bigcup \{\ell | d, \mathrm{ord}_\ell \det T \;\; even\}.$$

If $|\mathrm{Diff}(T)| \geqslant 1$ *and* $\mathrm{Diff}(T) \neq \{2\}$*, then*

$$\widehat{\deg}\, \mathfrak{Z}(T) q^T = \frac{2h(k)}{|o_k^\times|} \mathcal{E}'_T(z, 0),$$

where $h(k)$ *is the class number of* k*, and, for* $z \in \mathfrak{H}_2$*, we set* $q^T := e(tr(Tz))$*.*

The novel aspects of this theorem emerge when $\mathrm{Diff}(T) = \{p\}$ is a single inert prime p dividing d. In this case, the cycle $\mathfrak{Z}(T)$ is supported in the fibre $\mathcal{M}_p$. This fibre in turn bears a close relationship to the *Drinfeld upper half-plane* $\mathcal{D}$ which, as we recall in § 2, admits an interpretation as a moduli space of p-divisible groups.

Our first task, carried out in § 2, is therefore to consider a family of *local Kudla–Rapoport* divisors on $\mathcal{D}$ defined in terms of deformations of p-divisible groups, and study their intersection behaviour. Explicit equations for these divisors were found in [14]; by combining that information with a combinatorial description of the reduced locus $\mathcal{D}_{red}$ in terms of the Bruhat–Tits building for $SL_2(\mathbb{Q}_p)$, we arrive at an explicit, and surprisingly simple, formula for the intersection number of two local Kudla–Rapoport divisors; see Corollary 2.17.

In § 3, we show that one can express the same formula in terms of *local representation densities* and their derivatives, which play an essential role in the determination of the Fourier coefficients of Eisenstein series. The key tool is the development of a closed-form

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

expression, in the particular case that we need, of Hironaka's general formula [2] for Hermitian representation densities in the unramified setting; see Proposition 3.1.

Finally, we connect the local calculations with the global setting and prove the main theorem; our approach here follows [8, §§ 7–10] quite closely.

The first half of § 4 concerns structural results regarding the geometry of $\mathcal{M}$ and the special cycles $\mathfrak{Z}(T)$; in particular, a *p-adic uniformization* result allows us to express $\widehat{\deg}\,\mathfrak{Z}(T)$ as a product of a local factor, corresponding to a local intersection number as calculated in § 2, and a global factor that is essentially a lattice point count.

We then turn to calculating the right-hand side of our main theorem; after recalling some general facts about Siegel–Weil Eisenstein series and their Fourier coefficients, we describe the particular choice of parameters that give rise to the Eisenstein series that figures in our main theorem, and compute the Tth Fourier coefficient of its derivative in Theorem 4.13. The formula we derive also decomposes as the product of a local factor, expressed in terms of representation densities, with a global lattice point count. A direct comparison of the two formulae yields the proof of the main theorem; see Corollary 4.15.

Notation. Throughout this paper, k will be a fixed imaginary quadratic field, with ring of integers o_k and discriminant $\Delta < 0$. We denote the non-trivial Galois operator on k by $a \mapsto a'$.

Let $\widehat{\mathbb{Z}} := \prod_\ell \mathbb{Z}_\ell$, and, for any prime p, we put $\widehat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$. If M is a $\mathbb{Z}$-module, we set

$$\widehat{M} := M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \quad \text{and} \quad \widehat{M}^p := M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p.$$

2. Local Kudla–Rapoport cycles on the Drinfeld upper half-plane

In this section, we fix an inert prime $p \neq 2$. Let k_p denote the completion of k at p, and $o_{k,p} \subset k_p$ the ring of integers. Fix an algebraic closure $\mathbb{F} = \overline{\mathbb{F}_p}$ and an embedding

$$\tau_0 \colon o_{k,p}/(p) \hookrightarrow \mathbb{F}.$$

Denote the non-trivial Galois operator on k_p by $a \mapsto a'$, and let $\tau_1(a) := \tau_0(a')$ be the conjugate embedding; if $W = W(\mathbb{F})$ is the ring of Witt vectors, then τ_0 and τ_1 lift uniquely to embeddings

$$\tau_i : o_{k,p} \to W.$$

Finally, we let **Nilp** denote the category of W-schemes such that p is locally nilpotent, and, for $S \in \mathbf{Nilp}$, we set $\overline{S} := S \times_W \mathbb{F}$.

We begin by recalling the construction of the Drinfeld upper half-plane as a moduli space for p-divisible groups, following [9].

Definition 2.1. Let $S \in \mathbf{Nilp}$. An *almost-principally polarized p-divisible group* over S is a triple (X, i, λ) consisting of

(i) a p-divisible group X over S of height 4 and dimension 2;

(ii) an action $i : o_{k,p} \to \mathrm{End}(X)$ satisfying the following *signature (1, 1) condition*: for every $a \in o_{k,p}$, the characteristic polynomial of $i(a)$ on the Lie algebra $\mathsf{Lie}(X)$ is

$$\det(T - i(a)|_{\mathsf{Lie}(X)}) = (T - a)(T - a') \in \mathcal{O}_S[T]; \textit{ and}$$

(iii) a polarization λ such that

$$p \cdot \ker(\lambda) = 0, \quad |\ker(\lambda)| = p^2,$$

and such that the induced Rosati involution $*$ satisfies

$$i(a)^* = i(a') \quad \text{for all } a \in o_{k,p}.$$

The following lemma asserts that there is a single isogeny class of such tuples over $\mathbb{F}$.

Lemma 2.2. *Suppose that* $(\mathbb{X}, i_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *is a triple over* $\mathbb{F} = \overline{\mathbb{F}_p}$ *as above. Then* $\mathbb{X}$ *is supersingular (i.e., its Dieudonné module is isoclinic of slope* $1/2$*). Moreover, the data* $(\mathbb{X}, i_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *is unique up to isogeny; that is, given another triple* $(\mathbb{X}', i_{\mathbb{X}'}, \lambda_{\mathbb{X}'})$*, there exists an* $o_{k,p}$*-linear isogeny* $\mathbb{X} \to \mathbb{X}'$ *such that the pullback of* $\lambda_{\mathbb{X}'}$ *is* $\lambda_{\mathbb{X}}$*.*

Proof. As $\mathbb{X}$ has height 4, dimension 2, and is polarizable, its rational Dieudonné module $M_{\mathbb{Q}} := M(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to one of the following three possibilities:

$$M_{\mathbb{Q}} \simeq \begin{cases} \mathbb{D}_{1/2} \oplus \mathbb{D}_{1/2} \\ (\mathbb{D}_0 \oplus \mathbb{D}_0) \oplus (\mathbb{D}_1 \oplus \mathbb{D}_1) \\ \mathbb{D}_0 \oplus \mathbb{D}_{1/2} \oplus \mathbb{D}_1. \end{cases}$$

Here, $\mathbb{D}_\mu$ is the simple isoclinic rational Dieudonné module of slope μ. Since there are no non-zero maps between isoclinic Dieudonné modules of different slopes, the action of $o_{k,p}$ decomposes into an action on each isoclinic component. In particular, the third case is impossible: there is no action of $o_{k,p}$ on either $\mathbb{D}_0$ or $\mathbb{D}_1$, as $\mathrm{End}(\mathbb{D}_0) = \mathrm{End}(\mathbb{D}_1) = \mathbb{Q}_p$.

We exclude the second possibility by contradiction. Suppose that $M_{\mathbb{Q}} = (\mathbb{D}_0 \oplus \mathbb{D}_0) \oplus (\mathbb{D}_1 \oplus \mathbb{D}_1)$. Upon identifying $\mathbb{D}_0^\vee \simeq \mathbb{D}_1$, the polarization $\lambda = (\lambda_0, \lambda_1)$ decomposes as a pair of endomorphisms, where

$$\lambda_0 \colon (\mathbb{D}_0)^2 \to (\mathbb{D}_1^\vee)^2 \simeq (\mathbb{D}_0)^2 \quad \text{and} \quad \lambda_1 \colon (\mathbb{D}_1)^2 \to (\mathbb{D}_0^\vee)^2 \simeq (\mathbb{D}_1)^2,$$

and such that $\lambda_0^\vee = \lambda_1$. By the assumptions on the kernel of λ, we must have

$$\mathrm{ord}_p \det \lambda_1 = \mathrm{ord}_p \det \lambda_2 = 1.$$

On the other hand, both maps anti-commute with the corresponding $o_{k,p}$ action; since $\mathrm{End}((\mathbb{D}_0)^2) \simeq \mathrm{End}((\mathbb{D}_1)^2) \simeq M_2(\mathbb{Q}_p)$, this implies that the determinants of the maps λ_i lie in $N(k_p^\times)$, and in particular have even valuations. This contradiction establishes the fact that $M(\mathbb{X})_{\mathbb{Q}} \simeq (\mathbb{D}_{1/2})^2$, i.e., that $\mathbb{X}$ is supersingular.

For the second part of the lemma, note that the slopes of the operator pV^{-2} on $M(\mathbb{X})$ are 0, and so

$$M(\mathbb{X}) = \Lambda \otimes_{\mathbb{Z}_{p^2}} W, \quad \text{where } \Lambda := M(\mathbb{X})^{pV^{-2}}.$$

The embedding $\tau_0 \colon o_{k,p} \to W$ makes Λ into an $o_{k,p}$-module of rank 4. We define a pairing $h(\cdot, \cdot)$ on $M(\mathbb{X})$ by the formula

$$h(x, y) := \tau_0(\delta)^{-1} \langle x, Fy \rangle_\lambda,$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

where $\langle\cdot,\cdot\rangle_\lambda$ is the alternating form on $M(\mathbb{X})$ corresponding to λ, and $\delta \in o_{k,p}^\times$ is any element such that $\delta' = -\delta$. The form h restricts to an $o_{k,p}$-Hermitian form on Λ, which we also denote by $h(\cdot,\cdot)$. Furthermore, the action $i_{\mathbb{X}}\colon o_{k,p} \to \mathrm{End}(\mathbb{X})$ induces an orthogonal splitting $\Lambda = \Lambda_0 \oplus \Lambda_1$ into rank-2 Hermitian $o_{k,p}$ modules, where

$$\Lambda_i := \{x \in \Lambda \mid i_{\mathbb{X}}(a)\cdot x = \tau_i(a)x \text{ for all } a \in o_{k,p}\}.$$

Note that one can recover the pair $\{i_{\mathbb{X}}, \lambda_{\mathbb{X}}\}$ from the data of the Hermitian form h on Λ together with the splitting $\Lambda = \Lambda_0 \oplus \Lambda_1$. In particular, the isogeny class of $(\mathbb{X}, i_{\mathbb{X}}, \lambda_{\mathbb{X}})$ depends only on the isometry classes of

$$C_0 := \Lambda_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \quad \text{and} \quad C_1 := \Lambda_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

as k_p-Hermitian spaces of dimension 2. As V restricts to an $o_{k,p}$-antilinear isomorphism $V\colon C_1 \xrightarrow{\sim} C_0$ of vector spaces with

$$h(Vx, Vy) = ph(x,y)^\sigma,$$

and the isometry class of a local vector space $\mathfrak{V}$ is determined by its local invariant $\det(\mathfrak{V}) \in \mathbb{Q}_p^\times/N(k_p^\times)$, we find that $C_1 \simeq C_0$. To conclude the proof, we now show C_0 is always split.

Let $M^\vee = M(\mathbb{X})^\vee$ denote the W−linear dual of $M = M(\mathbb{X})$ with respect to the polarization λ, and set $M_i = \Lambda_i \otimes_{o_{k,p}} W$. Then, by the assumptions on λ and the signature condition,

$$M_0 \subsetneq M_1^\vee \subsetneq p^{-1}M_0 \quad \text{and} \quad M_0 \subsetneq V^{-1}M_1 \subsetneq p^{-1}M_0.$$

On the other hand, an easy calculation gives

$$V(\Lambda_1^\sharp) \otimes_{o_{k,p}} W = M_1^\vee,$$

where $\Lambda_1^\sharp = \{x \in \Lambda_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \mid h(x,\Lambda) \subset o_{k,p}\}$. Thus

$$\Lambda_0 \subsetneq L \subset p^{-1}\Lambda_0, \quad \text{where } L := V(\Lambda_1^\sharp) + V^{-1}(\Lambda_1).$$

Since $\dim_{\mathbb{F}_{p^2}}(p^{-1}\Lambda_0/\Lambda_0) = 2$, either (i) $L = V(\Lambda_1^\sharp) = V^{-1}(\Lambda_1)$; or (ii) $L = p^{-1}\Lambda_0$. One easily checks that the lattice L is self-dual in the first case, or satisfies $L^\sharp = pL$ in the second. Since p is inert, the existence of a lattice $L \subset C_0$ satisfying one of these two properties implies that C_0 is split. □

We fix a triple $(\mathbb{X}, i_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\mathbb{F}$ once and for all, which will serve as a base point for the following moduli problem.

Definition 2.3. Let $\mathcal{D}$ denote the following moduli problem over **Nilp**: for a base scheme $S \in$ **Nilp**, the points $\mathcal{D}(S)$ parameterize isomorphism classes of tuples

$$\mathcal{D}(S) := \{\underline{X} = (X, i_X, \lambda_X, \rho_X)\}_{/\simeq},$$

where (X, i_X, λ_X) is an almost-principally polarized p-divisible group over S, and

$$\rho_X\colon X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}} \overline{S}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

is a height-0 quasi-isogeny of p-divisible groups over $\overline{S} := S \times_W \mathbb{F}$ that is equivariant with respect to the action of o_k, and such that

$$\rho_X^*(\lambda_{\mathbb{X},\overline{S}}) = \lambda_{X,\overline{S}}. \tag{2.1}$$

Two such tuples $\underline{X} = (X, i_X, \lambda_X, \rho_X)$ and $\underline{X}' = (X', i_{X'}, \lambda_{X'}, \rho_{X'})$ are isomorphic if there is an $o_{k,p}$-linear isomorphism $\varphi\colon X \to X'$ such that $\rho_X = \rho_{X'} \circ (\varphi_{\overline{S}})$ and $\varphi^*\lambda_{X'} = \lambda_X$.

This moduli problem is representable by (a formal model of) the Drinfeld upper half-plane; see [9]. In particular, it is a regular formal scheme over $\mathrm{Spf}(W)$.

Remark 2.4. The moduli problem considered in [9] is *a priori* more general. There, the authors consider the moduli space $\mathcal{N}_k$ whose points valued in a connected base scheme $S \in$ **Nilp** parameterize isomorphism classes of tuples $\underline{X} = (X/S, i_X, \lambda_X, \rho_X)$ as above, except they impose the condition that

$$\rho_X^*\lambda_{\mathbb{X},\overline{S}} = c\lambda_{X,\overline{S}} \quad \text{for some } c \in \mathbb{Z}_p^\times,$$

instead of the equality imposed in (2.1). Two such tuples $\underline{X}$ and $\underline{X}'$ are deemed isomorphic in $\mathcal{N}_k(S)$ if there is an $o_{k,p}$ linear isomorphism of p-divisible groups $\phi\colon X \to X'$ such that $\rho_X = \rho_{X'} \circ (\phi_S)$ and $\phi^*\lambda_{X'} \in \mathbb{Z}_p^\times\lambda_X$.

As $o_{k,p}$ is an unramified extension of $\mathbb{Z}_p$, the norm map $N\colon o_{k,p}^\times \to \mathbb{Z}_p^\times$ is surjective, and so every isomorphism class contains a representative for which we have $c = 1$. To be precise, given a point $(X, i, \lambda, \rho) \in \mathcal{N}_k(S)$ as above with $c^{-1} = N(u)$, the map $\phi = i(u^{-1})$ gives rise to an isomorphic tuple $\underline{X}' = (X, i, \lambda, \rho')$, where $\rho' = \rho \circ i(u)$ satisfies $(\rho')^*\lambda_{\mathbb{X},\overline{S}} = \lambda_{X,S}$.

Thus, in the unramified setting, the moduli problem $\mathcal{D}$ coincides with $\mathcal{N}_k$, and so the representability follows from [9, Theorem 1.2]. ⋄

Let $\mathbb{Y}$ be supersingular p-divisible group $\mathbb{Y}$ over $\mathbb{F}$ of dimension 1 and height 2 (i.e., the p-divisible group of a supersingular elliptic curve). We also fix an action $i_{\mathbb{Y}}\colon o_{k,p} \to \mathrm{End}(\mathbb{Y})$, and a principal polarization $\lambda_{\mathbb{Y}}$ such that the induced Rosati involution acts by Galois conjugation on the image $i_{\mathbb{Y}}(o_{k,p})$.

Following [7], we define the space of *special local homomorphisms*:

$$\mathbb{V} := \mathrm{Hom}_{o_{k,p}}(\mathbb{Y}, \mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{2.2}$$

This space comes equipped with a natural Hermitian form: for $b_1, b_2 \in \mathbb{V}$, put

$$(b_1, b_2) := \lambda_{\mathbb{Y}}^{-1} \circ b_2^\vee \circ \lambda_{\mathbb{X}} \circ b_1 \in \mathrm{End}_{o_{k,p}}(\mathbb{Y}) \otimes \mathbb{Q}_p \simeq k_p.$$

It turns out that with this Hermitian form, $\mathbb{V}$ is split; see [14, Remark 3.4].

Definition 2.5.

(i) Suppose that $\Lambda \subset \mathbb{V}$ is an $o_{k,p}$-lattice, and let $\Lambda^\sharp$ denote the dual lattice. We say that Λ is a 'vertex lattice' of type 0 (respectively, type 2) if $\Lambda^\sharp = \Lambda$ (respectively, $\Lambda^\sharp = p\Lambda$). In what follows, we shall use the term 'lattice' to mean a vertex lattice of type 0 or 2.

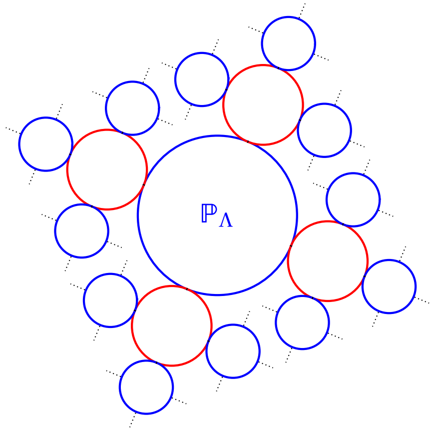

Figure 1. A portion of $\mathcal{D}_{red}$ for $p = 3$ as a union of projective lines indexed by vertex lattices.

(ii) Let $\mathscr{B}$ denote the Bruhat–Tits tree for $SU(\mathbb{V})$, which is a graph with the following description. The vertices are the vertex lattices, and edges only occur between lattices of differing type. Two lattices Λ and Λ' of type 0 and 2 respectively are connected by an edge if and only if

$$p\Lambda' \subset \Lambda \subset \Lambda',$$

where the successive quotients are $\mathbb{F}_{p^2}$-vector spaces of dimension 1. In particular, this graph is a $(p+1)$-regular tree.

The reduced locus $\mathcal{D}_{red}$ can be described by the Bruhat–Tits tree $\mathscr{B}$ as follows. Each irreducible component of $\mathcal{D}_{red}$ is a projective line $\mathbb{P}_\Lambda$ over $\mathbb{F}$ indexed by a vertex lattice. Two such lines $\mathbb{P}_\Lambda$ and $\mathbb{P}_{\Lambda'}$ intersect at at most one point, which we call a 'superspecial' point, and this happens if and only if Λ and Λ' are neighbours in $\mathscr{B}$. On a given component $\mathbb{P}_\Lambda$, the superspecial points are precisely the $\mathbb{F}_p$-rational points, of which there are $p+1$; see Figure 1.

Definition 2.6. Let $\mathcal{D}_0$ be the moduli space on **Nilp** that, for a scheme $S \in$ **Nilp**, parameterizes isomorphism classes of tuples

$$\mathcal{D}_0(S) := \{\underline{Y} = (Y, i_Y, \lambda_Y, \rho_Y)\}/_\simeq;$$

here, Y is a p-divisible group over S of dimension 1 equipped with an action $i_Y : o_{k,p} \to \mathrm{End}(Y)$, and λ_Y is a compatible principal polarization. Finally,

$$\rho_Y : Y \times_S \overline{S} \to \mathbb{Y} \times_{\mathbb{F}} \overline{S}$$

is an $o_{k,p}$-linear quasi-isogeny of height 0.

Note that the moduli functor $\mathcal{D}_0$ is trivial; i.e., it is represented by $\mathrm{Spf}(W)$. Indeed, by using Gross' theory of (quasi-)canonical liftings for example, see [1], one may show that, for any $S \in$ **Nilp**, there is a *unique* lift $\underline{\mathbb{Y}}_S$ of $\underline{\mathbb{Y}}$ that is determined by the action $i_{\mathbb{Y}}$.

We turn now to the local *Kudla–Rapoport* cycles, which are parameterized by elements in $\mathbb{V}$.

Definition 2.7. Let $b \in \mathbb{V}$. We define the *local Kudla–Rapoport cycle* Z(b) as the closed formal subscheme of $\mathcal{D}_0 \times_W \mathcal{D}$ defined by the following moduli problem: for $S \in \mathbf{Nilp}$, the set of S-points $Z(b)(S)$ is the locus of pairs $(\underline{Y}, \underline{X}) \in (\mathcal{D}_0 \times_W \mathcal{D})(S)$ such that the quasi-morphism

$$\rho_X^{-1} \circ b \circ \rho_Y : Y \times_S \overline{S} \to X \times_S \overline{S}$$

lifts to a morphism $Y \to X$ over S.

Similarly, if $\mathbf{b} = [b_1, b_2] \in \mathbb{V}^2$, then we define the cycle $Z(\mathbf{b})$ to be the locus $(\underline{Y}, \underline{X})$ where $\rho_X^{-1} \circ b_i \circ \rho_Y$ lifts for $i = 1, 2$.

These cycles were studied in detail in [14], where it was shown that the cycles $Z(b)$ corresponding to a single element $b \in \mathbb{V}$ are in fact divisors; i.e., they are locally cut out by a single non-zero equation. There is an explicit decomposition of these divisors into irreducible components as described below; note that, since $\mathcal{D}_0 \simeq \operatorname{Spf} W$, we shall henceforth implicitly identify the formal schemes

$$\mathcal{D}_0 \times_{\operatorname{Spf}(W)} \mathcal{D} \simeq \mathcal{D}$$

over $\operatorname{Spf}(W)$.

Definition 2.8. Let $b \in \mathbb{V}$ with $\operatorname{ord}_p(b, b) = m \geqslant 0$. Set $t = \lfloor \frac{m+1}{2} \rfloor$ and $\beta := p^{-t} b$, so $\operatorname{ord}_p(\beta, \beta)$ is either 0 or -1. Then by [14, Lemma 3.8], there exists a unique vertex lattice Λ such that $\beta \in \Lambda \setminus p\Lambda$, and moreover Λ is of type 0 (respectively, type 2) if m is even (respectively, odd). We call this lattice the *central lattice* for b.

Theorem 2.9 [14, Theorem 3.14]. *Let $b \in \mathbb{V}$, such that $m := \operatorname{ord}_p(b, b) \geqslant 0$. Then, as a cycle on $\mathcal{D}$,*

$$Z(b) = Z(b)^h + \sum_{\Lambda \in \mathcal{B}(b)} m(b, \Lambda) \mathbb{P}_\Lambda =: Z(b)^h + Z(b)^v,$$

where

(i) *$Z(b)^h$ is a horizontal divisor isomorphic to* $\operatorname{Spf}(W)$ *that meets the special fibre of $\mathcal{D}$ at a single non-superspecial point on the component $\mathbb{P}_{\Lambda_0}$ corresponding to the central lattice Λ_0;*

(ii) *$\mathcal{B}(b) := \{\Lambda$ vertex lattice $\mid b \in \Lambda\}$; and*

(iii) *the multiplicities $m(b, \Lambda)$ are given by the formula*

$$m(b, \Lambda) = \begin{cases} t - \lfloor d(\Lambda, \Lambda_0)/2 \rfloor, & \text{if } m = 2t \text{ is even} \\ t - \lfloor (d(\Lambda, \Lambda_0) + 1)/2 \rfloor, & \text{if } m = 2t - 1 \text{ is odd;} \end{cases}$$

here, $d(\Lambda, \Lambda_0)$ is the distance between the two vertex lattices in the Bruhat–Tits tree. □

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Remark 2.10.

(i) For notational consistency, if $b \in \mathbb{V}$ with $\operatorname{ord}_p(b, b) < 0$, we define $Z(b) = 0$ and $\mathcal{B}(b) = \emptyset$.

(ii) By [14, Lemma 3.12], $\Lambda \in \mathcal{B}(b) \iff d(\Lambda, \Lambda_0) \leqslant m$. In other words, $\mathcal{B}(b)$ is simply the ball of radius m in the Bruhat–Tits tree, centred at Λ_0.

(iii) Note that, for vertex lattices that lie on the boundary of $\mathcal{B}(b)$, i.e., those Λ for which $d(\Lambda, \Lambda_0) = m$, we have $m(b, \Lambda) = 0$. While such vertices contribute nothing to $Z(b)$, it will make our formulae somewhat neater if we include them in $\mathcal{B}(b)$.

(iv) The following reformulation of the multiplicities will also be useful:

$$m(b, \Lambda) = \frac{1}{2} \cdot \begin{cases} m - d(\Lambda, \Lambda_0), & \text{if } m \equiv d(\Lambda, \Lambda_0) \pmod 2 \\ m - d(\Lambda, \Lambda_0) + 1, & \text{if } m \not\equiv d(\Lambda, \Lambda_0) \pmod 2. \end{cases} \tag{2.3}$$

2.1. Local intersection numbers

Given two formal closed subschemes Z and Z' of $\mathcal{D}$ such that the sum of their defining ideals is open in $\mathcal{O}_{\mathcal{D}}$, we define their intersection number to be

$$\langle Z, Z' \rangle := \chi(\mathcal{O}_Z \otimes^{\mathbb{L}} \mathcal{O}_{Z'}),$$

where the tensor product is taken in the derived sense, and χ is the Euler characteristic of the resulting complex; see [6, § 4].

If $Z(\mathbf{b})$ is a cycle corresponding to a pair $\mathbf{b} = [b_1, b_2] \in \mathbb{V}^2$, we set

$$\deg Z(\mathbf{b}) := \langle Z(b_1), Z(b_2) \rangle.$$

The aim of this section is to compute the intersection $\langle Z(b_1), Z(b_2) \rangle$ of two local cycles attached to linearly independent vectors $b_1, b_2 \in \mathbb{V}$, where $\operatorname{ord}_p(b_i, b_i) \geqslant 0$. Via Theorem 2.9, the intersection pairing $\langle Z(b_1), Z(b_2) \rangle$ can be expanded as

$$\begin{aligned} \langle Z(b_1), Z(b_2) \rangle &= \langle Z(b_1)^h, Z(b_2) \rangle + \sum_{\Lambda \in \mathcal{B}(b_1) \cap \mathcal{B}(b_2)} m(b_1, \Lambda) \langle \mathbb{P}_\Lambda, Z(b_2) \rangle \\ &= \langle Z(b_1)^h, Z(b_2)^h \rangle + \langle Z(b_1)^h, Z(b_2)^v \rangle + \sum_{\Lambda \in \mathcal{B}(b_1) \cap \mathcal{B}(b_2)} m(b_1, \Lambda) \langle \mathbb{P}_\Lambda, Z(b_2) \rangle. \end{aligned} \tag{2.4}$$

Lemma 2.11. *Let Λ_1 and Λ_2 denote the central lattices for b_1 and b_2, respectively, in the notation of Theorem 2.9. Then*

$$\langle Z(b_1)^h, Z(b_2)^v \rangle = \begin{cases} m(b_2, \Lambda_1), & \text{if } \Lambda_1 \in \mathcal{B}(b_2), \\ 0, & \text{otherwise.} \end{cases}$$

For any vertex lattice Λ, we also have

$$\langle \mathbb{P}_\Lambda, Z(b_2) \rangle = \begin{cases} 1, & \text{if } \Lambda \in \mathcal{B}(b_2) \text{ and } d(\Lambda, \Lambda_2) \equiv \operatorname{ord}_p(b_2, b_2) \pmod 2 \\ -p, & \text{if } \Lambda \in \mathcal{B}(b_2) \text{ and } d(\Lambda, \Lambda_2) \not\equiv \operatorname{ord}_p(b_2, b_2) \pmod 2 \\ 0 & \text{if } \Lambda \notin \mathcal{B}(b_2). \end{cases}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Proof. By [6, Equation (4.7)], we have that, for any vertex lattice Λ,

$$\langle Z(b_1)^h, \mathbb{P}_\Lambda\rangle = \begin{cases} 1, & \text{if } \Lambda = \Lambda_1 \\ 0, & \text{otherwise.} \end{cases}$$

The first statement in the proposition follows immediately.

Next, we recall the following formula; see [6, Lemma 4.7]: if Λ and Λ' are any two vertex lattices, then

$$\langle \mathbb{P}_\Lambda, \mathbb{P}_{\Lambda'}\rangle = \begin{cases} -(p+1), & \text{if } \Lambda = \Lambda' \\ 1, & \text{if } d(\Lambda, \Lambda') = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose that Λ is any vertex lattice, so that

$$\langle \mathbb{P}_\Lambda, Z(b_2)\rangle = \langle \mathbb{P}_\Lambda, Z(b_2)^h\rangle + \sum_{\substack{\Lambda' \in \mathcal{B}(b_2) \\ d(\Lambda', \Lambda) \leqslant 1}} m(b_2, \Lambda')\langle \mathbb{P}_\Lambda, \mathbb{P}_{\Lambda'}\rangle.$$

When $\Lambda \notin \mathcal{B}(b_2)$, it follows immediately that $\langle \mathbb{P}_\Lambda, Z(b_2)\rangle = 0$; see Remark 2.10 for the case when Λ is of distance 1 from the boundary.

Suppose next that $\Lambda \in \mathcal{B}(b_2)$ and $\Lambda \neq \Lambda_2$. Then Λ has one neighbour, say $\Lambda^\sharp$, that is strictly closer to Λ_2 than Λ is. Suppose further that $d(\Lambda, \Lambda_2) \equiv m \pmod 2$. Then (2.3) implies that

$$m(b_2, \Lambda^\sharp) = m(b_2, \Lambda) + 1.$$

For any other neighbour $\Lambda^\flat$ of Λ, of which there are p,

$$m(b_2, \Lambda^\flat) = m(b_2, \Lambda).$$

Therefore, we obtain

$$\begin{aligned} \langle \mathbb{P}_\Lambda, Z(b_2)\rangle &= m(b_2, \Lambda^\#)\langle \mathbb{P}_\Lambda, \mathbb{P}_{\Lambda^\sharp}\rangle + m(b_2, \Lambda)\langle \mathbb{P}_\Lambda, \mathbb{P}_\Lambda\rangle + \sum_{\Lambda^\flat} m(b_2, \Lambda^\flat)\langle \mathbb{P}_\Lambda, \mathbb{P}_{\Lambda^\flat}\rangle \\ &= (m(b_2, \Lambda) + 1) + m(b_2, \Lambda)(-p-1) + pm(b_2, \Lambda) \\ &= 1. \end{aligned}$$

The case where $d(\Lambda, \Lambda_2) \not\equiv m \pmod 2$ follows from similar considerations.

Finally, suppose that $\Lambda = \Lambda_2$. If $\mathrm{ord}_p(b_2, b_2) = 2t$ is even; then we see by (2.3) that Λ_2 and all of its neighbours occur in $Z(b_2)$ with the same multiplicity t, so that

$$\begin{aligned} \langle \mathbb{P}_{\Lambda_2}, Z(b_2)\rangle &= \langle \mathbb{P}_{\Lambda_2}, Z(b_2)^h\rangle + t\cdot\langle \mathbb{P}_{\Lambda_2}, \mathbb{P}_{\Lambda_2}\rangle + \sum_{d(\Lambda^\flat, \Lambda_2)=1} t\langle \mathbb{P}_{\Lambda_2}, \mathbb{P}_{\Lambda^\flat}\rangle \\ &= 1 + t(-p-1) + t\cdot(p+1) = 1. \end{aligned}$$

On the other hand, if $\mathrm{ord}_p(b_2, b_2) = 2t-1$ is odd, then

$$m(b_2, \Lambda_2) = t = m(b_2, \Lambda^\flat) + 1$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

for every neighbour $\Lambda^\flat$. Thus

$$\begin{aligned}\langle \mathbb{P}_{\Lambda_2}, Z(b_2)\rangle &= \langle \mathbb{P}_{\Lambda_2}, Z(b_2)^h\rangle + t\cdot\langle \mathbb{P}_{\Lambda_2}, \mathbb{P}_{\Lambda_2}\rangle + \sum_{d(\Lambda^\flat,\Lambda_2)=1} (t-1)\langle \mathbb{P}_{\Lambda_2}, \mathbb{P}_{\Lambda^\flat}\rangle \\ &= 1 + t(-p-1) + (t-1)\cdot(p+1) = -p. \end{aligned}$$ □

Lemma 2.12. *Let $b_1, b_2 \in \mathbb{V}$, with corresponding central lattices Λ_1 and Λ_2, respectively, and assume that*

$$m_1 := \operatorname{ord}_p(b_1, b_1) \leqslant \operatorname{ord}_p(b_2, b_2) =: m_2.$$

Suppose further that $\mathcal{B}(b_1) \cap \mathcal{B}(b_2)$ is non-empty, and let $\mathfrak{A}$ denote the unique shortest path between Λ_1 and Λ_2. Then the intersection $\mathcal{B}(b_1) \cap \mathcal{B}(b_2)$ is the ball of radius

$$r := \min\left(\frac{m_1 + m_2 - d(\Lambda_1, \Lambda_2)}{2}, m_1\right) \tag{2.5}$$

around the unique vertex lattice $\Gamma \in \mathfrak{A}$ such that

$$d(\Lambda_1, \Gamma) = m_1 - r, \quad \text{and} \quad d(\Lambda_2, \Gamma) = d(\Lambda_1, \Lambda_2) - (m_1 - r). \tag{2.6}$$

Proof. This follows easily from the fact that $\mathscr{B}$ is a tree, and that $\mathcal{B}(b_1)$ and $\mathcal{B}(b_2)$ are balls of radius m_1 and m_2, respectively. □

Remark 2.13. Recall that the *type* of the central lattice Λ attached to $b \in \mathbb{V}$ is determined by the parity of $\operatorname{ord}_p(b, b)$, as in Definition 2.8. As lattices of differing type are always at an odd distance apart,

$$m_1 + m_2 \equiv d(\Lambda_1, \Lambda_2) \pmod 2, \tag{2.7}$$

and so $r \in \mathbb{Z}$. ◇

We now come to the main theorem of this section.

Theorem 2.14. *Suppose that $b_1, b_2 \in \mathbb{V}$ are linearly independent vectors, with $m_1 := \operatorname{ord}_p(b_1, b_1)$ and $m_2 := \operatorname{ord}_p(b_2, b_2)$ and central lattices Λ_1 and Λ_2, respectively. Assume that $0 \leqslant m_1 \leqslant m_2$.*

(i) *If $\mathcal{B}(b_1) \cap \mathcal{B}(b_2) = \emptyset$, then $\langle Z(b_1), Z(b_2)\rangle = 0$.*

(ii) *If $\mathcal{B}(b_1) \subset \mathcal{B}(b_2)$, then*

$$\langle Z(b_1), Z(b_2)\rangle = \frac{m_1 + m_2 - d(\Lambda_1, \Lambda_2)}{2} - p\left(\frac{p^{m_1} - 1}{p - 1}\right) + \langle Z(b_1)^h, Z(b_2)^h\rangle.$$

(See Proposition 2.15 below for the calculation of the final 'h–h' term.)

(iii) *Suppose that $\mathcal{B}(b_1) \cap \mathcal{B}(b_2) \neq \emptyset$ but $\mathcal{B}(b_1) \not\subset \mathcal{B}(b_2)$, and let r be as in (2.5). Then*

$$\langle Z(b_1), Z(b_2)\rangle = r - p\left(\frac{p^r - 1}{p - 1}\right).$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Proof. (i) When $\mathcal{B}(b_1)\cap\mathcal{B}(b_2)=\emptyset$, one sees immediately from (2.4) that the intersection pairing vanishes.

Case (ii): $\mathcal{B}(b_1)\subset\mathcal{B}(b_2)$

In this case, we have $\mathcal{B}(b_1)\cap\mathcal{B}(b_2)=\mathcal{B}(b_1)$, which is the ball of radius m_1 centred at Λ_1. Let

$$Z(b_1)^v=\sum_{\Lambda\in\mathcal{B}(b_1)} m(b_1,\Lambda)\mathbb{P}_\Lambda$$

denote the 'vertical' part of $Z(b_1)$. We may express its contribution to the intersection by expanding radially from Λ_1:

$$\begin{aligned}\langle Z(b_1)^v, Z(b_2)\rangle &= m(b_1,\Lambda_1)\cdot\langle\mathbb{P}_{\Lambda_1},Z(b_2)\rangle+\sum_{\substack{\Lambda\\ d(\Lambda_1,\Lambda)=1}} m(b_1,\Lambda)\cdot\langle\mathbb{P}_\Lambda,Z(b_2)\rangle\\ &\quad+\cdots+\sum_{\substack{\Lambda\\ d(\Lambda_1,\Lambda)=m_1}} m(b_1,\Lambda)\cdot\langle\mathbb{P}_\Lambda,Z(b_2)\rangle.\end{aligned}$$

Suppose first that m_1 is even. Then

$$m(b_1,\Lambda)=\frac{m_1}{2}-\lfloor d(\Lambda_1,\Lambda)/2\rfloor$$

and, by (2.7), we have that $d(\Lambda_1,\Lambda_2)\equiv m_2 \pmod 2$. Therefore, applying (2.3),

$$\begin{aligned}\langle Z(b_1)^v, Z(b_2)\rangle &= \frac{m_1}{2}\cdot(1)+(p+1)\frac{m_1}{2}\cdot(-p)+p(p+1)\left(\frac{m_1}{2}-1\right)\cdot(1)\\ &\quad+\cdots+p^{m_1-2}(p+1)\left(\frac{m_1}{2}-[(m_1-1)/2]\right)(-p)\\ &\quad+p^{m_1-1}(p+1)\left(\frac{m_1}{2}-[m_1/2]\right)\\ &=\frac{m_1}{2}+(p+1)(-p-p^3-\cdots-p^{m_1-1})\\ &=\frac{m_1}{2}-p\left(\frac{p^{m_1}-1}{p-1}\right)\end{aligned}$$

On the other hand, using (2.3) and Lemma 2.11,

$$\langle Z(b_1)^h, Z(b_2)^v\rangle=m(b_2,\Lambda_1)=\frac{m_2-d(\Lambda_1,\Lambda_2)}{2},$$

and so

$$\begin{aligned}\langle Z(b_1), Z(b_2)\rangle &= \langle Z(b_1)^h, Z(b_2)^h\rangle+\langle Z(b_1)^h, Z(b_2)^v\rangle+\langle Z(b_1)^v, Z(b_2)\rangle\\ &=\frac{m_1+m_2-d(\Lambda_1,\Lambda_2)}{2}-p\left(\frac{p^{m_1}-1}{p-1}\right)+\langle Z(b_1)^h, Z(b_2)^h\rangle,\end{aligned}$$

as required.

Next, if m_1 is odd, then setting $t_1=(m_1+1)/2$ gives

$$m(b_1,\Lambda)=t_1-\left\lfloor\frac{d(\Lambda_1,\Lambda)+1}{2}\right\rfloor,\quad \text{for any } \Lambda\in\mathcal{B}(b_1).$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Since $d(\Lambda_1, \Lambda_2) \not\equiv m_2 \pmod 2$,

$$\begin{aligned}\langle Z(b_1)^v, Z(b_2)\rangle &= m(b_1, \Lambda_1)\cdot\langle \mathbb{P}_{\Lambda_1}, Z(b_2)\rangle + \sum_{\substack{\Lambda \\ d(\Lambda_1,\Lambda)=1}} m(b_1, \Lambda)\cdot\langle \mathbb{P}_{\Lambda}, Z(b_2)\rangle \\ &\quad + \cdots + \sum_{\substack{\Lambda \\ d(\Lambda_1,\Lambda)=m_1-2}} m(b_1, \Lambda)\cdot\langle \mathbb{P}_{\Lambda}, Z(b_2)\rangle \\ &= t_1\cdot(-p) + (p+1)(t_1-1)\cdot(1) + p(p+1)(t_1-1)\cdot(-p) \\ &\quad + \cdots + p^{m_1-3}(p+1)(t_1-[(m_1-1)/2]) \\ &\quad + p^{m_1-2}(p+1)(t_1-[m_1/2])(-p) \\ &= t_1 - (p+1)(1+p^2+p^4+\cdots+p^{m_1-1}) \\ &= \frac{m_1+1}{2} - \left(\frac{p^{m_1+1}-1}{p-1}\right).\end{aligned}$$

By (2.3), we have

$$\langle Z(b_1)^h, Z(b_2)^v\rangle = m(b_2, \Lambda_1) = \frac{m_2 - d(\Lambda_1, \Lambda_2)+1}{2},$$

so that

$$\begin{aligned}\langle Z(b_1), Z(b_2)\rangle &= \frac{m_1+m_2-d(\Lambda_1,\Lambda_2)}{2} + 1 - \left(\frac{p^{m_1+1}-1}{p-1}\right) + \langle Z(b_1)^h, Z(b_2)^h\rangle \\ &= \frac{m_1+m_2-d(\Lambda_1,\Lambda_2)}{2} - p\left(\frac{p^{m_1}-1}{p-1}\right) + \langle Z(b_1)^h, Z(b_2)^h\rangle,\end{aligned}$$

as required.

Case (iii): $\mathcal{B}(b_1) \not\subset \mathcal{B}(b_2)$

Recall that the intersection $\mathcal{B}(b_1)\cap\mathcal{B}(b_2)$ is a ball of radius

$$r = \frac{m_1+m_2-d(\Lambda_1,\Lambda_2)}{2},$$

centred at a vertex Γ along the geodesic $\mathfrak{A}$ connecting Λ_1 and Λ_2. We start by calculating

$$(Z(b_1)^v, Z(b_2))$$

as follows. Consider taking a walk along $\mathfrak{A}$, starting from Γ towards Λ_1. We stop walking either after a distance of r units, or when we arrive at Λ_1, whichever comes first; i.e., we travel a distance of $\min(r, d(\Lambda_1, \Gamma))$. For each vertex we encounter, say after k steps, we add up the contributions of all the lattices branching off of that vertex *away from* $\mathfrak{A}$, and call that sum $F(k)$. More precisely, let $\Gamma^{(k)}$ denote the vertex in $\mathfrak{A}$ which is a distance of $d(\Lambda_1, \Gamma) - k$ away from Λ_1 and a distance of k away from Γ. Then the sum

$$F(k) = \sum_{\Lambda\in\mathcal{F}(k)} m(b_1, \Lambda)\langle \mathbb{P}_{\Lambda}, Z(b_2)\rangle$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

is taken over the set $\mathcal{F}(k)$ consisting of lattices $\Lambda \in \mathcal{B}(b_1) \cap \mathcal{B}(b_2)$ such that the unique shortest path between Λ and Λ_1 first meets $\mathfrak{A}$ at $\Gamma^{(k)}$. The points $\Gamma = \Gamma^{(0)}$ and Λ_1 (if indeed we end up walking that far, as the vertex Λ_1 contributes if and only if $d(\Lambda_1, \Gamma) \leqslant r$) are different from the rest, because at these points there are p directions leading *away* from the path $\mathfrak{A}$, whereas at all the intermediate vertices there are $p-1$.

For convenience, set $\mu_k := m(b_1, \Gamma^{(k)})$ and $d_k = d(\Lambda_1, \Gamma^{(k)})$. Our first step is to prove the following calculation for $F(0)$:

$$F(0) = \begin{cases} \mu_0 - p^2\left(\dfrac{p^r-1}{p^2-1}\right), & \text{if } r \text{ is even,} \\ -p\left(\dfrac{p^{r+1}-1}{p^2-1}\right), & \text{if } r \text{ is odd.} \end{cases}$$

To prove this, first suppose that r is even. Then $d_0 = m_1 - r$ has the same parity as m_1. By (2.3), a vertex Λ contributing in $F(0)$ that is s units away from Γ appears in $Z(b_1)$ with multiplicity $m(b_1, \Lambda) = \mu_0 - \lfloor s/2 \rfloor$. Therefore,

$$\begin{aligned} F(0) = {} & \mu_0 \cdot \langle \mathbb{P}_\Gamma, Z(b_2)\rangle + \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=1}} \mu_0 \cdot \langle \mathbb{P}_\Lambda, Z(b_2)\rangle + \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=2}} (\mu_0 - 1) \cdot \langle \mathbb{P}_\Lambda, Z(b_2)\rangle \\ & + \cdots + \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=r}} (\mu_0 - (r/2)) \cdot \langle \mathbb{P}_\Lambda, Z(b_2)\rangle. \end{aligned}$$

From (2.6) and (2.7), it follows that $d(\Lambda_2, \Gamma^{(0)}) \equiv m_2 \pmod 2$, and so, applying Lemma 2.11,

$$\begin{aligned} F(0) &= \mu_0 \cdot (1) + \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=1}} \mu_0 \cdot (-p) + \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=2}} (\mu_0 - 1) \cdot (1) + \cdots + \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=r}} (\mu_0 - (r/2)) \cdot (1) \\ &= \mu_0 + p \cdot \mu_0(-p) + p^2 \cdot (\mu_0 - 1) + p^3 \cdot (\mu_0 - 1)(-p) + \cdots + p^r \cdot (\mu_0 - r/2) \\ &= \mu_0 - p^2 - p^4 \cdots - p^r \\ &= \mu_0 - p^2\left(\frac{p^r-1}{p^2-1}\right), \end{aligned}$$

as required.

Similarly, when r is odd,

$$\begin{aligned} F(0) &= \mu_0 \cdot (-p) + \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=1}} (\mu_0 - 1) \cdot (1) + \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=2}} (\mu_0 - 1)(-p) \\ &\quad + \cdots \sum_{\substack{\Lambda\in\mathcal{F}(0) \\ d(\Lambda,\Gamma)=r}} (\mu_0 - (r+1)/2) \cdot (1) \\ &= \mu_0 \cdot (-p) + p(\mu_0 - 1) + p^2(\mu_0 - 1) \cdot (-p) + p^3(\mu_0 - 2) + \cdots + p^r(\mu_0 - (r+1)/2) \\ &= -p - p^3 - \cdots p^r \end{aligned}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

$$= -p\left(\frac{p^{r+1}-1}{p^2-1}\right),$$

as required.

Next, when $0 < k < d_0 = d(\Lambda_1, \Gamma^{(0)})$ and $k \leqslant r$, we have the following calculation:

$$F(k) = \begin{cases} \mu_k - (p-1)(p)\left(\dfrac{p^{r-k}-1}{p^2-1}\right), & \text{if } r \equiv k \pmod 2 \\ -\mu_k - (p-1)\left(\dfrac{p^{r-k+1}-1}{p^2-1}\right), & \text{if } r \not\equiv k \pmod 2. \end{cases}$$

To prove this claim, we first consider the case $r \equiv k \pmod 2$. Then

$$d(\Lambda_1, \Gamma^{(k)}) = d(\Lambda_1, \Gamma^{(0)}) - k = m_1 - r - k \equiv m_1 \pmod 2,$$

and

$$d(\Lambda_2, \Gamma^{(k)}) = d(\Lambda_2, \Gamma^{(0)}) + k \equiv m_2 - r + k \equiv m_2 \pmod 2,$$

as well. Hence, for $\mu_k = m(b_1, \Gamma^{(k)})$, we have

$$\begin{aligned} F(k) &= \mu_k \cdot (1) + \sum_{\substack{\Lambda \in \mathcal{F}(k) \\ d(\Lambda,\Gamma)=1}} \mu_k \cdot (-p) + \sum_{\substack{\Lambda \in \mathcal{F}(k) \\ d(\Lambda,\Gamma)=2}} (\mu_k - 1) \cdot (1) \\ &\quad + \cdots + \sum_{\substack{\Lambda \in \mathcal{F}(k) \\ d(\Lambda,\Gamma)=r-k}} \left(\mu_k - \frac{r-k}{2}\right) \cdot (1) \\ &= \mu_k + (p-1)\mu_k \cdot (-p) + p(p-1)(\mu_k - 1) + p^2(p-1)(\mu_k - 1) \cdot (-p) \\ &\quad + \cdots + p^{r-k-1}(p-1)\left(\mu_k - \frac{r-k}{2}\right) \cdot (1) \\ &= \mu_k + (p-1)(-p - p^3 - \cdots - p^{r-k-1}) \\ &= \mu_k - p(p-1)\frac{p^{r-k}-1}{p^2-1}. \end{aligned}$$

The case when $r \not\equiv k \pmod 2$ follows from similar considerations, and we omit the proof.

Finally, we consider the case $k = d_0 = d(\Lambda_1, \Gamma^{(0)})$, which only arises when $r \geqslant d_0$. The calculation is almost identical to the case $k = 0$, and so we omit it; the result is

$$F(d_0) = \begin{cases} \mu_{d_0} - p^2\left(\dfrac{p^{r-d_0}-1}{p^2-1}\right), & \text{if } r \equiv d_0 \pmod 2 \\ -p\left(\dfrac{p^{r-d_0+1}-1}{p^2-1}\right), & \text{if } r \not\equiv d_0 \pmod 2. \end{cases}$$

Now, we consider the sum over all the contributions $F(k)$. Observe that, if $k > 0$ and $k \equiv r \pmod 2$, then

$$F(k) + F(k+1) = \mu_k - \mu_{k+1} - (p^{r-k} - 1). \tag{2.8}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

We first assume that $r < d_0$ and r is even. The first assumption implies that $\Lambda_1 \notin \mathcal{B}(b_1) \cap \mathcal{B}(b_2)$, and so $\langle Z(b_1)^h, Z(b_2)\rangle = 0$ in this case. Thus

$$\begin{aligned}\langle Z(b_1), Z(b_2)\rangle &= \langle Z(b_1)^v, Z(b_2)\rangle = \sum_{k=0}^{r} F(k)\\ &= F(0) + F(1) + \left(\sum_{n=1}^{r/2-1} F(2n) + F(2n+1)\right) + F(r)\\ &= \sum_{k=0}^{r}(-1)^k \mu_k - (p^2+p-1)\left(\frac{p^r-1}{p^2-1}\right) - \sum_{n=1}^{r/2}(p^{r-2n}-1)\\ &= \sum_{k=0}^{r}(-1)^k \mu_k - (p^2+p)\left(\frac{p^r-1}{p^2-1}\right) + \frac{r}{2}.\end{aligned}$$

Since r is even, we have $d_0 \equiv m_1 \pmod 2$, and so $\mu_k = \mu_0 + \lfloor (k+1)/2 \rfloor$. Thus

$$\sum_{k=0}^{r}(-1)^k \mu_k = \mu_0 - (\mu_0+1) + (\mu_0+1) - (\mu_0+2)\cdots + (\mu_0 + r/2) - (\mu_0 + r/2) = \mu_0$$

and, recalling (2.6) and (2.3), we have $\mu_0 = r/2$. Therefore

$$\langle Z(b_1), Z(b_2)\rangle = r - p\left(\frac{p^r-1}{p-1}\right),$$

which proves the proposition for $r < d_0$ with r even. A similar calculation gives the same result when $r < d_0$ and r is odd.

Next, we consider the case that $r \geqslant d_0$. Then we have

$$\langle Z(b_1), Z(b_2)\rangle = \langle Z(b_1)^h, Z(b_2)\rangle + \langle Z(b_1)^v, Z(b_2)\rangle.$$

The proof proceeds by a further case-by-case analysis, depending on the parity of r and d_0. Suppose that both are even. Then

$$\begin{aligned}\langle Z(b_1)^v, Z(b_2)\rangle &= F(0) + F(1) + \left[\sum_{n=1}^{d_0/2-1} F(2n) + F(2n+1)\right] + F(d_0)\\ &= \mu_0 - p^2\left(\frac{p^r-1}{p^2-1}\right) - \mu_1 - (p-1)\left(\frac{p^r-1}{p^2-1}\right)\\ &\quad + \sum_{n=1}^{d_0/2-1} \mu_{2n} - \mu_{2n+1} - (p^{r-2n}-1) + \mu_{d_0} - p^2\left(\frac{p^{r-d_0}-1}{p^2-1}\right)\\ &= \frac{d_0}{2} - 1 + \sum_{k=0}^{d_0}(-1)^k \mu_k - (p^2+p-1)\left(\frac{p^r-1}{p^2-1}\right) - \frac{p^r - p^{r-d_0+2}}{p^2-1}\\ &\quad - p^2\left(\frac{p^{r-d_0}-1}{p^2-1}\right).\end{aligned}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Using the fact that in this case $\sum_k (-1)^k \mu_k = \mu_0 = r/2$ and simplifying the above expression, we obtain

$$\langle Z(b_1)^v, Z(b_2)\rangle = \frac{d_0}{2} + \frac{r}{2} - p\left(\frac{p^r-1}{p^2-1}\right) = \frac{m_1}{2} - p\left(\frac{p^r-1}{p^2-1}\right).$$

On the other hand, we note that

$$\langle Z(b_1)^h, Z(b_2)\rangle = m(b_2, \Lambda_0).$$

Combining (2.6) with the assumptions that r and d_0 are even, we have $d(\Lambda_1, \Lambda_2) \equiv m_2 \pmod 2$, and so, by (2.3),

$$m(b_2, \Lambda_1) = \frac{m_2 - d(\Lambda_1, \Lambda_2)}{2} = r - \frac{m_1}{2}.$$

Therefore

$$\langle Z(b_1), Z(b_2)\rangle = r - p\left(\frac{p^r-1}{p-1}\right),$$

as desired, in the case $r \equiv d_0 \equiv 0 \pmod 2$. Again, the remaining cases are entirely analogous, and so we omit the proofs. □

We turn to the 'horizontal–horizontal' terms appearing in Theorem 2.14(ii). As usual, let $b_1, b_2 \in \mathbb{V}$ be linearly independent, with

$$m_1 = \mathrm{ord}_p(b_1, b_1) \leqslant m_2 := \mathrm{ord}_p(b_2, b_2)$$

and central lattices Λ_1 and Λ_2, respectively. For $i = 1, 2$, set

$$t_i := \lfloor \frac{m_i + 1}{2} \rfloor, \quad \text{and} \quad \beta_i := p^{-t_i} b_i,$$

so that $\beta_i \in \Lambda_i \setminus p\Lambda_i$ by the property characterizing central lattices.

Proposition 2.15. *With notation as in the previous paragraph,*

$$\langle Z(b_1)^h, Z(b_2)^h\rangle = \begin{cases} 0, & \text{if } \Lambda_1 \neq \Lambda_2 \\ \mathrm{ord}_p(\beta_2, \beta_1'), & \text{if } \Lambda_1 = \Lambda_2, \end{cases}$$

where $\beta_1' \in \Lambda_1$ is any vector such that $\{\beta_1, \beta_1'\}$ forms an orthogonal basis for Λ_1.

Proof. Recall from Theorem 2.9 that $Z(b_i)^h$ meets the special fibre at a single non-superspecial point in the component $\mathbb{P}_{\Lambda_i}$. Hence, if $\Lambda_1 \neq \Lambda_2$, the pairing clearly vanishes.

Thus we may assume that $\Lambda_1 = \Lambda_2 = \Lambda$, and we suppose further that Λ is self-dual, so that $m_1 = 2t_1$ and $m_2 = 2t_2$ are even. We may fix a basis $\{e, f\}$ for Λ with respect to which the Hermitian form is

$$h \sim \begin{pmatrix} & \delta \\ -\delta & \end{pmatrix},$$

where $\delta \in o_{k,p}^\times$ is a generator for $k_p/\mathbb{Q}_p$ with $\delta' = -\delta$. Writing

$$\beta_1 = p^{-t_1} b_1 = r_1 e + s_1 f, \quad \beta_2 = p^{-t_2} b_2 = r_2 e + s_2 f,$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

we note that

$$(\beta_i, \beta_i) = \delta(r_i s_i' - r_i' s_i) \in \mathbb{Z}_p^{\times},$$

and so r_i, s_i, and $r_i s_i' - r_i' s_i$ are all units in $o_{k,p}$.

Set $\beta_1' = r_1' e + s_1' f$, so that $\{\beta_1, \beta_1'\}$ forms an orthogonal basis for Λ. Then we may write

$$\beta_2 = \mu\beta_1 + \epsilon\beta_1', \quad \text{for some } \mu, \epsilon \in o_k, \epsilon \neq 0.$$

The local equations for these cycles are described explicitly in [14, § 3]; in particular, by Proposition 3.7 there,

$$\begin{aligned} Z(b_1)^h(\mathbb{F}) = Z(b_2)^h(\mathbb{F}) &\iff \beta_1 \text{ and } \beta_2 \text{ are collinear mod } p\Lambda \\ &\iff \mu \in o_k^{\times} \text{ and } \epsilon \in (p). \end{aligned}$$

If $\epsilon \in o_{k,p}^{\times}$, then $Z(b_1)^h$ and $Z(b_2)^h$ do not intersect, and so

$$\langle Z(b_1)^h, Z(b_2)^h \rangle = 0 = \mathrm{ord}_p\, \epsilon = \mathrm{ord}_p(\beta_2, \beta_1'),$$

as required.

Finally, we consider the situation $Z(b_1)^h(\mathbb{F}) = \{x\} = Z(b_2)^h(\mathbb{F})$. As described in [14, Proposition 3.10], the point x has a formal affine neighbourhood

$$\mathrm{Spf}\ W[T, (T^p - T)^{-1}]^{\vee} =: \mathrm{Spf}\ R \tag{2.9}$$

such that the two cycles $Z(b_1)^h$ and $Z(b_2)^h$ are given by

$$\mathrm{Spf}\ R/(r_1 T - s_1) \quad \text{and} \quad \mathrm{Spf}\ R/(r_2 T - s_2),$$

respectively. As $\epsilon \in o_{k,p}^{\times}$, we may write

$$r_2 T - s_2 = \mu \cdot (r_1 T - s_1 + \epsilon\mu^{-1}(r_1' T - s_1')),$$

and, since the cycles intersect properly, it follows immediately that

$$\begin{aligned} \chi(\mathcal{O}_{Z(b_1)^h} \otimes^{\mathbb{L}} \mathcal{O}_{Z(b_2)^h}) &= \mathrm{length}(R/(r_1 T - s_1) \otimes_R R/(r_2 T - s_2))_x \\ &= \mathrm{length}\ W/(\epsilon\mu^{-1}(r_1' s_1 - r_1 s_1')) \\ &= \mathrm{ord}_p(\epsilon), \end{aligned}$$

as required.

When Λ is a type-2 vertex lattice, we may fix a basis $\{e, f\}$ such that $h \sim p^{-1}\left(\begin{smallmatrix} & \delta \\ -\delta & \end{smallmatrix}\right)$. Writing

$$\beta_i = r_i e + s_i f$$

as before, Proposition 3.11 of [14] tells us that the local equation for $Z(b_i)^h$ is given by

$$s_i' T + r_i' = 0$$

for $i = 1, 2$. The result follows from similar considerations to the previous case. □

We have now completed the computation of the pairing $\langle Z(b_1), Z(b_2)\rangle$. The next step is to show that this pairing depends only the matrix of inner products, as in the following lemma.

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Lemma 2.16. *Suppose that* $\mathbf{b} = [b_1, b_2] \in \mathbb{V}$ *is a linearly independent pair of vectors, and let*

$$T = (\mathbf{b}, \mathbf{b}) = \begin{pmatrix} (b_1, b_1) & (b_1, b_2) \\ (b_2, b_1) & (b_2, b_2) \end{pmatrix}$$

denote the matrix of inner products. Set $m_i = \mathrm{ord}_p(b_i, b_i)$, *and let* Λ_1, Λ_2 *denote the central lattices of* b_1 *and* b_2, *respectively, with* $d := d(\Lambda_1, \Lambda_2)$. *Replacing* T *by a* $GL_2(o_{k,p})$*-conjugate if necessary, we may further assume that* $m_1 \leqslant m_2$. *Then the following hold.*

(i) $T \in \mathrm{Herm}_2(o_{k,p})$ *if and only if* $\mathcal{B}(b_1) \cap \mathcal{B}(b_2) \neq \emptyset$.

(ii) *If* $\mathcal{B}(b_1) \subset \mathcal{B}(b_2)$ *and* $\Lambda_1 \neq \Lambda_2$, *then* T *is* $GL_2(o_{k,p})$*-conjugate to the matrix* $\mathsf{diag}(p^a, p^b)$ *with*

$$a = m_2 - d, \quad b = m_1.$$

(iii) *If* $\mathcal{B}(b_1) \not\subset \mathcal{B}(b_2)$ *and* $\mathcal{B}(b_1) \cap \mathcal{B}(b_2) \neq \emptyset$, *then* $T \sim \mathsf{diag}(p^r, p^r)$, *where*

$$r = \frac{m_1 + m_2 - d}{2}.$$

(iv) *If* $\Lambda_1 = \Lambda_2$, *then* $T \sim \mathsf{diag}(p^a, p^b)$, *where*

$$a = m_2 + 2\ \mathrm{ord}_p(\beta_2, \beta_1'), \quad b = m_1;$$

here, β_2 *and* β_1' *are as in Proposition 2.15.*

Proof. If $m_1 < 0$, then clearly $T \notin \mathrm{Herm}_2(o_{k,p})$, and, by definition $\mathcal{B}(b_1) = \emptyset$, so the lemma holds trivially in this case. Hence we assume that $0 \leqslant m_1 \leqslant m_2$.

Case 1: $\Lambda_1 \neq \Lambda_2$

First suppose that Λ_1 is self-dual, and so m_1 is even. We can assume that Λ_1 and Λ_2 lie on the 'standard lattice chain'. In other words, there exists a basis $\{e_1, f_1\}$ for Λ_1 such that

(a) with respect to this basis, $h \sim \left(\begin{smallmatrix} & \delta \\ -\delta & \end{smallmatrix} \right)$, where $\delta \in o_{k,p}^\times$ with $\delta' = -\delta$, and

(b) the lattice Λ_2 has basis $\{e_2, f_2\}$, where

$$\{e_2, f_2\} = \begin{cases} \{p^{-k} e_1,\ p^k f_1\}, & \text{if } d = 2k \text{ is even} \\ \{p^{-k-1} e_1,\ p^k f_1\}, & \text{if } d = 2k+1 \text{ is odd.} \end{cases}$$

For a proof of the existence of such a basis, see [15, Proposition 4.10].

Set $t_1 = \frac{m_1}{2}$ and $t_2 = \left\lfloor \frac{m_2+1}{2} \right\rfloor$. Then T has the form

$$T = \begin{pmatrix} p^{m_1} \cdot (\text{unit}) & p^s \cdot (\text{unit}) \\ p^s \cdot (\text{unit}) & p^{m_2} \cdot (\text{unit}) \end{pmatrix}, \tag{2.10}$$

where

$$s := \begin{cases} t_1 + t_2 - k & \text{if } d = 2k > 0 \text{ is even,} \\ t_1 + t_2 - k - 1 & \text{if } d = 2k+1 \text{ is odd.} \end{cases}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Since m_1 is even, we have $m_2 \equiv d \pmod 2$, and so in fact

$$s = \frac{m_1 + m_2 - d}{2}$$

in both cases. Hence

$$T \in \mathrm{Herm}_2(o_k) \iff d \leqslant m_1 + m_2 \iff \mathcal{B}(b_1) \cap \mathcal{B}(b_2) \neq \emptyset,$$

which proves (i), at least when $\Lambda_1 \neq \Lambda_2$ (on the other hand, if $\Lambda_1 = \Lambda_2$, then (i) is trivial).

Note that, in general, if $T \sim \mathsf{diag}(p^a, p^b)$ with $a \geqslant b \geqslant 0$, the numbers a and b are characterized by the facts that

- b is the lowest valuation appearing among the entries of T; and
- $a + b = \mathrm{ord}_p \det T$.

We also observe that

$$\mathcal{B}(b_1) \subset \mathcal{B}(b_2) \iff m_1 \leqslant m_2 - d \iff m_1 \leqslant s.$$

Thus statements (ii) and (iii) of the lemma follow from a moment's contemplation of (2.10), and, begging the reader's forbearance for yet another instance of this refrain, the proof when Λ_1 is a type-2 lattice is entirely analogous.

Case 2: $\Lambda_1 = \Lambda_2$

Abbreviate $\Lambda_1 = \Lambda_2 = \Lambda$. Recall that we defined $t_i = \lfloor \frac{m_i+1}{2} \rfloor$ and $\beta_i = p^{-t_i} b_i$ as in Proposition 2.15, so that $\beta_i \in \Lambda - p\Lambda$ and $\mathrm{ord}_p(\beta_i, \beta_i)$ is equal to 0 or -1, depending on whether or not m_1 and m_2 are both even or both odd.

We may choose an element $\beta_1' \in \Lambda - p\Lambda$ such that $(\beta_1, \beta_1') = 0$ and $(\beta_1', \beta_1') = (\beta_1, \beta_1)$ with $\{\beta_1, \beta_1'\}$ forming a basis for Λ. Write

$$\beta_2 = \mu \cdot \beta_1 + \epsilon \cdot \beta_1',$$

where $\mu, \epsilon \in o_{k,p}$ with at least one of them being a unit. Then

$$T = (\beta_1, \beta_1) \cdot \begin{pmatrix} p^{2t_1} & p^{t_1+t_2}\mu' \\ p^{t_1+t_2}\mu & p^{2t_2}(n(\mu) + n(\epsilon)) \end{pmatrix}.$$

The smallest valuation appearing is

$$2t_1 + \mathrm{ord}_p(\beta_1, \beta_1) = m_1,$$

and the determinant of T has valuation

$$\mathrm{ord}_p \det(T) = 2\,\mathrm{ord}_p(\beta_1, \beta_1) + 2(t_1 + t_2) + \mathrm{ord}_p\, n(\epsilon) = m_1 + m_2 + 2\ \mathrm{ord}_p(\beta_2, \beta_1').$$

This proves (iv). □

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Corollary 2.17. *Suppose that* $\mathbf{b} = [b_1, b_2] \in \mathbb{V}^2$ *is a linearly independent pair of vectors with* $T = (\mathbf{b}, \mathbf{b})$ *as above. Then the pairing* $\deg Z(\mathbf{b}) := \langle Z(b_1), Z(b_2)\rangle$ *depends only on the* $GL_2(o_{k,p})$*-conjugacy class of* T*. More precisely, if* $T \in \mathrm{Herm}_2(o_k)$ *and* $T \sim \mathsf{diag}(p^a, p^b)$ *with* $a \geqslant b$*, then*

$$\deg Z(\mathbf{b}) = \langle Z(b_1), Z(b_2)\rangle = \frac{a+b}{2} - p\left(\frac{p^b - 1}{p-1}\right) =: \mu_p(T). \tag{2.11}$$

Proof. This follows immediately from Theorem 2.14 and Lemma 2.16. □

3. A closed-form formula for certain representation densities

Suppose that $S \in \mathrm{Herm}_m(o_{k,p})$ and $T \in \mathrm{Herm}_n(o_{k,p})$ for some integers m and n, where we continue to work with the localization $o_{k,p}$ at an unramified prime p. Then we may consider the *representation density*

$$\alpha(S, T) := \lim_{k\to\infty} p^{-kn(2m-n)} \#\{x \in M_{m,n}(o_{k,p}/p^k) \mid {}^t(x')Sx \equiv T \pmod{p^k}\}, \tag{3.1}$$

which will play a crucial role in our determination of the Fourier coefficients of Eisenstein series. As this quantity depends only on the $GL_n(o_{k,p})$-conjugacy classes (respectively, $GL_m(o_{k,p})$-conjugacy classes) of T and S, respectively, we may suppose that they are diagonal. Moreover, for a fixed $S \in \mathrm{Herm}_m(o_{k,p})$, we write

$$S_r := \begin{pmatrix} S & \\ & \mathtt{Id}_r \end{pmatrix} \in \mathrm{Herm}_{m+r}(o_{k,p}).$$

As we will see shortly, there is a polynomial $F(S, T; X) \in \mathbb{Q}[X]$ such that

$$\alpha(S_r, T) = F(S, T; (-p)^{-r}).$$

The aim of this section is to prove the following closed-form expression for $F(S, T, X)$ when $T \in \mathrm{Herm}_2(o_{k,p})$ such that $\mathrm{ord}_p \det(T)$ is even, and $S = \mathsf{diag}(p, 1)$. This can be seen as the counterpart to Nagaoka's result [12], which considers the case $S = \mathtt{Id}_2$.

Proposition 3.1. *Set* $S = \mathsf{diag}(p, 1)$*. Then*

(i)

$$F(S, \mathtt{Id}_2; X) = \frac{(1-X)(X+p)}{p} \quad \textit{and}$$
$$F\left(S, \begin{pmatrix} p & \\ & p\end{pmatrix}; X\right) = \frac{(1-X)(X+p)}{p}(X^2 - (p^2-p)X + 1).$$

(ii) *Let* $T = \mathsf{diag}(p^a, p^b)$*, where* $a \geqslant b \geqslant 0$ *and* $a + b$ *is even. Set*

$$\epsilon = \begin{cases} 0, & \textit{if } b \textit{ is even,} \\ 1, & \textit{if } b \textit{ is odd,} \end{cases}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

and define

$$F_\epsilon(X) := F(S,\ p^\epsilon \cdot \mathtt{Id}_2; X) = \frac{(1-X)(X+p)}{p}(X^2-(p^2-p)X+1)^\epsilon.$$

Then

$$\begin{aligned} F(S,T;X) \;=\; & F_\epsilon(X) + \frac{(X-1)(X+p)}{X-p}\Bigg\{(pX)(1-p)\frac{(pX)^b-(pX)^\epsilon}{pX-1} \\ & + X^2(p-p^{-1}X)\frac{X^{2b}-X^{2\epsilon}}{X^2-1} \\ & + (-p^{b+1}(X-1)+pX^{b+1}-p^{-1}X^{b+2})\cdot\frac{X^{a+1}-X^{b+1}}{X^2-1}\Bigg\}. \end{aligned}$$

The proof of this proposition, which appears at the end of this subsection, amounts to specializing Hironaka's explicit formula [2, Theorem II] to the case at hand. We shall briefly review Hironaka's notation in the general case. First, given integers $u \geqslant v \geqslant 0$, we define a symbol

$$\begin{bmatrix} u \\ v \end{bmatrix} := \frac{\prod_{i=1}^{u}(1-(-p)^{-i})}{\prod_{i=1}^{v}(1-(-p)^{-i})\prod_{i=1}^{u-v}(1-(-p)^{-i})}.$$

Let

$$\mathbb{Z}^k_{\succeq 0} := \{a = (a_1, \dots a_k) \in \mathbb{Z}^k \mid a_1 \geqslant a_2 \geqslant \cdots a_k \geqslant 0\}$$

denote the set of non-increasing non-negative vectors in $\mathbb{Z}^k$. Given $a = (a_1, \dots, a_k) \in \mathbb{Z}^k_{\succeq 0}$, we set

$$\tilde{a} := (a_1+1, \dots, a_k+1) \in \mathbb{Z}^k_{\succeq 0},$$

and, for any $i \geqslant 1$, we let

$$a'_i := \#\{j \mid a_j \geqslant i\}.$$

Next, suppose that $\lambda, \mu \in \mathbb{Z}^k_{\succeq 0}$. For an integer $j \geqslant 1$, we define[1]

$$I_j(\mu,\lambda) := \sum_{i=\mu'_{j+1}}^{\min((\tilde{\lambda})'_{j+1},\mu'_j)} (-p)^{i(2(\tilde{\lambda})'_{j+1}+1-i)/2} \cdot \begin{bmatrix} (\tilde{\lambda})'_{j+1}-\mu'_{j+1} \\ (\tilde{\lambda})'_{j+1}-i \end{bmatrix} \cdot \begin{bmatrix} (\tilde{\lambda})'_j - i \\ (\tilde{\lambda})'_j-\mu'_j \end{bmatrix}.$$

We also define a partial order on $\mathbb{Z}^k_{\succeq 0}$ by declaring

$$a \leqslant b \iff a_i \leqslant b_i \text{ for all } i = 1, \dots, k.$$

Finally, we put

$$|a| := \sum_{i=1}^k a_i, \quad \text{and} \quad n(a) := \sum_{i=1}^k (i-1)a_i.$$

With all of this notation in place, we can state Hironaka's formula.

[1]There is a typographical error in the statement of Theorem II of [2]: the corresponding formula in appears without the necessary tilde in the exponent of $(-p)$.

Theorem 3.2 ([2, Theorem II]). *Let $\lambda \in \mathbb{Z}^n_{\succeq 0}$ and $\xi \in \mathbb{Z}^m_{\succeq 0}$ with $m \geqslant n$. Suppose that $T_\lambda \in \mathrm{Herm}_n(o_k)$ is $GL_n(o_k)$-equivalent to $\mathsf{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})$, and that S_ξ is equivalent to $\mathsf{diag}(p^{\xi_1}, \ldots, p^{\xi_m})$. Then*

$$\alpha(S_\xi, T_\lambda) = \sum_{\substack{\mu \in \mathbb{Z}^n_{\succeq 0} \\ \mu \leqslant \tilde{\lambda}}} (-1)^{|\mu|} (-p)^{-n(\mu)+(n-m-1)|\mu|+\langle \xi', \mu' \rangle} \cdot \prod_{j \geqslant 1} I_j(\mu, \lambda),$$

where $\langle \xi', \mu' \rangle = \sum_{i \geqslant 1} \xi'_i \mu'_i$. □

We now specialize this formula to our case of interest: take $n = 2$, $m = r + 2$, $\lambda = (a, b)$ with $a + b$ even, and $\xi = (1, 0, \ldots, 0) \in \mathbb{Z}^{r+2}_{\succeq 0}$. If we put

$$T = T_\lambda := \mathsf{diag}(p^a, p^b), \quad \text{and} \quad S = \mathsf{diag}(p, 1),$$

then in particular $S_r = S \oplus 1_r = S_\xi$. Taking $X := (-p)^{-r}$ in Hironaka's theorem gives us the expression

$$F(S, T; X) = \sum_{c=0}^{a+1} \sum_{d=0}^{\min(c, b+1)} (-1)^d p^{-2d-c} X^{c+d} (-p)^{\epsilon_c + \epsilon_d} \cdot \prod_{j \geqslant 1} I_j\left(\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right), \lambda\right), \tag{3.2}$$

where ϵ_c is equal to 0 if $c = 0$, and is equal to 1 if $c \geqslant 1$; we define ϵ_d likewise.

Our first step towards giving a closed-form expression for (3.2) is the following table of values for $I_j(\cdot, \cdot)$, which is easily proven by explicit computation.

Lemma 3.3. *Suppose that $\ell = (\alpha, \beta)$, and $(c, d) \leqslant (\alpha + 1, \beta + 1)$ are integers with $c \geqslant d$.*

(i) *If $c > \beta + 1 \geqslant d$, then*

$$I_j\left(\begin{pmatrix} c \\ d \end{pmatrix}, \ell\right) = \begin{cases} -p^3, & 1 \leqslant j < d \\ p^2 - p^3, & j = d, d < \beta + 1 \\ p^2, & d + 1 \leqslant j < \beta + 1 \\ -p, & \beta + 1 \leqslant j < c \ (\textit{including } d = \beta + 1) \\ 1 - p, & j = c < \alpha + 1 \\ 1, & j = c = \alpha + 1 \ \textit{or} \ j > c. \end{cases}$$

(ii) *If $\beta \geqslant c > d$, then*

$$I_j\left(\begin{pmatrix} c \\ d \end{pmatrix}, \ell\right) = \begin{cases} -p^3, & 1 \leqslant j < d \\ p^2 - p^3, & j = d \\ p^2, & d + 1 \leqslant j < c \\ (1 + p^2)(1 - p^{-1}), & j = c \\ 1, & j > c. \end{cases}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

(iii) *If* $c = d \leqslant \beta$, *then*

$$I_j\left(\binom{d}{d}, \ell\right) = \begin{cases} -p^3, & 1 \leqslant j < d \\ (1+p^2)(1-p), & j = d \\ 1, & j > d. \end{cases}$$

(iv) *If* $c = \beta + 1$, $d \leqslant \beta$, *then*

$$I_j\left(\binom{\beta+1}{d}, \ell\right) = \begin{cases} -p^3, & 1 \leqslant j < d \\ p^2 - p^3, & j = d \\ p^2, & d+1 \leqslant j < \beta+1 \\ -p^{-1} + 1 - p, & j = \beta+1 < \alpha+1 \\ 1 - p^{-1}, & j = \beta+1 = \alpha+1 \\ 1, & j > \beta+1. \end{cases}$$

(v) *If* $c = d = \beta + 1$, *then*

$$I_j\left(\binom{\beta+1}{\beta+1}, \ell\right) = \begin{cases} -p^3, & 1 \leqslant j < \beta+1 \\ 1-p, & j = \beta+1 < \alpha+1 \\ 1, & j > \beta+1 \text{ or } j = \beta+1 = \alpha+1. \end{cases}$$ □

Next, we give a pair of lemmas describing inductive formulae for the representation densities.

Lemma 3.4. *Suppose that* $T^+ = \mathrm{diag}(p^{a+2}, p^b)$ *and* $T = \mathrm{diag}(p^a, p^b)$ *for a pair of integers* a, b *such that* $a + b$ *is even. Then*

$$F(S, T^+; X) - F(S, T; X) = \frac{X^{a+1}(X+p)(X-1)}{X-p}(-p^{b+1}(X-1) + pX^{b+1} - p^{-1}X^{b+2}),$$

where $S = \mathrm{diag}(p, 1)$.

Proof. Let $\Lambda = (a+2, b)$ and $\lambda = (a, b)$, and abbreviate

$$F(\Lambda) := F(S, T^+, X) \quad \text{and} \quad F(\lambda) := F(S, T, X).$$

Note that, if $c \leqslant a$, then $I_j\left(\binom{c}{d}, \Lambda\right) = I_j\left(\binom{c}{d}, \lambda\right)$, and so

$$\begin{aligned} F(\Lambda) - F(\lambda) &= -p^{-a}X^{a+1}\sum_{d=0}^{b+1}(-1)^d p^{-2d}X^d(-p)^{\epsilon_d} \\ &\quad \cdot \left\{\prod_j I_j\left(\binom{a+1}{d}, \Lambda\right) - \prod_j I_j\left(\binom{a+1}{d}, \lambda\right)\right\} \end{aligned}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

$$
\begin{aligned}
&- p^{-(a+1)} X^{a+2} \sum_{d=0}^{b+1} (-1)^d p^{-2d} X^d (-p)^{\epsilon_d} \prod_j I_j\left(\binom{a+2}{d}, \Lambda\right) \\
&- p^{-(a+2)} X^{a+3} \sum_{d=0}^{b+1} (-1)^d p^{-2d} X^d (-p)^{\epsilon_d} \prod_j I_j\left(\binom{a+3}{d}, \Lambda\right) \\
= &- p^{-a} X^{a+1} \sum_{d=0}^{b+1} (-1)^d (-p)^{-2d+\epsilon_d} X^d \\
&\times \left\{ \prod_j I_j\left(\binom{a+1}{d}, \Lambda\right) - \prod_j I_j\left(\binom{a+1}{d}, \lambda\right) \right. \\
&\left. + \frac{X}{p} \prod_j I_j\left(\binom{a+2}{d}, \Lambda\right) + \frac{X^2}{p^2} \prod_j I_j\left(\binom{a+3}{d}, \Lambda\right) \right\}.
\end{aligned}
$$

The term in curly braces can be computed explicitly using Lemma 3.3, and the result readily follows. □

Lemma 3.5. *Suppose that* $T^+ = \mathsf{diag}(p^{b+2}, p^{b+2})$ *and* $T = \mathsf{diag}(p^{b+2}, p^b)$, *and set* $S = \mathsf{diag}(p, 1)$. *Then*

$$
\begin{aligned}
F(S, T^+; X) - F(S, T; X) = &\ \frac{X^{b+1}(X+p)(X-1)}{X-p} \\
&\times \left[((1+p-p^2)X - p)p^{b+1} + \frac{p^2 - X}{p} X^{b+3} \right].
\end{aligned}
$$

Proof. Set $\Lambda := (b+2, b+2)$ and $\lambda = (b+2, b)$, and abbreviate

$$
F(\Lambda) := F(S, T^+; X), \quad F(\lambda) := F(S, T; X).
$$

Note that, for $c \leqslant b$ and any j, we have $I_j\left(\binom{c}{d}, \Lambda\right) = I_j\left(\binom{c}{d}, \lambda\right)$, and so

$$
\begin{aligned}
F(\Lambda) - F(\lambda) = &\sum_{d=0}^{b+1} (-1)^d (-p)^{-2d+1+e_d} X^d \\
&\times \left\{ \sum_{c=b+1}^{b+3} p^{-c} X^c \left(\prod_{j=1}^{c} I_j\left(\binom{c}{d}, \Lambda\right) - \prod_{j=1}^{c} I_j\left(\binom{c}{d}, \lambda\right) \right) \right\} \\
&+ (-1)^{b+2} (-p)^{-2b-2} X^{b+2} \left\{ \sum_{c=b+2}^{b+3} p^{-c} X^c \prod_{j=1}^{c} I_j\left(\binom{c}{b+2}, \Lambda\right) \right\} \\
&+ (-1)^{b+3} (-p)^{-2b-4} X^{b+3} \left\{ p^{-(b+3)} X^{b+3} \prod_{j=1}^{b+3} I_j\left(\binom{b+3}{b+3}, \Lambda\right) \right\},
\end{aligned}
$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

where

$$e_d := \begin{cases} 0, & \text{if } d = 0, \\ 1, & \text{if } d > 0. \end{cases}$$

The terms in curly braces can again be computed via Lemma 3.3, and the proposition follows after straightforward algebraic manipulations. □

Proof of Proposition 3.1. First, we note that the formulae for $F(S, \mathtt{Id}_2; X)$ and $F(S, p\mathtt{Id}_2; X)$, which correspond to the cases $(a, b) = (0, 0)$ and $(a, b) = (1, 1)$, respectively, can be verified directly via (3.2), proving (i).

Next, for notational convenience, set

$$F(r, s) := F\left(S, \begin{pmatrix} p^r & \\ & p^s \end{pmatrix}; X\right),$$

where, as usual, $r \geqslant s \geqslant 0$, and $r + s$ is even. By applying Lemmas 3.4 and 3.5 in sequence, we have that, for any $r \geqslant 0$,

$$\begin{aligned} F(r+2, r+2) - F(r, r) &= \frac{X^{r+1}(X+p)(X-1)}{X-p} \\ &\quad \times \{p^{r+1}(1-p)(pX+1) + X^{r+1}(1+X^2)(p - p^{-1}X)\}, \end{aligned}$$

which upon repeated application yields the formula

$$\begin{aligned} F(b, b) &= F(\epsilon, \epsilon) + \sum_{i=1}^{(b-\epsilon)/2} F(2i+\epsilon, 2i+\epsilon) - F(2i-2+\epsilon, 2i-2+\epsilon) \\ &= F(\epsilon, \epsilon) + \frac{(X+p)(X-1)}{X-p}\Bigg[(pX)(1-p)\frac{(pX)^b - (pX)^\epsilon}{pX-1} \\ &\quad + X^2(p - p^{-1}X)\frac{X^{2b} - X^{2\epsilon}}{X^2-1}\Bigg]. \end{aligned}$$

On the other hand, for $a \geqslant b \geqslant 0$ with $a + b$ even, the repeated application of Lemma 3.4 yields the relation

$$F(a, b) = F(b, b) + \frac{(X+p)(X-1)}{X-p}\left[-p^{b+1}(X-1) + pX^{b+1} - p^{-1}X^{b+2}\right] \cdot \frac{X^{a+1} - X^{b+1}}{X^2-1},$$

which then implies the proposition after a little straightforward algebra. □

The motivation for our calculations so far is to facilitate computing the derivative

$$\alpha'(S, T) := -\left[\frac{\partial}{\partial X} F(S, T; X)\right]_{X=1}.$$

Corollary 3.6. *Let $T \in \mathrm{Herm}_2(o_{k,p})$ such that $T \sim \mathsf{diag}(p^a, p^b)$, where $a \geqslant b \geqslant 0$ and $a + b$ is even. Then*

$$\frac{p}{(p+1)^2}\left[\alpha'\left(\begin{pmatrix} p & \\ & 1 \end{pmatrix}, T\right) + \frac{p^2}{1-p^2}\alpha(\mathtt{Id}_2, T)\right] = \frac{a+b}{2} - p\left(\frac{p^b - 1}{p-1}\right) =: \mu_p(T).$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Proof. We recall Nagaoka's formula [12] for the representation density for $S = \mathtt{Id}_2$:

$$F(\mathtt{Id}_2, T; X) = (1 + p^{-1}X)(1 - p^{-2}X) \sum_{\ell=0}^{b} (pX)^{\ell} \sum_{k=0}^{a+b-2\ell} (-X)^k.$$

Thus

$$\alpha(\mathtt{Id}_2, T) = F(\mathtt{Id}_2, T; X)_{X=1} = (1 + p^{-1})(1 - p^{-2}) \frac{p^{b+1} - 1}{p - 1}.$$

On the other hand, the derivative $\alpha'\left(\begin{pmatrix}1 & \\ & p\end{pmatrix}, T\right)$ can be computed directly from Proposition 3.1, and the proposition follows via straightforward algebraic manipulation. □

4. Global cycles and Eisenstein series

In this section, we turn to global aspects: we describe the Shimura varieties of interest and their global cycles, the construction of the relevant Eisenstein series, and prove our main theorem relating the two. Our presentation and approach is closely modelled on the account given by Kudla and Rapoport in [8]: we shall refer freely to the results therein and content ourselves in the present work to describing the necessary modifications to their arguments as the need arises.

4.1. Preliminaries on Hermitian spaces

Here we recall some basic notions about Hermitian spaces. Let V be a Hermitian space over k of signature (r, s). To every rational place $\ell \leqslant \infty$, there is an associated *invariant*

$$\mathrm{inv}_\ell(V) := (\det(V), \Delta)_\ell = \begin{cases} 1, & \text{if } \det(V) \in N(k_\ell^\times), \\ -1, & \text{if } \det(V) \notin N(k_\ell^\times), \end{cases}$$

where $\det(V) = \det((\mathbf{v}, \mathbf{v})) \in \mathbb{Q}_\ell^\times / N(k_\ell^\times)$ is the determinant of the matrix of inner products of any basis $\mathbf{v} = \{v_1, \ldots, v_{r+s}\}$ of V. In particular, $\mathrm{inv}_\infty(V) = (-1)^s$, and, if ℓ is split, then $\mathrm{inv}_\ell(V) = 1$. These invariants satisfy the product formula

$$1 = \prod_{\ell \leqslant \infty} \mathrm{inv}_\ell(V).$$

Using the same definition, we may also define the local invariant $\mathrm{inv}_\ell(V_\ell) = (\det V_\ell, \Delta)_\ell$ for a Hermitian space V_ℓ over k_ℓ. When ℓ is a finite prime, two local Hermitian vector spaces are isometric if and only if they have the same dimension and their invariants are equal. If $\ell = \infty$, there is a unique isometry class for each signature.

Suppose that we are given a collection of signs $(a_p)_{p \leqslant \infty}$, almost all of which are equal to 1, satisfying the product formula $\prod_{p \leqslant \infty} a_p = 1$. Then, for any pair of integers (r, s) such that $(-1)^s = a_\infty$, there exists a Hermitian space V over k of signature (r, s) such that $\mathrm{inv}_\ell(V) = a_\ell$ for all ℓ, and furthermore V is unique up to isometry; in other words, the Hasse principle holds for Hermitian vector spaces.

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Finally, let L be an o_k-Hermitian lattice, i.e., a projective o_k-module of finite rank equipped with an o_k-Hermitian form, and set $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. The *genus* $[L]$ is the set of isomorphism classes of lattices $M \subset V$ such that

$$M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \simeq L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$$

as Hermitian $\widehat{o_k}$-modules. In other words, $M \in [L]$ if and only if $M_{\mathbb{Q}} = V$, and, for every finite prime ℓ, there exists an element $g_\ell \in U(V_\ell)$ such that $g_\ell(M_\ell) = L_\ell$.

Suppose that either $\ell \neq 2$ or $\ell = 2$ is unramified in k, and that Λ is a self-dual $o_{k,\ell}$-lattice. Then, by [4], the set of self-dual lattices in $V_\ell = \Lambda_{\mathbb{Q}_\ell}$ forms a single $U(V_\ell)$-orbit. Furthermore, if ℓ is inert, then the existence of a self-dual lattice forces $\mathrm{inv}_\ell(V_\ell) = 1$.

When $\ell \neq 2$ is an inert prime, we say that an $o_{k,\ell}$ lattice Λ is *almost self-dual* if $\Lambda^\# / \Lambda \simeq \mathbb{F}_{\ell^2}$, where $\Lambda^\#$ is the dual lattice. In this case, $\mathrm{inv}(\Lambda_{\mathbb{Q}_\ell}) = -1$, and the set of almost self-dual lattices in $\Lambda_{\mathbb{Q}_\ell}$ again forms a single orbit under the action of $U(\Lambda_{\mathbb{Q}_\ell})$.

4.2. Global moduli problems and p-adic uniformizations

Fix an odd squarefree integer d whose prime factors are all inert in k. We define the moduli space of almost-principally polarized abelian surfaces as follows.

Definition 4.1. Let $\mathrm{M}^d_{(1,1)}$ denote the Deligne–Mumford stack over $\mathrm{Spec}(o_k)$ defined by the following moduli problem: for a scheme S over $\mathrm{Spec}(o_k)$, the points $\mathrm{M}^d_{(1,1)}(S)$ parametrize the category of tuples $\underline{A} = (A, i_A, \lambda_A)$, where

(i) A is an abelian surface over S;

(ii) $i_A : o_k \to \mathrm{End}(A)$ is an o_k-action satisfying the following *signature* $(1,1)$ *condition*: on (the locally free $\mathcal{O}_S$-module) $\mathsf{Lie}(A)$, the induced action has characteristic polynomial

$$\det(T - i_A(a)|_{\mathsf{Lie}(A)}) = (T - a)(T - a') \in \mathcal{O}_S[T] \quad \text{for all } a \in o_k; \textit{ and} \tag{4.1}$$

(iii) λ_A is a polarization such that the induced Rosati involution $*$ satisfies

$$i_A(a)^* = i_A(a').$$

In addition, we require that

$$\ker(\lambda_A) \subset A[d], \quad \text{and} \quad |\ker(\lambda_A)| = d^2.$$

Proposition 4.2. $\mathrm{M}^d_{(1,1)}$ *is flat over* $\mathrm{Spec}(o_k)$ *and smooth over* $\mathrm{Spec}\, o_k[(d \cdot \Delta)^{-1}]$.

Proof. This follows from combining the results of [8, § 2], for primes away from d, with [9] for those primes dividing d. □

We also set $\mathcal{E}$ to be the DM stack over $\mathrm{Spec}(o_k)$ that parameterizes principally polarized elliptic curves with multiplication by o_k. More precisely, for a scheme S/o_k, the points $\mathcal{E}(S)$ parameterize tuples $\underline{E} = (E, i_E, \lambda_E)$, where

(i) E/S is an elliptic curve;

(ii) $i_E : o_k \to \mathrm{End}(E)$ satisfies the *signature* $(1,0)$ *condition*: concretely, this means that on $\mathsf{Lie}(E)$,

$$i_E(a)|_{\mathsf{Lie}(E)} = \tau_S(a) \quad \text{for all } a \in o_k,$$

where $\tau_S : o_k \to \mathcal{O}_S$ is the structural morphism; and

(iii) λ_E is a principal polarization whose corresponding Rosati involution induces Galois conjugation on $i_E(o_k)$.

This is the stack denoted by $\mathcal{M}(1,0)^{naive}$ in the notation of [8]; in particular, it is proper over $\mathrm{Spec}(o_k)$ of relative dimension 0.

Finally, we set

$$\mathcal{M} := \mathcal{E} \times_{\mathrm{Spec}\, o_k} \mathrm{M}^d_{(1,1)}.$$

Next, we describe the *Kudla–Rapoport* cycles on $\mathcal{M}$, as introduced in [8, § 2]. Given a scheme S/o_k and a point $(\underline{E}, \underline{A}) = (E, i_E, \lambda_E, A, i_A, \lambda_A) \in \mathcal{M}(S)$, the space of *special homomorphisms*

$$\mathrm{Hom}_{o_k,S}(E, A)$$

can be equipped with a positive-definite Hermitian form, defined by the formula

$$(x, y) := \lambda_E^{-1} \circ y^\vee \circ \lambda_A \circ x \in \mathrm{End}_{o_k}(E) \simeq o_k. \tag{4.2}$$

Definition 4.3. (i) Let $m \in \mathbb{Z}_{>0}$. We define the special cycle $\mathfrak{Z}(m)$ to be the moduli space over $\mathrm{Spec}\, o_k$ whose S points parameterize triples

$$\mathfrak{Z}(m)(S) = \{(\underline{E}, \underline{A}; y) \mid (\underline{E}, \underline{A}) \in \mathcal{M}(S) \text{ and } y \in \mathrm{Hom}_{o_k,S}(E, A) \text{ with } (y, y) = m\}.$$

(ii) Suppose that $T \in \mathrm{Herm}_2(o_k)$. We define $\mathfrak{Z}(T)$ to be the moduli space over $\mathrm{Spec}\, o_k$ whose S points parameterize tuples

$$\mathfrak{Z}(T)(S) = \{(\underline{E}, \underline{A}; \mathbf{y}) \mid (\underline{E}, \underline{A}) \in \mathcal{M}(S) \text{ and } \mathbf{y} \in \mathrm{Hom}_{o_k,S}(E, A)^2 \text{ with } (\mathbf{y}, \mathbf{y}) = T\},$$

where, for $\mathbf{y} = (y_1, y_2)$, the matrix $(\mathbf{y}, \mathbf{y})$ is the matrix of inner products

$$(\mathbf{y}, \mathbf{y}) := \begin{pmatrix} (y_1, y_1) & (y_1, y_2) \\ (y_2, y_1) & (y_2, y_2) \end{pmatrix} \in \mathrm{Herm}_2(o_k).$$

Both moduli problems are represented by DM stacks. Furthermore, the natural forgetful maps to $\mathcal{M}$ are finite and unramified, see [8, Proposition 2.9], and so their images can be viewed as cycles on $\mathcal{M}$. In what follows, we shall abuse notation and use the symbols $\mathfrak{Z}(m)$ and $\mathfrak{Z}(T)$ to denote both the representing stacks and the corresponding cycles on $\mathcal{M}$, and hope that the intended meaning can be inferred from the context.

The aim of this section is to compute the *arithmetic degree* of a cycle $\mathfrak{Z}(T)$, which we define as follows: suppose that

$$T = \begin{pmatrix} m_1 & a \\ a' & m_2 \end{pmatrix} \in \mathrm{Herm}_2(o_k),$$

where $m_1, m_2 \in \mathbb{Z}_{>0}$. A glance at the definitions above reveals that

$$\mathfrak{Z}(m_1) \times_{\mathcal{M}} \mathfrak{Z}(m_2) = \coprod_{T' = \begin{pmatrix} m_1 & * \\ * & m_2 \end{pmatrix}} \mathfrak{Z}(T') \supset \mathfrak{Z}(T).$$

Suppose that T is positive definite. As we shall shortly see (see Lemma 4.9), we have $\mathfrak{Z}(T)_{\mathbb{Q}} = \emptyset$, and so $\mathfrak{Z}(T)$ is supported in the fibres of finitely many finite primes. We then define the arithmetic degree in this setting to be the Serre intersection multiplicity

$$\widehat{\deg}\, \mathfrak{Z}(T) := \sum_{\mathfrak{p} \subset o_k} \chi(\mathfrak{Z}(T)_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{Z}(m_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathfrak{Z}(m_2)}) \log(N(\mathfrak{p}))$$

of $\mathfrak{Z}(m_1) \times \mathfrak{Z}(m_2)$ in $\mathfrak{Z}(T)$.

We begin by describing convenient decompositions of the space $\mathcal{M}$ and the special cycles, in terms of genera of Hermitian lattices.

Definition 4.4.

1. Let $\mathcal{R}_d$ denote the set of isomorphism classes of genera $[L]$, where L is a Hermitian o_k-lattice such that
   - (i) $V := L \otimes_{\mathbb{Z}} \mathbb{Q}$ is a Hermitian space of signature $(1, 1)$;
   - (ii) $L_\ell := L \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is a self-dual $o_{k,\ell}$-lattice for all $\ell \nmid d$; and
   - (iii) for every $\ell | d$, we require that L_ℓ is an almost self-dual lattice; equivalently, we require that L_ℓ is a maximal lattice in a non-split Hermitian space of dimension 2 over k_ℓ.
2. Let $\mathcal{R}_0$ denote the set of isomorphism classes of genera $[L_0]$, where L is a self-dual Hermitian o_k-lattice and $V_0 = L_{0,\mathbb{Q}}$ is of signature $(1, 0)$.

Here, we consider two genera $[L]$ and $[L']$ to be isomorphic if and only if there are representatives $L \in [L]$ and $L' \in [L']$ such that $L \simeq L'$ as Hermitian o_k-modules.

Suppose that p is a prime, let $\overline{\mathbb{F}}_p$ denote an algebraic closure of $\mathbb{F}_p$, and fix a trivialization

$$\widehat{\mathbb{Z}}^p(1) := \prod_{\ell \neq p} \mu_{\ell^\infty}(\overline{\mathbb{F}}_p) \simeq \widehat{\mathbb{Z}}^p$$

of the prime-to-p roots of unity over $\overline{\mathbb{F}}_p$. Given a geometric point $\underline{A} = (A, i_A, \lambda_A) \in \mathrm{M}^d_{(1,1)}(\overline{\mathbb{F}}_p)$, the prime-to-$p$ Tate module

$$\mathtt{Ta}^p(A) := \prod_{\ell \neq p} \mathtt{Ta}_\ell(A)$$

is an $\widehat{o_k}^p$-module via the action induced by i_A. The polarization λ_A determines a Weil pairing

$$e_{\lambda_A} : \mathtt{Ta}^p(A) \times \mathtt{Ta}^p(A) \to \widehat{\mathbb{Z}}^p(1) \simeq \widehat{\mathbb{Z}}^p,$$

which in turn induces a Hermitian form $(\cdot, \cdot)_{\lambda_A}$ by the formula

$$(x, y)_{\lambda_A} = \tfrac{1}{2}(e_{\lambda_A}(i(\sqrt{\Delta})(x), y) + \sqrt{\Delta} \cdot e_{\lambda_A}(x, y)).$$

Lemma 4.5. *Suppose that $p \neq 2$.*

(i) *For every $\underline{A} \in \mathrm{M}^d_{(1,1)}(\overline{\mathbb{F}}_p)$, there is a unique genus $[L(A)] \in \mathcal{R}_d$ such that*

$$\mathtt{Ta}^p(A) \simeq \widehat{L(A)}^p := L(A) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$$

as Hermitian $\widehat{o_k}^p$-modules.

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

(ii) *The genus* $[L(A)]$ *depends only on the connected component of* $\mathrm{M}^d_{(1,1)}$ *that contains the point* A.

(iii) *For each* $[L] \in \mathcal{R}_d$, *there exists a point* $\underline{A} \in \mathrm{M}^d_{(1,1)}(\overline{\mathbb{F}}_p)$ *with* $[L(A)] \simeq [L]$.

Proof. (i) To prove uniqueness, suppose that $L, L' \in \mathcal{R}_d$ with $\widehat{L}^p \simeq \widehat{L'}^p$. Since L and L' have the same signature, the completions $L \otimes \mathbb{R}$ and $L' \otimes \mathbb{R}$ are also isomorphic, and so the Hasse principle implies that

$$L \otimes_{\mathbb{Z}} \mathbb{Q}_p \simeq L' \otimes_{\mathbb{Z}} \mathbb{Q}_p.$$

By definition of $\mathcal{R}_d$, the localizations L_p and L'_p are self-dual lattices when $p \nmid d$, and are identified with the unique maximal lattice in the two-dimensional non-split Hermitian space over k_p when $p|d$. In either case, they are isometric, and so L and L' lie in the same genus.

The existence claim is proved in [8, Proposition 2.12] when $p \nmid d$. When $p|d$, the claim follows from [10, Proposition 3.5]; see also the proof of Theorem 6.1 of [10]. The idea is roughly as follows: one shows that there exists a point $\underline{A}' \in \mathrm{M}^d_{(1,1)}(\mathbb{C})$ such that

$$\mathtt{Ta}^p(A) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathtt{Ta}^p(A') \otimes_{\mathbb{Z}} \mathbb{Q}$$

as Hermitian $k \otimes \mathbb{A}_f^p$-modules. Note that the homology group $V = H_1(A', \mathbb{Q})$ is a Hermitian space of signature $(1, 1)$. We may then find a lattice $L \subset V$ by identifying $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$ with $\mathtt{Ta}^p(A)$, and insisting that L_p is self-dual (respectively, almost self-dual) when $p \nmid d$ (respectively, $p|d$). One then needs to verify that the genus $[L]$ is independent of all choices, and that $[L] \in \mathcal{R}_d$.

(ii) This follows immediately from the proof of [8, Proposition 2.12].

(iii) This follows from a straightforward modification of the proof of [8, Lemma 5.1]. □

Similar assertions hold for the stack $\mathcal{E}$: for each geometric point $\underline{E} \in \mathcal{E}(\overline{\mathbb{F}}_p)$, there is a unique $[L_0(E)] \in \mathcal{R}_0$ such that

$$\mathtt{Ta}^p(E) \simeq \widehat{L_0(E)}^p,$$

which depends only on the connected component containing $\underline{E}$, and every element of $\mathcal{R}_0$ appears in this way.

Our next task is to apply the *p-adic uniformizations* of Rapoport and Zink, which relate $\mathcal{M} = \mathcal{E} \times \mathrm{M}^d_{(1,1)}$ and the special cycles to the moduli spaces of p-divisible groups that appeared in § 2. In the following, we fix an odd prime p. Let $\widehat{\mathrm{M}}$ denote the formal completion of $\mathrm{M}^d_{(1,1)}$ along its fibre at p. By Lemma 4.5, we have a decomposition

$$\widehat{\mathrm{M}} = \coprod_{[L] \in \mathcal{R}_d} \widehat{\mathrm{M}}^{[L]} \tag{4.3}$$

into components that are characterized by the property that, for any geometric point $\underline{A} \in \widehat{\mathrm{M}}^{[L]}(\overline{\mathbb{F}}_p)$, there exists an isomorphism $\mathrm{Ta}^p(A) \simeq \widehat{L}^p$ of $\widehat{o_k}^p$-Hermitian modules.

Fix a lattice $L \in \mathcal{R}_d$ as above, and set $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $K^p = \mathrm{Stab}(\widehat{L}^p) \subset U(V)(\mathbb{A}_f^p)$ denote the stabilizer of $\widehat{L}^p$.

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Next, we let V' be the Hermitian space of dimension 2 over k whose invariants differ from those of V at exactly p and ∞. Fixing an isomorphism

$$V' \otimes_{\mathbb{Q}} \mathbb{A}_f^p \simeq V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$$

induces an embedding

$$U(V')(\mathbb{Q}) \hookrightarrow U(V')(\mathbb{A}_f^p) \simeq U(V)(\mathbb{A}_f^p),$$

and $U(V')(\mathbb{Q})$ acts on $U(V)(\mathbb{A}_f^p)/K^p$ by left multiplication.

Finally, fix a geometric point $\underline{\mathbf{A}} \in \mathrm{M}^d_{(1,1)}(\overline{\mathbb{F}}_p)$ with $[L(\mathbf{A})] = [L]$. When $p|d$, the corresponding p-divisible group

$$\underline{\mathbb{X}} := \underline{\mathbf{A}}[p^\infty],$$

together with the induced polarization and $o_{k,p}$-action, serves as a base point for the Drinfeld upper half-plane $\mathcal{D}$; see Definition 2.3. Noting that V'_p is split by construction, the proof of Lemma 2.2 gives an identification

$$U(V')(\mathbb{Q}_p) \simeq \{\phi \in (\mathrm{End}_{o_{k,p}}(\underline{\mathbb{X}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\times \mid \phi^*\lambda_{\mathbb{X}} = \lambda_{\mathbb{X}}\},$$

and consequently $U(V')(\mathbb{Q}_p)$ acts on $\mathcal{D}$ by the formula $\phi \cdot (X, i, \lambda, \rho) \mapsto (X, i, \lambda, \phi \circ \rho)$.

With this notation in place, we obtain the following p-adic uniformization theorem, which is a special case of the general results of [13]; details regarding this particular case can be found in [10, Theorem 6.11].

Theorem 4.6. *Suppose that $p|d$, so in particular $p \neq 2$, and let $W = W(\overline{\mathbb{F}}_p)$ denote the ring of Witt vectors. Then there is an isomorphism of formal stacks*

$$\widehat{\mathrm{M}}^{[L]} \times_{o_{k,p}} \mathrm{Spf}\, W \simeq [U(V')(\mathbb{Q}) \backslash \mathcal{D} \times (U(V)(\mathbb{A}_f^p)/K^p)]. \qquad \square \quad (4.4)$$

Completely analogous results hold for the stack $\mathcal{E}$ at an odd inert prime p: let $L_0 \in \mathcal{R}_0$ be a lattice, and let $\widehat{\mathcal{E}}^{[L_0]}$ be the corresponding component of the formal completion $\widehat{\mathcal{E}}$, in analogy with (4.3). Fixing a geometric point $\underline{\mathbf{E}} = (\mathbf{E}, i_{\mathbf{E}}, \lambda_{\mathbf{E}}) \in \mathcal{E}(\mathbb{F})$, we have isomorphisms

$$\{\phi \in \mathrm{End}_{o_k}(\mathbf{E})_{\mathbb{Q}}^\times \mid \phi^*\lambda_{\mathbf{E}} = \lambda_{\mathbf{E}}\} \simeq k^1 \simeq U(V_0)(\mathbb{Q}),$$

where k^1 is the group of norm-1 elements of k, and $V_0 := L_0 \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $\underline{\mathbb{Y}} = \underline{\mathbf{E}}[p^\infty]$ denote the p-divisible group attached to $\mathbf{E}$, which comes equipped with the induced $o_{k,p}$-action and principal polarization, and which serves as the base point for the moduli space $\mathcal{D}_0$; see Definition 2.6. The reader is reminded that $\mathcal{D}_0 \simeq \mathrm{Spf}\, W$.

By [8, Theorem 5.5], and in analogy with Theorem 4.6,

$$\widehat{\mathcal{E}}^{[L_0]} \times_{o_{k,p}} \mathrm{Spf}\, W \simeq [U(V_0)(\mathbb{Q}) \backslash \mathcal{D}_0 \times U(V_0)(\mathbb{A}_f^p)/K_0^p], \qquad (4.5)$$

where $K_0^p \subset U(V_0)(\mathbb{A}_f^p)$ is the stabilizer of $\widehat{L_0}^p$.

Remark 4.7. Fix an isomorphism $V_0 \simeq k$, under which the lattice L_0 is identified with a fractional ideal $\mathfrak{a}$, and such that the Hermitian form is given by

$$(x, y) \mapsto \frac{1}{N(\mathfrak{a})} x y', \quad x, y \in k;$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

note that L_0 determines the class of $\mathfrak{a}$ in the ideal class group $Cl(k)$. Under this identification, we have $U(V_0)(\mathbb{A}_f^p) \simeq (\mathbb{A}_{k,f}^p)^1$, and the stabilizer K_0^p of $\widehat{L_0}^p$ is identified with $\widehat{o_k}^{p,1}$. Recalling that $\mathcal{D}_0 \simeq \operatorname{Spf} W$,

$$\widehat{\mathcal{E}}^{[L_0]} \times_{o_{k,p}} \operatorname{Spf} W \simeq [k^1 \backslash \mathbb{A}_{k,f}^{p,1} / \widehat{o_k}^{p,1}]. \qquad \diamond$$

Let $\widehat{\mathcal{M}}$ denote the formal completion of $\mathcal{M} = \mathcal{E} \times_{o_k} \mathrm{M}^d_{(1,1)}$ along its fibre at p. We have a decomposition

$$\widehat{\mathcal{M}} = \coprod_{\substack{[L_0] \in \mathcal{R}_0 \\ [L] \in \mathcal{R}_d}} \widehat{\mathcal{M}}^{[L_0],[L]},$$

where $\widehat{\mathcal{M}}^{[L_0],[L]} := \widehat{\mathcal{E}}^{[L_0]} \times \widehat{\mathrm{M}}^{[L]}$. Combining (4.4) and (4.5) yields the p-adic uniformization

$$\begin{aligned}&\widehat{\mathcal{M}}^{[L_0],[L]} \times \operatorname{Spf} W \\ &\quad \simeq [U(V_0)(\mathbb{Q}) \times U(V')(\mathbb{Q}) \backslash (\mathcal{D}_0 \times U(V_0)(\mathbb{A}_f^p)/K_0^p) \times (\mathcal{D} \times U(V)(\mathbb{A}_f^p)/K^p)].\end{aligned}$$

We now turn to the p-adic uniformization of the special cycles. Suppose that $T \in \operatorname{Herm}_2(o_k)$, let $\widehat{\mathfrak{Z}}(T)$ be the formal completion along its fibre at p, and let

$$\widehat{\mathfrak{Z}}(T)^{[L_0],[L]} := \widehat{\mathfrak{Z}}(T) \ \times_{\widehat{\mathcal{M}}} \ \widehat{\mathcal{M}}^{[L_0],[L]}$$

denote the component corresponding to the pair of genera $([L_0],[L])$. As before, we fix a pair of base points $(\underline{\mathbf{E}}, \underline{\mathbf{A}}) \in \widehat{\mathcal{M}}^{[L_0],[L]}(\mathbb{F})$, with corresponding p-divisible groups $(\underline{\mathbb{Y}}, \underline{\mathbb{X}})$. By considering local invariants, one can check that there is an isomorphism of Hermitian spaces

$$V' \simeq \operatorname{Hom}_{o_k}(\mathbf{E}, \mathbf{A})_{\mathbb{Q}}, \tag{4.6}$$

where V' is the Hermitian space whose invariants differ from those of $V = L_{\mathbb{Q}}$ at exactly p and ∞, and the Hermitian form on $\operatorname{Hom}_{o_k}(\mathbf{E}, \mathbf{A})_{\mathbb{Q}}$ is given by (4.2).

Given an element $x \in V'$, we may use (4.6) and take the corresponding completions to obtain elements

$$\mathtt{Ta}^p(x) \in \operatorname{Hom}_{k \otimes \mathbb{A}_f^p}(\mathtt{Ta}^p(\mathbf{E})_{\mathbb{Q}}, \mathtt{Ta}^p(\mathbf{A})_{\mathbb{Q}}) \quad \text{and} \quad x_p \in \operatorname{Hom}_{o_{k,p}}(\mathbb{Y}, \mathbb{X})_{\mathbb{Q}_p} = \mathbb{V}.$$

The following proposition is proved in the same way as [8, Proposition 6.3].

Proposition 4.8. *Let $T \in \operatorname{Herm}_2(o_k)$, and as usual, suppose that $p|d$. Then there is an isomorphism*

$$\widehat{\mathfrak{Z}}(T)^{[L_0],[L]} \times_{o_{k,p}} \operatorname{Spf} W \simeq \left[U(V')(\mathbb{Q}) \times U(V_0)(\mathbb{Q}) \Big\backslash \coprod_{g} \coprod_{g_0} \coprod_{\mathbf{x} \in \Omega(T,g,g_0)} Z(\mathbf{x}_p) \right],$$

where g and g_0 range over $U(V)(\mathbb{A}_f^p)/K^p$ and $U(V_0)(\mathbb{A}_f^p)/K_0^p$, respectively,

$$\begin{aligned}&\Omega(T, g, g_0) \\ &\quad := \{\mathbf{x} = [x_1, x_2] \in (V')^2 \mid (\mathbf{x}, \mathbf{x}) = T \ \textit{and} \ g^{-1} \circ \mathrm{Ta}^p(x_i) \circ g_0 \in \operatorname{Hom}(L_0, L) \otimes \widehat{\mathbb{Z}}^p\},\end{aligned}$$

and $Z(\mathbf{x}_p)$ is the local cycle corresponding to $\mathbf{x}_p = [x_{1,p}, x_{2,p}]$ as in Definition 2.7. $\square$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Next, we collect some information regarding the support of a special cycle $\mathfrak{Z}(T)$.

Lemma 4.9. *Suppose that $T \in \mathrm{Herm}_2(o_k)$ is positive definite, and set*

$$\mathrm{Diff}(T) := \{\ell \nmid d \text{ inert}, \mathrm{ord}_\ell \det T \text{ odd}\} \coprod \{\ell \mid d, \mathrm{ord}_\ell \det T \text{ even}\}.$$

Also, let V_T denote the vector space k^2 with Hermitian form given by the matrix T.

(i) *The generic fibre $\mathfrak{Z}(T)_{\mathbb{Q}}$ is empty.*

(ii) *Suppose that $\mathfrak{Z}(T)(\overline{\mathbb{F}}_v) \neq \emptyset$. Then, for all inert $\ell \neq v$, we have $\mathrm{inv}_\ell V_T = -1$ if $\ell | d$ and $\mathrm{inv}_\ell V_T = 1$ if $\ell \nmid d$.*

(iii) *If $\#\mathrm{Diff}(T) > 1$, then $\mathfrak{Z}(T) = \emptyset$.*

(iv) *If $\mathrm{Diff}(T) = \{\ell\}$ is a single odd inert prime, then (a) the support of $\mathfrak{Z}(T)$ is contained in the fibre $\mathcal{M}_\ell$ at ℓ; and (b) we have a decomposition*

$$\widehat{\mathfrak{Z}}(T) = \coprod_{[L_0] \in \mathcal{R}_0} \coprod_{\substack{[L] \in \mathcal{R}_d \\ L \otimes \mathbb{A}_f^\ell \simeq V_T \otimes \mathbb{A}_f^\ell}} \widehat{\mathfrak{Z}}(T)^{[L_0],[L]}. \tag{4.7}$$

Proof. We argue as in [8, Proposition 2.22]. Suppose that F is an algebraically closed field, and that $(\underline{E}, \underline{A}, \mathbf{x}) \in \mathfrak{Z}(T)(F)$ is a geometric point. Since T is non-degenerate, the pair $\mathbf{x} = [x_1, x_2]$ determines an isomorphism

$$V_T \simeq \mathrm{Hom}_{o_k}(E, A)_{\mathbb{Q}}$$

of Hermitian spaces.

If $F = \mathbb{C}$, then we have an embedding

$$V_T \simeq \mathrm{Hom}_{o_k}(E, A)_{\mathbb{Q}} \hookrightarrow \mathrm{Hom}_{o_k}(H_1(E, \mathbb{Q}), H_1(A, \mathbb{Q})) \simeq H_1(A, \mathbb{Q}),$$

where $H_1(A, \mathbb{Q})$ is endowed with the unique Hermitian form $(\cdot, \cdot)$ such that the Riemann form $\langle \cdot, \cdot \rangle_{\lambda_A}$ induced by λ_A satisfies

$$\left\langle i(\sqrt{\Delta})x, y \right\rangle_{\lambda_A} = \tfrac{1}{2} tr_{k/\mathbb{Q}}(x, y).$$

The signature of $H_1(A, \mathbb{Q})$ is $(1, 1)$ by the signature condition (2.7), while the signature of V_T is $(2, 0)$ by assumption; hence we obtain a contradiction that proves (i).

Suppose next that F has characteristic $v > 0$, so that

$$V_T(\mathbb{A}_f^v) \simeq \mathrm{Hom}_{\mathbb{A}_{k,f}^v}(\mathrm{Ta}^v(E)_{\mathbb{Q}}, \mathrm{Ta}^v(A)_{\mathbb{Q}}). \tag{4.8}$$

Let $\ell \neq v$ be an inert prime. Then, if $\ell \nmid d$, the space $V_{T,\ell}$ contains a self-dual lattice, and hence $\mathrm{inv}_\ell V_T = 1$; if $\ell | d$, then $V_{T,\ell}$ is the non-split Hermitian space over k_ℓ, and so $\mathrm{inv}_\ell V_T = -1$. This proves (ii), from which (iii) and the statement regarding the support of $\mathfrak{Z}(T)$ in (iv) follow easily.

Finally, it follows from (4.8) and the definition of the component $\widehat{\mathfrak{Z}}(T)^{[L_0],[L]}$ that there is a decomposition

$$\widehat{\mathfrak{Z}}(T) = \coprod_{[L_0],[L]} \widehat{\mathfrak{Z}}(T)^{[L_0],[L]},$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

where the union is over genera $[L_0]$ and $[L]$ in $\mathcal{R}_0$ and $\mathcal{R}_d$, respectively, such that

$$\mathrm{Hom}(L_0, L) \otimes_{\mathbb{Z}} \mathbb{A}_f^{\ell} \simeq V_T \otimes_{\mathbb{Q}} \mathbb{A}_f^{\ell}.$$

However, note that there is an isomorphism of k-Hermitian spaces

$$\mathrm{Hom}(L_0, L)_{\mathbb{Q}} \simeq L_{\mathbb{Q}},$$

since

$$\det(\mathrm{Hom}(L_0, L)_{\mathbb{Q}}) = \det(L_{0,\mathbb{Q}})^2 \det(L_{\mathbb{Q}}) \equiv \det(L_{\mathbb{Q}}) \mod N(k^{\times}).$$

Thus

$$\mathrm{Hom}(L_0, L) \otimes_{\mathbb{Z}} \mathbb{A}_f^{\ell} \simeq V_T \otimes_{\mathbb{Q}} \mathbb{A}_f^{\ell} \iff L \otimes_{\mathbb{Z}} \mathbb{A}_f^{\ell} \simeq V_T \otimes_{\mathbb{Q}} \mathbb{A}_f^{\ell},$$

and the second part of (iv) follows immediately. □

Let $T \in \mathrm{Herm}_2(o_k)$ be positive definite with $\mathrm{Diff}(T) = \{p\}$, where $p|d$. By the previous lemma, we may restrict our attention to lattices L such that $L \otimes_{\mathbb{Z}} \mathbb{A}_f^p \simeq V_T \otimes_{\mathbb{Q}} \mathbb{A}_f^p$.

Viewing $\widehat{L}^p = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$ as an adèlic lattice in $V_T \otimes \mathbb{A}_f^p$, we define the Schwarz function

$$\varphi_L'^p \in \mathscr{S}((V_T \otimes \mathbb{A}_f^p)^2) := \text{characteristic function of } (\widehat{L}^p)^2.$$

Let $K'^p \subset U(V_T)(\mathbb{A}_f^p)$ be the stabilizer of $\widehat{L}^p$, and write

$$U(V_T)(\mathbb{A}_f^p) = \coprod_j U(V_T)(\mathbb{Q}) h_j K'^p, \quad h_j \in U(V_T)(\mathbb{A}_f^p). \tag{4.9}$$

Finally, we define $\Gamma_j' := U(V_T)(\mathbb{Q}) \cap h_j K'^p h_j^{-1}$.

Theorem 4.10. *Suppose that $T \in \mathrm{Herm}_2(o_k)$ is positive definite with $\mathrm{Diff}(T) = \{p\}$ for $p|d$. Let $[L] \in \mathcal{R}_d$ with $L \otimes \mathbb{A}_f^p \simeq V_T \otimes \mathbb{A}_f^p$, and fix any $[L_0] \in \mathcal{R}_0$. Then, with notation as in the previous paragraph,*

$$\widehat{\deg}\, \mathfrak{Z}(T)^{[L_0],[L]} = \frac{h(k)}{|o_k^{\times}| 2^{o(\Delta)-1}} \cdot \mu_p(T) \cdot \left(\sum_j \sum_{\substack{\mathbf{x} \in (V_T)^2 \\ h(\mathbf{x}) = T \\ \mathit{mod}\ \Gamma_j'}} \varphi_L'^p (h_j^{-1} \mathbf{x}) \right) \cdot \log p^2,$$

where $o(\Delta)$ is the number of prime factors of the discriminant Δ of k, and $\mu_p(T)$ is the quantity defined in (2.11).

Proof. We may fix isomorphisms:

$$\mathrm{Hom}(L_0, L) \otimes \mathbb{A}_f^p \simeq L \otimes \mathbb{A}_f^p \simeq V_T \otimes \mathbb{A}_f^p. \tag{4.10}$$

In particular, the invariants of V_T differ from those of $V := L \otimes \mathbb{Q}$ at exactly p and ∞. Thus $V' = V_T$ in the notation of Proposition 4.8, and so

$$\widehat{\mathfrak{Z}(T)}^{[L_0],[L]} \times \mathrm{Spf}\, W \simeq \left[U(V_T)(\mathbb{Q}) \times U(V_0)(\mathbb{Q}) \Big\backslash \coprod_{g, g_0} \coprod_{\mathbf{x} \in \Omega(T, g, g_0)} Z(\mathbf{x}_p) \right],$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

where, for $x \in V_T$, we let x^p denote the image of x in $\mathrm{Hom}(L_0, L) \otimes \mathbb{A}_f^p$ in (4.10) above, and

$$\Omega(T, g, g_0) := \{\mathbf{x} = [x_1, x_2] \in (V_T)^2 \mid (\mathbf{x}, \mathbf{x}) = T \text{ and } g^{-1} \circ (x_i)^p \circ g_0 \in \mathrm{Hom}(L_0, L) \otimes \widehat{\mathbb{Z}}^p\}.$$

For a given pair $\mathbf{x} = [x_1, x_2] \in \Omega(T, g, g_0)$, we have already computed the degree of the local cycle $Z(\mathbf{x}_p)$; indeed, Corollary 2.17 tells us that

$$\chi(\mathfrak{Z}(T)_p, \mathcal{O}_{Z(x_{1,p})} \otimes^{\mathbb{L}} \mathcal{O}_{Z(x_{2,p})}) = \langle Z(x_{1,p}), Z(x_{2,p}) \rangle = \mu_p(T),$$

which in particular depends only on T. Thus

$$\widehat{\deg}\, \mathfrak{Z}(T)^{[L_0],[L]} = \mu_p(T) \cdot \# \left[U(V_T)(\mathbb{Q}) \times U(V_0)(\mathbb{Q}) \Big\backslash \coprod_{g, g_0} \Omega(T, g, g_0) \right] \cdot \log p^2,$$

where, on the right, we need to compute the 'stacky cardinality'.

Fixing momentarily an element $g_0 \in U(V_0)(\mathbb{A}_f^p) = (\mathbb{A}_{k,f}^p)^{\times,1}$, we first compute the cardinality

$$\# \left[U(V_T)(\mathbb{Q}) \Big\backslash \coprod_{g \in U(V)(\mathbb{A}_f^p)/K^p} \Omega(T, g, g_0) \right] \tag{4.11}$$

where, as we recall, $K^p \subset U(V)(\mathbb{A}_f^p)$ is the stabilizer of $\widehat{L}^p = L \otimes \widehat{\mathbb{Z}}^p$, and hence is identified with $K'^p \subset U(V_T)(\mathbb{A}_f^p)$. Without loss of generality, we may normalize (4.10) so that $\widehat{L}^p$ is identified with

$$g_0 \cdot \mathrm{Hom}(L_0, L) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p,$$

and so the quantity in (4.11) then becomes

$$\# \left[U(V_T)(\mathbb{Q}) \Big\backslash \coprod_{g \in U(V_T)(\mathbb{A}_f^p)/K'^p} \{\mathbf{x} \in (V_T)^2 \mid (\mathbf{x}, \mathbf{x}) = T \text{ and } \mathbf{x} \in (g \cdot \widehat{L}^p)^2\} \right]. \tag{4.12}$$

Since T is non-degenerate, the stabilizer of any $\mathbf{x}$ appearing in (4.12) is trivial, and so

$$(4.12) = \sum_j \sum_{\substack{\mathbf{x} \in (V_T)^2 \\ (\mathbf{x}, \mathbf{x}) = T \\ \bmod \Gamma'_j}} \varphi_L'^p(h_j^{-1} \mathbf{x}).$$

Note that this quantity is independent of g_0, L_0, the choice of representative L in its genus $[L]$, and the choices of isomorphisms in (4.10).

On the other hand,

$$U(V_0)(\mathbb{Q}) \backslash U(V_0)(\mathbb{A}_f^p)/K_0^p \simeq k^{\times,1} \backslash (\mathbb{A}_{k,f}^p)^1 / \widehat{o_k}^{p,1},$$

and the latter double quotient is easily seen to be isomorphic to $Cl(k)^2$, where $Cl(k)$ is the class group. Since the 2-torsion in $Cl(k)$ has order $2^{o(\Delta)-1}$, the order of $Cl(k)^2$ is $h(k)/2^{o(\Delta)-1}$, and so (taking automorphisms into account)

$$\#[U(V_0)(\mathbb{Q}) \backslash U(V_0)(\mathbb{A}_f^p)/K^p] = \#[k^{\times,1} \backslash \mathbb{A}_{k,f}^{p,1} / \widehat{o_k}^{p,1}] = \frac{h(k)}{|o_k^\times| \, 2^{o(\Delta)-1}}.$$

Combining these calculations yields the proposition. □

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

4.3. Non-degenerate Fourier coefficients of Siegel Eisenstein series

In this section, we recall some general definitions and formulae for the Eisenstein series attached to unitary groups that are of interest for our main theorem. Let $G = U(n, n)$ be the quasi-split unitary group, and let P be the standard Siegel parabolic; we view both as algebraic groups defined over $\mathbb{Q}$.

Fix a multiplicative character $\eta\colon \mathbb{A}_k^\times \to \mathbb{C}^\times$ whose restriction to $\mathbb{A}_\mathbb{Q}^\times$ corresponds to the quadratic character χ_k attached to the field extension $k/\mathbb{Q}$. We also let $\psi\colon \mathbb{A} = \mathbb{A}_\mathbb{Q} \to \mathbb{C}^\times$ denote the standard additive character that is trivial on $\widehat{\mathbb{Z}}$ and $\mathbb{Q}$.

Central to our investigation is a family of degenerate principal series representations, parameterized by a variable $s \in \mathbb{C}$, and defined via smooth normalized induction:

$$I(s, \eta) := \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} (\eta \circ \det)|\det|_\mathbb{A}^s.$$

A *holomorphic section* is a family of vectors $\Phi(s, \cdot) \in I(s, \eta)$ parameterized by $s \in \mathbb{C}$ such that, for each $g \in G(\mathbb{A})$, the assignment

$$s \mapsto \Phi(s, g)$$

is holomorphic as a function of s; for such a section Φ, we form the *Eisenstein series*

$$\mathbf{E}(g, s, \Phi) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(s, \gamma g), \tag{4.13}$$

at least for $Re(s)$ sufficiently large.

The aim of this section is to collect information about the Fourier coefficient $E_T(g, s, \Phi)$ for a non-degenerate matrix $T \in \mathrm{Herm}_n(o_k)$. When $Re(s)$ is sufficiently large and $\Phi = \otimes\Phi_v$ is factorizable, there is a product expansion

$$\mathbf{E}_T(g, s, \Phi) = \prod_{v \leqslant \infty} W_{T,v}(g_v, s, \Phi_v) \tag{4.14}$$

in terms of the *local Whittaker functions*

$$W_{T,v}(g_v, s, \Phi_v) := \int_{\mathrm{Herm}_n(k_v)} \Phi_v\left(\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} \cdot b \cdot g_v, s\right) \psi_v(-tr(Tb))\, \mathrm{d}b,$$

where $\mathrm{d}b$ is the additive Haar measure on $\mathrm{Herm}_n(k_v)$ normalized to be self-dual with respect to the pairing $(n_1, n_2) \mapsto \psi_v(tr(n_1 n_2))$. Each Whittaker function is entire in s. Furthermore, for any sufficiently large finite set Σ of places, there is an L-function $L^\Sigma(s)$ such that the expression

$$\mathbf{E}_T(g, s, \Phi) = L^\Sigma(s)^{-1} \cdot \prod_{v \in \Sigma} W_{T,v}(g_v, s, \Phi_v) \tag{4.15}$$

furnishes a meromorphic continuation of (4.14) to all $s \in \mathbb{C}$ that is holomorphic for $Re(s) \geqslant 0$; see [8, § 8] or [16].

Note that any standard section Φ is determined by its value $\Phi(0, \cdot) \in I(0, \eta)$, and so we would like to understand this latter space as a representation of $G(\mathbb{A})$. We begin by

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

describing the local case: let V_v be an n-dimensional Hermitian space over the local field k_v, for some place v, and let $\mathcal{S}(V_v^n)$ denote the space of Schwarz functions on V_v^n. We obtain a map

$$R_v : \mathcal{S}(V_v^n) \to I_v(0, \eta_v) = \mathrm{Ind}_{P(k_v)}^{G(k_v)}(\eta_v(\det)), \quad \text{where } R_v(\varphi_v)(g_v) = (\omega_v(g_v)\varphi_v)(0),$$

and ω_v is the local Weil representation. If $R_v(V_v)$ is the image of this map, there is a decomposition

$$I_v(0, \eta_v) = \bigoplus_{V_v} R_v(V_v)$$

into irreducible components; see [11]. Here, the sum is over isomorphism classes of Hermitian space V_v of dimension n, and so there are either 1, 2, or $n+1$ summands, corresponding to the cases v split, v non-split, and $v = \infty$, respectively. Given a Schwarz function $\varphi_v \in \mathcal{S}(V_v^n)$, we let $\Phi_{\varphi_v}(s)$ denote the unique standard section of $I_v(s, \eta_v)$ such that $\Phi_{\varphi_v}(0, \cdot) = R_v(\varphi_v)$; the section Φ_{φ_v} is called the *Siegel–Weil standard section* attached to φ_v.

Proposition 4.11 [8, Proposition 10.1]. *Suppose that v is a finite prime, φ_v is the characteristic function of $(L_v)^n$ for an $o_{k,v}$ Hermitian lattice L_v of rank n, and $\Phi_v(s)$ is the associated Siegel–Weil standard section. Then, for $r \in \mathbb{Z}_{\geqslant 0}$,*

$$W_{T,v}(e, r, \Phi_v) = \gamma_v(V_v)^n \ |N(\det S)|_v^{n/2} |\Delta|_v^e \alpha_v(S_r, T),$$

where

- (i) *S is any matrix representing the Hermitian form on L_v;*
- (ii) *$\alpha_v(S_r, T)$ is the representation density as in (3.1);*
- (iii) *Δ is the discriminant of k and $e := \frac{1}{4}n(3n + 4r - 1)$; and*
- (iv) *$\gamma_v(V_v)$ is an eighth root of unity depending only on $V_v = L_v \otimes_{\mathbb{Z}} \mathbb{Q}$; see [8, Equation 10.3].*

In particular,

$$W'_{T,v}(e, 0, \Phi_v) = \gamma_v(V_v)^n |N(\det S)|_v^{n/2} |\Delta|_v^e \alpha'_v(S, T) \cdot \log v. \qquad \square$$

Passing to the global picture, we observe that $I(0, \eta) = \otimes' I_v(0, \eta_v)$ can be decomposed as a restricted tensor product: a pure tensor $\Phi = \otimes \Phi_v$ lies in $I(0, \eta)$ if and only if, for almost all v, the local component Φ_v is the indicator function of $G(\mathbb{Z}_v)$. As a consequence, we may write

$$I(0, \eta) = \bigoplus_{\mathcal{C}} R(\mathcal{C}),$$

where the sum on $\mathcal{C}$ is over isomorphism classes of Hermitian spaces over $\mathbb{A}_k$ of rank n, such that $\mathrm{inv}_v(\mathcal{C} \otimes_{\mathbb{A}_k} k_v) = 1$ for almost all v. If $\prod_v \mathrm{inv}_v(\mathcal{C}_v) = 1$, then there exists a Hermitian space $\mathcal{V}$ over k such that $\mathcal{C} = \mathcal{V} \otimes_k \mathbb{A}_k$, and in this case we say that $\mathcal{C}$ is *coherent*.

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Theorem 4.12 (Extended Siegel–Weil formula, [3, Theorem 4.2]). *Suppose that $\mathcal{V}$ is an n-dimensional positive-definite Hermitian space over k, and let $H = U(\mathcal{V})$. If $\varphi \in \mathscr{S}(\mathcal{V}(\mathbb{A})^n)$ is an adelic Schwarz function, we define*

$$\mathbf{I}(g, \varphi) := \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \left(\sum_{x \in \mathcal{V}(\mathbb{Q})^n} \omega(g)\varphi(h^{-1}x) \right) \mathrm{d}h, \quad g \in G(\mathbb{A}). \tag{4.16}$$

Here, the measure $\mathrm{d}h$ *is the Haar measure normalized so that* $\mathrm{vol}(H(\mathbb{Q})\backslash H(\mathbb{A}), \mathrm{d}h) = 1$. *Then*

$$\mathbf{E}(g, 0, \Phi_\varphi) = 2\, \mathbf{I}(g, \varphi),$$

where Φ_φ is the Siegel–Weil standard section corresponding to φ. □

Specializing to the case $n = 2$, we now describe the Eisenstein series that figures in our main theorem. Let $L \in \mathcal{R}_d$ be a lattice. We define a (non-standard) section $\Phi^*_L = \otimes \Phi^*_{L,\ell}$ as follows.

- If $\ell \nmid d$ is a finite prime, set $\Phi^*_{L,\ell}(s)$ to be the standard Siegel–Weil section attached to the characteristic function $\varphi_{L,\ell}$ of $(L_\ell)^2$.
- If $\ell = \infty$, take $\Phi^*_{L,\infty}(s)$ to be the Siegel–Weil section attached to the standard Gaussian on the positive-definite Hermitian space $(\mathbb{C}^n)^2$.
- Suppose that $\ell | d$, so that in particular ℓ is odd. Let $\Phi_{L,\ell}$ denote the Siegel–Weil section attached to the characteristic function of $(L_\ell)^2$, and recall that $\mathrm{inv}_\ell V_\ell = -1$. Let V^+_ℓ be the Hermitian space with $\mathrm{inv}_\ell V^+_\ell = 1$, and fix a self-dual lattice L^+_ℓ inside it. Denote its characteristic function of $(L^+_\ell)^2$ by $\varphi_{L^+_\ell}$, and let $\Phi_{L^+_\ell}(s)$ be the corresponding Siegel–Weil section. We then define

$$\Phi^*_\ell(s) := \Phi_{L_\ell}(s) + A_\ell(s)\Phi_{L^+_\ell}(s),$$

where[2]

$$A_\ell(s) := \frac{1}{2(1-\ell^2)}(\ell^s - \ell^{-s}).$$

Next, let

$$\mathbf{E}(g, s, [L]) := \mathbf{E}(g, s, \Phi^*_L)$$

be the corresponding Eisenstein series, which only depends on the genus $[L]$. We also consider the ‘classicalized’ Eisenstein series, as follows. Let

$$\mathfrak{H}_2 := \left\{ z \in M_2(\mathbb{C}) \mid v(z) := \frac{1}{2i}(z - {}^t\bar{z}) > 0 \right\}$$

denote the Hermitian upper half-space; for $z \in \mathfrak{H}_2$, we may write $v(z) = a \cdot {}^t\bar{a}$ for some $a \in GL_2(\mathbb{C})$, and we set $u(z) := \frac{1}{2}(z + {}^t\bar{z})$. Define elements

$$g_{z,\infty} = \begin{pmatrix} \mathrm{Id}_2 & u(z) \\ & \mathrm{Id}_2 \end{pmatrix} \begin{pmatrix} a & \\ & {}^t\bar{a}^{-1} \end{pmatrix} \in G(\mathbb{R}), \quad \text{and} \quad g_z = (g_{z,\infty}, 1, 1, \ldots) \in G(\mathbb{A}),$$

and set

$$\mathcal{E}(z, s, [L]) := \eta_\infty(\det a)^{-1} \det(v)^{-1} \mathbf{E}(g_z, s, [L]).$$

[2]It will turn out that for our purposes, only the values $A_\ell(0) = 0$ and $A'_\ell(0) = \frac{\log \ell}{1-\ell^2}$ will be relevant, and so one may instead take any other function that yields the same values when it and its derivative are evaluated at $s = 0$; we have chosen this particular function only for the sake of concreteness.

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

This normalization ensures that $\mathcal{E}(z, s, [L])$ transforms as a Hermitian modular form of weight 2 on $\mathfrak{H}_2$.

Let $T \in \mathrm{Herm}_2(o_k)$ be a positive-definite matrix. As was the case in the previous section, it turns out that we are most interested in genera $[L] \in \mathcal{R}_d$ such that $L \otimes_{\mathbb{Z}} \mathbb{A}_f^p \simeq V_T \otimes_{\mathbb{Q}} \mathbb{A}_f^p$; we fix such a lattice L and an isomorphism in the following discussion.

Let $V := L \otimes \mathbb{Q}$; i.e. V is, up to isomorphism, the unique Hermitian space of signature $(1, 1)$ whose invariants differ from those of V_T at exactly p and ∞. Fix a non-zero element $\mathtt{b} \in \bigwedge_{\mathbb{Q}}^4 \mathrm{Herm}_2(k)^*$, and non-zero elements

$$\mathtt{a}_V \in \bigwedge_{\mathbb{Q}}^8 (V^2)^* \quad \text{and} \quad \mathtt{a}_{V_T} \in \bigwedge_{\mathbb{Q}}^8 (V_T^2)^*.$$

As explained in, for example, [8, § 10], these elements determine gauge forms $\omega_V = \omega(\mathtt{a}_V, \mathtt{b})$ on V and $\omega_{V_T} = \omega(\mathtt{a}_{V_T}, \mathtt{b})$ on V_T, which in turn induce factorizations

$$\mathrm{d}h = \frac{1}{2} L(1, \chi_k)^{-1} \prod_v \mathrm{d}_v h \quad \text{and} \quad \mathrm{d}h_T = \frac{1}{2} L(1, \chi_k)^{-1} \prod_v \mathrm{d}_v h_T \tag{4.17}$$

of the Haar measures $\mathrm{d}h$ and $\mathrm{d}h_T$ on $U(V)(\mathbb{A})$ and $U(V_T)(\mathbb{A})$, respectively, in terms of Tamagawa measures; for example, each term $\mathrm{d}_v h$ is a measure on $U(V)(\mathbb{Q}_v)$ determined by ω_V. We say that the gauge forms (or the respective decompositions of Haar measures) are *matched* if $\omega_{V_T} = \gamma^* \omega_V$ for some isomorphism $\gamma\colon V_T \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$.

We briefly recall the notation that we had set up in the previous section: we view $\widehat{L}^p$ as an adelic lattice in $V_T \otimes \mathbb{A}_f^p$, and define a Schwarz function

$$\varphi_L'^p \in \mathscr{S}((V_T \otimes \mathbb{A}_f^p)^2) \tag{4.18}$$

as the characteristic function of $(\widehat{L}^p)^2$. Put

$$K'^p := \mathrm{Stab}_{U(V_T)(\mathbb{A}_f^p)}(\widehat{L}^p) \subset U(V_T)(\mathbb{A}_f^p),$$

and, writing

$$U(V_T)(\mathbb{A}_f^p) = \coprod_j U(V_T)(\mathbb{Q}) h_j K'^p, \tag{4.19}$$

set $\Gamma_j' = h_j K'^p h_j^{-1} \cap U(V_T)(\mathbb{Q})$.

Theorem 4.13. *Let $[L] \in \mathcal{R}_d$, and suppose that $T \in \mathrm{Herm}_2(o_k)$ is positive definite. Set*

$$\mathrm{Diff}(T) := \{\ell \nmid d \ \textit{inert}, \mathrm{ord}_\ell \det T \ \textit{odd}\} \bigcup \{\ell \mid d, \mathrm{ord}_\ell \det T \ \textit{even}\}.$$

(i) *Suppose that $p \in \mathrm{Diff}(T)$, but $L \otimes_{\mathbb{Z}} \mathbb{A}_f^p \not\simeq V_T \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. Then $\mathcal{E}_T'(z, 0, [L]) = 0$. Moreover, this is the case whenever $\#\mathrm{Diff}(T) \geqslant 2$.*

(ii) *If $\mathrm{Diff}(T) = \{p\}$ is an inert prime with $p|d$, and $L \otimes_{\mathbb{Z}} \mathbb{A}_f^p \simeq V_T \otimes_{\mathbb{Q}} \mathbb{A}_f^p$, then*

$$\mathcal{E}_T'(z, 0, [L]) = C_{[L]} \sum_j \sum_{\substack{\mathbf{x} \in V_T^2 \\ (\mathbf{x},\mathbf{x}) = T \\ \mathit{mod}\ \Gamma_j'}} \varphi_L'^p(h_j^{-1}\mathbf{x}) \cdot \mu_p(T) \log(p) q^T, \tag{4.20}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

where $q^T := e^{2\pi i\,\mathrm{Tr}(Tz)}$ *and*

$$C_{[L]} = L(1, \chi_k)^{-1}\ \mathrm{vol}(U(V_T)(\mathbb{R}), \mathrm{d}_\infty h_T)\ \mathrm{vol}(K_L, \mathrm{d}^\infty h).$$

Here, $K_L \subset U(V)(\mathbb{A}_f)$ *is the stabilizer of* $\widehat{L} = L \otimes \widehat{\mathbb{Z}}$*, and the measures* $\mathrm{d}_\infty h_T$ *and* $\mathrm{d}^\infty h = \prod_{v\neq\infty} \mathrm{d}_v h$ *are components of a matched decomposition as in (4.17). Moreover, the constant* $C_{[L]}$ *is independent of* T *and all choices appearing in the decompositions of Haar measures.*

Proof. Taking $g = g_z \in G(\mathbb{A})$ as above, let Σ be a sufficiently large finite set of primes containing all the primes in $\mathrm{Diff}(T)$ so that, upon taking the derivative in (4.15), we obtain

$$\mathbf{E}'_T(g, 0, [L]) = -\frac{L^{\Sigma,'}(0)}{L^{\Sigma}(0)^2} \prod_{v\in\Sigma} W_{T,v}(g_v, 0, \Phi^*_v) + L^{\Sigma}(0)^{-1} \cdot \left(\sum_{v\in\Sigma} W'_{T,v}(g_v, 0, \Phi^*_v) \prod_{\substack{v'\in\Sigma \\ v'\neq v}} W_{T,v'}(g_{v'}, 0, \Phi^*_{v'}) \right) \tag{4.21}$$

for $[L] \in \mathcal{R}_d$ and $\Phi^* = \Phi^*_L$. If v is a finite prime such that $L \otimes_{\mathbb{Z}} \mathbb{Q}_v \not\simeq V_T \otimes_{\mathbb{Q}} \mathbb{Q}_v$, then L_v does not represent T, and so the representation density $\alpha_v(S, T)$ vanishes, where S is any matrix representing the Hermitian form on L. By Proposition 4.11,

$$W_{T,v}(g_{z,v}, 0, \Phi^*_v) = W_{T,v}(e, 0, \Phi^*_v) = 0$$

as well. Thus, if there are at least two such primes, then $\mathbf{E}'_T(g, 0, [L]) = 0$, as each term in (4.21) vanishes.

By the definition of $\mathcal{R}_d$, we have that $L \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \not\simeq V_T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ for any prime $\ell \in \mathrm{Diff}(T)$, since $\mathrm{inv}_\ell(L \otimes \mathbb{Q}) = 1$ when $\ell \nmid d$, and $\mathrm{inv}_\ell(L \otimes \mathbb{Q}) = -1$ when $\ell \mid d$. This proves the first statement.

Thus, from this point on, we suppose that $\mathrm{Diff}(T) = \{p\}$ for some $p|d$, and we fix a lattice $[L] \in \mathcal{R}_d$ with an isomorphism $L \otimes_{\mathbb{Z}} \mathbb{A}^p_f \simeq V_T \otimes_{\mathbb{Q}} \mathbb{A}^p_f$. Then (4.21) gives

$$\mathbf{E}'_T(g, 0, [L]) = W'_{T,p}(e, 0, \Phi^*_p) \cdot L^{\Sigma}(0)^{-1} \prod_{\substack{v\in\Sigma \\ v\neq p}} W_{T,v}(g_v, 0, \Phi^*_v). \tag{4.22}$$

By using Proposition 4.11 and the definition of Φ^*_p,

$$\begin{aligned} W'_{T,p}(e, 0, \Phi^*_p) &= \gamma_p(V_p^-)^2 |p^2|_p \alpha'_p\left(\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right), T\right) \log p + \gamma_p(V_p^+)^2 A'_p(0) \alpha_p\left(\left(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right), T\right) \\ &= \log p \cdot \left(\frac{\gamma_p(V_p^-)^2}{p^2} \alpha'_p\left(\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right), T\right) + \frac{\gamma_p(V_p^+)^2}{1-p^2} \alpha_p\left(\left(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right), T\right) \right). \end{aligned}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

It is easily seen that $\gamma_p(V_p^+) = -\gamma_p(V_p^-)$; see [8, Equation (10.3)]. By Corollary 3.6,

$$W'_{T,p}(e, 0, \Phi^*_{L,p}) = \gamma_p(V_p^+)^2 \cdot \frac{(p+1)^2}{p^3} \cdot \mu_p(T) \cdot \log p. \tag{4.23}$$

We may compute the remaining terms in (4.22) via the Siegel–Weil formula. Let

$$\mathfrak{L} \subset V_T \tag{4.24}$$

be a lattice such that $\mathfrak{L} \otimes \widehat{\mathbb{Z}}^p$ is identified with $L \otimes \widehat{\mathbb{Z}}^p$ and $\mathfrak{L}_p$ is self-dual. We then have a Schwarz function

$$\varphi' = \varphi'_\infty \otimes \varphi'_f \in \mathscr{S}(V_T(\mathbb{A})^2),$$

where φ'_f is the indicator function of $(\mathfrak{L} \otimes \widehat{\mathbb{Z}})^2$ in $(V_T \otimes \mathbb{A}_f)^2$, and φ'_∞ is the standard Gaussian on the positive-definite space $(V_T \otimes \mathbb{R})^2 \simeq \mathbb{C}^4$. Note that $\varphi_f'^p = \varphi_L'^p$ as before.

Let $\Phi' = \otimes \Phi'_v$ be the corresponding standard Siegel–Weil section. It follows immediately from definitions that

$$W_{T,v}(g_v, 0, \Phi'_v) = W_{T,v}(g_v, 0, \Phi^*_v) \quad \text{for all } v \neq p.$$

Combining this fact with the extended Siegel–Weil formula (Theorem 4.12) and the product expansion (4.15), we have, for any $g \in G(\mathbb{A})$,

$$2\mathbf{I}_T(g, \varphi') = \mathbf{E}_T(g, 0, \Phi') = L^\Sigma(0)^{-1} \left(\prod_{\substack{v \in \Sigma \\ v \neq p}} W_{T,v}(g_v, 0, \Phi^*_v) \right) W_{T,p}(g_p, 0, \Phi'_p),$$

where

$$\mathbf{I}_T(g, \varphi') = \int_{H_T(\mathbb{Q}) \backslash H_T(\mathbb{A})} \left(\sum_{\substack{\mathbf{x} \in (V_T)^2 \\ (\mathbf{x},\mathbf{x}) = T}} \omega(g) \varphi'(h_T^{-1} x) \right) \mathrm{d}h_T$$

is the Tth Fourier coefficient of the theta integral (4.16), and, for ease of notation, we have abbreviated $U(V_T) = H_T$. Writing

$$H_T(\mathbb{A}) = \coprod_j H_T(\mathbb{Q}) \cdot h_j \cdot K'^p \cdot H_T(\mathbb{R}) \cdot H_T(\mathbb{Q}_p)$$

for a collection of elements $h_j \in H_T(\mathbb{A}_f^p)$ appearing in (4.19), we then have

$$\mathbf{I}_T(g, \varphi_T) = \sum_j \sum_{\substack{\mathbf{x} \in (V_T)^2 \\ (\mathbf{x},\mathbf{x}) = T \\ \mathrm{mod}\ \Gamma'_j}} \left[\int_{K'^p H_T(\mathbb{R}) H_T(\mathbb{Q}_p)} \omega(g) \varphi'(h_T^{-1} h_j^{-1} \mathbf{x})\, \mathrm{d}h_T \right].$$

We again specialize to the case

$$g = g_z = (g_{z,\infty}, 1, 1, \ldots) \in G(\mathbb{A}).$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

Recall that we had chosen matched factorizations

$$\mathrm{d}h_T = \frac{1}{2}L(1,\chi_k)^{-1}\prod_v \mathrm{d}_v h_T \quad \text{and} \quad \mathrm{d}h = \frac{1}{2}L(1,\chi_k)^{-1}\prod_v \mathrm{d}_v h$$

for the Haar measures $\mathrm{d}h_T$ and $\mathrm{d}h$ on $H_T(\mathbb{A})$ and $H(\mathbb{A})$, respectively. If we set $\mathrm{d}^{p,\infty} h_T = \prod_{v\neq p,\infty} \mathrm{d}_v h_T$, then, by definition of K'^p and φ',

$$\int_{K'^p} \varphi_f'^p(h^{-1}h_j^{-1}\mathbf{x})\,\mathrm{d}^{p,\infty}h = \varphi_f'^p(h_j^{-1}\mathbf{x})\ \mathrm{vol}(K'^p, \mathrm{d}^{p,\infty}h_T) = \varphi_L'^p(h_j^{-1}\mathbf{x})\ \mathrm{vol}(K_L^p, \mathrm{d}^{p,\infty}h),$$

where we use part (i) of Lemma 4.14 below in the second equality. On the other hand, a straightforward computation using the archimedean Weil representation, see [8, Equation (7.4)], yields

$$\int_{H_T(\mathbb{R})} \omega(g_{z,\infty})\varphi'_\infty(h_T^{-1}h_j^{-1}\mathbf{x})\,\mathrm{d}_\infty h_T = \mathrm{vol}(H_T(\mathbb{R}), \mathrm{d}_\infty h_T)\eta_\infty(\det a)\det(v)q^T.$$

Combining these calculations with part (ii) of Lemma 4.14 below gives

$$\mathbf{I}_T(g_z,\varphi') = C\sum_j \left(\sum_{\substack{\mathbf{x}\in(V_T)^2\\(\mathbf{x},\mathbf{x})=T\\ \mathrm{mod}\ \Gamma_j'}} \varphi_f'^p(\mathbf{x})\right)\eta_\infty(\det a)\det(v)W_{T,p}(e,0,\Phi'_p)q^T, \tag{4.25}$$

where

$$C = \tfrac{1}{2}L(1,\chi_k)^{-1}\ \mathrm{vol}(H_T(\mathbb{R}), \mathrm{d}_\infty h_T)\ \mathrm{vol}(K_L, \mathrm{d}^\infty h).$$

The proof of [8, Lemma 9.5] implies that C is independent of T as well as the choices involved in the decompositions of Haar measures; indeed this constant can be written as a ratio of volumes for which the independence is immediately evident.

Using (4.23) and (4.25),

$$\begin{aligned}\mathbf{E}'(g_z,0,[L]) &= W'_{T,p}(e,0,\Phi_p^*)\frac{2\,\mathbf{I}_T(g_z,\varphi')}{W_{T,p}(e,0,\Phi'_p)}\\ &= 2C\cdot\mu_p(T)\cdot\left(\sum_{\mathbf{x}}\varphi_f'^p(\mathbf{x})\right)\cdot\eta_\infty(\det a)\cdot\det(v)\cdot\log p\cdot q^T.\end{aligned}$$

The result now follows from rewriting the above in terms of the classicalized Eisenstein series $\mathcal{E}(z,0,[L])$. □

It remains to prove the following lemma.

Lemma 4.14. *With the notation as in the previous theorem, we have*

(i) $\mathrm{vol}(K_L^p, d^{p,\infty}h) = \mathrm{vol}(K'^p, d^{p,\infty}h_T)$; *and*

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

(ii)

$$\int_{H_T(\mathbb{Q}_p)} \varphi'(h_T^{-1}\mathbf{x}) \mathrm{d}_p h_T = \gamma_p(\mathbb{V}_p^+)^{-2} \frac{p^3}{(p+1)^2} \ \mathrm{vol}(K_{L,p}, \mathrm{d}_p h) \cdot W_{T,p}(e, 0, \Phi'_p),$$

where $\mathbf{x} \in (V_T \otimes \mathbb{Q}_p)^2$ *with* $(\mathbf{x}, \mathbf{x}) = T$.

Proof. We use the following expression, found in [8, Lemma 10.4], for the volume of the stabilizer $K_v = K_{L,v}$ of the localized lattice L_v at a finite place v. Fix a basis $\mathbf{e} = \{e_1, e_2\}$ for L_v, with $S = (\mathbf{e}, \mathbf{e})$ the corresponding matrix of inner products. We also fix a $\mathbb{Z}_v$-basis $\mathbf{f} = \{f_1, f_2, f_3, f_4\}$ for $o_{k,v}^2$, so that

$$\mathbf{e} \otimes \mathbf{f} := \{e_i \otimes f_j \mid i = 1, 2,\ j = 1, \dots, 4\}$$

is a $\mathbb{Z}_v$-basis for L_v^2. Finally, fix a $\mathbb{Z}_v$-basis $\mathbf{c}$ for $\mathrm{Herm}_2(o_{k,v})$ whose span is a self-dual lattice. Then

$$\mathrm{vol}(K_v, \mathrm{d}_v h) = L_v(1, \chi_k) \frac{|\mathtt{a}(\mathbf{e} \otimes \mathbf{f})|_v}{|\mathtt{b}(\mathbf{c})|_v} |\Delta|_v \alpha_v(S, S), \tag{4.26}$$

where $\mathtt{a} \in \bigwedge_{\mathbb{Q}}^8 (V^2)^*$ and $\mathtt{b} \in \bigwedge_{\mathbb{Q}}^4 \mathrm{Herm}_2(o_k)^*$ were the fixed non-zero vectors used to construct the local measure $\mathrm{d}_v h$ on $H(\mathbb{Q}_v) = U(V)(\mathbb{Q}_v)$. Similarly, if $K' \subset H_T(\mathbb{A}_f)$ is the stabilizer of the lattice $\mathfrak{L} \subset V_T$ as in (4.24), then

$$\mathrm{vol}(K'_v, \mathrm{d}_v h_T) = L_v(1, \chi_k) \frac{|\mathtt{a}'(\mathbf{e} \otimes \mathbf{f})|_v}{|\mathtt{b}(\mathbf{c})|_v} |\Delta|_v \alpha_v(S', S'),$$

where $\mathtt{a}' = \gamma^* \mathtt{a}$ is the pullback under an isomorphism $\gamma \colon V_T \otimes \overline{\mathbb{Q}} \xrightarrow{\sim} V \otimes \overline{\mathbb{Q}}$, and $S' = (\mathbf{e}', \mathbf{e}')$ is the matrix of inner products of any basis $\mathbf{e}'$ of $\mathfrak{L}_v$. Furthermore, Kudla and Rapoport compute

$$\frac{|\mathtt{a}'(\mathbf{e}' \otimes f)|_v}{|\mathtt{a}(\mathbf{e} \otimes f)|_v} = \frac{|\det S'|_v^2}{|\det S|_v^2}, \tag{4.27}$$

and then conclude in [8, Lemma 10.4] that

$$\frac{\mathrm{vol}(K'_v, \mathrm{d}_v h_T)}{\mathrm{vol}(K_v, \mathrm{d}_v h)} = \frac{|\det S'|_v^2 \alpha_v(S', S')}{|\det S|_v^2 \alpha_v(S, S)}. \tag{4.28}$$

If $v \neq p$, then $\mathfrak{L}_v \simeq L_v$. In particular, we may choose the bases $\mathbf{x}$ and $\mathbf{x}'$ so that $S = S'$ in the above display, and from this (i) follows immediately.

To prove (ii), we apply a standard calculation relating orbital integrals and Whittaker functionals, see the proof of [8, Lemma 10.4], and recall that here p is inert:

$$\begin{aligned}
&\int_{H_T(\mathbb{Q}_p)} \varphi'(h_T^{-1} x) \mathrm{d}_p h_T \\
&= \gamma_p(\mathbb{V}^+)^2 L_p(1, \chi_k) \frac{|\mathtt{a}'(\mathbf{e}' \otimes f)|_p}{|\det S'|_p^2 |\mathtt{b}(\mathbf{c})|_p} W_{T,p}(e, 0, \Phi'_p) \\
&= \gamma_p(\mathbb{V}^+)^2 L_p(1, \chi_k) \frac{|\mathtt{a}(\mathbf{e} \otimes f)|_p}{|\det S|_p^2 |\mathtt{b}(\mathbf{c})|_p} W_{T,p}(e, 0, \Phi'_p) && \text{[by (4.27)]} \\
&= \gamma_p(\mathbb{V}^+)^2 \cdot \mathrm{vol}(K_p, \mathrm{d}_p h) \cdot (|\det S|_p^2 \alpha_p(S, S))^{-1} \cdot W_{T,p}(e, 0, \Phi'_p) && \text{[by (4.26)].}
\end{aligned}$$

https://doi.org/10.1017/S1474748015000109 Published online by Cambridge University Press

By definition, L_p is the maximal lattice in the non-split Hermitian space of dimension 2 over k_p, and so we may choose a basis so that $S = \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$. A direct computation using Hironaka's formula (Theorem 3.2) gives the expression

$$\alpha_p\left(\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)\right) = p^{-1}(p+1)^2,$$

which, when combined with equation preceding it, yields the statement of the lemma. □

By combining the above calculation with the geometric computations of § 4.2, we obtain our main theorem, relating the arithmetic degree of a special cycle $\mathfrak{Z}(T)$ to the Eisenstein series

$$\mathcal{E}(z, s) := \sum_{[L] \in \mathcal{R}_d} (C_{[L]})^{-1} \mathcal{E}(z, s, [L]).$$

Corollary 4.15. *Suppose that* $T \in \mathrm{Herm}_2(o_k)$ *is positive definite and that* $|\mathrm{Diff}(T)| \geqslant 1$ *with* $\mathrm{Diff}(T) \neq \{2\}$. *Then*

$$\widehat{\deg}\, \mathfrak{Z}(T) q^T = \frac{2h(k)}{|o_k^\times|} \mathcal{E}'_T(z, 0).$$

Proof. If $|\mathrm{Diff}(T)| \geqslant 2$, then the right-hand side vanishes by Theorem 4.13(i), and the left-hand side vanishes by Lemma 4.9(iii).

Next, suppose that $\mathrm{Diff}(T) = \{p\}$ for some $p | d$. Then, comparing Theorems 4.10 and 4.13,

$$\widehat{\deg}\, \mathfrak{Z}(T)^{[L_0],[L]} q^T = \frac{4h(k)}{|o_k^\times| 2^{o(\Delta)}} C_{[L]}^{-1} \mathcal{E}'(z, 0, [L])$$

for any $[L_0] \in \mathcal{R}_0$ and $[L] \in \mathcal{R}_d$. Note that the right-hand side is independent of $[L_0]$. Given a Hermitian space V_0 of signature $(1, 0)$ such that $\mathrm{inv}_\ell(V_0) = 1$ for all inert ℓ, there is a single genus of self-dual lattices in V_0; conversely, if L_0 is self-dual, then $\mathrm{inv}_\ell(L_0 \otimes \mathbb{Q}) = 1$ for all inert ℓ. Thus, by counting the possibilities of the invariants of V_0,

$$\#\mathcal{R}_0 = \#\{V_0 \mid \mathrm{inv}_\ell V_0 = 1 \text{ for all } \ell \text{ inert}\}_{/\mathrm{isom.}} = 2^{o(d_k)-1},$$

and so

$$\begin{aligned} \widehat{\deg}\, \mathfrak{Z}(T) q^T &= \sum_{[L] \in \mathcal{R}_d} \sum_{[L_0] \in \mathcal{R}_0} \widehat{\deg}\, \mathfrak{Z}(T)^{[L_0],[L]} q^T \\ &= 2^{o(\Delta)-1} \sum_{[L] \in \mathcal{R}_d} \widehat{\deg}\, \mathfrak{Z}(T)^{[L_0],[L]} q^T \\ &= \frac{2h(k)}{|o_k^\times|} \mathcal{E}'_T(z, 0). \end{aligned}$$

Finally, when $\mathrm{Diff}(T) = \{p\}$ for some odd $p \nmid d$, the desired result is exactly [8, Theorem 11.9] applied to the case at hand. □

Acknowledgements. This work was supported by the SFB/TR 45 'Periods, Moduli Spaces, and Arithmetic of Algebraic Varieties' of the DFG (German Research Foundation).

References

1. B. Gross, On canonical and quasi-canonical liftings, *Invent. Math.* **84** (1986), 321–326.
2. Y. Hironaka, Local Zeta functions on Hermitian forms and its application to local densities, *J. Number Theory* **71** (1998), 40–64.
3. A. Ichino, A regularized Siegel–Weil formula for unitary groups, *Math. Z.* **247** (2004), 241–277.
4. R. Jacobowitz, Hermitian forms over local fields, *Amer. J. Math.* **84** (1962), 441–465.
5. S. Kudla, Special cycles and derivatives of Eisenstein series, in *Heegner Points and Rankin L-series*, Mathematical Sciences Research Institute Publications, Volume 49, pp. 243–270 (Cambridge University Press, Cambridge, 2004).
6. S. Kudla and M. Rapoport, Height pairings on Shimura curves and p-adic uniformization, *Invent. Math.* **142** (2000), 153–222.
7. S. Kudla and M. Rapoport, Special cycles on unitary Shimura varieties I: unramified local theory, *Invent. Math.* **184** (2010), 629–682.
8. S. Kudla and M. Rapoport, Special cycles on unitary Shimura varieties II: global theory, *J. Reine Angew. Math.* **697** (2014), 91–157.
9. S. Kudla and M. Rapoport, An alternative description of the Drinfeld upper half-plane, *Ann. Inst. Fourier* **64** (2014), 1203–1228.
10. S. Kudla and M. Rapoport, New cases of p-adic uniformization, Preprint. Available at: http://arxiv.org/abs/1302.3521.
11. S. Kudla and W. J. Sweet, Degenerate principal series representations for $U(n, n)$, *Israel J. Math.* **98** (1997), 253–306.
12. S. Nagaoka, An explicit formula for Siegel series, *Abh. Math. Semin. Univ. Hambg.* **59** (1989), 235–262.
13. M. Rapoport and Th. Zink, *Period Spaces for p-divisible Groups*, Annals of Mathematics Studies, Volume 141 (Princeton University Press, 1996).
14. S. Sankaran, Unitary cycles on Shimura curves and the Shimura lift I, *Doc. Math.* **18** (2013), 1403–1464.
15. G. Shimura, Arithmetic of unitary groups, *Ann. of Math.* (2) **79** (1964), 369–409.
16. V. Tan, Poles of Eisenstein series on $U(n, n)$, *Canad. Math. J.* **51** (1999), 164–175.