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ON COMPONENTS OF STABLE AUSLANDER–REITEN QUIVERS THAT CONTAIN HELLER LATTICES: THE CASE OF TRUNCATED POLYNOMIAL RINGS

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Abstract. Let A be a truncated polynomial ring over a complete discrete valuation ring $\mathcal{O}$, and we consider the additive category consisting of A-lattices M with the property that $M \otimes \mathcal{K}$ is projective as an $A \otimes \mathcal{K}$-module, where $\mathcal{K}$ is the fraction field of $\mathcal{O}$. Then, we may define the stable Auslander–Reiten quiver of the category. We determine the shape of the components of the stable Auslander–Reiten quiver that contain Heller lattices.

Introduction

The shape of Auslander–Reiten quivers is one of fundamental interests in representation theory of algebras. For algebras over a field, a wealth of examples are given in textbooks, [ASS] for example. Let $\mathcal{O}$ be a complete discrete valuation ring, ϵ a uniformizer, $\mathcal{K}$ its fraction field, $\kappa = \mathcal{O}/\epsilon\mathcal{O}$ its residue field. Let A be an $\mathcal{O}$-order, namely an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. If $A \otimes \mathcal{K}$ is a semisimple algebra, we may also find results in the literature. However, few results seem to be known for the case when $A \otimes \mathcal{K}$ is not a semisimple algebra. An exception is a famous work by Hijikata and Nishida, but their main focus is on a Bass order and $A \otimes \mathcal{K}$ needs to be a quasi-Frobenius radical square zero algebra for a Bass order [HN, Theorem 3.7.1].

Recall that an A-module is called an A-lattice or a Cohen–Macaulay A-module if it is free of finite rank as an $\mathcal{O}$-module. (Cohen–Macaulay A-modules are by definition finitely generated A-modules which are Cohen–Macaulay as $\mathcal{O}$-modules. Since $\mathcal{O}$ is regular here, Cohen–Macaulay $\mathcal{O}$-modules are free [Y, (1.5)] and vice versa.) Then, it is known that for any nonprojective A-lattice M with the property that $M \otimes \mathcal{K}$ is projective as an $A \otimes \mathcal{K}$-module, there is an almost split sequence ending at M, and dually,

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for any noninjective A-lattice M with the property that $M \otimes \mathcal{K}$ is injective as an $A \otimes \mathcal{K}$-module, there is an almost split sequence starting at M. See [AR] for example. Thus, if $A \otimes \mathcal{K}$ is self-injective, we may define the (stable) Auslander–Reiten quiver consisting of such A-lattices. Typical examples of such A-lattices are Heller lattices. For group algebras, Heller lattices were studied by Kawata [K], and it inspired us to study the components that contain Heller lattices for the case of orders in non-semisimple algebras.

In this article, we determine the shape of the components of the stable Auslander–Reiten quiver that contain Heller lattices, for the truncated polynomial rings $A = \mathcal{O}[X]/(X^n)$. As $\mathcal{O}[X]/(X^n)$ is a Gorenstein $\mathcal{O}$-order, that is, $\mathrm{Hom}_{\mathcal{O}}(A_A, \mathcal{O})$ is a projective A-module [I, Section 4], we explain explicit construction of almost split sequences for a Gorenstein $\mathcal{O}$-order, which generalizes construction of almost split sequences in [T], and use this construction to do necessary calculations. Main difficulty in the computation is the proof that certain direct summands of the middle terms of those almost split sequences are indecomposable. We use elementary brute force argument to overcome this difficulty. Then, some argument on tree classes which takes the possibility of the existence of loops in the stable Auslander–Reiten quiver into account proves the result. This argument is necessary because there may exist loops [W].

If $A \otimes \kappa$ is a special biserial algebra, we may calculate indecomposable $A \otimes \kappa$-modules and their Heller lattices. It is natural to consider the above problem in this setting. We will report some results in this direction in future work.

§1. Preliminaries

1.1 Gorenstein orders

We start by observing that $A = \mathcal{O}[X]/(X^n)$ is a symmetric $\mathcal{O}$-order. By abuse of notation, we write $1, X, \ldots, X^{n-1}$ for the standard $\mathcal{O}$-basis of A. Define $\theta_i \in \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$, for $0 \leqslant i \leqslant n-1$, by

$$\theta_i(X^j) = \begin{cases} 1 & \text{if } j = n-i-1, \\ 0 & \text{if } j \neq n-i-1. \end{cases}$$

Then we have the following lemma.

LEMMA 1.1. *$\theta_i \mapsto X^i$ induces an isomorphism of (A, A)-bimodules $\mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}) \simeq A$.*

Proof. As $X\theta_i = \theta_i X : X^j \mapsto \theta_i(X^{j+1}) = \delta_{j+1,n-i-1}$, we have $X\theta_i = \theta_i X = \theta_{i+1}$. ∎

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REMARK 1.2. A different definition of Gorenstein order is given in [CR, Section 37]: it requires not only that every exact sequence of A-lattices $0 \to A \to M \to N \to 0$ starting at A splits, but also that $A \otimes \mathcal{K}$ is a semisimple algebra. Perhaps the semisimplicity condition was added by some technical reasons.

REMARK 1.3. In [A1, Chapter I, Section 7], the definition of $\mathcal{O}$-order itself is different. If we restrict to the case when $\mathcal{O}$ is a Dedekind domain, A is an $\mathcal{O}$-order in his sense if A is not only a finitely generated projective $\mathcal{O}$-module but also $A \otimes \mathcal{K}$ is a self-injective $\mathcal{K}$-algebra.

Then, a Gorenstein $\mathcal{O}$-order is a Noetherian $\mathcal{O}$-algebra A which is Cohen–Macaulay as an $\mathcal{O}$-module and $\mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}) \simeq A$ as (A, A)-bimodules [A1, Chapter III, Section 1]. Nowadays, Gorenstein $\mathcal{O}$-orders in Auslander's sense are called symmetric $\mathcal{O}$-orders [IW, Definition 2.8].

Lemma 1.1 implies that $A = \mathcal{O}[X]/(X^n)$ is a symmetric $\mathcal{O}$-order. Note that A is also a Gorenstein ring, since depth $A = \dim A$ and if the parameter ideal ϵA is the intersection of two ideals I and J then either $I = \epsilon A$ or $J = \epsilon A$ holds.

LEMMA 1.4. *Let $A = \mathcal{O}[X]/(X^n)$, for $n \geqslant 2$. Then there are infinitely many pairwise nonisomorphic indecomposable A-lattices.*

Proof. If there were only finitely many, then [A2, Section 10] and [Y, (3.1), (4.22)] would imply that A is reduced, contradicting our assumption that $n \geqslant 2$. Below, we give an example of a family of infinitely many pairwise nonisomorphic indecomposable A-lattices.

For $r \in \mathbb{Z}_{\geqslant 0}$, let $L_r = \mathcal{O}\epsilon^r \oplus \mathcal{O}X \oplus \cdots \oplus \mathcal{O}X^{n-1} \subseteq A$. Then the representing matrix of the action of X on L_r with respect to the basis is given by the following matrix:

$$X = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \epsilon^r & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, we have $L_r \otimes \mathcal{K} \simeq A \otimes \mathcal{K}$ and $L_r \not\simeq L_s$ whenever $r \neq s$. In particular, L_r, for $r = 0, 1, 2, \ldots$, are pairwise nonisomorphic indecomposable A-lattices. $\square$

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Since $\mathcal{O}$ is a complete local ring, $\mathrm{End}_A(X)$ is a local $\mathcal{O}$-algebra for every indecomposable A-lattice X [CR, (6.10)(30.5)]. Thus, the Jacobson radical $\mathrm{Rad}\,\mathrm{End}_A(X)$ consists of all noninvertible endomorphisms of X. Another consequence is that A is semiperfect and every finitely generated A-module has a projective cover [CR, (6.23)].

In the next subsection, we assume that A is a Gorenstein $\mathcal{O}$-order and we explain a method to construct almost split sequences for A-lattices. Note that there exists an almost split sequence ending (resp. starting) at M if and only if $M \otimes \mathcal{K}$ is projective (resp. injective) [AR], [RR, Theorem 6].

1.2 Construction of almost split sequences

We recall several definitions.

Definition 1.5. Let A be an $\mathcal{O}$-order, M and N A-lattices. The *radical* $\mathrm{Rad}\,\mathrm{Hom}_A(M,N)$ of $\mathrm{Hom}_A(M,N)$ is the $\mathcal{O}$-submodule of $\mathrm{Hom}_A(M,N)$ consisting of $f \in \mathrm{Hom}_A(M,N)$ such that, for all indecomposable A-lattice X, we have $hfg \in \mathrm{Rad}\,\mathrm{End}_A(X)$, for any $g \in \mathrm{Hom}_A(X,M)$ and $h \in \mathrm{Hom}_A(N,X)$. It is equivalent to the condition that $1 - gf$ is invertible, for all $g \in \mathrm{Hom}_A(N,M)$, and to the condition that $1 - fg$ is invertible, for all $g \in \mathrm{Hom}_A(N,M)$.

Let $\mathcal{A}$ be an abelian category with enough projectives, $\mathcal{C}$ an additive full subcategory which is closed under extensions and direct summands. Then, $f \in \mathrm{Hom}_{\mathcal{C}}(M,N)$ in $\mathcal{C}$ is called *right minimal in* $\mathcal{C}$ if an endomorphism $h \in \mathrm{End}_{\mathcal{C}}(M)$ is an isomorphism whenever $f = fh$, *right almost split in* $\mathcal{C}$ if it is not a split epimorphism and for each $X \in \mathcal{C}$ and $h \in \mathrm{Hom}_{\mathcal{C}}(X,N)$ which is not a split epimorphism, there is $s \in \mathrm{Hom}_{\mathcal{C}}(X,M)$ such that $fs = h$. If f is both right minimal in $\mathcal{C}$ and right almost split in $\mathcal{C}$, f is called *minimal right almost split in* $\mathcal{C}$. Similarly, $g \in \mathrm{Hom}_{\mathcal{C}}(L,M)$ is called *left minimal in* $\mathcal{C}$ if an endomorphism $h \in \mathrm{End}_{\mathcal{C}}(M)$ is an isomorphism whenever $g = hg$, *left almost split in* $\mathcal{C}$ if it is not a split monomorphism and for each $Y \in \mathcal{C}$ and $h \in \mathrm{Hom}_{\mathcal{C}}(L,Y)$ which is not a split monomorphism, there is $t \in \mathrm{Hom}_{\mathcal{C}}(M,Y)$ such that $tg = h$, and if g is both left minimal in $\mathcal{C}$ and left almost split in $\mathcal{C}$, g is called *minimal left almost split in* $\mathcal{C}$. We have the following proposition in this general setting [A1, Chapter II, Proposition 4.4].

Proposition 1.6. *Suppose that $\mathcal{C}$ is an additive full subcategory of an abelian category $\mathcal{A}$ with enough projectives such that $\mathcal{C}$ is closed under extensions and direct summands. Let $L, M, N \in \mathcal{C}$. Then the following are* *equivalent for a short exact sequence*

$$0 \longrightarrow L \xrightarrow{g} M \xrightarrow{f} N \longrightarrow 0.$$

(a) *f is right almost split in $\mathcal{C}$ and g is left almost split in $\mathcal{C}$.*
(b) *f is minimal right almost split in $\mathcal{C}$.*
(c) *f is right almost split and* $\mathrm{End}_{\mathcal{C}}(L)$ *is local.*
(d) *g is minimal left almost split in $\mathcal{C}$.*
(e) *g is left almost split in $\mathcal{C}$ and* $\mathrm{End}_{\mathcal{C}}(N)$ *is local.*

We return to $\mathcal{O}$-orders over a complete discrete valuation ring $\mathcal{O}$. Among equivalent conditions in Proposition 1.6, we choose (c) as the definition of an almost split sequence for lattices over an $\mathcal{O}$-order.

DEFINITION 1.7. Let A be an $\mathcal{O}$-order, L, E, M A-lattices. A short exact sequence

$$0 \longrightarrow L \longrightarrow E \xrightarrow{p} M \longrightarrow 0$$

is called an *almost split sequence* (of A-lattices) ending at M if

(i) the epimorphism p does not split;
(ii) L and M are indecomposable;
(iii) the morphism $p : E \to M$ induces the epimorphism

$$\mathrm{Hom}_A(X, p) : \mathrm{Hom}_A(X, E) \longrightarrow \mathrm{Rad}\,\mathrm{Hom}_A(X, M),$$

for every indecomposable A-lattice X.

DEFINITION 1.8. Let $f : M \to N$ be a morphism between A-lattices. We say that f is an *irreducible morphism* if

(i) f is neither a split monomorphism nor a split epimorphism;
(ii) if there are $g \in \mathrm{Hom}_A(M, L)$ and $h \in \mathrm{Hom}_A(L, N)$ such that $f = hg$, then either g is a split monomorphism or h is a split epimorphism.

LEMMA 1.9. *Let A be an $\mathcal{O}$-order, L, E, M A-lattices. We suppose that an almost split sequence for A-lattices ending at M exists. Then, a short exact sequence*

$$0 \longrightarrow L \xrightarrow{\iota} E \xrightarrow{p} M \longrightarrow 0$$

is an almost split sequence if and only if ι and p are irreducible.

Proof. The arguments in [ARS, V. Theorem 5.3] and [ARS, V. Proposition 5.9] work without change in our setting. □

REMARK 1.10. The definitions of almost split sequences and irreducible morphisms are taken from [R2], although it is assumed that $A \otimes \mathcal{K}$ is a semisimple algebra there.

DEFINITION 1.11. Let A be an $\mathcal{O}$-order. For an indecomposable $A \otimes \kappa$-module N, we view N as an A-module, and take the projective cover $p : P \to N$. We denote $\mathrm{Ker}(p)$ by Z_N and direct summands of the A-lattice Z_N are called *Heller lattices* of N. Note that Z_N is uniquely determined up to isomorphism.

In the sequel, we consider an indecomposable A-lattice M with the property

$$(*) \qquad M \otimes \mathcal{K} \text{ is projective as an } A \otimes \mathcal{K}\text{-module,}$$

and show how to construct the almost split sequence ending at M.

REMARK 1.12. Heller lattices have the property $(*)$. Indeed, for an indecomposable $A \otimes \kappa$-module N, Z_N is an A-submodule of the projective A-module P, and we have $\epsilon P \subseteq Z_N$. Thus, $Z_N \otimes \mathcal{K} = P \otimes \mathcal{K}$ is projective and so are their direct summands.

Let $D = \mathrm{Hom}_{\mathcal{O}}(-, \mathcal{O})$ and define the *Nakayama functor* for A-lattices by

$$\nu = D(\mathrm{Hom}_A(-, A)) = \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}_A(-, A), \mathcal{O}).$$

LEMMA 1.13. *Let M be an A-lattice, $p : P \to M$ its projective cover. We define*

$$L = D(\mathrm{Coker}(\mathrm{Hom}_A(p, A))).$$

Then we have the exact sequence of A-lattices

$$0 \longrightarrow L \longrightarrow \nu(P) \xrightarrow{\nu(p)} \nu(M) \longrightarrow 0.$$

Proof. $\mathrm{Hom}_A(\mathrm{Ker}(p), A)$ is an A-lattice since $\mathrm{Ker}(p)$ and A are. Since the cokernel of $\mathrm{Hom}_A(p, A) : \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(P, A)$ is an A-submodule of $\mathrm{Hom}_A(\mathrm{Ker}(p), A)$, $\mathrm{Coker}(\mathrm{Hom}_A(p, A))$ is a free $\mathcal{O}$-module. Then, $\mathrm{Ext}^1_{\mathcal{O}}(\mathrm{Coker}(\mathrm{Hom}_A(p, A)), \mathcal{O}) = 0$ implies the result. □

REMARK 1.14. If we take a minimal projective presentation $Q \xrightarrow{q} P \xrightarrow{p} M$ of an A-lattice M, we have the short exact sequence

$$\begin{aligned} 0 &\to \mathrm{Coker}(\mathrm{Hom}_A(p, A)) \to \mathrm{Hom}_A(Q, A) \\ &\to \mathrm{Coker}(\mathrm{Hom}_A(q, A)) = \mathrm{Tr}(M) \to 0. \end{aligned}$$

Thus, $L = D(\mathrm{Coker}(\mathrm{Hom}_A(p, A)))$ represents the Auslander–Reiten translate $\tau(M) = D\Omega\,\mathrm{Tr}(M)$ of the A-lattice M.

Taking a suitable pullback of the exact sequence from Lemma 1.13, we may construct almost split sequences as follows. This generalizes the construction in [T]. We give the proof of Proposition 1.15 in the appendix, for the convenience of the reader.

The right and left minimality in Proposition 1.6 implies that the almost split sequence ending at M and the almost split sequence starting at L are uniquely determined by M and L respectively, up to isomorphism of short exact sequences. Thus, we may define the Auslander–Reiten translate τ and τ^- by $\tau(M) = L$ and $\tau^-(L) = M$.

PROPOSITION 1.15. *Suppose that A is a Gorenstein $\mathcal{O}$-order, M an indecomposable nonprojective A-lattice with the property (∗), and let $p: P \to M$ be its projective cover. For $\varphi \in \mathrm{Hom}_A(M, \nu(M))$, we consider the pullback diagram along φ:*

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & L \arrow[r] & \nu(P) \arrow[r, "\nu(p)"] & \nu(M) \arrow[r] & 0
\end{tikzcd}$$

Then the following (1) *and* (2) *are equivalent.*

(1) *The pullback $0 \to L \to E \to M \to 0$ is an almost split sequence.*
(2) *The following three conditions hold.*

  (i) *φ does not factor through $\nu(p)$.*
  (ii) *L is an indecomposable A-lattice.*
  (iii) *For all $f \in \mathrm{Rad}\,\mathrm{End}_A(M)$, φf factors through $\nu(p)$.*

If A is a symmetric $\mathcal{O}$-order, then we have functorial isomorphisms $\nu(X) \simeq X$, for A-lattices X. Hence, we pull back $0 \to L \to P \to M \to 0$ along $\varphi \in \mathrm{End}_A(M)$ in this case. Further, the left term $L = \tau(M)$ and the middle term E of the almost split sequence satisfy the property (∗).

1.3 Translation quivers and tree classes

In this subsection we recall fundamentals of translation quivers.

DEFINITION 1.16. Let $Q = (Q_0, Q_1)$, where Q_0 is the set of vertexes and Q_1 is the set of arrows, be a locally finite quiver, that is, there are only finitely many incoming and outgoing arrows for each vertex. If a map $v : Q_1 \to \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is given, we call the pair (Q, v) a *valued quiver*. Let $\tau : Q \to Q$ be a quiver automorphism. Then, we call the pair (Q, τ) a *stable translation quiver* if the following two conditions hold:

(i) Q has no loops and no multiple arrows.
(ii) For each vertex $x \in Q_0$, we have

$$\{y \in Q_0 \mid \tau x \to y \text{ in } Q_1\} = \{y \in Q_0 \mid y \to x \text{ in } Q_1\}.$$

We call the triple (Q, v, τ) a *valued stable translation quiver* if (Q, τ) is a stable translation quiver and if $v(x \to y) = (a, b)$ then $v(\tau(y) \to x) = (b, a)$.

DEFINITION 1.17. Let (Q, τ) be a stable translation quiver and C a full subquiver of Q. We call C a *component* of (Q, τ) if:

(i) C is stable under the quiver automorphism τ;
(ii) C is a disjoint union of connected components of the underlying undirected graph;
(iii) there is no proper subquiver of C that satisfies (i) and (ii).

Note that components are also stable translation quivers.

EXAMPLE 1.18. Let (Δ, v) be a valued quiver without loops and multiple arrows. Then, the set $\mathbb{Z} \times \Delta$ becomes a valued stable translation quiver by defining as follows:

- arrows are $(n, x) \to (n, y)$ and $(n-1, y) \to (n, x)$, for $x \to y$ in Δ and $n \in \mathbb{Z}$;
- if $v(x \to y) = (a, b)$, for $x \to y$ in Δ, then

$$v((n, x) \to (n, y)) = (a, b) \qquad \text{and} \qquad v((n-1, y) \to (n, x)) = (b, a).$$

- $\tau((n, x)) = (n-1, x)$.

We denote the valued stable translation quiver by $\mathbb{Z}\Delta$.

Now we recall Riedmann's structure theorem [B, Theorem 4.15.6]. For the definition of admissible subgroups, see [B, Definition 4.15.4].

Definition-Theorem 1.19. Let (Q, τ) be a stable translation quiver and C a component of (Q, τ). Then there is a directed tree T and an admissible subgroup $G \subseteq \mathrm{Aut}(\mathbb{Z}T)$ such that $C \simeq \mathbb{Z}T/G$ as a stable translation quiver. Moreover,

(1) the underlying undirected graph $\overline{T}$ of T is uniquely determined by C.
(2) G is unique up to conjugation in $\mathrm{Aut}(\mathbb{Z}T)$.

The underlying tree $\overline{T}$ is called the *tree class* of C.

DEFINITION 1.20. Let (Δ, v) be a valued quiver without loops and multiple arrows. For $x \to y$ in Δ, we write $v(x \to y) = (d_{xy}, d_{yx})$. If there is no arrow between x and y, we understand that $d_{xy} = d_{yx} = 0$. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers.

(i) A *subadditive function* on (Δ, v) is a $\mathbb{Q}_{>0}$-valued function f on the set of vertexes of Δ such that $2f(x) \geqslant \sum_{y \neq x} d_{yx} f(y)$, for each vertex x.
(ii) An *additive function* on (Δ, v) is a $\mathbb{Q}_{>0}$-valued function f on the set of vertexes of Δ such that $2f(x) = \sum_{y \neq x} d_{yx} f(y)$, for each vertex x.

The following lemma is well known. See [B, Theorem 4.5.8], for example.

LEMMA 1.21. *Let (Δ, v) be a valued quiver without loops and multiple arrows, and we assume that the underlying undirected graph $\overline{\Delta}$ is connected.*

(1) *Suppose that (Δ, v) admits a subadditive function.*

   (i) *If Δ has a finite number of vertexes, then $\overline{\Delta}$ is one of finite or affine Dynkin diagrams.*
   (ii) *If Δ has infinite number of vertexes, then $\overline{\Delta}$ is one of infinite Dynkin diagrams A_∞, B_∞, C_∞, D_∞ or A_∞^∞.*

(2) *If (Δ, v) admits a subadditive function which is not additive, then $\overline{\Delta}$ is either a finite Dynkin diagram or A_∞.*
(3) *(Δ, v) does not admit a bounded subadditive function if and only if $\overline{\Delta}$ is A_∞.*

1.4 AR quivers

We define the stable Auslander–Reiten quiver for symmetric $\mathcal{O}$-orders as follows.

DEFINITION 1.22. Let A be a symmetric $\mathcal{O}$-order over a complete discrete valuation ring $\mathcal{O}$. The *stable Auslander–Reiten quiver* of A is a valued quiver such that:

- vertexes are isoclasses of nonprojective A-lattices M such that $M \otimes \mathcal{K}$ is projective;

- valued arrows $M \overset{(a,b)}{\to} N$ for irreducible morphisms $M \to N$, where the value (a, b) of the arrow is given as follows.
  (a) For a minimal right almost split morphism $f : E \to N$, M appears a times in E as a direct summand.
  (b) For a minimal left almost split morphism $g : M \to E$, N appears b times in E as a direct summand.

A component of the stable Auslander–Reiten quiver is defined in the similar way as the stable translation quiver.

LEMMA 1.23. *Let A be a symmetric $\mathcal{O}$-order over a complete discrete valuation ring $\mathcal{O}$, and let C be a component of the stable Auslander–Reiten quiver of A. Assume that C satisfies the following conditions:*

(i) *There exists a τ-periodic indecomposable A-lattice in C.*
(ii) *The number of vertexes in C is infinite.*

Then C has no loops. In particular, C is a valued stable translation quiver.

Proof. As in the proof of [B, Theorem 4.16.2], we know that all indecomposable A-lattices in C are τ-periodic. Thus, we may choose $n_X \geqslant 2$, for each $X \in C$, such that $\tau^{n_X}(X) \simeq X$. Define a $\mathbb{Q}_{>0}$-valued function f on C by

$$f(X) = \frac{1}{n_X} \sum_{i=0}^{n_X - 1} \operatorname{rank} \tau^i(X).$$

C does not have multiple arrows by definition. For each indecomposable N, there is an irreducible morphism $M \to N$ if and only if there is an irreducible morphism $\tau(N) \to M$ by the existence of the almost split sequence $0 \to \tau(N) \to E \to N \to 0$. The condition on valued arrows may also be checked. Thus, $C \setminus \{\text{loops}\}$ is a valued stable translation quiver, and we may apply the Riedmann structure theorem. We write $C \setminus \{\text{loops}\} = \mathbb{Z}T/G$, for a directed tree T and an admissible subgroup G. Then f is a $\mathbb{Q}_{>0}$-valued function on T. For $X \in T$, one can show that

$$\sum_{X \to Y} d_{YX} \operatorname{rank} Y \leqslant \operatorname{rank} X + \operatorname{rank} \tau(X),$$

which implies that f is a subadditive function.

We now suppose that C has a loop. Then, f is not additive. Thus, Lemma 1.21 and our assumption (ii) imply that $\overline{T} = A_\infty$. Thus, we may assume without loss of generality that T is a chain of irreducible maps

$$X_1 \to X_2 \to \cdots \to X_r \to \cdots.$$

Then, for any $Y \in C$, there is a unique r such that Y is in the τ-orbit through X_r. We may assume that X_r has a loop, for some r. The almost split sequence starting at X_r is

$$0 \to X_r \to X_r^{\oplus l} \oplus X_{r+1} \oplus \tau^-(X_{r-1}) \to \tau^-(X_r) \to 0,$$

where $l \geqslant 1$. In particular, we have

$$f(X_r) \geqslant (2-l)f(X_r) \geqslant f(X_{r+1}) + f(X_{r-1}) \geqslant f(X_{r+1}).$$

We show that $f(X_m) \geqslant f(X_{m+1})$, for $m \geqslant r$. Suppose that $f(X_{m-1}) \geqslant f(X_m)$ holds. The same argument as above shows $2f(X_m) \geqslant f(X_{m-1}) + f(X_{m+1})$, and the induction hypothesis implies $f(X_{m-1}) + f(X_{m+1}) \geqslant f(X_m) + f(X_{m+1})$. Hence $f(X_m) \geqslant f(X_{m+1})$. Thus, f is bounded. But $\overline{T} = A_\infty$ does not admit a bounded subadditive function. Hence, we conclude that C has no loops and C is a valued stable translation quiver. $\square$

1.5 No loop theorem

In this subsection, we show an analogue of Auslander's theorem and use this to show "no loop theorem".

LEMMA 1.24. *Let A be an $\mathcal{O}$-order, M an indecomposable A-lattice. Then, there exists an integer s such that $M/\epsilon^k M$ is an indecomposable $A/\epsilon^k A$-module, for all $k \geqslant s$.*

Proof. An $\mathcal{O}$-linear map $D : A \to \mathrm{End}_{\mathcal{O}}(M)$ is called a derivation if

$$D(xy) = xD(y) + D(x)y$$

for all $x, y \in A$. We denote by $\mathrm{Der}(A, \mathrm{End}_{\mathcal{O}}(M))$ the $\mathcal{O}$-module of derivations. Note that $\mathrm{Der}(A, \mathrm{End}_{\mathcal{O}}(M))$ is an $\mathcal{O}$-order since A and M are.

Let k be a positive integer. For $f \in \mathrm{End}_{\mathcal{O}}(M)$ such that $af(m + \epsilon^k M) = f(am + \epsilon^k M)$, for $a \in A$ and $m \in M$, we define $D_f \in \mathrm{Hom}_{\mathcal{O}}(A, \mathrm{End}_{\mathcal{O}}(M))$ as follows.

$$D_f(a)(m) = \epsilon^{-k}(f(am) - af(m)), \quad \text{for } a \in A \text{ and } m \in M.$$

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The following computation shows that D_f is a derivation.

$$\begin{aligned} D_f(xy)(m) &= \epsilon^{-k}(f(xym) - xy(m)) \\ &= \epsilon^{-k}(xf(ym) - xyf(m)) + \epsilon^{-k}(f(xym) - xf(ym)) \\ &= xD_f(y)(m) + D_f(x)(ym). \end{aligned}$$

Let $\mathrm{Der}(k)$ be the $\mathcal{O}$-submodule of $\mathrm{Der}(A, \mathrm{End}_{\mathcal{O}}(M))$ which is generated by all such D_f, and we define $\mathrm{Der}(\infty) = \sum_{k\geqslant 1} \mathrm{Der}(k)$. Since $\mathrm{Der}(A, \mathrm{End}_{\mathcal{O}}(M))$ is a finitely generated $\mathcal{O}$-module, there exists an integer s such that

$$\mathrm{Der}(\infty) = \sum_{k=1}^{s-1} \mathrm{Der}(k).$$

We show that the algebra homomorphism $\mathrm{End}_A(M) \to \mathrm{End}_A(M/\epsilon^k M)$ is surjective, for all $k \geqslant s$. Let $\theta \in \mathrm{End}_A(M/\epsilon^k M)$, for $k \geqslant s$. We fix $f \in \mathrm{End}_{\mathcal{O}}(M)$ such that

$$f(m + \epsilon^k M) = \theta(m + \epsilon^k M), \quad \text{for } m \in M.$$

Then, there exist $c_i \in \mathcal{O}$ and $f_i \in \mathrm{End}_{\mathcal{O}}(M)$ that satisfy

$$\begin{aligned} &f_i(m + \epsilon^{l_i} M) = \theta_i(m + \epsilon^{l_i} M), \\ &\qquad\qquad\qquad \text{for some } 1 \leqslant l_i \leqslant s-1 \text{ and } \theta_i \in \mathrm{End}_A(M/\epsilon^{l_i} M), \end{aligned}$$

such that $D_f = \sum_{i=1}^N c_i D_{f_i}$. More explicitly, we have

$$f(am) - af(m) = \sum_{i=1}^N \epsilon^{k-l_i} c_i(f_i(am) - af_i(m)), \quad \text{for } a \in A \text{ and } m \in M.$$

It implies that $f - \sum_{i=1}^N \epsilon^{k-l_i} c_i f_i \in \mathrm{End}_A(M)$. Since it coincides with θ if we reduce modulo ϵ, we have proved

$$\mathrm{Im}(\mathrm{End}_A(M) \to \mathrm{End}_A(M/\epsilon^k M)) + \epsilon\, \mathrm{End}_A(M/\epsilon^k M) = \mathrm{End}_A(M/\epsilon^k M).$$

Thus, Nakayama's lemma implies that $\mathrm{End}_A(M) \to \mathrm{End}_A(M/\epsilon^k M)$ is surjective, and we have an isomorphism of algebras $\mathrm{End}_A(M)/\epsilon^k\, \mathrm{End}_A(M) \simeq \mathrm{End}_A(M/\epsilon^k M)$. As $\mathcal{O}$ is a complete local ring, the lifting idempotent argument works [CR, (6.7)]. Hence, if $M/\epsilon^k M$ is decomposable, so is M. □

We recall the Harada–Sai lemma from [ARS, VI. Corollary 1.3].

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LEMMA 1.25. *Let B be an Artin algebra, $\{N_i \mid 1 \leqslant i \leqslant 2^m\}$ a collection of indecomposable B-modules such that the length of composition series of N_i is less than or equal to m, for all i. If none of $f_i \in \mathrm{Hom}_B(N_i, N_{i+1})$ $(1 \leqslant i \leqslant 2^m - 1)$ is an isomorphism, then*

$$f_{2^m-1} \cdots f_1 = 0.$$

PROPOSITION 1.26. *Let A be a symmetric $\mathcal{O}$-order over a complete discrete valuation ring $\mathcal{O}$, and assume that A is indecomposable as an $\mathcal{O}$-algebra. Let C be a component of the stable Auslander–Reiten quiver of A. Assume that the number of vertexes in C is finite. Then C exhausts all nonprojective indecomposable A-lattices.*

Proof. We add indecomposable projective A-lattices to the stable Auslander–Reiten quiver of A to obtain the Auslander–Reiten quiver of A. We show that if C is a finite component of the Auslander–Reiten quiver then C exhausts all indecomposable A-lattices. Assume that M is an indecomposable A-lattice which does not belong to C. It suffices to show

$$\mathrm{Hom}_A(M, N) = 0 = \mathrm{Hom}_A(N, M), \quad \text{for all } N \in C.$$

To see that it is sufficient, let P be a direct summand of the projective cover of $N \in C$. Then, $P \in C$ by $N \in C$ and $\mathrm{Hom}_A(P, N) \neq 0$. As A is indecomposable as an algebra, there is no indecomposable projective A-lattice Q with the property that

$$\mathrm{Hom}_A(Q, R) = 0 = \mathrm{Hom}_A(R, Q),$$

for all indecomposable projective A-lattices $R \in C$. It implies that any direct summand Q of the projective cover of M belongs to C. Then $\mathrm{Hom}_A(Q, M) \neq 0$ implies that $M \in C$, which contradicts our assumption. Thus, C exhausts all indecomposable A-lattices.

Assume that there exists a nonzero morphism $f \in \mathrm{Hom}_A(M, N)$. As $M \notin C$ and $N \in C$, f is not a split epimorphism. We consider the almost split sequence of A-lattices ending at N, and we denote by $N_1, \ldots, N_r$ the indecomposable direct summands of the middle term of the almost split sequence. Let

$$g_i^{(1)} : N_i \longrightarrow N$$

be irreducible morphisms. Then, there exist $f_i \in \mathrm{Hom}_A(M, N_i)$ such that

$$f = \sum_{i=1}^{r} g_i^{(1)} f_i.$$

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If N_i is nonprojective, we apply the same procedure to f_i. If N_i is projective, f_i factors through the Heller lattice Rad N_i of the irreducible $A \otimes \kappa$-module $N_i/\mathrm{Rad}(N_i)$. Thus, we apply the procedure after we replace N_i with Rad N_i. After repeating n times, we obtain,

$$f = \sum g_i^{(1)} \cdots g_i^{(n)} h_i,$$

such that $g_i^{(j)}$ are morphisms among indecomposable A-lattices in C, h_i are morphisms $M \to X_i$, where X_i are indecomposable A-lattices in C and they are not isomorphisms.

Since the number of vertexes in C is finite, there exists an integer s such that $X/\epsilon^s X$ is indecomposable, for all $X \in C$. Let m be the maximal length of $A/\epsilon^s A$-modules $X/\epsilon^s X$, for $X \in C$. Applying Lemma 1.25 to the Artin algebra $A/\epsilon^s A$ with $n = 2^m - 1$, we obtain

$$\mathrm{Hom}_A(M, N) = \epsilon^s \,\mathrm{Hom}_A(M, N),$$

and Nakayama's Lemma implies $\mathrm{Hom}_A(M, N) = 0$. The proof of $\mathrm{Hom}_A(N, M) = 0$ is similar. We start with a nonzero morphism $f \in \mathrm{Hom}_A(N, M)$ and consider the almost split sequence of A-lattices starting at N. Let $N_1, \ldots, N_r$ be the indecomposable direct summands of the middle term of the almost split sequence as above, and let

$$g_i^{(1)} : N \longrightarrow N_i$$

be irreducible morphisms. If N_i is projective, then we replace N_i with Rad N_i. Then, after repeating the procedure n times, we obtain

$$f = \sum h_i g_i^{(n)} \cdots g_i^{(1)},$$

where h_i are morphisms from indecomposable A-lattices in C to M. Then, we may deduce $\mathrm{Hom}_A(N, M) = 0$ by the Harada–Sai lemma and Nakayama's lemma as before. □

THEOREM 1.27. *Let A be a symmetric $\mathcal{O}$-order over a complete discrete valuation ring $\mathcal{O}$, and let C be a component of the stable Auslander–Reiten quiver of A. Suppose that:*

(i) *there exists a τ-periodic indecomposable A-lattice in C;*
(ii) *the stable Auslander–Reiten quiver of A has infinitely many vertexes.*

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Then, the number of vertexes in C is infinite and C is a valued stable translation quiver.

Proof. As in the proof of Lemma 1.23, C admits a subadditive function by the condition (i). Hence, the tree class of the valued stable translation quiver $C \setminus \{\text{loops}\}$ is one of finite, affine or infinite Dynkin diagrams. In the first two cases, the number of vertexes in C is finite, since all vertexes in C are τ-periodic. Then we may apply Proposition 1.26 and it contradicts the condition (ii). Thus, the tree class is one of infinite Dynkin diagrams and the number of vertexes in C is infinite. Then, Lemma 1.23 implies that there is no loop in C and C is a valued stable translation quiver. $\square$

§2. The case $A = \mathcal{O}[X]/(X^n)$

2.1 Heller lattices

Let $M_i = \kappa[X]/(X^{n-i})$, for $1 \leqslant i \leqslant n-1$. They form a complete set of isoclasses of nonprojective indecomposable $A \otimes \kappa$-modules. We realize M_i as the $A \otimes \kappa$-submodule $X^i A + \epsilon A/\epsilon A$ of $A \otimes \kappa = A/\epsilon A$. We view M_i as an A-module. Then, $p : A \twoheadrightarrow M_i$ defined by $f \mapsto X^i f + \epsilon A$ is the projective cover of M_i. Therefore, the Heller lattice Z_i of M_i, which is an A-submodule of A, is given as follows:

$$Z_i = \mathcal{O}\epsilon \oplus \mathcal{O}\epsilon X \oplus \cdots \mathcal{O}\epsilon X^{n-i-1} \oplus \mathcal{O}X^{n-i} \oplus \mathcal{O}X^{n-i+1} \oplus \cdots \oplus \mathcal{O}X^{n-1}.$$

Then the representing matrix of the action of X on Z_i with respect to the above basis is given by the following matrix:

$$X = \begin{array}{r c}
 & \begin{array}{cccccccc} & & & n-i & & & & \end{array} \\
\begin{array}{r} \\ \\ \\ \\ n-i+1 \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
0 & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & 0 \\
1 & \ddots & & \vdots & & & & \vdots \\
 & \ddots & 0 & \vdots & & & & \\
 & & 1 & 0 & & & & \vdots \\
\cdots & \cdots & \cdots & \epsilon & 0 & & & \vdots \\
 & & & & 1 & 0 & & \\
\vdots & & & & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & & & & 1 & 0
\end{pmatrix}
\end{array}$$

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Thus, $\mathrm{End}_A(Z_i) \simeq \{M \in \mathrm{Mat}(n, \mathcal{O}) \mid MX = XM\}$ is a local $\mathcal{O}$-algebra, since the right hand side is contained in

$$\left\{ \begin{pmatrix} a & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & * & \ddots & 0 \\ & & & a \end{pmatrix}, \quad a \in \mathcal{O} \right\}.$$

It follows the next lemma. Note that $\rho \in \mathrm{End}_A(Z_i)$ is determined by $\rho(\epsilon) \in Z_i$.

Lemma 2.1. *We have the following.*

(1) *The Heller lattices Z_i are pairwise nonisomorphic indecomposable A-lattices.*

(2) *If $\rho \in \mathrm{Rad}\,\mathrm{End}_A(Z_i)$ then $\rho(\epsilon)$ has the form*

$$\rho(\epsilon) = a_0\epsilon + \cdots + a_{n-i-1}\epsilon X^{n-i-1} + a_{n-i}X^{n-i} + \cdots + a_{n-1}X^{n-1},$$

where $a_i \in \mathcal{O}$, for $1 \leqslant i \leqslant n-1$, and $a_0 \in \epsilon\mathcal{O}$.

We now consider the following pullback diagram:

$$\begin{tikzcd} 0 \arrow[r] & Z_{n-i} \arrow[r] \arrow[d, equal] & E_i \arrow[r] \arrow[d] & Z_i \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & Z_{n-i} \arrow[r, "\iota"] & A \oplus A \arrow[r, "\pi"] & Z_i \arrow[r] & 0 \end{tikzcd}$$

where ϕ is defined by $\phi(\epsilon) = X^{n-1}$ and

$$\phi(\epsilon X) = \cdots = \phi(\epsilon X^{n-i-1}) = \phi(X^{n-i}) = \cdots = \phi(X^{n-1}) = 0,$$

$\pi(f, g) = X^{n-i}f - \epsilon g$, for $(f, g) \in A \oplus A$, and ι is given as follows.

$$\begin{aligned} \iota(\epsilon X^j) &= (\epsilon X^j, X^{n-i+j}) && \text{if } 0 \leqslant j \leqslant i-1, \\ \iota(X^j) &= (X^j, 0) && \text{if } i \leqslant j \leqslant n-1. \end{aligned}$$

Remark 2.2. Using the exact sequences

$$0 \to Z_{n-i} \to A \oplus A \to Z_i \to 0 \qquad \text{and} \qquad 0 \to Z_{n-1} \to A \to \kappa \to 0,$$

one computes

$$\mathrm{Ext}^i_A(\kappa, A) = \begin{cases} \kappa & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.3. *We have the following.*

(1) *ϕ does not factor through π.*
(2) *For any $\rho \in \mathrm{Rad}\,\mathrm{End}_A(Z_i)$, $\phi\rho$ factors through π.*

Proof. (1) If there is a morphism $\mu = (\mu_1, \mu_2) : Z_i \to A \oplus A$ such that $\pi\mu = \phi$, then we have $X^{n-i}\mu_1(\epsilon) - \epsilon\mu_2(\epsilon) = \epsilon(\mu_1(X^{n-i}) - \mu_2(\epsilon)) = X^{n-1}$. This is a contradiction.

(2) Write

$$\rho(\epsilon) = a_0\epsilon + \cdots + a_{n-i-1}\epsilon X^{n-i-1} + a_{n-i}X^{n-i} + \cdots + a_{n-1}X^{n-1}.$$

Then, by Lemma 2.1, there exists $a \in \mathcal{O}$ such that $a_0 = \epsilon a$. We define $\mu \in \mathrm{Hom}_A(Z_i, A \oplus A)$ by $\mu(\epsilon) = (0, -aX^{n-1})$. Then, it is easy to check that $\pi\mu = \phi\rho$ holds. ∎

By Proposition 1.15 and Lemma 2.3, we have an almost split sequence

$$0 \to Z_{n-i} \to E_i \to Z_i \to 0,$$

where $E_i = \{(f, g, h) \in A \oplus A \oplus Z_i \mid \pi(f, g) = \phi(h)\}$ is given by

$$\begin{aligned} E_i = {} & \mathcal{O}(\epsilon, X^{n-i}, 0) \oplus \mathcal{O}(\epsilon X, X^{n-i+1}, 0) \oplus \cdots \oplus \mathcal{O}(\epsilon X^{i-1}, X^{n-1}, 0) \\ & \oplus \mathcal{O}(X^i, 0, 0) \oplus \mathcal{O}(X^{i+1}, 0, 0) \oplus \cdots \oplus \mathcal{O}(X^{n-1}, 0, 0) \\ & \oplus \mathcal{O}(X^{i-1}, 0, \epsilon) \oplus \mathcal{O}(0, 0, \epsilon X) \oplus \cdots \oplus \mathcal{O}(0, 0, \epsilon X^{n-i-1}) \\ & \oplus \mathcal{O}(0, 0, X^{n-i}) \oplus \mathcal{O}(0, 0, X^{n-i+1}) \oplus \cdots \oplus \mathcal{O}(0, 0, X^{n-1}). \end{aligned}$$

To simplify the notation, we define $a_0 = b_0 = 0$ and

$$a_k = \begin{cases} (X^{n-k}, 0, 0) & \text{if } 1 \leqslant k \leqslant n-i, \\ (\epsilon X^{n-k}, X^{2n-k-i}, 0) & \text{if } n-i < k \leqslant n, \end{cases}$$

$$b_k = \begin{cases} (0, 0, X^{n-k}) & \text{if } 1 \leqslant k \leqslant i, \\ (0, 0, \epsilon X^{n-k}) & \text{if } i < k < n, \\ (X^{i-1}, 0, \epsilon) & \text{if } k = n. \end{cases}$$

Then, we have

$$Xa_k = \begin{cases} a_{k-1} & (k \neq n-i+1) \\ \epsilon a_{k-1} & (k = n-i+1) \end{cases}$$

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$$Xb_k = \begin{cases} b_{k-1} & (k \neq i+1, n) \\ \epsilon b_{k-1} & (k = i+1) \\ a_{n-i} + b_{n-1} & (k = n) \end{cases}$$

and

$$\mathrm{Ker}(X^k) = \bigoplus_{1 \leqslant j \leqslant k} (\mathcal{O}a_j \oplus \mathcal{O}b_j).$$

2.2 Almost split sequence ending at $\mathbf{Z_i}$

In this subsection, we show that the middle term E_i of the almost split sequence

$$0 \to Z_{n-i} \to E_i \to Z_i \to 0$$

is indecomposable, for $2 \leqslant i \leqslant n-1$.

PROPOSITION 2.4. *We have the following.*

(1) *A is an indecomposable direct summand of E_1.*
(2) *For $2 \leqslant i \leqslant n-1$, E_i are indecomposable A-lattices.*

Proof. (1) As $Z_{n-1} = \mathrm{Rad}\, A$, it follows from [A1, Chapter III, Theorem 2.5]. We also give more explicit computational proof here. Define $x_k, y_k \in E_1$, for $1 \leqslant k \leqslant n$, as follows:

$$x_k = \begin{cases} a_1 + \epsilon b_1 & \text{if } k = 1, \\ a_k + b_k & \text{if } 2 \leqslant k \leqslant n-1, \\ b_n & \text{if } k = n, \end{cases}$$

$$y_k = \begin{cases} b_k & \text{if } 1 \leqslant k \leqslant n-1, \\ a_n - \epsilon b_n & \text{if } k = n. \end{cases}$$

Then they form an $\mathcal{O}$-basis of E_1. Moreover, we have $Xx_1 = 0$ and $Xy_1 = 0$,

$$Xx_k = x_{k-1}, \quad \text{for } 2 \leqslant k \leqslant n, \qquad \text{and} \qquad Xy_k = \begin{cases} \epsilon y_1 & \text{if } k = 2, \\ y_{k-1} & \text{if } 3 \leqslant k \leqslant n-1, \\ -\epsilon y_{n-1} & \text{if } k = n. \end{cases}$$

Thus, the $\mathcal{O}$-span of $\{x_k \mid 1 \leqslant k \leqslant n\}$ is isomorphic to the indecomposable projective A-lattice A. In particular, A is an indecomposable direct summand of E_1, and the other direct summand is indecomposable, because it becomes $A \otimes \mathcal{K}$ after tensoring with $\mathcal{K}$.

(2) E_{n-1} does not have a projective direct summand by [A1, Chapter III, Theorem 2.5]. Thus, [A1, Chapter III, Propositions 1.7, 1.8] and (1) imply that $E_{n-1} \simeq \tau(E_1)$ is indecomposable. We assume $2 \leqslant i \leqslant n-2$ in the rest of the proof.

Suppose that $E_i = E' \oplus E''$ with $E', E'' \neq 0$. Since $Z_i \otimes \mathcal{K} = Z_{n-i} \otimes \mathcal{K} = A \otimes \mathcal{K}$, we have $E_i \otimes \mathcal{K} \simeq A \otimes \mathcal{K} \oplus A \otimes \mathcal{K}$, which implies that

$$E' \otimes \mathcal{K} \simeq A \otimes \mathcal{K} \simeq E'' \otimes \mathcal{K}.$$

In particular, rank $E' = n =$ rank E''. Since

$$0 \to E' \cap \operatorname{Ker}(X^k) \to E' \to \operatorname{Im}(X^k) \to 0$$

and $\operatorname{Im}(X^k)$ is a free $\mathcal{O}$-module, we have the increasing sequence of $\mathcal{O}$-submodules

$$0 \subsetneq \cdots \subsetneq E' \cap \operatorname{Ker}(X^k) \subsetneq E' \cap \operatorname{Ker}(X^{k+1}) \subsetneq \cdots \subsetneq E' \cap \operatorname{Ker}(X^n) = E'$$

such that all the $\mathcal{O}$-submodules are direct summands of E' as $\mathcal{O}$-modules. Thus, we may choose an $\mathcal{O}$-basis $\{e'_k\}_{1 \leqslant k \leqslant n}$ such that $e'_k \in E' \cap \operatorname{Ker}(X^k) \setminus \operatorname{Ker}(X^{k-1})$. Similarly, we may choose an $\mathcal{O}$-basis $\{e''_k\}_{1 \leqslant k \leqslant n}$ of E'' such that $e''_k \in E'' \cap \operatorname{Ker}(X^k) \setminus \operatorname{Ker}(X^{k-1})$. Write

$$\begin{aligned} e'_k &= \alpha_k a_k + \beta_k b_k + A'_k, && \text{for } \alpha_k, \beta_k \in \mathcal{O} \text{ and } A'_k \in \operatorname{Ker}(X^{k-1}), \\ e''_k &= \gamma_k a_k + \delta_k b_k + A''_k, && \text{for } \gamma_k, \delta_k \in \mathcal{O} \text{ and } A''_k \in \operatorname{Ker}(X^{k-1}). \end{aligned}$$

Without loss of generality, we may assume

$$A'_k \in \operatorname{Ker}(X^{k-1}) \cap E'', \qquad A''_k \in \operatorname{Ker}(X^{k-1}) \cap E'.$$

Since $\{e'_k, e''_k\}$ and $\{a_k, b_k\}$ are $\mathcal{O}$-bases of $\operatorname{Ker}(X^k)/\operatorname{Ker}(X^{k-1})$, we have $\alpha_k \delta_k - \beta_k \gamma_k \notin \epsilon \mathcal{O}$.

As $X e'_k \in \operatorname{Ker}(X^{k-1}) \cap E'$, there are $f^{(k)}_{k-1}, \ldots, f^{(k)}_1 \in \mathcal{O}$ such that

$$X e'_k = f^{(k)}_{k-1} e'_{k-1} + \cdots + f^{(k)}_1 e'_1.$$

Similarly, there are $g^{(k)}_{k-1}, \ldots, g^{(k)}_1 \in \mathcal{O}$ such that

$$X e''_k = g^{(k)}_{k-1} e''_{k-1} + \cdots + g^{(k)}_1 e''_1.$$

The coefficient of a_{k-1} in Xe'_k is given by

$$\begin{cases} \alpha_k & \text{if } k \neq n-i+1, \\ \epsilon\alpha_k & \text{if } k = n-i+1. \end{cases}$$

Thus, we have

$$f_{k-1}^{(k)}\alpha_{k-1} = \begin{cases} \alpha_k & \text{if } k \neq n-i+1, \\ \epsilon\alpha_k & \text{if } k = n-i+1. \end{cases}$$

Similarly, we have the following.

$$f_{k-1}^{(k)}\beta_{k-1} = \begin{cases} \beta_k & \text{if } k \neq i+1, \\ \epsilon\beta_k & \text{if } k = i+1. \end{cases}$$

$$g_{k-1}^{(k)}\gamma_{k-1} = \begin{cases} \gamma_k & \text{if } k \neq n-i+1, \\ \epsilon\gamma_k & \text{if } k = n-i+1. \end{cases}$$

$$g_{k-1}^{(k)}\delta_{k-1} = \begin{cases} \delta_k & \text{if } k \neq i+1, \\ \epsilon\delta_k & \text{if } k = i+1. \end{cases}$$

We shall deduce a contradiction in the following three cases and conclude that E_i is indecomposable, for $2 \leqslant i \leqslant n-2$.

(Case a) $2 \leqslant n-i < i$.

(Case b) $2 \leqslant i = n-i$.

(Case c) $2 \leqslant i < n-i$.

Suppose that we are in (Case a). We multiply each of e'_k and e''_k by suitable invertible elements to get new $\mathcal{O}$-bases of E' and E'' in order to have the equalities

$$f_{k-1}^{(k)} = \begin{cases} 1 & \text{if } k \neq n-i+1, \\ \epsilon & \text{if } k = n-i+1, \end{cases} \qquad \text{and} \qquad g_{k-1}^{(k)} = \begin{cases} 1 & \text{if } k \neq i+1, \\ \epsilon & \text{if } k = i+1, \end{cases}$$

in the new bases. For $k = 1$, we keep the original basis elements e'_1 and e''_1. Suppose that we have already chosen new e'_j and e''_j, for $1 \leqslant j \leqslant k-1$. If $k \neq n-i+1, i+1$, then

$$f_{k-1}^{(k)}g_{k-1}^{(k)}(\alpha_{k-1}\delta_{k-1} - \beta_{k-1}\gamma_{k-1}) = \alpha_k\delta_k - \beta_k\gamma_k$$

implies that $f_{k-1}^{(k)}$ and $g_{k-1}^{(k)}$ are invertible. Thus, multiplying e'_k and e''_k with their inverses respectively, we have $f_{k-1}^{(k)} = 1$, $g_{k-1}^{(k)} = 1$ in the new basis. Note that we have

$$\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix} = \begin{pmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{pmatrix} = \cdots\cdot = \begin{pmatrix}\alpha_{n-i} & \beta_{n-i}\\ \gamma_{n-i} & \delta_{n-i}\end{pmatrix}.$$

If $k = n-i+1$, then, by using $i \neq n-i$, we have

$$f_{n-i}^{(n-i+1)} g_{n-i}^{(n-i+1)} \alpha_{n-i}\delta_{n-i} = \epsilon\alpha_{n-i+1}\delta_{n-i+1},$$
$$f_{n-i}^{(n-i+1)} g_{n-i}^{(n-i+1)} \beta_{n-i}\gamma_{n-i} = \epsilon\beta_{n-i+1}\gamma_{n-i+1}.$$

It follows that $f_{n-i}^{(n-i+1)} g_{n-i}^{(n-i+1)} \in \epsilon\mathcal{O} \setminus \epsilon^2\mathcal{O}$, and we may assume

$$f_{n-i}^{(n-i+1)} = \epsilon, \qquad g_{n-i}^{(n-i+1)} = 1,$$

by swapping E' and E'' if necessary. Thus, we have

$$\begin{pmatrix}\alpha_{n-i} & \beta_{n-i}\\ \gamma_{n-i} & \delta_{n-i}\end{pmatrix} = \begin{pmatrix}\alpha_{n-i+1} & \epsilon^{-1}\beta_{n-i+1}\\ \epsilon\gamma_{n-i+1} & \delta_{n-i+1}\end{pmatrix} = \cdots = \begin{pmatrix}\alpha_i & \epsilon^{-1}\beta_i\\ \epsilon\gamma_i & \delta_i\end{pmatrix}.$$

Finally, if $k = i+1$, then the similar argument shows

$$f_i^{(i+1)} g_i^{(i+1)} \in \epsilon\mathcal{O} \setminus \epsilon^2\mathcal{O},$$

and we may assume that $(f_i^{(i+1)}, g_i^{(i+1)})$ is either $(\epsilon, 1)$ or $(1, \epsilon)$. In the former case,

$$\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix} = \begin{pmatrix}\alpha_i & \epsilon^{-1}\beta_i\\ \epsilon\gamma_i & \delta_i\end{pmatrix} = \begin{pmatrix}\epsilon^{-1}\alpha_{i+1} & \epsilon^{-1}\beta_{i+1}\\ \epsilon\gamma_{i+1} & \epsilon\delta_{i+1}\end{pmatrix},$$

which implies that $\alpha_{i+1}, \beta_{i+1} \in \epsilon\mathcal{O}$, a contradiction. Thus, we obtain

$$f_i^{(i+1)} = 1, \qquad g_i^{(i+1)} = \epsilon.$$

Therefore, we have obtained the desired formula. In particular, we have the following.

$$\alpha_{k-1} = \alpha_k, \qquad f_{k-1}^{(k)}\beta_{k-1} = g_{k-1}^{(k)}\beta_k, \qquad g_{k-1}^{(k)}\gamma_{k-1} = f_{k-1}^{(k)}\gamma_k, \qquad \delta_{k-1} = \delta_k,$$
$$Xa_k = f_{k-1}^{(k)}a_{k-1}, \qquad Xb_k = g_{k-1}^{(k)}b_{k-1} + \delta_{k,n}a_{n-i},$$

where $\delta_{k,n}$ is the Kronecker delta. Suppose that $1 \leqslant k \leqslant n-1$. Then, we have

$$XA'_k = X(e'_k - \alpha_k a_k - \beta_k b_k) = Xe'_k - f_{k-1}^{(k)}\alpha_k a_{k-1} - g_{k-1}^{(k)}\beta_k b_{k-1},$$

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$$\begin{aligned} f^{(k)}_{k-1}A'_{k-1} &= f^{(k)}_{k-1}(e'_{k-1} - \alpha_{k-1}a_{k-1} - \beta_{k-1}b_{k-1}) \\ &= f^{(k)}_{k-1}e'_{k-1} - f^{(k)}_{k-1}\alpha_k a_{k-1} - g^{(k)}_{k-1}\beta_k b_{k-1}. \end{aligned}$$

We compute $Xe'_k - f^{(k)}_{k-1}e'_{k-1}$ in two ways:

$$\begin{aligned} Xe'_k - f^{(k)}_{k-1}e'_{k-1} &= XA'_k - f^{(k)}_{k-1}A'_{k-1} \in E'', \\ Xe'_k - f^{(k)}_{k-1}e'_{k-1} &= f^{(k)}_{k-2}e'_{k-2} + \cdots + f^{(k)}_1 e'_1 \in E'. \end{aligned}$$

Thus, we have $Xe'_k = f^{(k)}_{k-1}e'_{k-1}$, for $1 \leqslant k \leqslant n-1$. Next suppose that $k = n$. Then, the similar computation shows

$$\beta_n a_{n-i} + XA'_n - f^{(n)}_{n-1}A'_{n-1} = Xe'_n - f^{(n)}_{n-1}e'_{n-1} = f^{(n)}_{n-2}e'_{n-2} + \cdots + f^{(n)}_1 e'_1.$$

We compute $X^{n-i+1}e'_n - f^{(n)}_{n-1}X^{n-i}e'_{n-1}$ in two ways as before, and we obtain

$$X^{n-i+1}A'_n - f^{(n)}_{n-1}X^{n-i}A'_{n-1} = f^{(n)}_{n-2}X^{n-i}e'_{n-2} + \cdots + f^{(n)}_1 X^{n-i}e'_1 = 0.$$

Hence, we have $f^{(n)}_{n-2} = \cdots = f^{(n)}_{n-i+1} = 0$. We define

$$z_n = e'_n, \qquad z_k = e'_k + X^{n-k-1}(f^{(n)}_{n-i}e'_{n-i} + \cdots + f^{(n)}_1 e'_1), \quad \text{for } 1 \leqslant k \leqslant n-1.$$

Then, $\{z_k \mid 1 \leqslant k \leqslant n\}$ is an $\mathcal{O}$-basis of E', since

$$X^{n-k-1}(f^{(n)}_{n-i}e'_{n-i} + \cdots + f^{(n)}_1 e'_1) \in \operatorname{Ker}(X^{k-1}).$$

Further, we have $z_k = e'_k$, for $1 \leqslant k \leqslant i-1$. In particular, $z_{n-i} = e'_{n-i}$ by $n-i \leqslant i-1$. Then, we can check that

$$Xz_k = \begin{cases} z_{k-1} & \text{if } k \neq n-i+1, \\ \epsilon z_{k-1} & \text{if } k = n-i+1. \end{cases}$$

Thus, we conclude that $E' \simeq Z_{n-i}$. Recall that the exact sequence

$$0 \to Z_{n-i} \to E_i \to Z_i \to 0$$

does not split. On the other hand, $E_i \simeq Z_{n-i} \oplus Z_i$ implies that it must split, by Miyata's theorem [M, Theorem 1]. Hence, E_i is indecomposable in (Case a).

Next assume that we are in (Case b). Then, $f^{(k)}_{k-1}$ and $g^{(k)}_{k-1}$, for $k \neq i+1$, are invertible as before, and we may choose

$$f^{(k)}_{k-1} = 1, \quad g^{(k)}_{k-1} = 1.$$

If $k = i+1$, note that

$$\begin{aligned} f^{(i+1)}_i \alpha_i &= \epsilon\alpha_{i+1}, & f^{(i+1)}_i \beta_i &= \epsilon\beta_{i+1}, \\ g^{(i+1)}_i \gamma_i &= \epsilon\gamma_{i+1}, & g^{(i+1)}_i \delta_i &= \epsilon\delta_{i+1}. \end{aligned}$$

Thus, $\alpha_i, \beta_i \in \epsilon\mathcal{O}$ if $f^{(i+1)}_i$ is invertible, and $\gamma_i, \delta_i \in \epsilon\mathcal{O}$ if $g^{(i+1)}_i$ is invertible. But both are impossible. Further,

$$f^{(i+1)}_i g^{(i+1)}_i (\alpha_i\delta_i - \beta_i\gamma_i) = \epsilon^2(\alpha_{i+1}\delta_{i+1} - \beta_{i+1}\gamma_{i+1})$$

implies $f^{(i+1)}_i g^{(i+1)}_i \in \epsilon^2\mathcal{O} \setminus \epsilon^3\mathcal{O}$. Thus, we may choose

$$f^{(i+1)}_i = \epsilon, \qquad g^{(i+1)}_i = \epsilon.$$

Hence, we may assume without loss of generality that

$$f^{(k)}_{k-1} = g^{(k)}_{k-1} = \begin{cases} 1 & \text{if } k \neq i+1, \\ \epsilon & \text{if } k = i+1. \end{cases}$$

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \cdots = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} = \begin{pmatrix} \alpha_{i+1} & \beta_{i+1} \\ \gamma_{i+1} & \delta_{i+1} \end{pmatrix} = \cdots = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}.$$

and $Xa_k = f^{(k)}_{k-1} a_{k-1}$, $Xb_k = g^{(k)}_{k-1} b_{k-1} + \delta_{k,n} a_i$. For $1 \leqslant k \leqslant n-1$, we have

$$XA'_k - f^{(k)}_{k-1} A'_{k-1} = Xe'_k - f^{(k)}_{k-1} e'_{k-1} = f^{(k)}_{k-2} e'_{k-2} + \cdots + f^{(k)}_1 e'_1,$$

and the same argument as before shows that

$$Xe'_k = \begin{cases} f^{(k)}_{k-1} e'_{k-1} & \text{if } k \neq n, \\ f^{(n)}_{n-1} e'_{n-1} + f^{(n)}_i e'_i + \cdots + f^{(n)}_1 e'_1 & \text{if } k = n. \end{cases}$$

Now, we compute

$$\begin{aligned} X^{i-1} e'_{n-1} &= f^{(n-1)}_{n-2} \cdots f^{(n-i+1)}_{n-i} e'_{n-i} = \epsilon e'_i, \\ X^i a_n &= f^{(n)}_{n-1} \cdots f^{(i+1)}_i a_i = \epsilon a_i, \end{aligned}$$

$$\begin{aligned} X^i b_n &= X^{i-1}(b_{n-1} + a_i) = g_{n-2}^{(n-1)} \cdots g_{n-i}^{(i+1)} b_i + f_{i-1}^{(i)} \cdots f_1^{(2)} a_1 \\ &= \epsilon b_i + a_1. \end{aligned}$$

Thus, we have

$$\begin{aligned} X^i e'_n - X^{i-1} e'_{n-1} &= X^i(\alpha_n a_n + \beta_n b_n + A'_n) - \epsilon e'_i \\ &= \epsilon(\alpha_n a_i + \beta_n b_i - e'_i) + X^i A'_n + \beta_n a_1. \end{aligned}$$

If $i+1 \leqslant k \leqslant n-1$ then $k-i+1 \leqslant n-i = i$ and

$$X^i e'_k = f_{k-1}^{(k)} \cdots f_{k-i}^{(k-i+1)} e'_{k-i} \in \epsilon E'.$$

Thus, $X^i A'_n \in \epsilon E'$. On the other hand, we have

$$\begin{aligned} X^i e'_n - X^{i-1} e'_{n-1} &= X^{i-1}(X e'_n - e'_{n-1}) = X^{i-1}(f_i^{(n)} e'_i + \cdots + f_1^{(n)} e'_1) \\ &= f_i^{(n)} X^{i-1} e'_i = f_i^{(n)} X^{i-1}(\alpha_i a_i + \beta_i b_i + A'_i) \\ &= f_i^{(n)}(\alpha_i X^{i-1} a_i + \beta_i X^{i-1} b_i) = f_i^{(n)}(\alpha_i a_1 + \beta_i b_1). \end{aligned}$$

Hence, we obtain $\beta_n a_1 \equiv f_i^{(n)}(\alpha_i a_1 + \beta_i b_1) \bmod \epsilon\mathcal{O}$. The similar computation using e''_k shows $\delta_n a_1 \equiv f_i^{(n)}(\gamma_i a_1 + \delta_i b_1) \bmod \epsilon\mathcal{O}$. If $f_i^{(n)}$ was invertible, it would imply $\beta_i, \delta_i \in \epsilon\mathcal{O}$, which contradicts $\alpha_i\delta_i - \beta_i\gamma_i \in \mathcal{O}^\times$. Thus, $f_i^{(n)} \in \epsilon\mathcal{O}$ and we have $\beta_n, \delta_n \in \epsilon\mathcal{O}$, which is again a contradiction. Hence, E_i is indecomposable in (Case b).

Finally, suppose that we are in (Case c). Since $E_i \simeq \tau(E_{n-i})$, for $2 \leqslant i \leqslant n-2$, and E_{n-i} is indecomposable by (Cases a), it follows from [A1, Chapter III, Propositions 1.7, 1.8] that E_i is indecomposable in (Case c). □

2.3 Almost split sequence ending at $\mathbf{E_i}$

We construct an almost split sequence ending at E_i, for $2 \leqslant i \leqslant n-2$. Define $\pi : A^{\oplus 4} \to E_i$, for $2 \leqslant i \leqslant n-2$, by

$$\pi(p, q, r, s) = (\epsilon p + X^{i-1} q, X^{n-i} p, \epsilon q + \epsilon X r + X^{n-i} s),$$

for $(p, q, r, s) \in A^{\oplus 4}$. Note that

$$\begin{aligned} \pi(1, 0, 0, 0) &= a_n, & \pi(0, 1, 0, 0) &= b_n, \\ \pi(0, 0, 1, 0) &= b_{n-1}, & \pi(0, 0, 0, 1) &= b_i. \end{aligned}$$

LEMMA 2.5. *Let $\pi : A^{\oplus 4} \to E_i$ be as above. Then,*

(1) *π is an epimorphism;*
(2) $\mathrm{Ker}(\pi) \simeq E_{n-i}$, *for* $2 \leqslant i \leqslant n-2$.

Proof. (1) It is easy to check that $a_k, b_k \in \mathrm{Im}(\pi)$, for $1 \leqslant k \leqslant n$. Note that E_i is generated by $\{a_n, b_n, b_{n-1}, b_i\}$ as an A-module and $a_{n-i} = Xb_n - b_{n-1}$.

(2) We define an A-module homomorphism $\iota : E_{n-i} \to A^{\oplus 4}$ by

$$\iota(f, g, h) = \left(g, -Xf + \frac{X^{n-i}h}{\epsilon}, f, -h\right), \quad \text{for } (f, g, h) \in E_{n-i}.$$

We write $h = h_0\epsilon + h_1\epsilon X + \cdots + h_{i-1}\epsilon X^{i-1} + h_i X^i + \cdots + h_{n-1}X^{n-1}$, for $h_i \in \mathcal{O}$. Then,

$$\frac{X^{n-i}h}{\epsilon} = h_0 X^{n-i} + h_1 X^{n-i+1} + \cdots + h_{i-1}X^{n-1}.$$

Note that $(f, g, h) \in A^{\oplus 3}$ belongs to E_{n-i} if and only if $h \in Z_{n-i}$ and $X^i f - \epsilon g = h_0 X^{n-1}$. It is clear that ι is a monomorphism and it suffices to show that $\mathrm{Im}(\iota) = \mathrm{Ker}(\pi)$. Since

$$\begin{aligned}
&\pi\iota(f, g, h) \\
&\quad = \left(\epsilon g - X^i f + \frac{X^{n-1}h}{\epsilon}, X^{n-i}g, \epsilon\left(-Xf + \frac{X^{n-i}h}{\epsilon}\right) + \epsilon Xf - X^{n-i}h\right) \\
&\quad = \left(\epsilon g - X^i f + \frac{X^{n-1}h}{\epsilon}, X^{n-i}g, 0\right) = (0, 0, 0),
\end{aligned}$$

we have $\mathrm{Im}(\iota) \subseteq \mathrm{Ker}(\pi)$. Let $(p, q, r, s) \in \mathrm{Ker}(\pi)$. Then we have

$$\begin{aligned}
\epsilon p + X^{i-1}q &= 0, \\
X^{n-i}p &= 0, \\
\epsilon q + \epsilon Xr + X^{n-i}s &= 0.
\end{aligned}$$

The third equation shows that the projective cover $A \twoheadrightarrow M_{n-i} = X^{n-i}A + \epsilon A/\epsilon A \subseteq A \otimes \kappa$ given by $f \mapsto X^{n-i}f + \epsilon A$ sends s to 0. Thus, we have $s \in Z_{n-i}$. Further,

$$\begin{aligned}
X^{n-1}s + \epsilon(-\epsilon p + X^i r) &= X^{n-1}s + \epsilon(X^{i-1}q + X^i r) \\
&= X^{i-1}(X^{n-i}s + \epsilon q + Xr) = 0
\end{aligned}$$

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implies $X^i r - \epsilon p = \frac{X^{n-1}(-s)}{\epsilon}$. Hence, we have $(r, p, -s) \in E_{n-i}$ and

$$\iota(r, p, -s) = \left(p, -Xr - \frac{X^{n-i}s}{\epsilon}, r, s\right) = (p, q, r, s).$$

Therefore, we have $\mathrm{Ker}(\pi) = \mathrm{Im}(\iota)$, which implies $\mathrm{Ker}(\pi) \simeq E_{n-i}$. ∎

We consider the following pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & E_{n-i} \arrow[r] \arrow[d, equal] & F_i \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & E_{n-i} \arrow[r, "\iota"] & A^{\oplus 4} \arrow[r, "\pi"] & E_i \arrow[r] & 0
\end{tikzcd}$$

where ι is the isomorphism $E_{n-i} \simeq \mathrm{Ker}(\pi)$ defined in the proof of Lemma 2.5, and

$$\begin{aligned}
\phi(a_k) &= 0 && \text{for } 1 \leqslant k \leqslant n,\\
\phi(b_k) &= 0 && \text{for } 1 \leqslant k \leqslant n-1,\\
\phi(b_n) &= b_1 && \text{for } k = n.
\end{aligned}$$

LEMMA 2.6. *Suppose that* $2 \leqslant i \leqslant n-i$. *Let* $\rho \in \mathrm{Rad}\,\mathrm{End}_A(E_i)$ *such that*

$$\rho(a_n) = \alpha a_n + \beta b_n + A, \qquad \rho(b_n) = \alpha' a_n + \beta' b_n + B,$$

where $\alpha, \beta, \alpha', \beta' \in \mathcal{O}$ *and* $A, B \in \mathrm{Ker}(X^{n-1})$. *Then we have the following.*

(1) $\beta \in \epsilon\mathcal{O}$, *and* $\alpha \in \epsilon\mathcal{O}$ *if and only if* $\beta' \in \epsilon\mathcal{O}$.
(2) $\alpha\beta' - \beta\alpha' \in \epsilon\mathcal{O}$.

Proof. (1) We compute $\rho(\epsilon X^{n-i}b_n - X^{n-1}a_n)$ in two ways. Since $X^{n-i}b_n = \epsilon b_i + a_1$ and $X^{n-1}a_n = \epsilon a_1$, we have $\rho(\epsilon X^{n-i}b_n - X^{n-1}a_n) = \epsilon^2\rho(b_i) \in \epsilon^2 E_i$. On the other hand, since $X^{n-i}b_n = \epsilon b_i + a_1$, we have

$$\begin{aligned}
&\rho(\epsilon X^{n-i}b_n - X^{n-1}a_n)\\
&\quad = \epsilon X^{n-i}(\alpha' a_n + \beta' b_n + B) - X^{n-1}(\alpha a_n + \beta b_n + A)\\
&\quad = \epsilon\alpha' X^{n-i}a_n + \epsilon^2\beta' b_i + \epsilon(\beta' - \alpha)a_1 - \epsilon\beta b_1 + \epsilon X^{n-i}B.
\end{aligned}$$

Then, $X^{n-i}a_k = \epsilon a_{k-n+i}$ and $X^{n-i}b_k = \epsilon b_{k-n+i}$, for $n-i+1 \leqslant k \leqslant n-1$, imply that $\epsilon X^{n-i}B \in \epsilon^2 E_i$. Hence, we may divide the both sides by ϵ. Reducing modulo ϵ, we have

$$(\beta' - \alpha)a_1 - \beta b_1 \equiv 0 \mod \epsilon E_i,$$

since $X^{n-i}a_n \equiv 0 \bmod \epsilon E_i$ if $2 \leqslant i \leqslant n-i$. Now, the claim is clear.

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(2) Since $\rho(a_k), \rho(b_k) \in \mathrm{Ker}(X^k)$, we may write

$$\begin{aligned}\rho(a_k) &= \alpha_k a_k + \beta_k b_k + A_k,\\ \rho(b_k) &= \alpha'_k a_k + \beta'_k b_k + B_k,\end{aligned}$$

where $\alpha_k, \beta_k, \alpha'_k, \beta'_k \in \mathcal{O}$ and $A_k, B_k \in \mathrm{Ker}(X^{k-1})$. We claim that

$$\alpha_k\beta'_k - \beta_k\alpha'_k = \alpha\beta' - \beta\alpha'.$$

To see this, observe that we have the following identities in $E_i/\mathrm{Ker}(X^{k-1})$.

$$\begin{cases}\alpha a_k + \beta b_k \equiv \rho(X^{n-k}a_n) \equiv \rho(a_k) \mod \mathrm{Ker}(X^{k-1}) & \text{if } k > n-i,\\ \alpha\epsilon a_k + \beta b_k \equiv \rho(X^{n-k}a_n) \equiv \epsilon\rho(a_k) \mod \mathrm{Ker}(X^{k-1}) & \text{if } i < k \leqslant n-i,\\ \alpha\epsilon a_k + \beta\epsilon b_k \equiv \rho(X^{n-k}a_n) \equiv \epsilon\rho(a_k) \mod \mathrm{Ker}(X^{k-1}) & \text{if } k \leqslant i,\end{cases}$$

$$\begin{cases}\alpha' a_k + \beta' b_k \equiv \rho(X^{n-k}b_n) \equiv \rho(b_k) \mod \mathrm{Ker}(X^{k-1}) & \text{if } k > n-i,\\ \alpha'\epsilon a_k + \beta' b_k \equiv \rho(X^{n-k}b_n) \equiv \rho(b_k) \mod \mathrm{Ker}(X^{k-1}) & \text{if } i < k \leqslant n-i,\\ \alpha'\epsilon a_k + \beta'\epsilon b_k \equiv \rho(X^{n-k}b_n) \equiv \epsilon\rho(b_k) \mod \mathrm{Ker}(X^{k-1}) & \text{if } k \leqslant i.\end{cases}$$

Thus, if we denote

$$\begin{aligned}(\overline{a}_k, \overline{b}_k) &= (a_k + \mathrm{Ker}(X^{k-1}), b_k + \mathrm{Ker}(X^{k-1})),\\ (\overline{a}'_k, \overline{b}'_k) &= (\rho(a_k) + \mathrm{Ker}(X^{k-1}), \rho(b_k) + \mathrm{Ker}(X^{k-1})),\end{aligned}$$

Then, we have

$$(\overline{a}_k, \overline{b}_k)\begin{pmatrix}\alpha_k & \alpha'_k\\ \beta_k & \beta'_k\end{pmatrix} = (\overline{a}'_k, \overline{b}'_k) = (\overline{a}_k, \overline{b}_k)\begin{pmatrix}\alpha & \alpha'\\ \beta & \beta'\end{pmatrix} \qquad \text{or}$$
$$(\overline{a}_k, \overline{b}_k)\begin{pmatrix}\alpha & \alpha'\epsilon\\ \beta\epsilon^{-1} & \beta'\end{pmatrix}.$$

Therefore, we have $\alpha_k\beta'_k - \beta_k\alpha'_k = \alpha\beta' - \beta\alpha'$. In particular, if $\alpha\beta' - \beta\alpha' \in \mathcal{O}^\times$, then ρ is surjective, which contradicts $\rho \in \mathrm{Rad}\,\mathrm{End}_A(E_i)$. ∎

LEMMA 2.7. *Suppose that $2 \leqslant i \leqslant n-i$, and let $\phi \in \mathrm{End}_A(E_i)$ be as in the definition of the pullback diagram. Then we have the following.*

(1) *ϕ does not factor through π.*
(2) *For any $\rho \in \mathrm{Rad}\,\mathrm{End}_A(E_i)$, $\phi\rho$ factors through π.*

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Proof. (1) Suppose that there exists

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4) : E_i \longrightarrow A \oplus A \oplus A \oplus A$$

such that $\pi\psi = \phi$. Then, we have

$$\begin{aligned} 0 &= \pi\psi(a_n) = \big(\epsilon\psi_1(a_n) + X^{i-1}\psi_2(a_n), X^{n-i}\psi_1(a_n), \epsilon\psi_2(a_n) \\ &\quad + \epsilon X\psi_3(a_n) + X^{n-i}\psi_4(a_n)\big), \\ b_1 &= \pi\psi(b_n) = \big(\epsilon\psi_1(b_n) + X^{i-1}\psi_2(b_n), X^{n-i}\psi_1(b_n), \epsilon\psi_2(b_n) \\ &\quad + \epsilon X\psi_3(b_n) + X^{n-i}\psi_4(b_n)\big). \end{aligned}$$

The first equality implies $\psi_4(X^{n-1}a_n) \in \epsilon^2 A$ by the following computation.

$$\begin{aligned} \psi_4(X^{n-1}a_n) &= X^{i-1}(X^{n-i}\psi_4(a_n)) = -X^{i-1}(\epsilon\psi_2(a_n) + \epsilon X\psi_3(a_n)) \\ &= -\epsilon X^{i-1}\psi_2(a_n) - \epsilon\psi_3(X^i a_n) = \epsilon^2\psi_1(a_n) - \epsilon^2\psi_3(a_{n-i}). \end{aligned}$$

Thus, we conclude $\psi_4(X^{n-i}b_n) \equiv 0 \bmod \epsilon A$ from

$$\begin{aligned} \epsilon\psi_4(X^{n-i}b_n) &= \epsilon\psi_4(X^{n-i-1}a_{n-i} + X^{n-i-1}b_{n-1}) = \epsilon\psi_4(a_1 + \epsilon b_i) \\ &= \psi_4(\epsilon a_1) + \epsilon^2\psi_4(b_i) = \psi(X^{n-1}a_n) + \epsilon^2\psi_4(b_i) \in \epsilon^2 A. \end{aligned}$$

On the other hand, using $b_1 = (0, 0, X^{n-1})$, the second equality implies

$$\epsilon\psi_2(b_n) + \epsilon X\psi_3(b_n) + X^{n-i}\psi_4(b_n) = X^{n-1},$$

and we have $\psi_4(X^{n-i}b_n) \not\equiv 0 \bmod \epsilon A$. Hence, we have reached a contradiction.

(2) Let $\rho \in \operatorname{Rad}\operatorname{End}_A(E_i)$. We write $\rho(a_n) = \alpha a_n + \beta b_n + A$ and $\rho(b_n) = \alpha' a_n + \beta' b_n + B$, where $\alpha, \beta, \alpha', \beta' \in \mathcal{O}$ and $A, B \in \operatorname{Ker}(X^{n-1})$. Then, $\phi\rho(a_n) = \beta b_1$ and $\phi\rho(b_n) = \beta' b_1$.

By Lemma 2.6(1), $\beta \in \epsilon\mathcal{O}$ and if β' was invertible then α would be invertible, which contradicts Lemma 2.6(2). Thus, $\beta, \beta' \in \epsilon\mathcal{O}$ and we may define $\psi_2 : E_i \to A$ by

$$(f, g, h) \mapsto \frac{\beta X^{n-1} f}{\epsilon^2} + \frac{\beta' X^{n-1} h}{\epsilon^2},$$

where $(f, g, h) \in A \oplus A \oplus Z_i$ with $X^{n-i}f - \epsilon g = X^{n-1}h/\epsilon$. This is well defined. Indeed, we have $\psi_2(a_k) = 0$ and $\psi_2(b_k) = 0$, for $1 \leqslant k \leqslant n-1$, and

$$\psi_2(a_n) = \frac{\beta}{\epsilon} X^{n-1}, \qquad \psi_2(b_n) = \frac{\beta'}{\epsilon} X^{n-1}.$$

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Then

$$\psi = (0, \psi_2, 0, 0) : E_i \to A \oplus A \oplus A \oplus A$$

satisfies $\pi\psi = (X^{i-1}\psi_2, 0, \epsilon\psi_2) = \phi\rho$. ∎

By Proposition 1.15 and Lemma 2.7, we have an almost split sequence

$$0 \to E_{n-i} \to F_i \to E_i \to 0,$$

where $F_i = \{(p, q, r, s, t) \in A^{\oplus 4} \oplus E_i \mid \pi(p, q, r, s) = \phi(t)\}$, for $2 \leqslant i \leqslant n - i$.

We define $z_k = (0, 0, 0, 0, a_k) \in F_i$, for $1 \leqslant k \leqslant n$, and $x_k, y_k, w_k \in F_i$, for $1 \leqslant k \leqslant n$, by

$$x_k = \begin{cases} (0, 0, 0, X^{n-k}, a_k) & \text{if } 1 \leqslant k \leqslant n - i, \\ (0, 0, -X^{2n-i-k-1}, \epsilon X^{n-k}, a_k) & \text{if } n - i < k \leqslant n. \end{cases}$$

$$y_k = \begin{cases} (0, 0, 0, 0, b_k) & \text{if } 1 \leqslant k \leqslant i, \\ (0, 0, 0, X^{n+i-k-1}, b_k + a_{k-i+1}) & \text{if } i < k < n, \\ (0, 0, 0, X^{i-1}, b_n) & \text{if } k = n. \end{cases}$$

$$w_k = \begin{cases} (0, -X^{n-k+1}, X^{n-k}, 0, 0) & \text{if } 1 \leqslant k \leqslant i, \\ (X^{n-k+i}, -\epsilon X^{n-k+1}, \epsilon X^{n-k}, 0, 0) & \text{if } i < k \leqslant n. \end{cases}$$

Note that $(p, q, r, s, t) \in F_i$ if and only if

$$(\epsilon p + X^{i-1}q, X^{n-i}p, \epsilon q + \epsilon X r + X^{n-i}s) = \beta_n b_1,$$

where $t = \sum_{k=1}^{n}(\alpha_k a_k + \beta_k b_k)$.

LEMMA 2.8. *$\{x_k, y_k, z_k, w_k \mid 1 \leqslant k \leqslant n\}$ is an $\mathcal{O}$-basis of F_i.*

Proof. It suffices to show that they generate F_i as an $\mathcal{O}$-module, since $\operatorname{rank} F_i = 4n$. Let F_i' be the $\mathcal{O}$-submodule generated by $\{x_k, y_k, z_k, w_k \mid 1 \leqslant k \leqslant n\}$. We show first that $(\operatorname{Ker}(\pi), 0) \subseteq F_i'$. Recall that any element of $(\operatorname{Ker}(\pi), 0) = (\operatorname{Im}(\iota), 0)$ has the form

$$\left(g, -Xf + \frac{X^{n-i}h}{\epsilon}, f, -h, 0\right),$$

where $(f, g, h) \in A \oplus A \oplus Z_{n-i}$ and $X^i f - \epsilon g = X^{n-1}h/\epsilon$. Thus, $X^{n-i}g = 0$ and g is an $\mathcal{O}$-linear combination of X^{n-k+i}, for $i < k \leqslant n$. Thus, subtracting the corresponding $\mathcal{O}$-linear combination of w_k, for $i < k \leqslant n$, we may assume

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$g = 0$. Since

$$h \in Z_{n-i} = \mathcal{O}\epsilon \oplus \cdots \oplus \mathcal{O}\epsilon X^{i-1} \oplus \mathcal{O}X^i \oplus \cdots \oplus \mathcal{O}X^{n-1},$$

we may further subtract an $\mathcal{O}$-linear combination of x_k, for $1 \leqslant k \leqslant n$, and we may assume $g = h = 0$ without loss of generality. Then, $(0, -Xf, f, 0, 0)$, for $f \in A$ with $X^i f = 0$, is an $\mathcal{O}$-linear combination of w_k, for $1 \leqslant k \leqslant i$. Hence, $(\mathrm{Ker}(\pi), 0) \subseteq F_i'$. Next we show that $(0, 0, 0, 0, \mathrm{Ker}(\phi)) \subseteq F'$. But it is clear from $(0, 0, 0, 0, a_k) = z_k$, for $1 \leqslant k \leqslant n$, and

$$(0, 0, 0, 0, b_k) = \begin{cases} y_k & \text{if } 1 \leqslant k \leqslant i, \\ y_k - x_{k-i+1} & \text{if } i < k < n. \end{cases}$$

Suppose that $(p, q, r, s, t) \in F_i$. Write $t = \beta b_n + t'$ such that $\beta \in \mathcal{O}$ and $t' \in \mathrm{Ker}(\phi)$. Then, to show that $(p, q, r, s, t) \in F_i'$, it is enough to see $(p, q, r, s, \beta b_n) \in F_i'$. Since

$$\epsilon q + \epsilon X r + X^{n-i} s = \beta X^{n-1},$$

we have $(p, q, r, s - \beta X^{i-1}) \in \mathrm{Ker}(\pi)$. Therefore, we deduce

$$(p, q, r, s, \beta b_n) = (p, q, r, s - \beta X^{i-1}, 0) + \beta(0, 0, 0, X^{i-1}, b_n) \in F_i',$$

because $(0, 0, 0, X^{i-1}, b_n) = y_n$. ∎

Let F_i' be the $\mathcal{O}$-span of $\{x_k, y_k, w_k \mid 1 \leqslant k \leqslant n\}$, F_i'' the $\mathcal{O}$-span of $\{z_k \mid 1 \leqslant k \leqslant n\}$. It is easy to compute as follows.

$$Xw_k = \begin{cases} w_{k-1} & \text{if } k \neq i+1, \\ \epsilon w_i & \text{if } k = i+1. \end{cases}$$

$$Xx_k = \begin{cases} x_{k-1} & \text{if } k \neq n-i+1, \\ \epsilon x_{n-i} - w_1 & \text{if } k = n-i+1. \end{cases}$$

$$Xy_k = \begin{cases} y_{k-1} & \text{if } k \neq i+1, \\ \epsilon y_i + x_1 & \text{if } k = i+1. \end{cases}$$

$$Xz_k = \begin{cases} z_{k-1} & \text{if } k \neq n-i+1, \\ \epsilon z_{n-i} & \text{if } k = n-i+1. \end{cases}$$

Hence, the direct summands F_i' and F_i'' of $F_i = F_i' \oplus F_i''$ are A-lattices and $F_i'' \simeq Z_{n-i}$.

LEMMA 2.9. *The middle term of the almost split sequence ending at E_i, for $2 \leqslant i \leqslant n-2$, is the direct sum of Z_{n-i} and an indecomposable direct summand.*

Proof. Since $\tau(Z_i) \simeq Z_{n-i}$ implies $\tau(E_i) \simeq E_{n-i}$, we may assume $2 \leqslant i \leqslant n-i$ without loss of generality. Let F_i' be the A-lattice as above. Then we have to show that F_i' is an indecomposable A-lattice. Suppose that F_i' is not indecomposable. Then, there exist A-sublattices Z and L such that $F_i' \simeq Z \oplus L$ and $Z \otimes \mathcal{K} \simeq A \otimes \mathcal{K}$. Since

$$\mathrm{Ker}(X^k) \cap F_i' = \bigoplus_{1 \leqslant j \leqslant k} (\mathcal{O}w_j + \mathcal{O}x_j + \mathcal{O}y_j),$$

we may choose an $\mathcal{O}$-basis $\{e_k \mid 1 \leqslant k \leqslant n\}$ of Z such that

$$e_k = \alpha_k w_k + \beta_k x_k + \gamma_k y_k + A_k,$$

where $\alpha_k, \beta_k, \gamma_k \in \mathcal{O}$ with $(\alpha_k, \beta_k, \gamma_k) \notin (\epsilon\mathcal{O})^{\oplus 3}$ and $A_k \in \mathrm{Ker}(X^{k-1}) \cap L$. Then,

$$\mathrm{Ker}(X^k) \cap Z = \mathcal{O}e_1 \oplus \cdots \oplus \mathcal{O}e_k$$

and at least one of $\alpha_k, \beta_k, \gamma_k$ is invertible. Write

$$Xe_k = f_{k-1}^{(k)} e_{k-1} + \cdots + f_1^{(k)} e_1,$$

for $f_1^{(k)}, \ldots, f_{k-1}^{(k)} \in \mathcal{O}$. We first assume that $2 \leqslant i < n-i$. Note that

$$Xe_k = \begin{cases} \alpha_k w_{k-1} + \beta_k x_{k-1} + \gamma_k y_{k-1} + XA_k & \text{if } k \neq i+1, n-i+1, \\ \alpha_{n-i+1} w_{n-i} + \beta_{n-i+1}(\epsilon x_{n-i} - w_1) & \\ \quad + \gamma_{n-i+1} y_{n-i} + XA_{n-i+1} & \text{if } k = n-i+1, \\ \alpha_{i+1}\epsilon w_i + \beta_{i+1} x_i & \\ \quad + \gamma_{i+1}(\epsilon y_i + x_1) + XA_{i+1} & \text{if } k = i+1. \end{cases}$$

Thus, we have

$$\begin{aligned} & f_{k-1}^{(k)}(\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}) \\ & \quad = \begin{cases} (\alpha_k, \beta_k, \gamma_k) & \text{if } k \neq i+1, n-i+1, \\ (\alpha_{n-i+1}, \epsilon\beta_{n-i+1}, \gamma_{n-i+1}) & \text{if } k = n-i+1, \\ (\epsilon\alpha_{i+1}, \beta_{i+1}, \epsilon\gamma_{i+1}) & \text{if } k = i+1. \end{cases} \end{aligned}$$

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We may assume one of the following two cases occurs.

(1) $f_{k-1}^{(k)} = 1$ $(k \neq n-i+1)$, $f_{n-i}^{(n-i+1)} = \epsilon$.
(2) $f_{k-1}^{(k)} = 1$ $(k \neq i+1)$, $f_i^{(i+1)} = \epsilon$.

In fact, since at least one of $\alpha_k, \beta_k, \gamma_k$ is invertible, if $k \neq n-i+1, i+1$ then $f_{k-1}^{(k)}$ is invertible. We multiply its inverse to e_k, and we obtain

$$f_1^{(2)} = \cdots = f_{i-1}^{(i)} = 1 \qquad \text{and} \qquad (\alpha_1, \beta_1, \gamma_1) = \cdots = (\alpha_i, \beta_i, \gamma_i)$$

in the new basis. By the same reason, we have $f_{k-1}^{(k)} \notin \epsilon^2 \mathcal{O}$, for all k. Suppose that both $f_{n-i}^{(n-i+1)}$ and $f_i^{(i+1)}$ are invertible. Then, we may reach

$$\begin{aligned}(\alpha_i, \beta_i, \gamma_i) &= (\epsilon\alpha_{i+1}, \beta_{i+1}, \epsilon\gamma_{i+1}) = \cdots = (\epsilon\alpha_{n-i}, \beta_{n-i}, \epsilon\gamma_{n-i}) \\ &= (\epsilon\alpha_{n-i+1}, \epsilon\beta_{n-i+1}, \epsilon\gamma_{n-i+1}),\end{aligned}$$

which is a contradiction. Suppose that both $f_{n-i}^{(n-i+1)}$ and $f_i^{(i+1)}$ are not invertible. Then,

$$\begin{aligned}(\alpha_i, \beta_i, \gamma_i) &= (\alpha_{i+1}, \epsilon^{-1}\beta_{i+1}, \gamma_{i+1}) = \cdots = (\alpha_{n-i}, \epsilon^{-1}\beta_{n-i}, \gamma_{n-i}) \\ &= (\epsilon^{-1}\alpha_{n-i+1}, \epsilon^{-1}\beta_{n-i+1}, \epsilon^{-1}\gamma_{n-i+1}),\end{aligned}$$

which implies that none of $\alpha_{n-i+1}, \beta_{n-i+1}, \gamma_{n-i+1}$ is invertible. Thus, we have proved that we are in case (1) or case (2). Suppose that we are in case (1). Then, we have

$$\begin{aligned}Xe_k - f_{k-1}^{(k)} e_{k-1} &= f_{k-2}^{(k)} e_{k-2} + \cdots + f_1^{(k)} e_1 \\ &= \begin{cases} XA_k - A_{k-1} & \text{if } k \neq n-i+1, i+1, \\ XA_{n-i+1} - \epsilon A_{n-i} - \beta_{n-i+1} w_1 & \text{if } k = n-i+1, \\ XA_{i+1} - A_i + \gamma_{i+1} x_1 & \text{if } k = i+1. \end{cases}\end{aligned}$$

Since $A_k \in \operatorname{Ker}(X^k) \cap L$, we obtain that

$$Xe_k = \begin{cases} e_{k-1} & \text{if } k \neq n-i+1, i+1, \\ \epsilon e_{n-i} + f_1^{(n-i+1)} e_1 & \text{if } k = n-i+1, \\ e_i + f_1^{(i+1)} e_1 & \text{if } k = i+1, \end{cases}$$

and $XA_{n-i+1} = X^2A_{n-i+2} = \cdots = X^iA_n$. As we are in case (1),

$$\begin{aligned}(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2) &= \cdots = (\alpha_i, \beta_i, \gamma_i) \\ &= (\epsilon\alpha_{i+1}, \beta_{i+1}, \epsilon\gamma_{i+1}) = \cdots = (\epsilon\alpha_{n-i}, \beta_{n-i}, \epsilon\gamma_{n-i}) \\ &= (\alpha_{n-i+1}, \beta_{n-i+1}, \gamma_{n-i+1}) = \cdots = (\alpha_n, \beta_n, \gamma_n),\end{aligned}$$

so that we may write

$$e_k = \begin{cases} \epsilon\alpha w_k + \beta x_k + \epsilon\gamma y_k + A_k & \text{if } 1 \leqslant k \leqslant i \text{ or } n-i+1 \leqslant k \leqslant n, \\ \alpha w_k + \beta x_k + \gamma y_k + A_k & \text{if } i+1 \leqslant k \leqslant n-i, \end{cases}$$

with $\alpha, \gamma \in \mathcal{O}$ and $\beta \in \mathcal{O}^\times$. Then, $Xe_{n-i+1} = \epsilon e_{n-i} + f_1^{(n-i+1)}e_1$ implies

$$\begin{aligned}\epsilon\alpha w_{n-i} + \beta(\epsilon x_{n-i} - w_1) + \epsilon\gamma y_{n-i} + X^iA_n \\ = \epsilon e_{n-i} + f_1^{(n-i+1)}(\epsilon\alpha w_1 + \beta x_1 + \epsilon\gamma y_1).\end{aligned}$$

We equate the coefficients of w_1 on both sides. Since contribution from X^iA_n comes from $X^iw_{i+1} = \epsilon w_1$ only, we conclude that $\beta \in \epsilon\mathcal{O}$, which contradicts $\beta \in \mathcal{O}^\times$.

Suppose that we are in case (2). Then, the same argument as above shows that

$$Xe_k = \begin{cases} e_{k-1} & \text{if } k \neq n-i+1, i+1, \\ e_{n-i} + f_1^{(n-i+1)}e_1 & \text{if } k = n-i+1, \\ \epsilon e_i + f_1^{(i+1)}e_1 & \text{if } k = i+1. \end{cases}$$

We define an $\mathcal{O}$-basis $\{e''_k\}$ of Z as follows:

(i) $e''_k = e_k$ $(1 \leqslant k \leqslant i)$;
(ii) $e''_{n-i} = e_{n-i} - f_1^{(i+1)}e_{n-2i+1} + f_1^{(n-i+1)}e_1$;
(iii) $e''_{n-1} = e_{n-1} - f_1^{(i+1)}e_{n-i} - f_1^{(i+1)}f_1^{(n-i+1)}e_1$;
(iv) $e''_k = e_k - f_1^{(i+1)}e_{k-i+1}$ $(i+1 \leqslant k \leqslant n, k \neq n-i, n-1)$.

Then, we have $Z \simeq Z_i$. To summarize, we have proved that if there is a direct summand of rank n then it must be isomorphic to Z_i. As there is an irreducible morphism $Z_i \to E_i$, E_i must be a direct summand of E_{n-i} and we conclude $E_i \simeq E_{n-i}$. Then there exist $a'_k, b'_k \in E_{n-i}$, for $1 \leqslant k \leqslant n$, such that

$$\begin{aligned}a_n &= \alpha a'_n + \beta b'_n + A, \\ b_n &= \gamma a'_n + \delta b'_n + B,\end{aligned}$$

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where $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ with $\alpha\delta - \beta\gamma \in \mathcal{O}^{\times}$, $A, B \in \operatorname{Ker}(X^{n-1})$, and

$$Xa'_k = \begin{cases} a'_{k-1} & (k \neq n-i+1) \\ \epsilon a'_{k-1} & (k = n-i+1), \end{cases}$$

$$Xb'_k = \begin{cases} b'_{k-1} & (k \neq i+1, n) \\ \epsilon b'_{k-1} & (k = i+1) \\ a'_{n-i} + b'_{n-1} & (k = n). \end{cases}$$

We compute $X^{n-i}a_n$ and $X^{n-i}b_n$ as follows.

$$\begin{aligned} \epsilon a_i &= \epsilon(\alpha a'_i + \beta b'_i) + \beta a'_1 + X^{n-i}A, \\ \epsilon b_i &= \epsilon(\gamma a'_i + \delta b'_i) + \delta a'_1 + X^{n-i}B. \end{aligned}$$

Since $X^{n-i}A, X^{n-i}B \in \epsilon E_{n-i}$ by $2 \leqslant i < n-i$, we have $\beta, \delta \in \epsilon\mathcal{O}$, which is a contradiction.

Thus, F'_i is indecomposable if $2 \leqslant i < n-i$. It remains to consider $2 \leqslant i = n-i$. We choose an $\mathcal{O}$-basis $\{e_k \mid 1 \leqslant k \leqslant n\}$ of Z and write

$$e_k = \alpha_k w_k + \beta_k x_k + \gamma_k y_k + A_k,$$

as before. Then, we have

$$Xe_k = \begin{cases} \alpha_k w_{k-1} + \beta_k x_{k-1} + \gamma_k y_{k-1} + XA_k & \text{if } k \neq i+1, \\ \alpha_{i+1}\epsilon w_i + \beta_{i+1}(\epsilon x_i - w_1) + \gamma_{i+1}(\epsilon y_i + x_1) + XA_{i+1} & \text{if } k = i+1, \end{cases}$$

and it follows that

$$f^{(k)}_{k-1}(\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}) = \begin{cases} (\alpha_k, \beta_k, \gamma_k) & \text{if } k \neq i+1, \\ (\epsilon\alpha_{i+1}, \epsilon\beta_{i+1}, \epsilon\gamma_{i+1}) & \text{if } k = i+1. \end{cases}$$

Hence, we may assume $f^{(k)}_{k-1} = 1$, for $k \neq i+1$, and $f^{(i+1)}_i = \epsilon$, without loss of generality. Since $A_k \in \operatorname{Ker}(X^{k-1}) \cap L$, we obtain from the computation of $Xe_k - f^{(k)}_{k-1}e_{k-1}$ that

$$Xe_k = \begin{cases} e_{k-1} & \text{if } k \neq i+1, \\ \epsilon e_i + f^{(i+1)}_1 e_1 & \text{if } k = i+1, \end{cases}$$

and $XA_{i+1} = X^2A_{i+2} = \cdots = X^iA_n$. Let λ, μ, ν be the coefficient of w_{n-i+1}, x_{n-i+1}, y_{n-i+1} in A_n, respectively. Then the coefficient of w_1, x_1, y_1 in

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XA_{i+1} are $\epsilon\lambda$, $\epsilon\mu$, $\epsilon\nu$. Since $f_1^{(i+1)}e_1 = XA_{i+1} - \epsilon A_i - \beta_{i+1}w_1 + \gamma_{i+1}x_1$, we have

$$f_1^{(i+1)}\alpha_1 \equiv -\beta_{i+1} \mod \epsilon\mathcal{O}, \qquad f_1^{(i+1)}\beta_1 \equiv \gamma_{i+1} \mod \epsilon\mathcal{O},$$
$$f_1^{(i+1)}\gamma_1 \equiv 0 \mod \epsilon\mathcal{O}.$$

We may show that $f_1^{(i+1)}$ is not invertible, but whenever it is invertible or not,

$$\gamma_1 = \gamma_2 = \cdots = \gamma_n \qquad \text{and} \qquad \beta_1 = \beta_2 = \cdots = \beta_n$$

imply that $\beta_k \equiv 0 \bmod \epsilon\mathcal{O}$ and $\gamma_k \equiv 0 \bmod \epsilon\mathcal{O}$, for $1 \leqslant k \leqslant n$. It follows that we may choose an $\mathcal{O}$-basis $\{a'_k, b'_k \mid 1 \leqslant k \leqslant n\}$ of L in the following form.

$$\begin{aligned} a'_k &= \lambda'_k w_k + x_k + A'_k, \\ b'_k &= \lambda''_k w_k + y_k + B'_k, \end{aligned}$$

where $\lambda', \lambda'' \in \mathcal{O}$ and $A'_k, B'_k \in \operatorname{Ker}(X^{k-1}) \cap Z$. Write

$$Xa'_k = \sum_{j=1}^{k-1} (g_j^{(k)} a'_j + h_j^{(k)} b'_j).$$

Multiplying $a'_k = \lambda'_k w_k + x_k + A'_k$ with X, we obtain

$$Xa'_k = \begin{cases} \lambda'_k w_{k-1} + x_{k-1} + XA'_k & \text{if } k \neq i+1, \\ \epsilon\lambda'_{i+1} w_i + \epsilon x_i - w_1 + XA'_{i+1} & \text{if } k = i+1. \end{cases}$$

Thus, $g_{k-1}^{(k)} = 1$, for $k \neq i+1$, $g_i^{(i+1)} = \epsilon$, and $h_{k-1}^{(k)} = 0$, for all k. Further, we have

$$Xa'_k - g_{k-1}^{(k)} a'_{k-1} = \begin{cases} XA'_k - A'_{k-1} & \text{if } k \neq i+1, \\ XA'_{i+1} - \epsilon A'_i - w_1 & \text{if } k = i+1. \end{cases}$$

We obtain $Xa'_k - a'_{k-1} = 0$ if $k \neq i+1$, and if $k = i+1$ then $Xa'_{i+1} - \epsilon a'_i$ is equal to

$$g_1^{(i+1)} a'_1 + h_1^{(i+1)} b'_1 = XA'_{i+1} - \epsilon A'_i - w_1.$$

Since $XA'_{i+1} = X^2 A'_{i+2} = \cdots = X^{n-i} A'_n$, the coefficient of x_1 in XA'_{i+1} is in $\epsilon\mathcal{O}$. Thus,

$$(\lambda'_1 g_1^{(i+1)} + \lambda''_1 h_1^{(i+1)} + 1) w_1 + g_1^{(i+1)} x_1 + h_1^{(i+1)} y_1 \equiv 0 \mod \epsilon F'_i.$$

We must have $g_1^{(i+1)}, h_1^{(i+1)} \in \epsilon\mathcal{O}$, but then $w_1 \equiv 0 \mod \epsilon F'_i$, which is impossible. Hence, F'_i is indecomposable if $2 \leqslant n - i = i$. ∎

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§3. Main result

In this section, we prove the main result of this article.

THEOREM 3.1. *Let $\mathcal{O}$ be a complete discrete valuation ring, $A = \mathcal{O}[X]/(X^n)$, for $n \geqslant 2$. Then, the component of the stable Auslander–Reiten quiver of A which contains Z_i and Z_{n-i} is $\mathbb{Z}A_\infty/\langle\tau^2\rangle$ if $2i \neq n$, and $\mathbb{Z}A_\infty/\langle\tau\rangle$ that is, homogeneous tube if $2i = n$.*

Proof. Let C be a component of the stable Auslander–Reiten quiver of A that contains a Heller lattice. As Heller lattices are τ-periodic for $A = \mathcal{O}[X]/(X^n)$, Theorem 1.27 and Lemma 1.4 implies that C is a valued stable translation quiver and its tree class is one of $A_\infty, B_\infty, C_\infty, D_\infty$ and A_∞^∞. If $i = 1$ or $i = n-1$, then Proposition 2.4(1) implies that the subadditive function considered in the proof of Lemma 1.23 is not additive. Thus, the tree class of C is A_∞. We now assume that $i \neq 1, n-1$. Proposition 2.4(2) implies that the Heller lattices Z_i and Z_{n-i} are on the boundary of the stable Auslander–Reiten quiver, and the tree class can not be A_∞^∞. If the tree class was one of B_∞, C_∞ and D_∞, then F_i or F_{n-i} would have at least three indecomposable direct summands. But it contradicts Lemma 2.9. Therefore, the tree class is A_∞. Then, the component C must be a tube, and the rank is the period of the Heller lattices Z_i and Z_{n-i}, which is two if $n - i \neq i$, one if $n - i = i$. ∎

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Appendix

In this appendix, we prove Proposition 1.15. The proof uses arguments from [Bu] and [R1]. As it is clear that (1) implies (2), we show that (2) implies (1). Let us consider the injective resolution of $\mathcal{O}$ as an $\mathcal{O}$-module:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\iota} \mathcal{K} \xrightarrow{d} \mathcal{K}/\mathcal{O} \longrightarrow 0.$$

Since $\operatorname{Ext}^1_{\mathcal{O}}(X, \mathcal{O}) = 0$ for any free $\mathcal{O}$-modules of finite rank X, we have

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{O}) \longrightarrow \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{K}) \longrightarrow \operatorname{Hom}_{\mathcal{O}}(X, \mathcal{K}/\mathcal{O}) \longrightarrow 0.$$

In particular, if we define functors $D' = \mathrm{Hom}_{\mathcal{O}}(-, \mathcal{K})$ and $D'' = \mathrm{Hom}_{\mathcal{O}}(-, \mathcal{K}/\mathcal{O})$, then we have the short exact sequence

$$0 \longrightarrow D(\mathrm{Hom}_A(M, -)) \longrightarrow D'(\mathrm{Hom}_A(M, -)) \longrightarrow D''(\mathrm{Hom}_A(M, -)) \longrightarrow 0,$$

for any A-lattice M. We define functors

$$\nu' = D' \,\mathrm{Hom}_A(-, A), \quad \nu'' = D'' \,\mathrm{Hom}_A(-, A),$$

which we also call Nakayama functors. Applying the Nakayama functors ν, ν', ν'' to M, we obtain the following exact sequences

$$0 \longrightarrow \nu(M) \longrightarrow \nu'(M) \longrightarrow \nu''(M) \longrightarrow 0,$$

and $0 \longrightarrow \mathrm{Hom}_A(-, \nu(M)) \longrightarrow \mathrm{Hom}_A(-, \nu'(M)) \longrightarrow \mathrm{Hom}_A(-, \nu''(M))$. Let λ be the functorial isomorphism defined by

$$\begin{aligned} D(\mathrm{Hom}_A(M, A) \otimes_A -) &= \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}_A(M, A) \otimes_A -, \mathcal{O}) \\ &\simeq \mathrm{Hom}_A(-, \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}_A(M, A), \mathcal{O})) \\ &= \mathrm{Hom}_A(-, \nu(M)). \end{aligned}$$

We define λ' and λ'' in the similar manner by replacing ν with ν' and ν''. Let

$$\mu_M : \mathrm{Hom}_A(M, A) \otimes_A - \longrightarrow \mathrm{Hom}_A(M, -)$$

be the natural transformation defined by $\phi \otimes x \mapsto (m \mapsto \phi(m)x)$. Then, it induces the following three morphisms of functors

$$\begin{aligned} D\mu_M : D\,\mathrm{Hom}_A(M, -) &\longrightarrow D(\mathrm{Hom}_A(M, A) \otimes_A -), \\ D'\mu_M : D'\,\mathrm{Hom}_A(M, -) &\longrightarrow D'(\mathrm{Hom}_A(M, A) \otimes_A -), \\ D''\mu_M : D''\,\mathrm{Hom}_A(M, -) &\longrightarrow D''(\mathrm{Hom}_A(M, A) \otimes_A -). \end{aligned}$$

Then, we have the following commutative diagram of functors on A-lattices.

$$\begin{tikzcd} 0 \arrow[r] & D\,\mathrm{Hom}_A(M, -) \arrow[r] \arrow[d, "\lambda\circ D\mu_M"] & D'\,\mathrm{Hom}_A(M, -) \arrow[r, "d_*"] \arrow[d, "\lambda'\circ D'\mu_M"] & D''\,\mathrm{Hom}_A(M, -) \arrow[r] \arrow[d, "\lambda''\circ D''\mu_M"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_A(-, \nu(M)) \arrow[r, "\iota_*"] & \mathrm{Hom}_A(-, \nu'(M)) \arrow[r] & \mathrm{Hom}_A(-, \nu''(M)) & \end{tikzcd}$$

with exact rows, where ι_* and d_* are given by compositions of ι and d on the left.

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LEMMA A.1. *Let X be an A-lattice. If $M \otimes \mathcal{K}$ is a projective $A \otimes \mathcal{K}$-module, then:*

(i) *$D'\mu_M(X)$ is an isomorphism and natural in X;*
(ii) *$D\mu_M(X)$ is a monomorphism and natural in X;*
(iii) *if M is a projective A-module, then $D\mu_M(X)$ is an isomorphism;*
(iv) *$D''\mu_M(X)$ is an epimorphism and natural in X.*

Moreover, the sequence

$$D\operatorname{Hom}_A(M, X) \xrightarrow{\lambda\circ D\mu_M(X)} \operatorname{Hom}_A(X, \nu(M)) \xrightarrow{d_*\circ(\lambda'\circ D'\mu_M(X))^{-1}\circ\iota_*} D''\mathsf{m}\operatorname{Hom}_A(M, X)$$

is exact.

Proof. Observe that we have an isomorphism

$$\operatorname{Hom}_{A\otimes\mathcal{K}}(M\otimes\mathcal{K}, A\otimes\mathcal{K})\otimes_A X \simeq \operatorname{Hom}_{A\otimes\mathcal{K}}(M\otimes\mathcal{K}, X\otimes\mathcal{K}),$$

since $M\otimes\mathcal{K}$ is a projective $A\otimes\mathcal{K}$-module. Thus, $\operatorname{Coker}(\mu_M(X))$ is a torsion $\mathcal{O}$-module and $D'\operatorname{Coker}(\mu_M(X)) = 0$. Then,

$$0 \to D'\operatorname{Hom}_A(M, X) \xrightarrow{D'\mu_M(X)} D'(\operatorname{Hom}_A(M, A)\otimes_A X) \to \operatorname{Ext}^1_A(\operatorname{Coker}(\mu_M(X)), \mathcal{K}) = 0,$$

proving (i). As $\operatorname{Coker}(\mu_M(X))$ is a torsion $\mathcal{O}$-module, (ii) also follows. The proof of (iii) is the same as (i). The proof of (iv) is similar. By chasing the diagram above, (i) implies the exact sequence. ∎

LEMMA A.2. *Let M be an A-lattice, $p : P \to M$ the projective cover, and we define*

$$L = D(\operatorname{Coker}(\operatorname{Hom}_A(p, A))).$$

Then, we have the following exact sequence of functors.

$$0 \longrightarrow D\operatorname{Hom}_A(M, -) \xrightarrow{\lambda\circ D\mu_M(-)} \operatorname{Hom}_A(-, \nu(M)) \longrightarrow \operatorname{Ext}^1_A(-, L) \longrightarrow 0.$$

Proof. We recall the short exact sequence

$$0 \to L \longrightarrow \nu(P) \longrightarrow \nu(M) \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_A(X, -)$, for an A-lattice X, we obtain

$$\begin{aligned}\mathrm{Hom}_A(X, \nu(P)) \longrightarrow \mathrm{Hom}_A(X, \nu(M)) \longrightarrow \mathrm{Ext}^1_A(X, L)\\ \longrightarrow \mathrm{Ext}^1_A(X, \nu(P)) = 0,\end{aligned}$$

since $\nu(P)$ is an injective A-lattice. Thus, we have the following diagram with exact rows:

$$\begin{tikzcd}
 & \mathrm{Hom}_A(X, \nu(P)) \arrow[r, "\nu(p)_*"] & \mathrm{Hom}_A(X, \nu(M)) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_A(X, L) \arrow[r] & 0 \\
0 \arrow[r] & D\,\mathrm{Hom}_A(M, X) \arrow[r] & \mathrm{Hom}_A(X, \nu(M)) \arrow[r] & D''\,\mathrm{Hom}_A(M, X) &
\end{tikzcd}$$

We show that $\nu(p)_*$ factors through $\lambda \circ D\mu_M(X) : D\,\mathrm{Hom}_A(M, X) \to \mathrm{Hom}_A(X, \nu(M))$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(M, A) \otimes_A X \arrow[r, "p^* \otimes \mathrm{id}_X"] \arrow[d, "\mu_M(X)"] & \mathrm{Hom}_A(P, A) \otimes_A X \arrow[d, "\mu_P(X)"] \\
\mathrm{Hom}_A(M, X) \arrow[r, "p^*"] & \mathrm{Hom}_A(P, X).
\end{tikzcd}$$

By dualizing the diagram, we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(X, \nu(M)) & \mathrm{Hom}_A(X, \nu(P)) \arrow[l, "\nu(p)_*"'] \\
D\,\mathrm{Hom}_A(M, X) \arrow[u, "\lambda \circ D(\mu_M(X))"] & D\,\mathrm{Hom}_A(P, X), \arrow[l, "Dp^*"'] \arrow[u, "\lambda \circ D(\mu_P(X))"']
\end{tikzcd}$$

and $\lambda \circ D(\mu_P(X))$ is an isomorphism. Therefore, $\nu(p)_*$ factors through $D\,\mathrm{Hom}_A(M, X)$. Since $\mathrm{Coker}(p^*)$ is an $\mathcal{O}$-submodule of $\mathrm{Hom}_A(\mathrm{Ker}(p), X)$, it is a free $\mathcal{O}$ module of finite rank and $\mathrm{Ext}^1_{\mathcal{O}}(\mathrm{Coker}(p^*), \mathcal{O}) = 0$. It follows that Dp^* is an epimorphism. This implies that $\mathrm{Im}(\nu(p)_*) = \mathrm{Im}(\lambda \circ D(\mu_M(X)))$, and we get the desired exact sequence. $\square$

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By Lemma A.2, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}((\nu(p)_*) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(X, \nu(M)) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_A(X, L) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Im}(\lambda \circ D(\mu_M(X))) \arrow[r] & \mathrm{Hom}_A(X, \nu(M)) \arrow[r] & D'' \, \mathrm{Hom}_A(M, X),
\end{tikzcd}$$

which implies that there exists a monomorphism $\mathrm{Ext}^1_A(X, L) \to D'' \, \mathrm{Hom}_A(M, X)$.

We set $X = M$. Then $0 \to \mathrm{Ext}^1_A(M, L) \to D'' \, \mathrm{End}_A(M)$. Since M is indecomposable, $\mathrm{Soc}(D'' \, \mathrm{End}_A(M))$ is a simple $\mathrm{End}_A(M)$-module, and hence there exists an isomorphism

$$\mathrm{Soc}(\mathrm{Ext}^1_A(M, L)) \simeq \{f \in D''(\mathrm{End}_A(M)) \mid f(\mathrm{Rad}\,\mathrm{End}_A(M)) = 0\}.$$

We are ready to prove that (2) implies (1) in Proposition 1.15. By the condition (2)(i), $0 \to L \to E \to M \to 0$ does not split. As L and M are indecomposable by the condition (2)(ii), we show that every $f \in \mathrm{Rad}\,\mathrm{Hom}_A(X, M)$ factors through E under the condition (2)(iii). Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & L \arrow[r] & \nu(P) \arrow[r, "\nu(p)"'] & \nu(M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where F is the pullback of X and E over M. Let ξ be an element in $\mathrm{Ext}^1_A(M, L)$ which represents the second sequence. Then, the condition (2)(iii) implies that $\mathrm{Rad}\,\mathrm{End}_A(M)\xi = 0$ and $\xi \in \mathrm{Soc}(\mathrm{Ext}^1_A(M, L))$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_A(M, L) \arrow[r] \arrow[d, "\mathrm{Ext}^1_A(f, L)"] & D'' \, \mathrm{Hom}_A(M, M) \arrow[d, "D'' \, \mathrm{Hom}_A(M, f)"] \\
0 \arrow[r] & \mathrm{Ext}^1_A(X, L) \arrow[r] & D'' \, \mathrm{Hom}_A(M, X).
\end{tikzcd}$$

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Let ξ' be the image of ξ under $\mathrm{Ext}^1_A(M, L) \to D'' \mathrm{Hom}_A(M, M)$. Since

$$D'' \mathrm{Hom}_A(M, f)(\xi')(\psi) = \xi'(f\psi) \in \xi'(\mathrm{Rad}\,\mathrm{End}_A(M)) = 0$$

for $\psi \in \mathrm{Hom}_A(M, X)$, we have $\mathrm{Ext}^1_A(f, L)(\xi) = 0$. Hence, $0 \to L \to F \to X \to 0$ splits. Then, it implies that f factors through E.

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