

## A CO-ANALYTIC COHEN-INDESTRUCTIBLE MAXIMAL COFINITARY GROUP

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**Abstract.** Assuming that every set is constructible, we find a  $\Pi_1^1$  maximal cofinitary group of permutations of  $\mathbb{N}$  which is indestructible by Cohen forcing. Thus we show that the existence of such groups is consistent with arbitrarily large continuum. Our method also gives a new proof, inspired by the forcing method, of Kasternans' result that there exists a  $\Pi_1^1$  maximal cofinitary group in  $\mathbf{L}$ .

**§1. Introduction.** (A) We denote the group of permutations (bijections) of  $\mathbb{N}$  by  $S_\infty$ , and its unit element by  $\text{id}_\mathbb{N}$ . An element of  $S_\infty$  is cofinitary if and only if it has only finitely many fixed points, and  $\mathcal{G}$  is called a *cofinitary group* precisely if  $\mathcal{G} \leq S_\infty$  and all elements of  $\mathcal{G} \setminus \{\text{id}_\mathbb{N}\}$  are cofinitary.

A cofinitary group is said to be *maximal* if and only if it is maximal under inclusion among cofinitary groups.

Various aspects of maximal cofinitary groups (or short, *mcs*) have long been studied (see e.g., [1, 3, 4, 10, 20, 26, 27]), including possible sizes of mcs; their relation to maximal almost disjoint (or *mad*) families, of which they are examples; as well as inequalities relating  $a_g$ , i.e., the least size of a mcs, to other cardinal invariants of the continuum; see e.g., [2, 8, 12, 19, 28, 29]. Analogous questions about permutation groups on  $\kappa$ , where  $\kappa$  is an uncountable cardinal, have also been studied; see e.g., [6]. The isomorphism types of mcs have been investigated in [17].

The line of research to which this paper belongs concerns the *definability* of mcs.

(B) Since the existence of mcs relies on the axiom of choice, the question of whether a mcs can be *definable* has drawn considerable interest.

It was shown by Truss [26] and Adeleke [1] that no mcs can be countable; this was improved by Kasternans' result [16, Theorem 10] that no mcs can be  $K_\sigma$ . On the other hand, Gao and Zhang [9] showed that assuming  $\mathbf{V} = \mathbf{L}$ , there is a mcs with a co-analytic generating set. This, too, was improved by Kasternans with the following theorem.

**THEOREM 1.1 ([16]).** *If  $\mathbf{V} = \mathbf{L}$  there is a  $\Pi_1^1$  (i.e., effectively co-analytic) mcs.*

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The previous theorem immediately raises the question of whether the existence of a  $\Pi_1^1$  mcg is consistent with  $\mathbf{V} \neq \mathbf{L}$ , or even with the negation of the continuum hypothesis. In this paper we answer these questions in the positive:

**THEOREM 1.2.** *The existence of a  $\Pi_1^1$  mcg is consistent with arbitrarily large continuum (assuming the consistency of ZFC).*

At the same time we give a new proof of Kastermans' Theorem 1.1. This is worthwhile for several reasons: Firstly, our method shows that in  $\mathbf{L}$ , any countable cofinitary group is contained in a co-analytic mcg. Secondly, the 'coding technique' which ensures that the group is co-analytic, described in Definition 3.6, is much more straightforward than the one in [16]. Thirdly, this method seems open to a wider range of variation, allowing to construct mcgs with additional properties.

An example of such a property is *Cohen-indestructibility*, which we now define. For this, first observe that if  $\mathcal{G}$  is a cofinitary group, then clearly it remains so in any extension of the universe.

**DEFINITION 1.3.** Let  $\mathcal{G}$  be a mcg and let  $\mathbb{C}$  denote Cohen forcing. We say  $\mathcal{G}$  is *Cohen-indestructible* if and only if  $\Vdash_{\mathbb{C}} \check{\mathcal{G}}$  is maximal.

A Cohen-indestructible mcg was first obtained by Zhang [29]. The following is our main result; Theorem 1.2 is clearly a corollary.

**THEOREM 1.4.** *If  $\mathbf{V} = \mathbf{L}$ , there is a  $\Pi_1^1$  Cohen-indestructible mcg.*

To prove the theorem, we first find a forcing which, given a cofinitary group  $\mathcal{G}$  and  $z \in 2^{\mathbb{N}}$ , adds a generic cofinitary group  $\mathcal{G}'$  such that  $\mathcal{G} \leq \mathcal{G}'$  and with the property that  $z$  is computable from (or 'is coded by', see Definition 3.6) each element from a sufficiently large subset of  $\mathcal{G}'$ . To find this forcing, we refine Zhang's forcing from [28] (also see [8] and [6] for variations).

We then use this to give a new proof of Kastermans' Theorem 1.1, building our group from permutations which are generic over certain countable initial segments of  $\mathbf{L}$ . We use ideas from [6] to see that the group produced in this manner is Cohen-indestructible.

(C) The paper is structured as follows. In Section 2, we establish basic terminology for Section 3. In Section 3.1 we give a streamlined presentation of Zhang's forcing  $\mathbb{Q}_{\mathcal{G}}$ , in order to simplify the definition and discussion of our forcing  $\mathbb{Q}_{\mathcal{G}}^z$ , which follows in Section 3.2. In Section 4, we prove our main result, Theorem 1.4, in a slightly more general form (Theorem 4.1), after a short review of facts of effective descriptive set theory and fine structure theory used in the proof. We close in Section 5 by listing some questions which remain open.

*Addendum.* After the appearance of this paper, Horowitz-Shelah [11] showed in ZF that there is a Borel maximal cofinitary group. Note that our result remains interesting, as the group we construct is in  $\mathbf{L}$  and of size  $\omega_1$  but remains maximal after adding any number of Cohen reals, while the group from [11] always has size  $2^{\omega}$ .

**§2. Notation and preliminaries.** We start by reviewing the necessary definitions and introduce convenient terminology, in particular the notion of a *path*.

(A) Since we build a generic element of  $S_\infty$  from finite approximations, we shall work with partial functions. We write  $\text{par}(\mathbb{N}, \mathbb{N})$  for the set of partial functions from  $\mathbb{N}$  to  $\mathbb{N}$ , and  $\text{fin}(\mathbb{N}, \mathbb{N})$  for the set of finite such functions. For  $a \in \text{par}(\mathbb{N}, \mathbb{N})$ , when we write  $a(n) = k$  it is clearly implied that  $n \in \text{dom}(a)$  except when we say  $a(n)$  is undefined, which means that  $n \in \text{dom}(a)$ . For the set of *fixed points of a* we write

$$\text{fix}(a) = \{n \in \mathbb{N} : a(n) = n\}.$$

The set  $\text{par}(\mathbb{N}, \mathbb{N})$  is naturally equipped with the operation of composition of partial functions

$$(fg)(n) = m \iff f(g(n)) = m,$$

making it an associative monoid.

Let  $\mathcal{G}$  be an arbitrary group. By  $\mathbb{F}(X)$  we denote the *free group with single generator X*. We identify the group  $\mathcal{G} * \mathbb{F}(X)$ , i.e., the free product of  $\mathcal{G}$  and  $\mathbb{F}(X)$ , with the set  $W_{\mathcal{G}, X}$  of reduced words from the alphabet  $(\mathcal{G} \setminus \{1_{\mathcal{G}}\}) \cup \{X, X^{-1}\}$ , equipped with the familiar ‘concatenate and reduce’ operation (see e.g., [21, Normal Form Theorem]). The neutral element  $1_{\mathcal{G}}$  is therefore identified with the empty word, which we denote by  $\emptyset$ .

By a *circular shift* of a nonempty word  $w = w_n \dots w_1$  we mean the result of reducing the word  $w_{\sigma(n)} \dots w_{\sigma(1)}$ , where  $\sigma$  is a permutation of  $\{1, \dots, n\}$  such that for some  $k \in \mathbb{N}$ ,

$$\sigma(i) = i + k \pmod n.$$

Thus, e.g., *cdab* is a circular shift of *abcd* (in the free group generated by  $\{a, b, c, d\}$ ). By a subword of  $w$  we mean a contiguous subword  $w_i \dots w_j$  for  $n \geq i \geq j \geq 1$ , or the empty word. Of course, the empty word is both the only circular shift and the only subword of itself.

We call a group homomorphism  $\rho: \mathcal{G} \rightarrow S_\infty$  a *cofinitary representation of  $\mathcal{G}$*  if and only  $\rho[\mathcal{G}]$  is cofinitary. Clearly, if  $\rho$  is injective (i.e., a *faithful representation*), we may identify  $\mathcal{G}$  with the cofinitary group  $\rho[\mathcal{G}] \leq S_\infty$ .

For the remainder of this section assume  $\mathcal{G} \leq S_\infty$ . Choosing an arbitrary  $s \in \text{par}(\mathbb{N}, \mathbb{N})$  gives rise to a unique homomorphism of monoids

$$\rho: \mathcal{G} * \mathbb{F}(X) \rightarrow \text{par}(\mathbb{N}, \mathbb{N})$$

such that  $\rho(X) = s$  and  $\rho$  is the identity on  $\mathcal{G}$ . It can be defined by induction on the length of words in the obvious way. Let’s denote this homomorphism by  $\rho_{\mathcal{G}, s}$ , departing from [8] (where it is precisely the map  $w \mapsto e_w(s)$ ). Its image is the compositional closure  $\langle \mathcal{G}, s \rangle$  of  $\mathcal{G} \cup \{s\}$  in  $\text{par}(\mathbb{N}, \mathbb{N})$ .

CONVENTION 2.1. We adopt the convention to denote  $\rho_{\mathcal{G}, s}(w)$  by  $w[s]$ , for any  $w \in W_{\mathcal{G}, X}$  (we ‘substitute  $s$  for  $X$  in  $w$ ’; as e.g., in [16, 28]).

Observe that slightly awkwardly by this convention  $\emptyset[s] = \text{id}_{\mathbb{N}}$  for any  $s \in \text{par}(\mathbb{N}, \mathbb{N})$ .

(B) We define the notion of a *path*, which will be extremely useful in the next section. Fix  $s \in \text{par}(\mathbb{N}, \mathbb{N})$ . Say  $w \in W_{\mathcal{G}, X}$ , and in reduced form

$$w = a_n \dots a_1.$$

We define the *path under*  $(w, s)$  of  $m$ , also called the  $(w, s)$ -*path of*  $m$  and written

$$\text{path}(w, s, m) = \langle m_i : i \in \alpha \rangle,$$

to be the following sequence of natural numbers:  $m_0 = m$  and for  $l \in \mathbb{N}$  and  $i \in \mathbb{N}$  such that  $0 < i \leq n$ ,

$$m_{nl+i} = a_i \dots a_1 w^l[s](m_0)$$

and  $\alpha \in \omega + 1$  is maximal such that all of these expressions are defined. That is we simply iterate applying all the letters of  $w$  as they appear from right to left and record the outcome until we reach an undefined expression.

We can represent such a path e.g., as follows (where  $i = 1 + (k \bmod n)$ ):

$$\dots m_{k+1} \xleftarrow{a_i} m_k \dots \xleftarrow{a_2} m_{n+1} \xleftarrow{a_1} m_n \xleftarrow{a_n} \dots \xleftarrow{a_2} m_1 \xleftarrow{a_1} m_0,$$

or more simply, we shall represent it as  $\langle \dots, m_n, \dots, m_1, m_0 \rangle$ .

For  $k < \alpha$  and  $i = 1 + (k \bmod n)$ , we say  $a_i$  *occurs* at step  $k + 1$  in the above path; if  $m_{k+1}$  is defined we also say  $a_i$  *is applied (to  $m_k$ )* at this step. If  $m_{k+1}$  is undefined, we say the path *terminates after  $\alpha - 1 = k$  steps with last value  $m_k$*  or that it *terminates before (an occurrence of) the letter  $a_i$* .

Sometimes we are interested in the path merely as a set, rather than as a sequence; so let

$$\text{use}(w, s, m) = \{m_i : i < \alpha\}.$$

(C) Of course, we identify  $\mathbb{N}$  and  $\omega$ , but prefer to denote this set as  $\mathbb{N}$  in the context of permutations. We denote by  $|A|$  the cardinality of  $A$ , for any set  $A$ . We sometimes, but not always, decorate names in the forcing language with dots and checks, with the goal of aiding the reader. Notation regarding forcing is as in [13].

**§3. Coding into a generic group extension.** Fix, for this section, a cofinitary group  $\mathcal{G} \leq S_\infty$ . We want to enlarge it by  $\sigma^* \in S_\infty$ , such that  $\langle \mathcal{G}, \sigma^* \rangle$  is cofinitary. This can be done using a forcing invented by Zhang [28]; for some of its applications see [2, 9, 12, 18, 19, 29, 30].

In Section 3.2, we introduce a new forcing  $\mathbb{Q}_{\mathcal{G}}^z$ , such that in addition to the above, every permutation in  $\langle \mathcal{G}, \sigma^* \rangle$  with certain property ‘codes’ a given, fixed  $z \in 2^{\mathbb{N}}$ , in a certain sense (see below).

Before we introduce this new forcing notion, we define our own version of Zhang’s forcing,  $\mathbb{Q}_{\mathcal{G}}$  in Section 3.1, differing slightly from [28]. We then analyse carefully how paths behave when conditions in  $\mathbb{Q}_{\mathcal{G}}$  are extended, facilitating the treatment of  $\mathbb{Q}_{\mathcal{G}}^z$ .

Note that in the case of countable  $\mathcal{G}$ , Zhang’s  $\mathbb{Q}_{\mathcal{G}}$  from [28], our version of  $\mathbb{Q}_{\mathcal{G}}$  described in Section 3.1, and the forcing  $\mathbb{Q}_{\mathcal{G}}^z$  are all countable, i.e., particular presentations of Cohen forcing.

**3.1. Zhang’s forcing revisited.** We now turn to our definition of the forcing to add a generic faithful cofinitary representation of  $\mathcal{G} * \mathbb{F}(X)$ .

DEFINITION 3.1 (The forcing  $\mathbb{Q}_{\mathcal{G}}$ ).

- (a) Conditions of  $\mathbb{Q}_{\mathcal{G}}$  are pairs  $p = (s^p, F^p)$ , where  $s \in \text{fin}(\mathbb{N}, \mathbb{N})$  is injective and  $F^p \subseteq W_{\mathcal{G}, X}$  is finite and closed under taking subwords.

- (b)  $(s^q, F^q) \leq_{\mathbb{Q}_G} (s^p, F^p)$  if and only if  $s^q \supseteq s^p, F^q \supseteq F^p$  and for all  $w \in F^p \setminus \mathcal{G}$ , if  $m \in \text{fix}(w[s^q])$ , then there is a nonempty subword  $w'$  of  $w$  such that  $\text{use}(w, s^q, m) \cap \text{fix}(w'[s^p]) \neq \emptyset$ .

The reader is invited to check that  $\leq_{\mathbb{Q}_G}$  is transitive—it is for this reason that we require  $F^p$  to be closed under taking subwords.

We write any condition  $p \in \mathbb{Q}_G$  as  $(s^p, F^p)$  if we want to refer to the components of that condition.

If  $G$  is  $(\mathbf{V}, \mathbb{Q}_G)$ -generic, we let

$$\sigma_G = \bigcup_{p \in G} s^p.$$

As we shall see,  $\rho_{G, \sigma_G} : \mathcal{G} * \mathbb{F}(X) \rightarrow \langle \mathcal{G}, \sigma_G \rangle$  is a faithful cofinitary representation; moreover,  $\langle \mathcal{G}, \sigma_G \rangle$  is maximal with respect to the ground model, that is, for no  $\tau \in (S_\infty \setminus \mathcal{G}) \cap \mathbf{V}$  is  $\mathcal{G} \cup \{\sigma_G, \tau\}$  contained in a cofinitary group.

The role of the requirement regarding fixed points in (3.1) is to guarantee that  $\langle \mathcal{G}, \sigma_G \rangle$  is cofinitary (as will be seen Lemma 3.12).

As is pointed out in [28, p. 42f.], one cannot replace (3.1) by the simpler

- (b)'  $(s^q, F^q) \leq_{\mathbb{Q}_G} (s^p, F^p)$  if and only if  $s^q \supseteq s^p, F^q \supseteq F^p$  and for all  $w \in F^p$ , we have  $\text{fix}(w[s^q]) \subseteq \text{fix}(w[s^p])$ .

For with this simpler definition, supposing  $g \in \mathcal{G}$  and  $n \in \text{fix}(g)$ , the condition  $(\emptyset, \{X^{-1}gX\}) \in \mathbb{Q}_G$  has no extension  $q \in \mathbb{Q}_G$  with  $n \in \text{ran}(s^q)$ . It is fixed points of conjugate subwords that give rise to this problem. Accordingly, if desired one could replace the phrase ‘nonempty subword’ in (3.1) by the phrase ‘conjugate subword’.

In a previous article [8] by two of the present authors, allowing only certain words in  $F^p$  made it possible to define  $\leq_{\mathbb{Q}_G}$  as in (b)'. The present definition helps to simplify proofs in comparison with both [8] and Zhang’s original version [28].

We now prove two versions of the Domain Extension Lemma and a crucial lemma concerning the length of certain paths (Lemma 3.5). This will considerably clean up the presentation when we deal with the more complicated forcing  $\mathbb{Q}_G^z$ .

The following is implicit in [28]; for the convenience of the reader, we include a new, very short proof.

**LEMMA 3.2 (Contingent Domain Extension for  $\mathbb{Q}_G$ ).** *Suppose  $s \in \text{fin}(\mathbb{N}, \mathbb{N})$  is injective,  $w \in W_{\mathcal{G}, X}$  and  $n \in \mathbb{N}$  is such that for any nonempty subword  $w'$  of  $w$ ,  $n \notin \text{fix}(w'[s])$ . Then for a cofinite set of  $n'$ , letting  $s' = s \cup \{(n, n')\}$ , we have that  $s'$  is injective and  $\text{fix}(w[s']) = \text{fix}(w[s])$ .*

**PROOF.** Let  $W^*$  be the set of subwords of circular shifts of  $w$  and pick  $n'$  arbitrary such that

$$\begin{aligned} n' &\notin \bigcup \{ \text{fix}(w'[s]) : w' \in W^* \setminus \{\emptyset\} \}, \\ n' &\notin \bigcup \{ w'[s]^i(n) : i \in \{-1, 1\}, w' \in W^* \}, \text{ and} \\ n' &\notin \text{ran}(s). \end{aligned} \tag{3.1}$$

The role of the last requirement is to ensure that  $s'$  is injective.

Assume towards a contradiction that  $m_0 \in \text{fix}(w[s']) \setminus \text{fix}(w[s])$ . As the  $(w, s)$ -path of  $m_0$  differs from the  $(w, s')$ -path, the latter must contain an application of  $X$  to  $n$  or of  $X^{-1}$  to  $n'$ . Write this latter path as

$$\dots m_{k(3)} \xleftarrow{w''} m_{k(2)} \xleftarrow{X^j} m_{k(1)} \xleftarrow{w'} m_{k(0)} = m_0, \tag{3.2}$$

where  $j \in \{-1, 1\}$  and  $m_{k(1)} = n$  when  $j = 1$ ,  $m_{k(1)} = n'$  when  $j = -1$ ; moreover we ask that  $w', w'' \in W_{G,X}$  are the maximal subwords of  $w$  such that from  $m_{k(0)}$  to  $m_{k(1)}$  and  $m_{k(2)}$  to  $m_{k(3)}$ , the path contains no application of  $X$  to  $n$  or of  $X^{-1}$  to  $n'$  (allowing either of  $w', w''$  to be empty). Thus,  $w'$  and  $w''$  correspond to path segments where  $s$  and  $s'$  agree:

$$\begin{aligned} w'[s](m_{k(0)}) &= w'[s'](m_{k(0)}) = m_{k(1)}, \\ w''[s](m_{k(2)}) &= w''[s'](m_{k(2)}) = m_{k(3)}. \end{aligned}$$

We show that we can assume  $w = w''X^jw'$ . Otherwise, by maximality of  $w''$ , at step  $k(3)$  again  $X$  is applied to  $n$  or  $X^{-1}$  to  $n'$ . Write the path as

$$\dots \xleftarrow{X^{j'}} m_{k(3)} \xleftarrow{w''} m_{k(2)} \xleftarrow{X^j} m_{k(1)} \xleftarrow{w'} m_{k(0)} = m_0$$

with  $j' \in \{-1, 1\}$  and observe:

1.  $m_{k(2)} = m_{k(3)}$ ; for otherwise,  $n' = (w'')^i[s](n)$  for some  $i \in \{-1, 1\}$ , contradicting the choice of  $n'$ .
2. Thus,  $w'' \neq \emptyset$ , since on one side of  $w''$  we have  $X$  and on the other  $X^{-1}$  and  $w$  is in reduced form.
3. As  $n' \notin \text{fix}(w''[s])$ , we have that  $m_{k(2)} = m_{k(3)} = n$ .
4. So  $n \in \text{fix}(w''[s])$ , contradicting the hypothesis of the lemma.

Thus,  $w = w''X^jw'$  and  $m_0 = m_{k(3)}$ . We infer that  $n' = (w'w'')^{-j}[s](n)$ , again contradicting the choice of  $n'$ . ⊥

REMARK 3.3. Note as this is easily overlooked, that (3.1) implies  $n' \neq n$ , since  $\emptyset \in W^*$  and by Convention 2.1. Also note that the requirements in (3.1) were chosen to be easy-to-state rather than minimal (as will be the case for (3.3), (3.4), (3.5), and (3.7) below).

This enables us to give a short proof of the analogue of [28, Lemma 2.2]:

LEMMA 3.4 (Domain Extension for  $\mathbb{Q}_G$ ). *For any  $n \in \mathbb{N}$ , the set of  $q$  such that  $n \in \text{dom}(s^q)$ , is dense in  $\mathbb{Q}$ .*

PROOF. Fix  $p \in \mathbb{Q}$ ; we shall find a stronger condition  $q \in \mathbb{Q}$  such that  $n \in \text{dom}(s^q)$ . Analogously to the previous proof, let  $F^*$  consist of the empty word together with all subwords of circular shifts of words in  $F^p$ , and let  $n'$  be arbitrary such that

$$\begin{aligned} n' &\notin \bigcup \{ \text{fix}(w[s]) : w \in F^* \setminus \{ \emptyset \} \}, \\ n' &\notin \bigcup \{ w[s]^i(n) : i \in \{-1, 1\}, w \in F^* \}, \text{ and} \\ n' &\notin \text{ran}(s). \end{aligned} \tag{3.3}$$

Note that (3.3) excludes only finitely many values for  $n'$ . Define  $s' = s \cup \{(n, n')\}$  and  $q = (s', F^p)$ . As in the proof of Lemma 3.2,  $s'$  is injective.

Given  $w \in F^p$  and supposing  $m_0 \in \text{fix}(w[s']) \setminus \text{fix}(w[s])$ , the proof of the previous lemma shows that there is a subword  $w'$  of  $w$  such that  $n \in \text{fix}(w'[s])$  and  $n$  appears in the  $(w, s')$ -path of  $m_0$ . In other words,

$$\text{use}(w, s', m) \cap \text{fix}(w'[s]) \neq \emptyset,$$

whence  $q \leq_{\mathbb{Q}_G} p$ . ⊣

A crucial observation for the following discussion of  $\mathbb{Q}_G^z$  is that when extending the domain of  $s^p$  for a given condition  $p$ , we have fine control over the length of paths that result from this extension.

**LEMMA 3.5 (Lengths of paths).** *Fix  $w \in W_{G,X}$  and  $m \in \mathbb{N}$ . Moreover, let  $s \in \text{fin}(\mathbb{N}, \mathbb{N})$  and  $n \in \mathbb{N} \setminus \text{dom}(s)$  be given.*

*Then for cofinitely many  $n' \in \mathbb{N}$ , if we let  $s' = s \cup \{(n, n')\}$  the following holds:*

1. *If the  $(w, s)$ -path of  $m$  does not terminate with last value  $n$  before an occurrence of  $X$ ,  $\text{path}(w, s', m) = \text{path}(w, s, m)$ .*
2. *If  $w$  is not conjugate to any of its proper subwords,  $w' \in G$ , and the  $(w, s)$ -path of  $m$  terminates with last value  $n$  before an occurrence of  $X$ , the  $(w; s')$ -path of  $m$  contains exactly one more application of  $X$  than does the  $(w, s)$ -path and no further application of  $X^{-1}$ .*

**PROOF.** Let  $E = \text{dom}(s) \cup \text{ran}(s) \cup \{n\} \cup \{m\}$  and  $W^*$  be the set of subwords of circular shifts of  $w$ . Suppose  $n'$  is arbitrary such that

$$\begin{aligned} n' \notin \bigcup \{ \text{fix}(w'[s]) : w' \in W^* \setminus \{\emptyset\} \} \text{ and} \\ n' \notin \bigcup \{ w'[s]^i[E] : i \in \{-1, 1\}, w' \in W^* \}. \end{aligned} \tag{3.4}$$

In Case 1 of the lemma, show  $\text{path}(w, s', m) = \text{path}(w, s, m)$ . Suppose that the  $(w, s)$ -path of  $m$  terminates after  $k$  steps with last value  $m_k$  before an occurrence of  $X^j$ ,  $j \in \{-1, 1\}$ . If  $j = 1$ ,  $m_k \neq n$  by assumption; and as the  $(w, s)$ -path terminates with  $m_k$ , we have  $m_k \notin \text{dom}(s) \cup \{n\}$ , so the  $(w, s')$ -path terminates as well.

So assume towards a contradiction that  $j = -1$  and  $m_{k+1}$  in the  $(w, s')$ -path of  $m$  is defined. As the  $(w, s)$ -path terminates with  $m_k$ , before an occurrence of  $X^{-1}$ , while  $(w, s')$ -path does not terminate,  $m_k = n'$ . Thus,  $n' \in w'[s][E]$  for a subword  $w'$  of  $w$ , contradicting (3.4).

For Case 2 of the lemma, suppose that the  $(w, s)$ -path of  $m$  terminates after  $k$  steps with last value  $m_k = n$  before an occurrence of  $X$ . In the  $(w, s')$ -path we have on the contrary that  $m_{k+1} = n'$ .

If the letter occurring at step  $k + 2$  in this path is again  $X$ , the path terminates after  $k + 1$  steps with last value  $n' = m_{k+1}$ , since  $n' \notin \text{dom}(s')$  by (3.4).

If the letter occurring at step  $k + 2$  is  $g \in \mathcal{G} \setminus \{\text{id}_{\mathbb{N}}\}$  followed by  $X$  or  $X^{-1}$  at step  $k + 3$ , the path terminates after  $k + 2$  steps with last value  $m_{k+2} = g(n')$ , as otherwise,  $n' \in g^{-1}[\text{dom}(s') \cup \text{ran}(s')]$ , which implies  $n' \in g^{-1}[E]$  or  $n' \in \text{fix}(g)$ , contradicting (3.4).

Lastly, suppose the letter occurring at step  $k + 2$  is  $g \in \mathcal{G} \setminus \{\text{id}_{\mathbb{N}}\}$  and  $g$  is the last letter of  $w$ . Further suppose the following letter at step  $k + 3$  (the first letter in  $w$ ) is  $h \in \mathcal{G} \setminus \{\text{id}_{\mathbb{N}}, g^{-1}\}$  followed by  $X$  or  $X^{-1}$  at step  $k + 4$ . Then the path terminates after  $k + 3$  steps with last value  $m_{k+3} = h(g(n'))$ , as otherwise,



$n' \in (hg)^{-1}[\text{dom}(s') \cup \text{ran}(s')]$ , which implies  $n' \in (hg)^{-1}[E]$  or  $n' \in \text{fix}(hg)$ , contradicting (3.4).

As  $w$  has no proper conjugate subwords and is reduced no other combination of letters can occur at the next steps. ←

All other proofs regarding  $\mathbb{Q}_{\mathcal{G}}$  will be omitted (but note that they can be inferred from their counterparts for  $\mathbb{Q}_{\mathcal{G}}^z$  in the next section).

**3.2. Coding into a generic cofinitary group extension.** Our next goal is to define, given  $z \in 2^{\mathbb{N}}$ , a forcing  $\mathbb{Q}_{\mathcal{G}}^z$  such that whenever  $G$  is  $(\mathbf{V}, \mathbb{Q}_{\mathcal{G}}^z)$ -generic the following holds: There exists  $\sigma_G \in S_{\infty}$  such that for a large enough set of  $\sigma \in \langle \mathcal{G}, \sigma_G \rangle$  (i.e., large enough for our application),  $z$  is computable from  $\sigma$  (i.e., the characteristic function of  $z$  is computable by a Turing machine using  $\sigma$  as an oracle).

First, we describe the algorithm by which  $z$  is computed from an element of  $\langle \mathcal{G}, \sigma_G \rangle \setminus \mathcal{G}$ . Since our forcing uses finite approximations to  $\sigma_G$ , we must also define the coding for elements of  $\text{fin}(\mathbb{N}, \mathbb{N})$ .

DEFINITION 3.6 (Coding).

1. We say that  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  codes  $z \in 2^{\mathbb{N}}$  with parameter  $m$  if and only if

$$(\forall k \in \mathbb{N}) \sigma^{k+1}(m) \equiv z(k) \pmod{2}.$$

2. Let  $t \in \{0, 1\}^l$ , where  $l \in \mathbb{N}$ . We say that  $\sigma \in \text{fin}(\mathbb{N}, \mathbb{N})$  exactly codes  $t$  with parameter  $m \in \mathbb{N}$  if and only if

$$(\forall k < l) \sigma^{k+1}(m) \equiv t(k) \pmod{2}$$

and in addition,  $\sigma^{l+1}(m)$  is undefined.

Note that using this algorithm when  $\sigma \in S_{\infty}$  is conjugate to an element of  $\mathcal{G}$  with only finite orbits, we can't expect that arbitrary  $z \in 2^{\mathbb{N}}$  be coded—the sequence coded by such  $\sigma$  will always be periodic. Fortunately we can exclude such  $\sigma$  from consideration for the proof of our main result

In fact it suffices for our application to restrict our attention to those  $\sigma$  which can be represented as a word in  $W_{\mathcal{G}, \sigma_G} \setminus \mathcal{G}$  which is not conjugate to any of its proper subwords (we will see below that  $\rho_{\mathcal{G}, \sigma_G}$  is injective so in fact this representation is unique).<sup>1</sup>

For the rest of this section, fix  $z \in 2^{\mathbb{N}}$ . Now we can define the forcing.

DEFINITION 3.7 (Definition of  $\mathbb{Q} = \mathbb{Q}_{\mathcal{G}}^z$ ).

- (A) Conditions of  $\mathbb{Q}$  are triples  $p = (s^p, F^p, \bar{m}^p)$  s.t.

- (1)  $(s^p, F^p) \in \mathbb{Q}_{\mathcal{G}}$ .
- (2)  $\bar{m}^p$  is a partial function into  $\mathbb{N}$  from the set of words  $w \in W_{\mathcal{G}, X} \setminus \mathcal{G}$  such that no proper subword  $w'$  of  $w$  is conjugate to  $w$ .
- (3) For any  $w \in \text{dom}(\bar{m}^p)$  there is  $l \in \omega$  such that  $w[s^p]$  exactly codes  $z \upharpoonright l$  with parameter  $\bar{m}^p(w)$ .
- (4) If  $w, w' \in \text{dom}(\bar{m}^p)$  and  $w \neq w'$ ,

$$\text{use}(w, s^p, \bar{m}^p(w)) \cap \text{use}(w', s^p, \bar{m}^p(w')) = \emptyset.$$

---

<sup>1</sup>Devising a forcing which similarly codes  $z$  into all words except those conjugate to elements of  $\mathcal{G}$  with only finite orbits is possible, but more involved for several reasons.



- (B)  $(s^q, F^q, \bar{m}^q) \leq (s^p, F^p, \bar{m}^p)$  if and only if
  - (1)  $(s^q, F^q) \leq_{\mathbb{Q}_G} (s^p, F^p)$ ,
  - (2)  $\bar{m}^q$  extends  $\bar{m}^p$  as a function.

We write any condition  $p \in \mathbb{Q}$  as  $(s^p, F^p, \bar{m}^p)$  if we want to refer to the components of that condition.

Note that (A3) ensures that for any  $p \in \mathbb{Q}$  and  $w \in \text{dom}(\bar{m}^p)$  the path under  $(w, s^p)$  of  $\bar{m}^p(w)$  is finite (although other paths may be eventually periodic and thus infinite). Also note that by (A4),  $|\mathcal{G}|$  is collapsed to  $\omega$  by  $\mathbb{Q}$  whenever  $\mathcal{G}$  is uncountable in the ground model.

For a  $(V, \mathbb{Q})$ -generic  $G$ , as in the previous section we let

$$\sigma_G = \bigcup_{p \in G} s^p.$$

We now show in a series of lemmas that  $\rho_{G, \sigma_G} : \mathcal{G} * \mathbb{F}(X) \rightarrow \langle \mathcal{G}, \sigma_G \rangle$  is a faithful cofinitary representation and for any  $w \in W_{G, X} \setminus \mathcal{G}$  which does not have a proper conjugate subword,  $w[\sigma_G]$  codes  $z$ . The reader should note that this immediately implies that for any  $g \in \mathcal{G} \setminus \{\text{id}_{\mathbb{N}}\}$  and any  $\sigma \in \langle \mathcal{G}, \sigma_G \rangle$  either  $z$  is coded by  $\sigma$  or by  $g\sigma$ .

We begin with a Lemma showing that  $\sigma_G$  is forced by  $\mathbb{Q}$  to be totally defined on  $\mathbb{N}$ .

LEMMA 3.8 (Domain Extension).

1. For any  $n \in \mathbb{N}$ , the set  $D_n^d = \{q \in \mathbb{Q} : n \in \text{dom}(s^q)\}$  is dense in  $\mathbb{Q}$ .
2. In fact, suppose  $p \in \mathbb{Q}$ ,  $n \in \mathbb{N}$ ,  $w^* \in W_{G, X} \setminus \mathcal{G}$  is not conjugate to any of its proper subwords,  $m^* \in \mathbb{N}$ ,  $n$  is the last value of  $\text{path}(w^*, s^p, m^*)$  and this path terminates before an occurrence of  $X$ . Then one can find  $q \in \mathbb{Q}$  such that  $q \leq p$  and  $\text{path}(w^*, s^q, m^*)$  contains exactly one more application of  $X$ , and no further application of  $X^{-1}$ , than does  $\text{path}(w^*, s^p, m^*)$ .

Before we prove the lemma, to avoid repetition, we introduce the following terminology: For  $w \in W_{G, X}$  and  $j \in \{-1, 1\}$ , call an occurrence of  $X^j$  in  $w$  *critical* if there is no occurrence of  $X$  or  $X^{-1}$  in  $w$  to its left. Otherwise, we call it an *uncritical occurrence*. Clearly, it is through a critical occurrence of  $X$  (resp.  $X^{-1}$ ) in some word in  $\text{dom}(\bar{m})$  that the coding requirements from (A3) restrict our possibilities to extend  $\text{dom}(s^p)$  (resp.  $\text{ran}(s^p)$ ).

PROOF OF LEMMA 3.8. Let  $p \in \mathbb{Q}$ ,  $w^* \in W_{G, X}$ , and  $m^*, n \in \mathbb{N}$  be as in the statement of the lemma and suppose  $n \notin \text{dom}(s^p)$ . We will find  $n'$  such that for  $s' = s^p \cup \{(n, n')\}$ ,  $q = (s', F^p, \bar{m}^p)$  is a condition stronger than  $p$ .

Write  $s$  for  $s^p$  and  $\bar{m}$  for  $\bar{m}^p$ . Let

$$E = \text{ran}(s) \cup \text{dom}(s) \cup \{n\} \cup \text{ran}(\bar{m}) \cup \{m^*\},$$

and let  $F^*$  consist of all words which are a subword of a circular shift of a word in  $F^p \cup \{w^*\}$ . The first requirement we make is that  $n'$  be chosen such that

$$\begin{aligned} n' \notin \bigcup \{ \text{fix}(w[s]) : w \in F^* \setminus \{\emptyset\} \} \text{ and} \\ n' \notin \bigcup \{ g^{-1}w^i[s][E] : i \in \{-1, 1\}, w \in F^*, g \in F^* \cap \mathcal{G} \}. \end{aligned} \tag{3.5}$$

Note that (3.5) excludes only finitely many possible values for  $n'$ . The taking of preimages under  $g \in F^* \cap \mathcal{G}$  in (3.5) serves the sole purpose of ensuring that the following holds:

$$g(n') \notin \bigcup \{ \text{use}(w, s, \bar{m}(w)) : w \in \text{dom}(\bar{m}) \} \tag{3.6}$$

for  $g = \text{id}_{\mathbb{N}}$  as well as for all  $g \in \mathcal{G} \setminus \{ \text{id}_{\mathbb{N}} \}$  occurring in a word from  $F^p$ .

Now suppose for some  $w \in \text{dom}(\bar{m})$ ,  $n$  appears in the  $(w, s)$ -path of  $\bar{m}(w)$  before a critical occurrence of  $X$ . In this case, we must make an additional requirement: Noting that by (A4) there is at most one such  $w \in \text{dom}(\bar{m})$ , let this unique word be denoted by  $w^{**}$ . Let  $l$  be such that  $w^{**}[s]$  exactly codes  $z \upharpoonright l$  with parameter  $\bar{m}(w^{**})$ . Further, suppose  $w^{**} = gXw'$ , where  $w' \in W_{\mathcal{G}, X}$  and we allow  $g \in \mathcal{G}$  to be  $\text{id}_{\mathbb{N}}$  but no cancellation in  $Xw'$ . Now in addition to (3.5), require that  $g(n')$  be even if  $z(l) = 0$  and odd if  $z(l) = 1$ . This is possible as (3.5) excludes only finitely many values. Note that by choice of  $n'$  and Lemma 3.5, only for this one  $w \in \text{dom}(\bar{m}^q)$  does the  $(w, s')$ -path differ from the  $(w, s)$ -path, and as  $w$  has no proper conjugate subword, the  $(w, s')$ -path does not go past the next critical occurrence of  $X$  or  $X^{-1}$ . Thus (A3) holds of  $q$  by construction.

To see that  $q$  is a condition, it remains to verify (A4). We have seen in the proof of Lemma 3.5 that at most one path starting in  $\text{dom}(\bar{m}^p)$  acquires new values when passing to  $q$ , and these new values are  $n'$  and possibly  $g(n')$ , where  $g \in \mathcal{G}$  is a subword of a cyclic shift of a word in  $\text{dom}(\bar{m})$ . Thus requirement (A4) holds of  $q$  by (3.6).

We end the proof of Part 1 of the lemma by quoting the proof of the Domain Extension Lemma for  $\mathbb{Q}_{\mathcal{G}}$  to conclude that  $q \leq p$ .

For Part 2 of the lemma note that by the proof of Lemma 3.5 indeed the  $(w^*, s)$ -path of  $m^*$  contains exactly one more application of  $X$ , and no further application of  $X^{-1}$ , than does the  $(w^*, s')$ -path of  $m^*$ . ⊖

The next lemma shows that  $\mathbb{Q}$  forces  $\sigma_G$  to be onto  $\mathbb{N}$ .

LEMMA 3.9 (Range Extension).

1. For any  $n \in \mathbb{N}$ , the set  $D_n^r = \{ q \in \mathbb{Q} : n \in \text{ran}(s^q) \}$  is dense in  $\mathbb{Q}$ .
2. In fact, suppose  $p \in \mathbb{Q}$ ,  $n \in \mathbb{N}$ , and  $m^* \in \mathbb{N}$  are such that for some  $w^* \in W_{\mathcal{G}, X} \setminus \mathcal{G}$  not conjugate to any of its proper subwords,  $n$  is the last value of  $\text{path}(w^*, s^p, m^*)$  and this path terminates before an occurrence of  $X^{-1}$ . Then one can find  $q \in \mathbb{Q}$  such that  $q \leq p$  and  $\text{path}(w^*, s^q, \bar{m}^*)$  contains exactly one more application of  $X^{-1}$  and no further application of  $X$  than does  $\text{path}(w^*, s^p, \bar{m}^*)$ .

PROOF. The lemma is entirely symmetrical to the Domain Extension Lemma. By symmetry, the proofs of Lemmas 3.2, 3.4, 3.5, and 3.8 can easily be adapted. ⊖

By the previous lemma  $\Vdash_{\mathbb{Q}} \sigma_G \in S_{\infty}$ . By the next two lemmas it is forced that for all  $w \in W_{\mathcal{G}, X} \setminus \mathcal{G}$  which are not conjugate to any of their proper subwords,  $w[\sigma_G]$  codes  $z$ , as promised:

LEMMA 3.10 (Generic coding). For any  $w \in W_{\mathcal{G}, X} \setminus \mathcal{G}$  without a proper conjugate subword and any  $l \in \mathbb{N}$ , let  $D_{w,l}^{\text{code}}$  denote the set of  $q \in \mathbb{Q}$  such that  $w \in \text{dom}(\bar{m}^q)$  and for some  $l' \geq l$ ,  $q$  exactly codes  $z \upharpoonright l'$  with parameter  $\bar{m}^q(w)$ . Then  $D_{w,l}^{\text{code}}$  is dense in  $\mathbb{Q}$ .

PROOF. Let  $w$  and  $l$  be as above, and fix  $p \in \mathbb{Q}$ . We may assume  $w \in \text{dom}(\bar{m}^p)$ : Otherwise, find  $n' \in \mathbb{N}$  such that  $n' \equiv z(0) \pmod{2}$  and (3.5) holds with  $E = \text{dom}(s^p) \cup \text{ran}(s^p) \cup \text{ran}(\bar{m}^p)$  and  $F^*$  equal to the set of subwords of circular shifts of words in  $F^p \cup \{w\}$  and let  $p' = (s^p, F^p, \bar{m}^p \cup \{(w, n')\})$ . By (3.5) and by the proof of Lemma 3.8, (A4) is satisfied for  $p'$  and the  $(w, s^p)$ -path of  $n'$  will terminate before the right-most application of  $X$  or  $X^{-1}$  in  $w$ . As  $s^p = s^{p'}$ , this suffices to show  $p'$  is a condition stronger than  $p$ .

So supposing  $w \in \text{dom}(\bar{m}^p)$ , let  $m$  be the last value of the  $(w, s^p)$ -path of  $\bar{m}(w)$  and assume this path terminates before an occurrence of the letter  $X$ . By the Domain Extension Lemma, we may find  $q \leq p$  such that  $m \in \text{dom}(s^q)$  and the  $(w, s^q)$ -path at  $\bar{m}(w)$  terminates either at the next step or after one further application of an element of  $\mathcal{G} \setminus \{\text{id}_{\mathbb{N}}\}$ .

If instead the  $(w, s^p)$ -path of  $\bar{m}(w)$  terminates before an occurrence of the letter  $X^{-1}$ , argue similarly using the Range Extension Lemma.

Repeating the argument if necessary, we obtain a condition  $q$  such that  $s^q$  exactly codes  $z \upharpoonright l$ . ⊢

By the next two lemmas,  $\langle \mathcal{G}, \sigma_{\mathcal{G}} \rangle$  is forced to be cofinitary. The reader may care to notice that the proofs of the remaining lemmas in this section are almost identical (or in the case of Lemma 3.15 at least not very different) for  $\langle \mathbb{Q}_{\mathcal{G}}, \leq_{\mathcal{G}} \rangle$ —excepting of course Lemma 3.16(II) which doesn't hold for  $\langle \mathbb{Q}_{\mathcal{G}}, \leq_{\mathcal{G}} \rangle$ .

LEMMA 3.11. *If  $w \in W_{\mathcal{G}, X}$ , the set  $\{q \in \mathbb{Q} : w \in F^q\}$  is dense in  $\mathbb{Q}$ .*

PROOF. Given  $p \in \mathbb{Q}_{\mathcal{G}}^z$ , let  $q = (s^p, F^p \cup W, \bar{m}^p)$ , where  $W$  is the set of subwords of  $w$ . Then  $q \in \mathbb{Q}_{\mathcal{G}}^z$  and  $q \leq p$ . ⊢

LEMMA 3.12. *If  $p \in \mathbb{Q}$  and  $w \in F^p$ , there is  $N$  such that  $p \Vdash \text{fix}(w[\sigma_{\mathcal{G}}])$  has size at most  $N$ .*

PROOF. Suppose  $p$  and  $w \in F^p$ , and let the length of  $w$  be  $k$ . Let  $N$  be the number of triples  $(u, m, l)$  such that  $u$  is a subword of  $w$ ,  $m \in \text{fix}(u[s^p])$ , and  $l \leq k$ . Towards a contradiction, assume  $q \leq p$  forces that  $(w[\sigma_{\mathcal{G}}])$  has more than  $N$  fixed points. By strengthening  $q$  if necessary, we may assume that  $|\text{fix}(w[s^q])| > N$ .

As  $q \leq p$ , by (3.1) in Definition 3.1, for each  $n \in \text{fix}(w[s^q])$  there is a nonempty subword  $w'$  of  $w$  satisfying

$$\text{use}(w, s^q, n) \cap \text{fix}(w'[s^p]) \neq \emptyset.$$

So letting  $\langle \dots, m_1, m_0 \rangle$  be the  $(w, s^q)$ -path of  $n$ , pick a subword  $u = u(n)$  of  $w$ ,  $m = m(n) \in \text{fix}(u[s^p])$  and  $l = l(n) \leq k$  so that  $m_l = m$ , for each such  $n$ .

Note that the function  $n \mapsto (u(n), m(n), l(n))$  is injective (since  $w[s^q]$  is); but it maps  $\text{fix}(w[s^q])$  to a set of size  $N$ , contradiction. ⊢

The next lemma shows that our construction yields a group which is maximal with respect to permutations from the ground model: For any  $\tau \in S_{\infty} \setminus \mathcal{G}$ ,  $\Vdash_{\mathbb{Q}_{\mathcal{G}}} \langle \mathcal{G}, \sigma_{\mathcal{G}}, \cdot \rangle \cup \{\check{\tau}\}$  is not contained in a cofinitary group. In fact, the Lemma shows: For any  $\tau \in S_{\infty}$  such that  $\langle \mathcal{G}, \tau \rangle$  is cofinitary,  $\Vdash_{\mathbb{Q}_{\mathcal{G}}} \text{“}\{n : \sigma_{\mathcal{G}}(n) = \check{\tau}(n)\} \text{ is infinite”}$ . We will prove a generalization, the  $\mathbb{P}$ -generic Hitting Lemma below.

LEMMA 3.13 (Generic hitting). *Given  $m \in \mathbb{N}$  and  $\tau \in S_{\infty}$  such that  $\langle \mathcal{G}, \tau \rangle$  is cofinitary, the set  $D_{\tau, m}^{\text{hit}} = \{q \in \mathbb{Q} : (\exists n \geq m) s^q(n) = \tau(n)\}$  is dense.*

PROOF. Let  $p \in \mathbb{Q}$  and  $m \in \mathbb{N}$  be given. Find  $n \in \mathbb{N}$  such that  $n \geq m$ , and such that letting  $n' = \tau(n)$ ,  $n'$  satisfies (3.5) with  $s = s^p$ ,  $E = \text{dom}(s^p) \cup \text{ran}(s^p) \cup \{n\} \cup \text{ran}(\bar{m})$  and  $F^*$  equal to the set of subwords of circular shifts of words in  $F^p$ .

To see this is possible, note that (3.5) holds for  $n' = \tau(n)$  if and only if for  $E' = \text{dom}(s) \cup \text{ran}(s) \cup \text{ran}(\bar{m})$ ,

$$\begin{aligned} n &\notin \tau^{-1} \left[ \bigcup \{ \text{fix}(w[s]) : w \in F^* \setminus \{\emptyset\} \} \right], \\ n &\notin \tau^{-1} \left[ \bigcup \{ g^{-1}w'[s]^i[E'] : i \in \{-1, 1\}, w' \in F^*, g \in F^* \cap \mathcal{G} \} \right], \text{ and} \quad (3.7) \\ n &\notin \bigcup \{ \text{fix}(\tau^{-1}g^{-1}w'[s]^i) : i \in \{-1, 1\}, w' \in F^*, g \in F^* \cap \mathcal{G} \}. \end{aligned}$$

The first two requirements obviously exclude only finitely many  $n$ ; the same holds for the last requirement in case  $w' \notin \mathcal{G}$ , when  $w'[s]$  is finite. Lastly, when  $w' \in \mathcal{G}$ , the last requirement excludes only finitely many  $n$  as  $\langle \mathcal{G}, \tau \rangle$  is cofinitary. Thus,  $n$  as above can indeed be found.

By the proof of the Domain Extension Lemma, letting  $s' = s^p \cup \{(n, \tau(n))\}$ ,  $q = (s', F^p)$  is a condition stronger than  $p$ . ⊣

REMARK 3.14. By standard properties of product forcing, the previous lemma is easily seen to imply the following: if  $G$  is  $(\mathbf{V}, \mathbb{Q}_G^z)$ -generic and  $H$  is  $(\mathbf{V}[G], \mathbb{P})$ -generic for a forcing  $\mathbb{P} \in \mathbf{V}$  then  $\langle \mathcal{G}, \sigma_G \rangle$  is maximal with respect to  $S_\infty \cap \mathbf{V}[H]$ , i.e., if  $\tau \in (S_\infty \setminus \mathcal{G}) \cap \mathbf{V}[H]$  there is no cofinitary group  $\mathcal{G}' \in \mathbf{V}[G][H]$  such that  $\mathcal{G} \cup \{\sigma_G, \tau\} \subseteq \mathcal{G}'$ . This observation inspired our construction of a Cohen-indestructible mcg in the next section.

The following immediately implies that  $\mathbb{Q}$  forces that  $\rho_{\mathcal{G}, \sigma_G}$  is injective.

LEMMA 3.15. *For any  $w \in W_{\mathcal{G}, X} \setminus \{\text{id}_{\mathbb{N}}\}$ , the set  $D_w^{1-1}$  of  $q$  such that  $q \Vdash_{\mathbb{Q}} w[\sigma_G] \neq \text{id}_{\mathbb{N}}$  is dense.*

PROOF. Fix  $p \in \mathbb{Q}$  and  $w \in W_{\mathcal{G}, X} \setminus \{\text{id}_{\mathbb{N}}\}$ . We may assume  $w \notin \mathcal{G}$  and  $w$  has no proper conjugate subwords. Find  $l$  such that  $w^l \notin \text{dom}(\bar{m}^p)$  and  $q \in \mathbb{Q}$  stronger than  $p$  such that for some  $m \not\equiv z(0) \pmod{2}$  we have  $m = \bar{m}^q(w^l)$  (this is possible as  $m$  can be chosen from a cofinite set). Then  $q \Vdash_{\mathbb{Q}} w[\sigma_G] \neq \text{id}_{\mathbb{N}}$  ( $\text{id}_{\mathbb{N}}$  codes the sequence with constant value  $m \pmod{2}$  with parameter  $m$ ). ⊣

Anticipating that in the next section we apply our forcing  $\mathbb{Q}_G^z$  over countable initial segments of  $\mathbf{L}$  satisfying only a very weak fragment of ZFC, we list crucial properties of  $\mathbb{Q}_G^z$  when forcing over such models:

LEMMA 3.16. *Suppose  $M$  is a transitive  $\in$ -model such that  $\mathbb{Q}_G^z \in M$ , and for any  $m \in \omega$ ,  $\tau \in S_\infty \cap M$  such that  $\langle \mathcal{G}, \tau \rangle$  is cofinitary, and  $w \in W_{\mathcal{G}, X} \setminus \mathcal{G}$  we have  $\{D_m^d, D_m^r, D_{\tau, m}^{\text{hit}}, D_w^{1-1}\} \subseteq M$  and  $D_{w, m}^{\text{code}} \in M$  when  $w$  has no proper conjugate subwords. Then for any  $(M, \mathbb{Q}_G^z)$ -generic filter  $G$ , letting*

$$\sigma_G = \bigcup_{p \in G} s^p$$

the following holds:

- (I)  $\rho_{\mathcal{G}, \sigma_G} : \mathcal{G} * \mathbb{F}(X) \rightarrow \langle \mathcal{G}, \sigma_G \rangle$  is a faithful cofinitary representation.
- (II) For any word  $w \in W_{\mathcal{G}, X} \setminus \mathcal{G}$  which does not have a proper subword  $w0$  which is conjugate to  $w$ , we have that  $w[\sigma_G]$  codes  $z$  in the sense of Definition 3.6.

(III) For any  $\tau \in \text{cofin}(S_\infty) \cap M$  such that  $\tau \notin \mathcal{G}$ , there is no cofinitary group  $\mathcal{G}'$  such that  $\langle \mathcal{G}, \sigma_G \rangle \cup \{\tau\} \subseteq \mathcal{G}'$ .

PROOF. As  $G$  intersects  $D_m^d$  for each  $m \in \mathbb{N}$ ,  $\sigma_G$  is total. By analogous arguments, (I), (II) and (III) are obtained using  $D_m^r$ ,  $D_{w,m}^{\text{code}}$ ,  $D_{\tau,m}^{\text{hit}}$ , and  $D_w^{l-1}$ . ⊥

The next lemma, like its precursor [6, Theorem 4.1], will help us to show Cohen-indestructibility in Section 4, in a situation where as above, the ground model satisfies only a weak fragment of ZFC (see also Remark 3.14).

LEMMA 3.17 ( $\mathbb{P}$ -generic hitting). *Let an arbitrary forcing  $\mathbb{P}$ , a  $\mathbb{P}$ -name  $\dot{\tau}$ , a condition  $(p, q) \in \mathbb{P} \times \mathbb{Q}$  and  $k \in \omega$  be given and suppose  $p \Vdash_{\mathbb{P}} \text{“}\dot{\tau} \in S_\infty \text{ and } \langle \dot{\mathcal{G}}, \dot{\tau} \rangle \text{ is cofinitary”}$ . Then there is  $(p', q') \in \mathbb{P} \times \mathbb{Q}$  such that  $(p', q') \leq_{\mathbb{P} \times \mathbb{Q}} (p, q)$  and*

$$(p', q') \Vdash_{\mathbb{P} \times \mathbb{Q}} (\exists n \in \mathbb{N}) n > k \wedge \sigma_{\dot{\mathcal{G}}}(n) = \dot{\tau}(n).$$

PROOF. Fix  $\mathbb{P}$ ,  $\dot{\tau}$ ,  $(p, q) \in \mathbb{P} \times \mathbb{Q}$  and  $k$  as in the statement of the lemma. Let  $G$  be  $(\mathbb{P}, \mathbf{V})$ -generic and such that  $p \in G$  and let  $\tau = \dot{\tau}^G$ .

Working in  $\mathbf{V}[G]$ , follow the proof of the Generic Hitting Lemma: Find  $n$  such that for  $s = s^q$ ,  $F^*$  equal to the set of subwords of circular shifts of a word in  $F^q$  and  $E' = \text{dom}(s^q) \cup \text{ran}(s^q) \cup \text{ran}(\bar{m})$ , (3.7) holds; in addition, require  $\tau(n) > k$ . Letting  $n' = \tau(n)$ , find  $p' \in G$  extending  $p$  such that

$$p' \Vdash_{\mathbb{P}} \dot{\tau}(\check{n}) = \check{n}'.$$

Just as in the proof of the Generic Hitting Lemma, by choice of  $n$ , we have that for  $E = E' \cup \{n\}$ , (3.5) holds in  $\mathbf{V}$ . By the proof of the Domain Extension Lemma we can extend  $q$  to  $q' \in \mathbb{Q}$  such that

$$q' \Vdash_{\mathbb{Q}} \sigma_{\dot{\mathcal{G}}}(\check{n}) = \check{n}',$$

and we are done. ⊥

**§4. A co-analytic Cohen-indestructible mcg.** We now use the ideas from the previous section to prove the main result of this paper. At the same time, we give a new proof of Kastermans’ result that there is a  $\Pi_1^1$  mcg in  $\mathbf{L}$ , based on the idea of finding generics over countable models.

THEOREM 4.1. *Assume  $\mathbf{V} = \mathbf{L}$ . Let  $\mathcal{G}_0$  be any countable cofinitary group, and fix  $c \in 2^{\mathbb{N}}$  such that an enumeration of  $\mathcal{G}_0$  is arithmetical in  $c$  as a subset of  $N_x \mathbb{N}^{\mathbb{N}}$ . Then there is a Cohen-indestructible  $\Pi_1^1(c)$  maximal cofinitary group which contains  $\mathcal{G}_0$  as a subgroup.*

Note that for appropriately chosen  $\mathcal{G}_0$ , our method will produce a group which is isomorphic to Kastermans’ group from [16]. On the other hand, they are not identical as subgroups of  $S_\infty$ , as one should not expect that his construction produce a Cohen-indestructible group.

Our argument is of the same type as Miller’s classical construction given in [24, Section 7]. A very detailed exposition of this technique can be found in [7, Section 3]; the present account is parallel where possible. We start by fixing notation and reviewing some facts, in the course of which we also give a sketch of the proof of Theorem 4.1.

The canonical well-ordering of  $\mathbf{L}$  is denoted by  $\leq_{\mathbf{L}}$ . A *formula* is always a formula in the language of set theory ([15, p. 1]); likewise for *sentences*.

Given  $x \in 2^{\mathbb{N}}$ , let  $E_x \subseteq \omega^2$  be the binary relation defined by

$$m E_x n \iff x(2^m 3^n) = 0.$$

If it is the case that  $E_x$  is well-founded and extensional, we denote by  $M_x$  the set and by  $\pi_x$  the map such that  $\pi_x: \langle \omega, E_x \rangle \rightarrow \langle M_x, \in \rangle$  is the unique isomorphism of  $\langle \omega, E_x \rangle$  with a transitive  $\in$ -model. Recall:

FACT 4.2 (see [15, 13.8]). *If  $E_x$  is well-founded and extensional and  $\phi$  is a formula with  $k$  free variables, the following relations are arithmetical in  $x$ :*

$$\{(m_1, \dots, m_k) \in \mathbb{N} \times \dots \times \mathbb{N} : \langle M_x, \in \rangle \models \phi(m_1, \dots, m_k)\},$$

$$\{(y, m) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} : \pi_x(m) = y\}.$$

Recall that for  $x, y \in 2^{\mathbb{N}}$ ,  $y \in \Delta_1^1(x)$  means that  $y$  is a Borel subset of  $\mathbb{N}$  with a code recursive in  $x$  (i.e.,  $y$  is hyperarithmetical in  $x$ ; see e.g., [22]). We shall use the following facts:

FACT 4.3. *If  $M_z = \mathbf{L}_\delta$ , there is  $y \in \Delta_1^1(z)$  such that  $M_y = \mathbf{L}_{\delta+\omega+\omega}$  (an elementary proof is implicitly given in [18, 3.6]).*

FACT 4.4 (Mansfield-Solovay, see [22, Corollary 4.19, p. 53]). *For any  $\Pi_1^1(z)$  formula  $\Psi(x)$ , the formula  $(\exists x \in \Delta_1^1(z)) \Psi(x)$  is equivalent to a  $\Pi_1^1(z)$  formula.*

We can now give a brief sketch of our construction of a co-analytic Cohen-indestructible family:

If  $\mathbf{V} = \mathbf{L}$ , one may use the definable well-ordering to enlarge a given countable cofinitary group  $\mathcal{G}_0$  to a mcg: Assuming by induction we have constructed  $\mathcal{G}_\xi$ , where  $\xi < \omega_1$ , let  $\sigma_\xi$  be the  $\leq_{\mathbf{L}}$ -least  $\sigma \in S_\infty \setminus \mathcal{G}_\xi$  such that  $\mathcal{G}_{\xi+1} = \langle \mathcal{G}_\xi, \sigma \rangle$  is cofinitary. Then  $\mathcal{G} = \bigcup_{\xi < \omega_1} \mathcal{G}_\xi$  will be maximal cofinitary. Using the above correspondence of countable models with elements of  $2^{\mathbb{N}}$ , one finds  $\mathcal{G}$  to be  $\Sigma_2^1$  in a code for  $\mathcal{G}_0$ .

We can ‘bound the existential quantifier’ by altering the above construction so that at step  $\xi$ , an initial segment of  $\mathbf{L}$  witnessing the fact that  $\sigma_\xi \in \mathcal{G}$  can be found *effectively* in some data  $z$  (i.e., is hyperarithmetical in  $z$ —here we use Fact 4.3); at the same time applying ideas from the previous section to ensure that  $z$ , in turn, can be found effectively in  $\sigma_\xi$ . The resulting mcg is shown to be  $\Pi_1^1$  using Fact 4.4.

The ‘data’ alluded to in the above sketch will be some  $z \in 2^{\mathbb{N}}$  such that  $M_z$  ‘sees’ the first  $\xi$  steps of the construction. The question of how to uniquely pick  $z$  leads naturally to the notion of *projectum*: We say  $\delta$  *projects to*  $\omega$  if and only if there is a surjection from  $\omega$  onto  $\mathbf{L}_\delta$  which is  $\Sigma_1$ -definable (with a parameter) in  $\mathbf{L}_\delta$ .

FACT 4.5 (see [14]). *The set of  $\delta$  which project to  $\omega$  is cofinal in  $\omega_1$ . Moreover, if  $\delta$  projects to  $\omega$ , so does  $\delta + \omega$ .*

Supposing  $\delta$  projects to  $\omega$ , let  $p$  be  $\leq_{\mathbf{L}}$ -least parameter such that there is a surjection as above which is  $\Sigma_1$  in  $p$ . Call the surjection  $f$  so obtained the *canonical surjection from  $\omega$  onto  $\mathbf{L}_\delta$* .

Finally, we proceed to prove our main theorem.

PROOF OF THEOREM 4.1. Assume  $\mathbf{V} = \mathbf{L}$ , and fix  $\mathcal{G}_0$  and  $c$  as in the theorem. Since the argument relativizes to the parameter  $c$ , we may suppress it and assume that  $\mathcal{G}_0$  is arithmetic. We may also suppose that  $\mathcal{G}_0 \neq \{\text{id}_{\mathbb{N}}\}$ —otherwise, just replace it by a larger arithmetic cofinitary group.

We construct a sequence  $\vec{\mathcal{G}} = \langle (\delta_\xi, z_\xi, \mathcal{G}_\xi, \sigma_\xi) : \xi < \omega_1 \rangle$  such that

- $\delta_\xi$  is a countable ordinal,
- $z_\xi \in 2^{\mathbb{N}} \cap \mathbf{L}_{\delta_\xi + \omega}$ ,
- $\mathcal{G}_\xi$  is a countable cofinitary group, and
- $\sigma_\xi \in S_\infty$ ,

and so that the following hold for each  $\xi < \omega_1$ :

- (i)  $\delta_\xi$  is the least ordinal  $\delta > \sup_{\nu < \xi} \delta_\nu$  such that  $\delta$  is a limit of limit ordinals and projects to  $\omega$ .
- (ii)  $z_\xi$  is obtained from the canonical surjection  $f$  from  $\omega$  onto  $\mathbf{L}_{\delta_\xi}$  as follows: for  $k \in \mathbb{N}$ ,  $z_\xi(k) = 0 \iff (k = 2^m 3^n \wedge f(m) \in f(n))$ .
- (iii)  $\mathcal{G}_\xi$  is the compositional closure of  $\{\sigma_\nu : \nu < \xi\} \cup \mathcal{G}_0$  in  $S_\infty$ .
- (iv)  $\mathcal{G}_\xi$  is cofinitary and  $\mathcal{G}_\xi \in \mathbf{L}_{\delta_\xi}$ .
- (v)  $\sigma_\xi = \sigma_G$ , where  $G$  is the unique  $(\mathbf{L}_{\delta_\xi + \omega}, \mathbb{Q}_{\mathcal{G}_\xi}^{\delta_\xi})$ -generic obtained by hitting dense subsets of  $\mathbb{Q}_{\mathcal{G}_\xi}^{\delta_\xi}$  in the order in which they are enumerated by the canonical surjection from  $\omega$  onto  $\mathbf{L}_{\delta_\xi + \omega}$ .
- (vi) Letting  $\delta(\xi) = (\sup\{\delta_\nu + \omega + \omega : \nu < \xi\}) + \omega$  for limit  $\xi$  and  $\delta(\xi) = \delta_{\xi-1} + \omega + \omega$  for successor  $\xi < \omega_1$  we have

$$\langle (\delta_\nu, z_\nu, \mathcal{G}_\nu, \sigma_\nu) : \nu < \xi \rangle \in \mathbf{L}_{\delta(\xi)}.$$

Obtaining such a sequence is straight-forward: Having constructed  $\vec{\mathcal{G}} \upharpoonright \xi$ , (i) determines  $\delta_\xi$  from  $\langle \delta_\nu : \nu < \xi \rangle$  and (ii) determines  $z_\xi \in \mathbf{L}_{\delta_\xi + \omega}$ . Note that  $M_{z_\xi} = \mathbf{L}_{\delta_\xi}$ , and that by Fact 4.5,  $\delta_\xi + \omega$  projects to  $\omega$ , as well.

Noting  $\mathcal{G}_0 \in \mathbf{L}_{\delta_0}$  for the case  $\xi = 0$ , assume (iv) by induction. Then (v) uniquely determines  $\sigma_\xi$  from  $\delta_\xi, z_\xi$  and  $\mathcal{G}_\xi$ . That  $\mathcal{G}_{\xi+1}$  is a cofinitary group follows by Lemma 3.16. Moreover, as the canonical surjection from  $\omega$  to  $\mathbf{L}_{\delta_\xi + \omega}$  is in an element of  $\mathbf{L}_{\delta_\xi + \omega + \omega}$ , so are  $\sigma_\xi$  and  $\mathcal{G}_{\xi+1}$ . It follows that  $\mathcal{G}_{\xi+1} \in \mathbf{L}_{\delta_{\xi+1}}$  and the induction hypothesis (iv) is verified for  $\xi + 1$ .

It remains to verify (iv) for limit  $\xi < \omega_1$ ; for this, first show (vi). Observe that the inductive definition of  $\vec{\mathcal{G}}$  by (i), (ii) and (v) involves only concepts which are absolute for initial segments of the  $\mathbf{L}$ -hierarchy. In fact, let  $\Psi(x)$  be the formula that states that  $x$  is the sequence obtained using said inductive definition restricted to ordinals in  $\text{dom}(x)$  and that there is no largest limit ordinal in the universe; then  $\vec{g}$  is equal to some initial segment of  $\vec{\mathcal{G}}$  if and only if for any ordinal  $\delta$  such that  $\vec{g} \in \mathbf{L}_\delta$  we have  $\mathbf{L}_\delta \models \Psi(\vec{g})$ . With this, (vi) follows by induction.

In particular, it follows that  $\mathcal{G}_\xi \in \mathbf{L}_{\delta_\xi}$  for limit  $\xi < \omega_1$ , verifying (iv). This finishes the inductive construction of  $\vec{\mathcal{G}}$ .

We have just shown the following fact, crucial to the proof that  $\mathcal{G}$  is  $\Pi_1^1$ :

**FACT 4.6.** *There is a formula  $\Psi(x)$  such that  $\vec{g}$  is equal to an initial segment  $\langle (\mathcal{G}_\nu, \delta_\nu, z_\nu, \sigma_\nu) : \nu < \xi \rangle$  of  $\vec{\mathcal{G}}$  if and only if there is  $\delta < \omega_1$  such that  $\vec{g} \in \mathbf{L}_\delta$  and  $\mathbf{L}_\delta \models \Psi(\vec{g})$ .*

Finally, we let

$$\mathcal{G} = \bigcup_{\xi < \omega_1} \mathcal{G}_\xi,$$

which is a cofinitary group by (iv) above.



To see that  $\mathcal{G}$  is  $\Pi_1^1$ , we first show a weaker statement:

CLAIM 4.7. *As a subset of  $\mathbb{N}^{\mathbb{N}}$ ,  $\mathcal{G}$  is  $\Sigma_2^1$ .*

PROOF OF CLAIM. By [23] and [5] (or by [14]), we may fix a sentence  $\Lambda$  such that whenever  $M$  is transitive and  $\langle M, \in \rangle \models \Lambda$ , we have  $M = \mathbf{L}_\delta$  for some  $\delta$ . By the Fact 4.6,  $\sigma \in \mathcal{G}$  if and only if

$$\begin{aligned} &\text{there exists a countable transitive set } M \text{ s.t. } \langle M, \in \rangle \models \Lambda \text{ and} \\ &\quad \text{for some } \vec{g} \in M \text{ of the form } \vec{g} = \langle \langle \delta'_\xi, z'_\xi, \mathcal{G}'_\xi, \sigma'_\xi \rangle : \xi \leq v \rangle \\ &\quad \text{we have } \sigma = w[\sigma'_v] \text{ for some } w \in W_{\mathcal{G}'_v, X} \text{ and } \langle M, \in \rangle \models \Psi(\vec{g}). \end{aligned} \tag{4.1}$$

By Fact 4.2, (4.1) is equivalent to a  $\Sigma_2^1$  formula

$$(\exists y \in 2^{\mathbb{N}}) \Phi(y, \sigma), \tag{4.2}$$

where  $\Phi(y, \sigma)$  is a  $\Pi_1^1$  formula expressing that  $E_y$  is well-founded and extensional and  $M_y$  witnesses (4.1), i.e.,  $\langle \omega, E_y \rangle \models \Lambda$  and for some  $n, m \in \omega$ ,  $\pi_y(m) = \sigma$  and  $\langle \omega, E_y \rangle \models \text{“}\Psi(n) \wedge n = \langle \langle \delta'_\xi, z'_\xi, \mathcal{G}'_\xi, \sigma'_\xi \rangle : \xi \leq v \rangle \wedge \sigma'_v = m\text{”}$ . ¬Claim 4.7.

LEMMA 4.8. *In fact,  $\mathcal{G}$  is  $\Pi_1^1$  (as a subset of  $\mathbb{N}^{\mathbb{N}}$ ).*

PROOF OF LEMMA. The proof rests on the fact that  $y$  as in (4.2) can be found effectively in  $\sigma$ . First fix some arbitrary  $h^* \in \mathcal{G}_0 \setminus \{\text{id}_{\mathbb{N}}\}$  (this is not necessary but serves to better emphasize the structure of the argument) noting that by assumption  $h^*$  is arithmetic.

For  $\sigma \in \mathcal{G} \setminus \mathcal{G}_0$ , by (4) we may take  $M$  in (4.1) to be  $\mathbf{L}_{\delta_v + \omega + \omega}$  where  $v$  is least such that  $\sigma \in \mathcal{G}_{v+1}$ . Thus by Fact 4.3,  $y$  in (4.2) can be chosen so that  $y \in \Delta_1^1(z_v)$ . We show that  $z_v$  is arithmetic in  $\sigma$ : Letting  $\sigma = w[\sigma_G]$ , this holds by construction if  $w$  has no conjugate proper subword. Otherwise,  $h^*w$  has no conjugate proper subword and so  $z_v$  is computable in  $h^*\sigma$  (any arithmetic element of  $\mathcal{G}_v \setminus \{\text{id}_{\mathbb{N}}\}$  would do here). In either case we obtain ‘ $\Rightarrow$ ’ in the following (‘ $\Leftarrow$ ’ is obvious):

$$\sigma \in \mathcal{G} \iff (\exists y \in \Delta_1^1(\sigma)) \Phi(y, \sigma).$$

By Fact 4.4, the right-hand side can be rendered as a  $\Pi_1^1$  formula, proving the Lemma. ¬Lemma 4.8.

Since any  $\tau \in S_\infty$  appears in some  $\mathbf{L}_{\delta_\xi}$ , maximality of  $\mathcal{G}$  follows from (III) of Theorem 3.16, and (v) above. In fact, we show the stronger statement:

LEMMA 4.9.  *$\mathcal{G}$  is Cohen-indestructible.*

PROOF OF LEMMA. Towards a contradiction, fix a  $\mathbb{C}$ -name  $\dot{\tau}$  and  $p \in \mathbb{C}$  such that  $p \Vdash_{\mathbb{C}} \langle \check{\mathcal{G}}, \dot{\tau} \rangle$  is cofinitary.

For each  $n \in \mathbb{N}$ , pick  $\vec{p}_n = \langle p_k^n : k \in \omega \rangle$  such that  $\{p_k^n : k \in \omega\}$  is a maximal antichain in  $\mathbb{C}$  of conditions deciding  $\dot{\tau}(n)$ : i.e., we may find, for each  $n \in \mathbb{N}$ , a sequence  $\vec{m}_n = \langle m_k^n : k \in \omega \rangle$  such that for each  $k$ ,

$$p_k^n \Vdash \dot{\tau}(\check{n}) = \check{m}_k^n.$$

Further, find  $\zeta < \omega_1$  such that for each  $n \in \mathbb{N}$ ,  $\{\vec{p}_n, \vec{m}_n\} \subseteq L_{\delta_\zeta}$ . We may assume (by strengthening  $p$  if necessary) that there is  $N$  such that

$$p \Vdash_{\mathbb{C}} |\{n \in \mathbb{N} : \check{\sigma}_\zeta(n) = \dot{\tau}(n)\}| = \check{N}. \tag{4.3}$$

By repeatedly using Lemma 3.17, we may extend any given condition in  $\mathbb{Q}_{\mathcal{G}_\xi}^{\check{z}_\xi}$  to a condition  $q \in \mathbb{Q}_{\mathcal{G}_\xi}^{\check{z}_\xi}$  such that for some  $p' \in \mathbb{C}$  stronger than  $p$  and for some set  $Z \subseteq \text{dom}(s^q)$  of size  $N + 1$  we have

$$(\forall n \in Z) p' \Vdash_{\mathbb{C}} \dot{\tau}(\check{n}) = \check{s}^q(\check{n}).$$

Just as in Lemma 3.16, we must circumvent the use of the forcing relation: Since each  $\bar{p}_n$  enumerates a maximal antichain, by extending  $p'$  finitely many times, we can assume that for every  $n \in Z$ ,  $p' \leq_{\mathbb{C}} p_k^n$  for some  $k$  such that  $m_k^n = s^q(n)$ .

Let  $D$  be the set of  $q \in \mathbb{Q}_{\mathcal{G}_\xi}^{\check{z}_\xi}$  such that for some  $p' \in \mathbb{C}$  stronger than  $p$  and for some set  $Z \subseteq \text{dom}(s^q)$  of size  $N + 1$  we have that for every  $n \in Z$ ,  $p' \leq_{\mathbb{C}} p_k^n$  for some  $k$  such that  $m_k^n = s^q(n)$ .

We have just shown that  $D$  is dense  $\mathbb{Q}_{\mathcal{G}_\xi}^{\check{z}_\xi}$ ; as  $D \in \mathbf{L}_{\delta_\xi + \omega}$ , the generic which gave rise to  $\sigma_\xi$  meets  $D$  and we conclude that for some  $p' \in \mathbb{C}$  stronger than  $p$  and for some set  $Z \subseteq \mathbb{N}$  of size  $N + 1$  we have

$$(\forall n \in Z) p' \Vdash_{\mathbb{C}} \dot{\tau}(\check{n}) = \check{\sigma}_\xi(\check{n}),$$

contradicting (4.3); thus,  $\mathcal{G}$  is Cohen-indestructible. ⊥<sub>Lemma 4.9.</sub>  
⊥

As an immediate corollary, we obtain Theorem 1.2:

**COROLLARY 4.10.** *The existence of a constructible  $\Pi_1^1$  maximal cofinitary group of size  $\omega_1$  is consistent with arbitrarily large continuum (relative to ZFC).*

**PROOF.** Note that if  $\Psi(x)$  is the  $\Pi_1^1$  formula which we constructed in Claim 4.8 and which defines membership in  $\mathcal{G}$ , then the following holds in  $\mathbf{V}$

$$(\forall \sigma \in \mathbb{N}^{\mathbb{N}}) \Psi(\sigma) \iff \sigma \in \mathbf{L} \wedge \Psi(\sigma)^{\mathbf{L}}.$$

This is because  $\Psi(\sigma)$  is easily seen to imply  $\sigma \in \mathbf{L}$ . ⊥

**§5. Questions.** Considering the many known models where some inequality holds between  $\alpha_g$  and another cardinal invariant of the continuum, the methods developed in the present paper suggest to consider the following definable analogue:

For  $\Gamma$  an arbitrary pointclass, let  $\alpha_g(\Gamma)$  be the least cardinal  $\kappa$  such that there is a mcg  $\mathcal{G} \in \Gamma$  of size  $\kappa$ ; if there is no mcg  $\mathcal{G} \in \Gamma$ , let  $\alpha_g(\Gamma) = \infty$ .

**QUESTION 5.1.** *Which inequalities involving  $\alpha_g(\Pi_1^1)$  and other cardinal invariants of the continuum can be shown to hold consistently?*

An earlier version of this paper asked whether there can be a Borel mcg; as mentioned in Section 1 this question has since been answered in the affirmative (see [11], in which the authors also state as their goal to show in a future paper that a closed mcg exists; also compare [25] which shows that there is a closed so-called maximal eventually different family).

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