Proceedings of the Royal Society of Edinburgh, **150**, 2905–2936, *2020*
DOI:10.1017/prm.2019.51

Existence of multi-travelling waves in capillary fluids

Corentin Audiard
Laboratoire Jacques-Louis Lions (LJLL), Sorbonne université, CNRS, Université de Paris, F75005, Paris, France
(corentin.audiard@upmc.fr)

(MS Received 3 September 2018; Accepted 22 July 2019)

We prove the existence of multi-soliton and kink-multi-soliton solutions of the Euler–Korteweg system in dimension one. Such solutions behave asymptotically in time like several travelling waves far away from each other. A kink is a travelling wave with different limits at $\pm\infty$. The main assumption is the linear stability of the solitons, and we prove that this assumption is satisfied at least in the transonic limit. The proof relies on a classical approach based on energy estimates and a compactness argument.

Keywords: traveling waves; Euler–Korteweg; stability

2010 *Mathematics subject classification:* Primary: 35Q35
Secondary: 35C07; 35Q53; 35Q31; 35B35

1. Introduction

The Euler–Korteweg model The Euler–Korteweg equations read

$$\begin{cases}\partial_t\rho+\partial_x(\rho v) &= 0,\\ \partial_t v+v\partial_x v+g'(\rho)\partial_x\rho &= \partial_x\left(K(\rho)\partial_x^2\rho+\frac{1}{2}K'(\rho)(\partial_x\rho)^2\right),\end{cases}\quad (x,t)\in\mathbb{R}\times\mathbb{R}^+. \tag{1.1}$$

They are a modification of the usual Euler equations that model capillary forces in non viscous fluids. The function $K(\rho)$ is supposed to be smooth $\mathbb{R}^{+*}\to\mathbb{R}^{+*}$. In some relevant cases it is not bounded near 0, in particular for $K=1/\rho$ there exists a change of variable, the Madelung transform, that converts at least formally solutions of (1.1) into solutions of the nonlinear Schrödinger equation (for details on this interesting feature see the review article [8]).

There is a formally conserved energy

$$H[\rho,v]=\int_{\mathbb{R}}\frac{1}{2}(\rho v^2+K(\rho)(\partial_x\rho)^2)+G(\rho)\,\mathrm{d}x,$$

where G is a primitive of g, and under appropriate functional settings, denoting δH the variational derivative of H, $V=\begin{pmatrix}\rho\\ v\end{pmatrix}$ (1.1) can be viewed as a

Hamiltonian system

$$\partial_t V = J\partial_x \delta H[V], \quad \text{with } J = \begin{pmatrix} 0 & -1 & -1 & 0 \end{pmatrix}. \tag{1.2}$$

(1.1) also has a formally conserved momentum $P(\rho, v) = \int_{\mathbb{R}} \rho v$, whose conservation is related to the identity $\delta P[V] = -JV$. Although formal these identities are used at least as notations in this article.

Due to the intricate quasilinear nature of (1.1), only local well-posedness was obtained so far in dimension one and, even for data close to a constant state, global well-posedness is an open problem (on well-posedness and stability in larger dimension, see also [**2**, **6**, **11**]).

It was proved in [**4**] by ODE technics that (1.1) admits travelling waves as solutions, namely solutions of the form $(\rho, v)(x - ct)$. There exists two classes of travelling waves: those such that $\lim_{+\infty}(\rho, v)(z) \neq \lim_{-\infty}(\rho, v)(z)$ are labelled as kinks, while solitons satisfy $\lim_{+\infty}(\rho, v)(z) = \lim_{-\infty}(\rho, v)(z)$. No quantity is assumed to be zero at infinity.

Both types of travelling waves are physically relevant, especially kinks are supposed to model phase transition in capillary fluid (e.g. liquid to vapour). While kinks are known to be always stable, solitons are not and a (conditional) stability criterion in the spirit of [**12**] was derived in [**4**].

This article is devoted to a related, yet different issue : the existence of multi-travelling waves, i.e. solutions that decouple as $t \to +\infty$ to a sum of travelling waves.

Multi-travelling waves in the literature The existence of multiple travelling waves is now a classical topic. While the first examples came from the field of integrable equations (e.g., see the pioneering work of Zakharov–Shabat [**18**]), flexible and powerful methods have since been developed to tackle non-integrable equation. In particular considerable progress was achieved for the KdV and nonlinear Schrödinger equations over the last twenty years.

In the framework of the nonlinear Schrödinger equation we refer to the work of Martel, Merle and coauthors [**10**, **15**], in particular based on the use of modulation parameters and a compactness argument, see also Le Coz and Tsai [**13**] for an approach based on dispersive estimates. To the best of our knowledge, the inclusion of a kink in the asymptotic profile is a rather rare feature in the field of multiple travelling waves, to the noticeable exception of the work of Le Coz–Tsai [**13**], see also [**14**] in higher dimension.

All those results share the fact that they are more conveniently applied to equations that have a 'good' well-posedness theory available (existence of global solutions in not too restrictive spaces). As such, their adaptation to quasilinear systems like the Euler–Korteweg system raises some difficulties. To explain it roughly, a key step of the compactness argument of Martel–Merle requires the existence of solutions for $t \in \mathbb{R}^+$, while the well-posedness theory for the Euler–Korteweg system only allows existence in finite time.

In the context of the water waves, this difficulty was overcome by Ming–Rousset–Tzvetkov [**16**] with the construction of global *approximate* solutions with some fast decay in time on the approximation error. This is also, for the main lines, the

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

approach that we follow here. More details are included in the paragraph 'scheme of proof' page 2908

The travelling waves A short description of the construction of travelling waves is provided in the appendix, for more details we refer to [4]. Their main features are the following:

- A travelling wave is a solution of (1.1) of the form $V(x-ct)$, c is its speed.
- All the travelling waves that we consider are smooth and bounded. Their derivatives are exponentially decreasing at $\pm\infty$. Consequently all travelling waves have limits at $\pm\infty$.
- Kinks are travelling waves that have different endstates at $\pm\infty$, say $(\rho_\pm, u_\pm)$. The function ρ is monotonous.
- Solitons are travelling waves with same endstates at $\pm\infty$. ρ changes monotony only once, when it reaches its unique extremum. In the appendix we only deal with the case where this extremum is a minimum. By analogy with the Schrödinger equation (see e.g. [3]), we label such solutions bubbles (note that for fluids such solutions correspond to a negative bump in the density, therefore the word bubble is consistent).
- For a fixed endstate solitons can be smoothly parametrized by their speed.

Main result Let $(V^{c_j})_{1\leqslant j\leqslant n}$ be travelling waves with ordered speeds $c_j < c_{j+1}$. We assume that all V^{c_j} are stable, and we consider one of the following two cases:

- V^{c_1} is a kink, and V^{c_j} are solitons with $\forall\, j \geqslant 2,\ \lim_{\pm\infty} V^{c_j} = \lim_{+\infty} V^{c_1}$.
- V^{c_j} are all solitons with the same endstate.

For V^c a soliton, we denote $U^c = V^c - \begin{pmatrix}\rho_+\\ v_+\end{pmatrix}$, and define the rescaled momentum

$$P(V^c) = P(\rho^c, v^c) = \int_{\mathbb{R}} (\rho^c - \rho_+)(v^c - v_+)\,\mathrm{d}x. \tag{1.3}$$

We assume that the solitons are stable in the following sense

$$\text{Stability condition: } \frac{\mathrm{d}}{\mathrm{d}c}\int (\rho^c - \rho_+)(v^c - v_+)\,\mathrm{d}x < 0.$$

We refer to the appendix A for the proof that our conditions can be met, where we also show that this stability criterion coincides with the one derived in [4].

We define the multi-soliton

$$S(x,t) = V^{c_1}(x - c_1 t) + \sum_{k=2}^{n} U^{c_k}\left(x - c_k t - \sum_{j=2}^{k} A_j\right),$$

$A_j \geqslant A$ a large constant to choose later. Our main result is that there exists a solution which converges to S as $t \to \infty$ (see § 2 for the definition of $\mathcal{H}^n$).

Theorem 1.1. *For A_0 large enough, $A \geqslant A_0$, and $n \geqslant 3$, there exists a global solution of* (1.1) *such that $V - S \in C(\mathbb{R}^+, \mathcal{H}^{2n})$ and*

$$\lim_{t\to\infty} \|V(t) - S(t)\|_{\mathcal{H}^{2n}} \to 0.$$

Remark 1.2. It may be tempting to think that theorem 1.1 hints towards the stability of multi-solitons. This is not correct as the solution constructed is quite peculiar: it is a pure soliton solution with no dispersive part. For NLS multi-solitons have been constructed in cases where each soliton is unstable[**9**, **10**].

Note however that in the case of the Gross–Pitaevskii equation, whose hydrodynamics formulation is a special case of (1.1) with $K = 1/\rho$, $g = \rho - 1$, nonlinear stability of multi-solitons was obtained by Béthuel–Gravejat–Smets [**7**]. It is expectable that a similar result holds (at least in some regime) for (1.1), however, due to the lack of global well-posedness, going beyond *conditional stability*, that is stability until blow-up, requires significant new ideas.

Remark 1.3. It is apparent from the proof that multiple travelling waves can be constructed in more complicated configurations, such as kink-soliton-kink, soliton-kink-kink etc. We chose not to aim at such results to keep a reasonably simple proof, and because configurations with multiple kinks and stable solitons might require very exotic pressure laws to exist.

Scheme of proof As in [**16**], the key is to construct an approximate solution V^a to (1.1) that satisfies

$$\partial_t V^a - J\partial_x \delta H[V^a] = f^a,$$

which is defined globally, converges as $t \to \infty$ to the multi-soliton, and such that the error term f^a decays rapidly in time. Once V^a is constructed, we use the local well-posedness theory from [**5**] (with some modified energy estimates) to construct a sequence of exact solutions V^k close to V^a, defined on $[0, k]$ with $V^k(k) = U^a(k)$. A compactness argument then provides a global solution of the Euler–Korteweg system which converges at $t \to \infty$ to the multi-soliton.

The construction of V^a is quite intricate, it requires fine resolvent estimates on the linearized operator $J\partial_x \delta^2 H[S]$, building upon a spectral decomposition of $\delta^2 H[V^{c_j}] - c_j \delta P$. This spectral decomposition sharpens some arguments from [**4**]. Once these estimates are proved, the approximate solution is constructed by a Newton iteration method[1]. As for the Taylor approach the loss of derivatives caused by the iterations is not an issue because travelling waves are smooth with a fast (exponential) decay of their derivatives.

Plan of the article In § 2 we define some notations and functional settings. The energy estimate for (1.1) are proved in § 3.

[1]Note that the method followed in [**16**] is rather based on a Taylor expansion of the linearized operator. The iterative method seems more natural as it only requires a first-order expansion of $\delta H[V]$ instead of a high-order Taylor expansion, this allows to keep a better track of the constants. Although it is not important here, this requires also less smoothness of the functional H

Section 4 is the core of the article. We first give a convenient spectral decomposition of the operator $\delta^2 E[V^{c_j}] - c_j\delta^2 P$. We deduce some estimates on the flow of $J\partial_x\delta^2 H$ that are not useful for this paper, but contain most of the ideas for the much more technical estimates on the flow of $J\partial_x\delta^2 H[S]$.

With these estimates at hand we construct in § 5 an approximate solution by following Newton's iteration method. The compactness argument that provides the multi-soliton solution is detailed in § 6.

Finally, as the existence of a 'kink-stable solitons' configuration is not obvious, we prove it in the appendix. The appendix is also used to recall how kinks and solitons for (1.1) are constructed.

2. Notations, functional spaces

Reference state of a solution Any solution V of (1.1) that we consider is of the form

$$V = V_{\text{ref}} + U, \tag{2.1}$$

where U vanishes at infinity, V_{ref} is a reference state which is a smooth function with finite limit at $\pm\infty$, and for any $k + j \geqslant 1$, $\partial_x^k\partial_t^j V_{\text{ref}}$ decays exponentially at $\pm\infty$.

The notation $V = V_{\text{ref}} + U$ will be used without explanation when the context is clear, in particular for a soliton of endstate (ρ_+, v_+) we always take $V_{\text{ref}} = (\rho_+, v_+)^t$. If any sub/superscript is present we denote $V^a = V^a_{\text{ref}} + U^a$, $V_j = V_{\text{ref},j} + U_j$ etc.

We always denote $V = \begin{pmatrix}\rho\\ v\end{pmatrix}$, $U = \begin{pmatrix}r\\ u\end{pmatrix}$, and similarly for V^a, U^a ...

Symbols and conventions of computation The constant C in inequalities $A \leqslant CB$ changes from line to line. Depending on the context, they are allowed to depend on some quantities, but for conciseness this dependency is not explicited. For example, when proving $A \leqslant C(\|u\|_\infty)B$, we write freely $|uv| \leqslant C|v|$.

The inequality $A \lesssim B$ means $A \leqslant CB$ for some constant $C > 0$, where the previous rule applies to C.

The L^2 scalar product for real vector valued functions is denoted $\langle\,\cdot,\,\cdot\rangle$.

Sobolev spaces Even for functions of one variable, we use the notation $u' = \partial_x u$. H^n is the usual L^2 based Sobolev space

$$H^n = \{u \in \mathcal{S}' :\ \forall\, 0 \leqslant k \leqslant n,\ \partial_x^k u \in L^2\},\ \|u\|^2_{H^n} = \sum_0^n \|\partial_x^k u\|^2_{L^2}.$$

We denote $\mathcal{C}^n_b$ the set of n times differentiable functions that are bounded as well as their derivatives. For a vector valued distribution $U = \begin{pmatrix}r\\ u\end{pmatrix}$, we also define

$$\|U\|^2_{\mathcal{H}^n} = \|r\|^2_{H^{n+1}} + \|u\|^2_{H^n}, \quad \text{and } \|U\|_{X^n} = \|U\|_{\mathcal{H}^{n+1}} + \|\partial_t U\|_{\mathcal{H}^n}.$$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

We have the interpolation property

$$\forall\, 0 \leqslant k \leqslant n,\ \|u\|_{H^k} \leqslant \|u\|_{L^2}^{1-k/n} \|u\|_{H^n}^{k/n},$$

the continuous embedding $H^n \subset \mathcal{C}_b^{n-1}$. For $n \geqslant 1$, H^n is a Banach algebra.

The following composition rules hold for $a \in \mathcal{C}_b^n + H^n,\ u \in H^n$ and F smooth on $\mathrm{Im}(a),\ \mathrm{Im}(a+u)$:

$$\|F(a+u) - F(a)\|_{H^n} \leqslant C(\|a\|_{\mathcal{C}_b^n + H^n} + \|u\|_{H^n})\|u\|_{H^n}, \tag{2.2}$$
$$\text{in particular if } F(0) = 0,\ \|F(u)\|_{H^n} \leqslant C(\|u\|_{H^n})\|u\|_{H^n}.$$

A similar second-order rule holds

$$\|F(a+u) - F(a) - uF'(a)\|_{H^n} \leqslant C(\|a\|_{\mathcal{C}_b^n + H^n} + \|u\|_{H^n})\|u\|_{H^n}^2. \tag{2.3}$$

Both are consequences of a combination of the Faa Di Bruno formula, Sobolev's embedding, Hölder's inequality and Taylor's formula.

3. Energy estimates

An essential step is to bound the distance between an exact solution and a smoother approximate solution $V^a = (\rho^a, v^a)$ satisfying

$$\partial_t V^a = J\partial_x \delta H[V^a] + f^a, \quad \text{for some remainder } f^a.$$

Due to the quasi-linear nature of the system the flow map is (probably) not Lipschitz even in high regularity Sobolev spaces, nevertheless Lipschitz bounds with harmless loss of derivatives on V^a can be obtained.

Energy estimates were obtained by Benzoni *et al.* [**5**] thanks to a change of variable (initially due to F. Coquel), and this section is actually more or less contained in [**5**]. Let us shortly describe the argument : if (ρ, v) is a smooth solution of (1.1) without vacuum, $n \geqslant 2$, set $w = \sqrt{K/\rho}\partial_x \rho$, and $z = v + iw$. Then z satisfies

$$\partial_t z + v\partial_x z + iw\partial_x z + i\partial_x(a\partial_x z) + g'(\rho)\partial_x \rho = 0, \tag{3.1}$$

with $a(\rho) = \sqrt{\rho K}$. This equation has a nice structure : $i\partial_x a \partial_x$ is antisymmetric, $v\partial_x$ too up to zero-order terms, $g'\partial_x \rho$ is of order zero since w is a derivative of ρ. The only bad term $iw\partial_x z$ is dealt with thanks to a gauge method.

A few preliminary notations : for $V = (\rho, v)$, solution of (1.1) and $V^a = (\rho^a, v^a)$ an approximate solution, we denote z and z^a the associated new variables. We assume that $V^a = V_{\mathrm{ref}} + U^a$ and $V = V_{\mathrm{ref}} + U$ (same reference state) so that $V - V^a = U - U^a$.

Generically for F a function of ρ we denote $\Delta F = F(\rho) - F(\rho^a)$, for F a function of v, $\Delta F = F(v) - F(v^a)$, etc. The gauge function of order n is $\varphi_n(\rho) := a^{n/2}\sqrt{\rho}$ and the modified norm

$$\widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}} := \|\Delta \rho\|_{L^2} + \|\sqrt{\rho}\Delta z\|_{L^2} + \|\varphi_n \partial_x^{2n} \Delta z\|_{L^2}.$$

This notation is quite incorrect as the 'norm' depends on V in a nonlinear way. Nevertheless, using the computation rules 2.2 and with constants depending continuously on $\|V\|_{\mathcal{H}^{2n}} + \|V^a\|_{\mathcal{H}^{2n}} + \|\rho + 1/\rho\|_{L^\infty} + \|\rho^a + 1/\rho^a\|_{L^\infty}$, we have

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

$\widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}} \sim \|\Delta\rho\|_{L^2} + \|\Delta z\|_{H^{2n}}$, and $\|(\Delta\rho, \Delta v)\|_{\mathcal{H}^{2n}} \sim \|\Delta z\|_{H^{2n}} + \|\Delta\rho\|_{L^2}$, so

$$\widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}} \sim \|\Delta V\|_{\mathcal{H}^{2n}}. \tag{3.2}$$

The main result is the following:

PROPOSITION 3.1. *Let V^a_{ref} be a reference state smooth, bounded with its derivatives rapidly decaying at infinity. Let $V^a = (\rho^a, v^a) = V^a_{\mathrm{ref}} + U^a$ be an approximate solution of* (1.1)

$$\partial_t V^a = J\partial_x \delta H[V^a] + f^a,$$

and V a solution of (1.1) *such that $U = V - V^a_{\mathrm{ref}} \in \mathcal{H}^{2n}$, $n \geqslant 1$. Then the estimate holds*

$$\left|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}}\right| \leqslant C\left(\|U\|_{\mathcal{H}^{2n}} + \|U^a\|_{\mathcal{H}^{2n+2}} + \left\|1/\rho + 1/\rho^a\right\|_{L^\infty}\right) \times (\widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}} + \|f^a\|_{\mathcal{H}^{2n}}),$$

with C a continuous, positive nondecreasing function $\mathbb{R}^+ \to \mathbb{R}^{+*}$.

Proof. We recall the convention of § 2; the hidden constants in $\lesssim, \sim$ are as the function C of the statement. If $f^a = (f^a_1, f^a_2)$, the equations on $\Delta\rho, \Delta z$ are

$$\begin{cases} \partial_t\Delta\rho + \partial_x(\Delta\rho v + \rho^a\Delta v) & = \quad f^a_1, \\ \partial_t\Delta z + v\partial_x\Delta z + \Delta v\partial_x z^a + iw\partial_x\Delta z + i\Delta w\partial_x z^a & \\ + \partial_x\Delta g + i\partial_x(a\partial_x\Delta z + \Delta a\partial_x z^a) & = \quad i\sqrt{\frac{K}{\rho^a}}\partial_x f^a_1 + f^a_2 := h^a. \end{cases}$$

$\|\Delta\rho\|_{L^2}$ is estimated by multiplying the first equation by $\Delta\rho$ and space integration

$$\begin{aligned} \left|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\rho\|^2_{L^2}\right| &\leqslant \|\partial_x\Delta\rho\|_{L^2}(\|\Delta\rho\|_{L^2}\|v\|_{L^\infty} + \|\rho^a\|_{L^\infty}\|\Delta v\|_{L^2}) + \|\Delta\rho\|_{L^2}\|f^a_1\|_{L^2} \\ &\lesssim (\|\Delta V\|_{\mathcal{H}^{2n}} + \|f^a\|^2_{\mathcal{H}^{2n}})\|\Delta V\|_{\mathcal{H}^{2n}}. \end{aligned} \tag{3.3}$$

The main issue is thus to control Δz. Let us first note that

$$\|h^a\|^2_{H^{2n}} \lesssim \|\partial_x f^a_1\|^2_{H^{2n}} + \|f^a_2\|^2_{H^{2n}} \leqslant \|f^a\|^2_{\mathcal{H}^{2n}}. \tag{3.4}$$

For $0 \leqslant k \leqslant n$, we apply $a^k\sqrt{\rho}\partial_x^{2k} := \varphi_k\partial_x^{2k}$ to the second equation. Denoting $\Delta z_k = \varphi_k\partial_x^{2k}\Delta z$ we find after some commutations

$$\begin{aligned} &\partial_t\Delta z_k + v\partial_x\Delta z_k + i\partial_x(a\partial_x\Delta z_k) + i(\varphi_k w + 2k\partial_x(a)\varphi_k - 2a\partial_x\varphi_k)\partial_x^{2k+1}\Delta z \\ &\quad + i\varphi_k\partial_x^{2k+1}(\Delta a\partial_x z^a) = R + \varphi_k\partial_x^{2k}h^a \end{aligned} \tag{3.5}$$

where R is a remainder term containing derivatives of Δz of order at most $2k$, and derivatives of z^a of order at most $2k+2$

$$\begin{aligned} R = {} & [z\partial_x, \varphi_k\partial_x^{2k}]\Delta z - i\varphi_k\partial_x^{2k}(\Delta v\partial_x z^a + i\Delta w\partial_x z^a) \\ & + i[\partial_x(a\partial_x\cdot), \varphi_k\partial_x^{2k}]\Delta z + 2ik\partial_x(a)\varphi_k\partial_x^{2k+1}\Delta z \\ & - \varphi_k\partial_x^{2k+1}(\Delta g) + i\varphi_k\partial_x^{2k+1}(\Delta a\partial_x z^a) - \varphi'_k\partial_x(\rho v)\partial_x^{2k}\Delta z. \end{aligned}$$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

By construction,

$$\begin{aligned}\varphi_k w + 2k\partial_x(a)\varphi_k - 2a\partial_x\varphi_k &= a^k\sqrt{K} + 2k\sqrt{\rho K}\\ &= \left(\sqrt{\frac{K}{\rho}}a^k\sqrt{\rho} + 2ka^k a'\sqrt{\rho} - 2ka^k a'\sqrt{\rho} - \frac{a^{k+1}}{\sqrt{\rho}}\right)\partial_x\rho\\ &= 0.\end{aligned}$$

Therefore, multiplying (3.5) by Δz_k and integrating,

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta z_k\|_{L^2}^2\right| \lesssim (\|v\|_{L^\infty}\|\Delta z_k\|_{L^2} + \|R\|_{L^2} + C\|z^a\|_{H^{2k+2}}\|\Delta z\|_{H^{2k}}\\ + \|\varphi_k\partial_x^{2k}h^a\|_{L^2})\|\Delta z_k\|_{L^2}.\end{aligned} \tag{3.6}$$

Using § 2.2 and Faa di Bruno formula $\|R\|_{L^2} \lesssim \|\Delta V\|_{\mathcal{H}^{2k}}$, moreover from (3.4) $\|\varphi_k\partial_x^{2k}h^a\|_{L^2} \lesssim \|f^a\|_{\mathcal{H}^{2k}}$, (3.6) rewrites

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta z_k\|_{L^2}^2\right| \lesssim \|\Delta V\|_{\mathcal{H}^{2k}}(\|\Delta V\|_{\mathcal{H}^{2k}} + +\|f^a\|_{\mathcal{H}^{2k}}).$$

Thanks to (3.2), $\|\Delta V\|_{\mathcal{H}^{2k}} \lesssim \widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}}$. Adding estimates (3.3) and (3.6) with $k = 0$ and $k = n$ we conclude

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}}^2\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta\rho\|_{L^2}^2 + \|\sqrt{\rho}\Delta z\|_{L^2}^2 + \|\varphi_n\partial_x^{2n}\Delta z\|_{L^2}^2)\right|\\ &\lesssim \widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}}(\widetilde{\|}\Delta V\|_{\mathcal{H}^{2n}} + \|f^a\|_{\mathcal{H}^{2n}}).\end{aligned}$$

□

4. Linear estimates

This section is devoted to estimates in $\mathcal{H}^n$ on the flows associated to $J\partial_x\delta^2 H[V^c]$ (V^c a travelling wave) and $J\partial_x\delta^2 H[S]$.

We recall the notation $\delta H[V] = \begin{pmatrix}-K\partial_x^2\rho - \frac{1}{2}K'(\partial_x\rho)^2 + g(\rho) + v^2/2\\ \rho v\end{pmatrix}$, in the same spirit

$$\delta^2 H[V]\begin{pmatrix}r\\ u\end{pmatrix} = \begin{pmatrix}(-K'(\rho)\partial_x^2\rho - \frac{1}{2}K''(\rho)(\partial_x\rho)^2 + g'(\rho))r - \partial_x(K\partial_x r) + uv\\ \rho u + rv\end{pmatrix}, \tag{4.1}$$

or in a matrix operator notation

$$\delta^2 H[V] = \begin{pmatrix}(-K'(\rho)\partial_x^2\rho - \frac{1}{2}K''(\rho)(\partial_x\rho)^2 + g'(\rho)) - \partial_x(K\partial_x\cdot) & v\\ v & \rho\end{pmatrix}.$$

As can be expected, $\delta^2 H$ is a symmetric operator. Recalling $\langle\cdot,\cdot\rangle$ is the L^2 scalar product, we shall use frequently that

$$\begin{aligned}\left\langle\delta^2 H[V]\begin{pmatrix}r\\ u\end{pmatrix}, \begin{pmatrix}r_1\\ u_1\end{pmatrix}\right\rangle = \int_{\mathbb{R}}\left(-K'(\rho)\partial_x^2\rho - \frac{1}{2}K''(\rho)(\partial_x\rho)^2 + g'(\rho)\right)rr_1\\ + K\partial_x r\partial_x r_1 + vur_1 + vru_1 + \rho uu_1\,\mathrm{d}x,\end{aligned} \tag{4.2}$$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

so that $\delta^2 H$ induces a continuous bilinear form on $\mathcal{H}^0$ if V is smooth enough.

4.1. Linear stability of a travelling wave

The case of a kink Let V^c be a kink of speed c. The system (1.1) linearized near V^c reads after the change of variables $x \to x - ct$

$$\partial_t U(x,t) = J\partial_x \delta^2 (H - cP)[V^c(x)]U(x,t).$$

We define a modified energy functional $E = H - cP$. According to lemma 3 in [4] (see also remark 2 in this reference) kinks are always stable in the following sense:

LEMMA 4.1. *For any $U \in \mathcal{H}$ there exists a unique orthogonal decomposition*

$$U = \alpha \partial_x V^c + W,\ \partial_x V^c \in Ker(\delta^2 E) \quad \text{and } \langle \delta^2 E[V^c]U, U\rangle \gtrsim \|W\|^2_{\mathcal{H}^0}. \tag{4.3}$$

For the link between linear stability and $\delta^2 E$ being definite positive, see e.g. theorem 3.1 of Pego–Weinstein [17].

The case of a soliton We consider a branch of solitons V^c. As it is more convenient here to work on U^c, we denote $P[U^c] = \int r^c u^c$ and abusively $\delta H[U^c] = \delta H[V^c]$. We recall (see (1.2)) that $J = \begin{pmatrix} 0 & -1 - 1 & 0 \end{pmatrix}$ so that $\delta^2 P = -J$. From $-c\partial_x U^c = J\partial_x \delta H[U^c]$ we have a number of useful identities

$$\forall U, V,\ \delta P[U] = \delta^2 P[V]U = -JU, \tag{4.4}$$

$$(\delta H - c\delta P)[U^c] := \delta E[U^c] \text{ is constant}, \tag{4.5}$$

$$\delta^2 E[U^c] \cdot \partial_x U^c = 0 \text{ (differentiation of (4.5) in } x), \tag{4.6}$$

$$\delta^2 E[U^c]\partial_c U^c - \delta P[U^c] = 0 \text{ (differentiation in } c) \tag{4.7}$$

$$\Leftrightarrow \delta^2 E[U^c]\partial_c U^c = -JU^c. \tag{4.8}$$

Stability assumption. We assume that U_c is stable, namely it satisfies :

$$\frac{\mathrm{d}P[U^s]}{\mathrm{d}s}|_{s=c} < 0.$$

(see the appendix for a link with the so-called Boussinesq momentum of instability). This also implies that $\partial_c U^c$ is an unstable direction in the sense that

$$\langle \delta^2 E[U^c]\partial_c U^c, \partial_c U^c\rangle = \langle \delta P[U^c], \partial_c U^c\rangle = \frac{\mathrm{d}}{\mathrm{d}c} P[U^c] < 0.$$

Let us first recall a result from [4] (proved for the formulation of the Euler–Korteweg system in Lagrangian coordinates, see also [1] appendix B for a proof in Eulerian coordinates).

LEMMA 4.2. *Under the stability assumption, the operator $\delta^2 E[U^c]$ is block diagonal on the orthogonal decomposition $\mathcal{H} = vect(U_-) \oplus_\perp vect(\partial_x U^c) \oplus_\perp \mathcal{G}$, where $\partial_x U^c$*

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

spans the kernel of $\delta^2 E$, U_- is a normalized eigenvector associated to the unique negative eigenvalue, and

$$\forall W \in \mathcal{G},\ \langle \delta^2 E[V^c]W, W\rangle \gtrsim \|W\|^2_{\mathcal{H}^0}.$$

LEMMA 4.3. *For $U \in \mathcal{H}$, there exists a unique orthogonal decomposition*

$$U = \alpha\delta P[U^c] + \beta\partial_x U^c + W,\ W \in (\delta P[U^c],\ \partial_x U^c)^\perp \quad \textit{and}$$
$$\langle \delta^2 EU, U\rangle \gtrsim \|W\|^2_{\mathcal{H}^0} - C\alpha^2. \tag{4.9}$$

REMARK 4.4. To underline the unity between this decomposition and (4.3) in the case of a kink, let us point out that since the reference state is constant

$$\partial_x V^c = \partial_x U^c.$$

Proof. The momentum being invariant by translation, $\langle \delta P[U^c], \partial_x U^c\rangle = 0$ and according to (4.6), $\partial_x U^c \in \mathrm{Ker}(\delta^2 E)$. Therefore the only thing to prove is $\langle \delta^2 EW, W\rangle \gtrsim \|W\|^2_{\mathcal{H}^0}$. By contradiction we assume the existence of $W \in (\delta P[U^c], \partial_x U^c)^\perp \setminus \{0\}$ such that $\langle \delta^2 EW, W\rangle \leqslant 0$, then for any $(\alpha, \beta, \gamma) \in \mathbb{R}^3$, using identities (4.6), (4.7)

$$\begin{aligned}
&\langle \delta^2 E(\alpha\partial_c U^c + \beta\partial_x U^c + \gamma W),\ \alpha\partial_c U_c + \beta\partial_x U^c + \gamma W\rangle\\
&\quad = \langle \delta^2 E(\alpha\partial_c U^c + \gamma W),\ \alpha\partial_c U^c + \gamma W\rangle\\
&\quad = \alpha^2\langle \delta P[U^c], \partial_c U^c\rangle + \gamma^2\langle \delta^2 EW, W\rangle + 2\alpha\gamma\langle \delta P[U^c], W\rangle\\
&\quad = \alpha^2\langle \delta P[U^c], \partial_c U^c\rangle + \gamma^2\langle \delta^2 EW, W\rangle,
\end{aligned}$$

by orthogonality. By definition, $\langle \partial_x U^c, W\rangle = 0$ and $\langle \partial_c U^c, \delta P[U^c]\rangle < 0$ therefore $(\partial_c U^c, \partial_x U^c, W)$ is free. But $\delta^2 E$ is thus nonpositive on a dimension 3 space, which contradicts lemma 4.2. As a consequence

$$\forall W \in (\delta P[U^c],\ \partial_x U^c)^\perp \setminus \{0\},\ \langle \delta^2 EW, W\rangle > 0. \tag{4.10}$$

The improved inequality $\langle \delta^2 EW, W\rangle \gtrsim \|W\|^2_{\mathcal{H}^0}$ follows from a (probably standard) compactness argument : consider a sequence V_n of $(\delta P[U^c],\ \partial_x U^c)^\perp$ such that $\|V_n\|_{\mathcal{H}^0} = 1$ and $\langle \delta^2 EV_n, V_n\rangle \to 0$. Using lemma 4.2 we write $V_n = \alpha_n U_- + \beta_n\partial_x U^c + W_n$, $W_n \in \mathcal{G}$. By assumption, $\beta_n = 0$ and up to an extraction, $\alpha_n \to_{n\to\infty} \alpha$, $W_n \rightharpoonup W \in \mathcal{G}$. Denoting $-\lambda_-$ the negative eigenvalue,

$$\begin{aligned}
&1 = \|V_n\|^2_{\mathcal{H}} = \alpha_n^2 + \|W_n\|^2_{\mathcal{H}^0} = \alpha^2 + \lim_n \|W_n\|^2_{\mathcal{H}},\\
&0 = \lim_{n\to\infty}\langle \delta^2 EV_n, V_n\rangle = \lim_{n\to\infty} -\lambda_-\alpha_n^2 + \langle \delta^2 EW_n, W_n\rangle\\
&\quad \geqslant -\lambda_-\alpha^2 + c\liminf_n \|W_n\|^2_{\mathcal{H}^0}.
\end{aligned}$$

This implies $\alpha \neq 0$. Let $V = \alpha U_- + W$, then by weak convergence

$$\langle \delta^2 EV, V\rangle = -\lambda_-\alpha^2 + \langle \delta^2 EW, W\rangle \leqslant -\lambda_-\alpha^2 + \lim_n\langle \delta^2 EW_n, W_n\rangle = 0,$$

but since V is the weak limit of V_n, it belongs to $(\delta P[U^c], \partial_x U^c)^\perp$, and (4.10) implies $V = 0$, which contradicts $\alpha \neq 0$. $\square$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

As a consequence, we deduce the following linear stability result, whose proof will be a guideline for the computations in the multi-soliton case.

THEOREM 4.5. *Under the stability assumption* (4.1), *the solution of*

$$\begin{cases}\partial_t U(x,t) = J\partial_x \delta^2 H[U^c(x-ct)]U(x,t),\\ U|_{t=0} = U_0,\end{cases}$$

satisfies for $t \in \mathbb{R}$

$$\|U(t)\|_{\mathcal{H}^0} \lesssim (1+|t|)\|U_0\|_{\mathcal{H}^0}.$$

Proof. For conciseness we write $\delta^2 H$ for $\delta^2 H[U^c]$. Using $\delta^2 P = -J$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle(\delta^2 H - c\delta^2 P)U, U\rangle &= \langle[\partial_t, \delta^2 H]U, U\rangle + 2\langle\delta^2 HJ\partial_x\delta^2 HU, U\rangle + c\langle[\partial_x, \delta^2 H]U, U\rangle\\ &= \langle[\partial_t + c\partial_x, \delta^2 H]U, U\rangle.\end{aligned}$$

Since the coefficients of the operator $\delta^2 H$ only depend on $x - ct$, $[\partial_t + c\partial_x, \delta^2 H] = 0$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle(\delta^2 H - c\delta^2 P)U, U\rangle = 0. \tag{4.11}$$

We use the decomposition (4.9) for the solution $U(t) = \alpha(t)\delta P[U^c(x-ct)] + \beta(t)\partial_x U^c + W(t)$. Since $\partial_x U^c \in \mathrm{Ker}(\delta^2 E)$

$$\alpha'(t) = \frac{\langle J\partial_x\delta^2 HU, \delta^2 PU^c\rangle + \langle U, -c\partial_x\delta^2 PU^c\rangle}{\langle\delta P[U_c], \delta P[u_c]\rangle} = \frac{\langle U, (\delta^2 H - c\delta^2 P)\partial_x U^c\rangle}{\langle\delta P[U_c], \delta P[u_c]\rangle} = 0.$$

By the conservation (4.11) and the continuity of $\delta^2 H$ as a bilinear form (4.2)

$$\begin{aligned}\langle\delta^2 EU(0), U(0)\rangle &= \langle\delta^2 EU(t), U(t)\rangle\\ &= \langle\delta^2 E(\alpha\delta P[U^c] + W), \alpha\delta P[U^c] + W\rangle \gtrsim \|W\|_{\mathcal{H}^0}^2 - C\alpha^2\\ &\Rightarrow \|W(t)\|_{\mathcal{H}^0}^2 \lesssim \alpha(0)^2 + \|U(0)\|_{\mathcal{H}^0}^2.\end{aligned}$$

Moreover $|\alpha(0)| = \langle U(0), \delta P[U^c]\rangle/\|\delta P[U^c]\|_{\mathcal{H}}^2\rangle \lesssim \|U(0)\|_{\mathcal{H}^0}$. The last term is estimated thanks to the bounds on α, V :

$$|\beta'(t)| = \left|\frac{d}{dt}\frac{\langle U(t), \partial_x U^c\rangle}{\|\partial_x U^c\|_{L^2}^2}\right| = \left|\frac{\langle J\partial_x\delta^2 E(\alpha\delta P[U^c] + V), \partial_x U^c\rangle}{\|\partial_x U^c\|_{L^2}^2}\right| \lesssim \|U(0)\|_{L^2},$$

and by integration, $|\beta(t)| \leqslant |\beta(0)| + |t|\|U(0)\|_{\mathcal{H}^0}$. □

REMARK 4.6. The linear growth in time is unavoidable, indeed we have

$$J\partial_x\delta^2 E\partial_c U^c = \partial_x J\delta P[U^c] = \partial_x U^c \in \mathrm{Ker}(J\partial_x\delta^2 E),$$

therefore $(\partial_c U^c, \partial_x U^c)$ is associated to a Jordan block of the eigenvalue 0 of $J\partial_x\delta^2 E$.

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

Remark 4.7. Of course to study the stability of a single soliton it is much more natural to do the Galilean change of variable $y = x - ct$ and consider the *autonomous* linear problem $\partial_t U = (\delta^2 H - c\delta^2 P)[U^c(y)]U$. The proof of theorem 4.5 is simplified in this frame. Nevertheless, when considering multi-soliton such a change variable is not available and this first simple case is a good warm up before the more technical computations of § 4.2.

4.2. Stability near multiple travelling waves

We recall that the multi-soliton is defined as

$$S(x,t) = V^{c_1}(x - c_1 t) + \sum_{k=2}^{n} U^{c_k}\left(x - c_k t - \sum_{j=2}^{k} A_j\right) = V_1 + \sum_{2}^{n} U_k,\ A_j \geqslant A,$$

where $V^{c_k} = V^{c_k}_{\text{ref}} + U^{c_k}$ are travelling waves, $c_1 < c_2 \cdots < c_n$. We have exponential decay

$$\forall\, p \geqslant 0,\ 1 \leqslant k \leqslant n,\ \exists\, \alpha > 0 :\ |\partial_x^p U_k(x,t)| \lesssim \mathrm{e}^{-\alpha|x - c_k t - \sum_2^k A_j|}. \tag{4.12}$$

The aim of this section is to get bounds on the flow associated to $J\partial_x \delta^2 H[S+\eta]$ where η is a small perturbation of limited smoothness that depends on x and t. When there is no ambiguity, we write $\delta^2 H$ for $\delta^2 H[S+\eta]$ and

$$\delta^2 E_k := \delta^2 H[V_k] - c_k \delta^2 P.$$

Lemma 4.8. *For $s \geqslant 0$, let U solve*

$$\begin{cases} \partial_t U = J\partial_x \delta^2 H[S+\eta]U, \\ U|_{t=s} = U_0, \end{cases} \tag{4.13}$$

with η a smooth perturbation. There exist C and ε_0 such that for $\varepsilon := 1/A + \|\eta\|_{X^1} \leqslant \varepsilon_0$,

$$\forall\, t \geqslant 0,\ \|U(t)\|_{\mathcal{H}^0} \leqslant C(1 + |t-s|)\mathrm{e}^{C\varepsilon^{1/4}|t-s|}\|U_0\|_{\mathcal{H}^0}.$$

Remark 4.9. The estimate is not true for $t \leqslant 0$ as the key argument is that the distance between the travelling waves must be (in some sense) larger than A.

The proof requires some preliminaries that will be used through the section. Let $2c_0 = \inf_{j<k} c_k - c_j$, and for $1 \leqslant k < n$, $c_{k+1/2} = (c_k + c_{k+1})/2$. We first define localizing functions : pick a nondecreasing $\chi \in C^\infty(\mathbb{R})$, $\operatorname{supp}(\chi) = [0,\infty]$, $\chi|_{[1/2,\infty)} = 1, 0 < \chi < 1$ on $(0, 1/2)$, and set

$$\varphi_1(x,t) = 1 - \chi\left(\frac{x - c_{1+1/2}t - A_2/2}{A_2}\right),$$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

$$\forall\, 2 \leqslant k < n,\ \varphi_k(x,t) = \chi\left(\frac{x - c_{k-1/2}t - \left(\sum_2^k A_j - A_k/2\right)}{A_k}\right) - \chi\left(\frac{x - c_{k+1/2}t - \left(\sum_2^k A_j + A_{k+1}/2\right)}{A_{k+1}}\right),$$

$$\varphi_n(x,t) = \chi\left(\frac{x - c_{n-1/2}t - (\sum_2^n A_j - A_n/2)}{A_n}\right).$$

It is easily seen that

$$\operatorname{supp}(\varphi_1) = (-\infty, c_{3/2}t + A_2],$$

$$\forall\, 2 \leqslant k \leqslant n-1,\ \operatorname{supp}(\varphi_k) = \left[c_{k-1/2}t + \sum_2^k A_j - \frac{A_k}{2}, c_{k+1/2}t + \sum_2^{k+1} A_j\right],$$

$$\operatorname{supp}(\varphi_n) = \left[c_{n-1/2}t + \sum_2^n A_j - A_n/2, \infty\right),$$

and $\sum_{k=1}^n \varphi_k^2 \geqslant c > 0$ for some constant independent of x. The localizing functions are then defined as

$$\chi_j = \frac{\varphi_j}{\sqrt{\sum_1^n \varphi_j^2}} \quad \text{so that } \sum \chi_j^2 = 1. \tag{4.14}$$

Note that φ_j and χ_j have same support. Thanks to (4.12) we have the following estimates, uniformly for A large

$$\|\partial_x^j \partial_t^k \chi_j\|_{L^\infty_{x,t}} = O(1/A^{k+j}) \quad \text{(slow variation)}, \tag{4.15}$$

$$\forall\, j \neq k,\ (p,q) \in \mathbb{N}^2,\ r \geqslant 1,\ \exists\, \alpha > 0 : \|\partial_x^p \partial_t^q U_k\|_{L^r_x(\operatorname{supp}(\chi_j))} = O(\mathrm{e}^{-\alpha c_0 t}/A), \tag{4.16}$$

$$\text{if } (p,q) \neq (0,0),\ r \geqslant 1,\ \|\partial_x^p \partial_t^q V_k\|_{L^r_x(\operatorname{supp}(\chi_j))} = O(\mathrm{e}^{-\alpha c_0 t}/A), \tag{4.17}$$

(support decorrelation).

Proof of lemma 4.8. In the spirit of the proof of theorem 4.5 we define the modified energy

$$\widetilde{E}(t) = \langle \delta^2 H[S+\eta]U(t), U(t)\rangle - \sum_{k=1}^n c_k \langle \delta^2 P \chi_k U(t), \chi_k U(t)\rangle. \tag{4.18}$$

Similarly to theorem 4.5, the proof has three steps : (1) Control of $\mathrm{d}\widetilde{E}/\mathrm{d}t$, (2) control of $\|U(t)\|^2_{\mathcal{H}^0}$ by $\widetilde{E}$ up to a finite number of parameters, (3) Control of these parameters.

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

Step 1*: Control of* $\mathrm{d}\widetilde{E}/\mathrm{d}t$. From basic computations, using $\delta^2 P = -J, J^2 = I$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E} &= \langle([\partial_t, \delta^2 H] + \delta^2 H J\partial_x\delta^2 H)U, U\rangle + \sum_{k=1}^{n} 2c_k\langle\chi_k' JU, \chi_k U\rangle \\
&\quad + \sum_{k=1}^{n} c_k(\langle\chi_k\partial_x\delta^2 HU, \chi_k U\rangle) + \big(\langle\chi_k U, \chi_k\partial_x\delta^2 HU\rangle\big) \\
&= \langle[\partial_t, \delta^2 H]U, U\rangle + \sum_{k=1}^{n} 2c_k\langle\chi_k' JU, \chi_k U\rangle \\
&\quad + \sum_{k=1}^{n} c_k(\langle[\chi_k^2, \partial_x\delta^2 H]U, U\rangle + \langle[\partial_x, \delta^2 H]\chi_k^2 U, U\rangle) \\
&= \sum_{k=1}^{n}\langle([\partial_t, \delta^2 H] + c_k[\partial_x, \delta^2 H])\chi_k^2 U, U\rangle \\
&\quad + c_k(2\langle\chi_k' JU, \chi_k U\rangle + c_k\langle[\chi_k^2, \partial_x\delta^2 H]U, U\rangle) \\
&= \sum_{k=1}^{n} C_{1,k}(t) + C_{2,k}(t) + C_{3,k}(t).
\end{aligned}$$

We first point out that X^1 controls the L^∞ norm, therefore for $\|\eta\|_{X^1}$ small enough the density of $S + \eta$ remains bounded away from 0 and the computations rules in (2.2), (2.3) can be applied.

$C_{2,k}$ and $C_{3,k}$ are not difficult to control : let us write $[\chi_k^2, \partial_x\delta^2 H] = (L_{i,j})_{1\leqslant i,j\leqslant 2}$ as a matrix of operators, $S + \eta = \begin{pmatrix}\rho_1\\ v_1\end{pmatrix}$ and detail the estimate for $\langle L_{1,1}r, r\rangle$

$$\begin{aligned}
\langle L_{1,1}r, r\rangle &= \langle[\chi_k^2, \partial_x((g' - K''(\partial_x\rho_1)^2 - K'\partial_x^2\rho_1) - \partial_x K\partial_x)]r,\ r\rangle \\
&= -\langle 2\chi_k\partial_x(\chi_k)(g' - K''(\partial_x\rho_1)^2 - K'\partial_x^2\rho_1)r, r\rangle \qquad (4.19)\\
&\quad - 2\langle\chi_k\partial_x\chi_k\partial_x(K\partial_x r), r\rangle - \langle[\chi_k^2, \partial_x K\partial_x]r, \partial_x r\rangle. \qquad (4.20)
\end{aligned}$$

Using the Sobolev estimates (2.2) and (4.15), we find (4.19) $\lesssim \|r\|_{H^1}^2/A$, the second one is estimated by an integration by part

$$|-2\langle\chi_k\partial_x\chi_k\partial_x(K\partial_x r), r\rangle| \leqslant \int_{\mathbb{R}} |K\partial_x r\partial_x(2\chi_k\partial_x\chi_k\, r)|\,\mathrm{d}x \lesssim \frac{1}{A}\|r(t)\|_{H^1}^2.$$

The last term in (4.20) is estimated similarly with the explicit commutator formula $[\chi_k^2, \partial_x K\partial_x]r = -2\partial_x(\chi_k^2)K\partial_x r - \partial_x(\chi_k^2)\partial_x(Kr)$. Similar computations eventually lead to

$$|C_{2,k} + C_{3,k}| \lesssim \frac{1}{A}\|U\|_{\mathcal{H}^0}^2. \qquad (4.21)$$

To bound $C_{1,k}$ we introduce the (bilinear) operator $\delta^3 H$ such that

$$[\partial_t, \delta^2 H[S + \eta]]U = \delta^3 H[S + \eta](U, \partial_t(S + \eta)). \qquad (4.22)$$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

This is merely a convenient notation, as writing $S+\eta = \begin{pmatrix}\rho_1\\ v_1\end{pmatrix}$, $\delta^3 H(\cdot, \partial_t(S+\eta))$ is explicitly

$$\delta^3 H[S+\eta](\cdot, \partial_t(S+\eta)) = \begin{pmatrix}\mathcal{M}_t & \partial_t v_1\\ \partial_t v_1 & \partial_t \rho_1\end{pmatrix},$$

$$\text{with } \mathcal{M}_{\text{tr}} = \left(g''\partial_t\rho_1 - \frac{K'''\partial_t\rho_1(\partial_x\rho_1)^2 + 2K''\partial_x\rho_1\partial_{xt}\rho_1}{2}\right) r \\ -(K''\partial_t\rho_1\partial_x^2\rho_1 + K'\partial_{xxt}\rho_1)r - \partial_x(K'\partial_t\rho_1\partial_x r), \quad (4.23)$$

and we use the same notation for $[\partial_x, \delta^2 H] := \delta^3 H[S](\cdot, \partial_x S)$. We can thus rewrite using $S = V_1 + \sum_{j=2}^n U_j$, with $\partial_t V_1 = -c_1\partial_x V_1$, $\partial_t U_j = -c_j\partial_x U_j$, for $k>1$

$$C_{1,k}(t) = \langle \delta^3 H(U, \partial_t S + c_k\partial_x S)\chi_k^2 U, U\rangle + \langle \delta^3 H(U, \partial_t\eta + c_k\partial_x\eta)\chi_k^2 U, U\rangle \\ = \langle \delta^3 H(U, -c_1\partial_x V_1 - \sum_{j\neq k} c_j\partial_x U_j)\chi_k^2 U, U\rangle + \langle \delta^3 H(U, \partial_t\eta + c_k\partial_x\eta)\chi_k^2 U, U\rangle,$$

and if $k=1$

$$C_{1,1}(t) = \left\langle \delta^3 H(U, -\sum_{j=2}^n c_j\partial_x U_j)\chi_1^2 U, U\right\rangle + \langle \delta^3 H(U, \partial_t\eta + c_1\partial_x\eta)\chi_1^2 U, U\rangle.$$

Now using the support decorrelation property (4.16) and the explicit form (4.23) of $\delta^3 H$ we obtain in both cases

$$|C_{1,k}(t)| \lesssim \frac{\mathrm{e}^{-\alpha c_0 t}}{A}\|U(t)\|^2_{\mathcal{H}^0} + (\|\eta(t)\|_{\mathcal{H}^2} + \|\partial_t\eta(t)\|_{\mathcal{H}^1})\|U(t)\|^2_{\mathcal{H}^0}. \quad (4.24)$$

Adding this estimate with (4.21) gives

$$\left|\widetilde{E}'(t)\right| \lesssim \varepsilon\|U(t)\|^2_{\mathcal{H}^0}. \quad (4.25)$$

Step 2: Lower bounds for $\widetilde{E}$. The key here is the decompositions (4.3) and (4.9). For $1 \leqslant k \leqslant n$ we set $\chi_k U(t) = \alpha_k(t)\delta P[U_k] + \beta_k(t)\partial_x V_k + W_k(t)$, with the convention that if V_1 is a kink, $\alpha_1 = 0$ (in this case the relevant decomposition is (4.3)). Using the translation invariance of the L^2 norm, the lower bound in (4.9) gives for some $m, C > 0$

$$\forall\, 1 \leqslant k \leqslant n,\ \langle \delta^2 E[U_k]\chi_k U, \chi_k U\rangle \geqslant m\|W_k\|^2_{\mathcal{H}^0} - C\alpha_k^2.$$

According to this, we split $\widetilde{E}$ as a sum of localized terms and remainders:

$$\widetilde{E} = \langle \delta^2 H[S+\eta]U, U\rangle - \sum_{k=1}^n c_k\langle \delta^2 P\chi_k U, \chi_k U\rangle \\ = \sum_{k=1}^n \langle \delta^2 H[S+\eta]\chi_k^2 U, U\rangle - c_k\langle \delta^2 P\chi_k U, \chi_k U\rangle,$$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

so commuting $\delta^2 H$ and χ_k we obtain

$$\begin{aligned}\widetilde{E}(t) &= \sum_{k=1}^{n} \langle \delta^2 E_k \chi_k U, \chi_k U\rangle + \langle (\delta^2 H[S+\eta] - \delta^2 H[U_k])\chi_k U, \chi_k U\rangle \\ &\quad + \langle [\delta^2 H[S+\eta], \chi_k]\chi_k U, U\rangle \\ &\geqslant \sum_{k=1}^{n} m\|W_k(t)\|^2_{\mathcal{H}^0} - C\alpha_k^2(t) + \langle (\delta^2 H[S+\eta] - \delta^2 H[V_k])\chi_k U, \chi_k U\rangle \\ &\quad + \langle [\delta^2 H, \chi_k]\chi_k U, U\rangle. \end{aligned} \tag{4.26}$$

The last term is estimated as in (4.20),

$$\big|\langle [\delta^2 H, \chi_k]\chi_k U(t), U(t)\rangle\big| \lesssim \varepsilon \|U(t)\|^2_{\mathcal{H}^0}. \tag{4.27}$$

Thanks to the support decorrelation (4.16) and calculus rules (2.2), one can check

$$\langle (\delta^2 H[S+\eta] - \delta^2 H[V_k])\chi_k U, \chi_k U\rangle \lesssim \left(\frac{\mathrm{e}^{-\alpha c_0 t}}{A} + \|\eta\|_{\mathcal{H}^2}\right)\|U(t)\|^2_{\mathcal{H}^0}. \tag{4.28}$$

For example, the term associated to $\partial_x(K\partial_x r)$ is controlled as follows

$$\begin{aligned}&|\langle \partial_x((K(S+\eta) - K(V_k))\partial_x(\chi_k r)),\, \chi_k r\rangle| \\ &\quad \leqslant |\|K(S+\eta) - K(V_k)\|_{L^\infty(\mathrm{supp}(\chi_k))}\|\partial_x(\chi_k r)\|^2_{L^2} \\ &\quad \lesssim \left(\frac{\mathrm{e}^{-\alpha c_0 t}}{A} + \|\eta(t)\|_{\mathcal{H}^0}\right)\|r(t)\|^2_{H^1}.\end{aligned}$$

Note that $\|U\|^2_{\mathcal{H}^0} \lesssim \sum_1^n \|W_k\|^2_{\mathcal{H}^0} + \alpha_k^2 + \beta_k^2$, so for ε small enough, from (4.26), (4.27),(4.28), there exists constants m, C_0, C_1 (m is not the same as in (4.26)) such that

$$\widetilde{E}(t) \geqslant \sum_{k=1}^{n} m\|W_k(t)\|^2_{\mathcal{H}^0} - C_0\alpha_k^2(t) - C_1\varepsilon\beta_k^2(t). \tag{4.29}$$

Step 3: Control of the parameters. Once more it is a matter of repeating the proof of theorem 4.5 with some commutators. Let us start with $(\alpha_k(t))_{1\leqslant k\leqslant n}$ ($k>2$ when V_1 is a kink):

$$\begin{aligned}\alpha_k'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle \chi_k U, \delta^2 P U_k\rangle}{\|\delta^2 P U_k\|^2_{L^2}} \\ &= \frac{\langle (\partial_t\chi_k)U, \delta^2 PU_k\rangle + \langle \chi_k J\partial_x\delta^2 H[S+\eta]U, \delta^2 PU_k\rangle + \langle \chi_k U, -c_k\delta^2 P\partial_x U_k\rangle}{\|\delta^2 PU_k\|^2_{L^2}} \\ &= \frac{\langle (\partial_t\chi_k)U, \delta^2 PU_k\rangle + \langle [\chi_k, J\partial_x\delta^2 H[S+\eta]]U, \delta^2 PU_k\rangle}{\|\delta^2 PU_k\|^2_{L^2}} \\ &\quad + \frac{\langle J\partial_x(\delta^2 H[S+\eta] - \delta^2 H[V_k])\chi_k U, \delta^2 PU_k\rangle + \langle \chi_k U, \delta^2 E_k\partial_x U_k\rangle}{\|\delta^2 PU_k\|^2_{L^2}}.\end{aligned}$$

There are four terms. The fourth one is actually 0, thanks to identity (4.6). From the same argument as for $C_{2,k}, C_{3,k}$ in (4.20), the first and second ones

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

are $O(\|U(t)\|_{\mathcal{H}^0}/A)$. Using integration by part, the smoothness of U_k and the same argument as for (4.28), the third one is $O(\|U(t)\|_{\mathcal{H}^0}(e^{-\alpha c_0 t}/A + \|\eta\|_{\mathcal{H}^2}))$. To summarize

$$\forall\, 1 \leqslant k \leqslant n,\ t \geqslant 0,\ |\alpha_k'(t)| \lesssim \varepsilon \|U(t)\|_{\mathcal{H}^0}. \tag{4.30}$$

We bound now $\beta_k(t)$:

$$\begin{aligned}\|\partial_x V_k\|_{L^2}^2 \beta_k'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \chi_k U, \partial_x V_k\rangle \\ &= \langle(\partial_t \chi_k)U, \partial_x V_k\rangle + \langle \chi_k J\partial_x \delta^2 HU, \partial_x V_k\rangle + \langle \chi_k U, -c_k \partial_x^2 V_k\rangle \\ &= \langle(\partial_t \chi_k)U, \partial_x U_k\rangle + \langle J\partial_x \delta^2 E_k(\chi_k U), \partial_x V_k\rangle \\ &\quad + \langle \chi_k J\partial_x(\delta^2 H[S+\eta] - \delta^2 H[V_k])U, \partial_x V_k\rangle \\ &\quad + \langle [\chi_k, J\partial_x \delta^2 H[V_k]]U, \partial_x V_k\rangle.\end{aligned}$$

Since $\chi_k U = \alpha_k \delta P[U_k] + \beta_k \partial_x V_k + W_k$, with $\delta^2 E_k \partial_x V_k = 0$, we have

$$|\langle J\partial_x \delta^2 E_k(\chi_k U(t)), \partial_x V_k(t)\rangle| \lesssim |\alpha_k(t)| + \|W_k(t)\|_{\mathcal{H}^0}.$$

The other terms are estimated as for α_k', leading to

$$|\beta_k'(t)| \lesssim |\alpha_k(t)| + \|W_k(t)\|_{\mathcal{H}^0} + \varepsilon\|U(t)\|_{\mathcal{H}^0}. \tag{4.31}$$

Conclusion. Let us rewrite (4.25), (4.30), (4.31) : there exists some $C > 0$ such that for ε small enough

$$\begin{aligned}&|\widetilde{E}'(t)| \leqslant C\varepsilon\|U(t)\|_{\mathcal{H}^0}^2\,\mathrm{d}s,\\ &|\alpha_k'(t)| \leqslant C\varepsilon\|U(t)\|_{\mathcal{H}^0},\\ &|\beta_k'(t)| \leqslant C(|\alpha_k(t)| + \|W_k(t)\|_{\mathcal{H}^0} + \varepsilon\|U(t)\|_{\mathcal{H}^0}).\end{aligned}$$

With the same constants as in (4.29), let $\widehat{E}(t) := \widetilde{E}(t) + \sum_1^n (C_0+m)\alpha_k^2 + \varepsilon^{1/2}\beta_k^2$, then for ε small enough $\widehat{E}(t) \gtrsim \sum_1^n \alpha_k^2 + \|W_k\|_{\mathcal{H}^0}^2 + \varepsilon^{1/2}\beta_k^2$, and

$$\begin{aligned}|\widehat{E}'(t)| &\leqslant \sum_{k=1}^n C\varepsilon(\|W_k\|_{\mathcal{H}^0}^2 + \alpha_k^2 + \beta_k^2) + C\varepsilon^{1/2}|\beta_k|(|\alpha_k| + \|W_k\|_{\mathcal{H}^0} + \varepsilon\|U\|_{\mathcal{H}^0}) \\ &\leqslant C\varepsilon^{1/2}\widetilde{E} + C\varepsilon^{1/2}\left(\frac{\alpha_k^2 + \|W_k\|_{\mathcal{H}^0}^2}{\varepsilon^{1/4}} + \varepsilon^{1/4}\beta_k^2\right) \\ &\leqslant C\varepsilon^{1/4}\widehat{E}(t).\end{aligned}$$

With Gronwall's lemma and thanks to (4.29) we get

$$\sum_{k=1}^n m(\|W_k(t)\|_{\mathcal{H}^0}^2 + \alpha_k^2(t)) + \varepsilon^{1/2}\beta_k^2(t) \leqslant \widehat{E}(t) \leqslant \widehat{E}(s)\,\mathrm{e}^{C\varepsilon^{1/4}|t-s|}.$$

We can assume $\varepsilon \leqslant 1/(4C_1^2)$ so that $\varepsilon^{1/2} - C_1\varepsilon \geqslant \varepsilon^{1/2}/2$, and

$$\sum_{k=1}^n (\|W_k(t)\|_{\mathcal{H}^0}^2 + \alpha_k(t)^2) \lesssim \|U(s)\|_{\mathcal{H}^0}^2\,\mathrm{e}^{C\varepsilon^{1/4}|t-s|} = \|U_0\|_{\mathcal{H}^0}^2\,\mathrm{e}^{C\varepsilon^{1/4}|t-s|}.$$

To bound β_k independently of ε, we plug the estimate above in the differential inequality (4.31)

$$\begin{aligned}\sum_{k=1}^n |\beta_k'(t)| &\leqslant M\left(\sum_{k=1}^n |\alpha_k(t)| + \|W_k(t)\|_{\mathcal{H}^0} + \varepsilon\|U(t)\|_{\mathcal{H}^0}\right)\\ &\leqslant M_1\|U_0\|_{\mathcal{H}^0}\,\mathrm{e}^{C\varepsilon^{1/4}|t-s|/2} + \frac{C\varepsilon^{1/4}}{2}\sum_{k=1}^n |\beta_k(t)|. \end{aligned} \tag{4.32}$$

(for ε small enough so that $M\varepsilon \leqslant C\varepsilon^{1/4}/2$). (4.32) has the form $(\mathrm{e}^{-\delta|t-s|}\varphi(t))' \leqslant Me^{\mu|t-s|}$, so by integration on $[s,t]$

$$\begin{aligned}\sum_{k=1}^n |\beta_k(t)| &\leqslant \sum_{k=1}^n |\beta_k(s)|\mathrm{e}^{C\varepsilon^{1/4}|t-s|/2} + \frac{2M_1\|U_0\|_{\mathcal{H}^0}}{C\varepsilon^{1/4}}(\mathrm{e}^{C\varepsilon^{1/4}|t-s|}-1)\\ &\lesssim \|U_0\|_{\mathcal{H}^0}(1+|t-s|)\mathrm{e}^{C\varepsilon^{1/4}|t-s|}.\end{aligned}$$

Combining this with the estimate on W_k, α_k, we conclude

$$\|U(t)\|_{\mathcal{H}^0} \lesssim \|U_0\|_{\mathcal{H}^0}(1+|t-s|)\,\mathrm{e}^{C\varepsilon^{1/4}|t-s|}. \qquad \square$$

It is useful to restate the result of lemma 4.8 in a slightly more abstract way:

Corollary 4.10. *Let $R_\eta(t,s)$ the resolvent operator associated to $\partial_t U = J\partial_x\delta^2 H[S+\eta]U$. There exists ε_0 and C such that if $1/A + \|\eta\|_{X^1} := \varepsilon \leqslant \varepsilon_0,\ t,s \geqslant 0$:*

$$\|R_\eta(t,s)\|_{\mathcal{L}(\mathcal{H}^0)} \leqslant (1+|t-s|)Ce^{C\varepsilon^{1/4}|t-s|}.$$

We deduce now similar estimates at any level of regularity.

Theorem 4.11 (Higher order estimates). *Let $\varepsilon := 1/A + \|\eta\|_{X^{2n}},\ n \in \mathbb{N}^*$. There exists ε_n, $C_{e,n}$, C_n such that for any $\varepsilon \leqslant \varepsilon_n$, we have the resolvent estimate*

$$\|R_\eta(t,s)\|_{\mathcal{L}(\mathcal{H}^{2n})} \leqslant (1+|t-s|)C_n\mathrm{e}^{C_{e,n}\varepsilon^{1/4}|t-s|}$$

Proof. The Hamiltonian structure is useless here, so we denote for conciseness $L := J\partial_x\delta^2 H[S+\eta]$. As for the energy estimate, an important issue is that $J\partial_x\delta^2 H[S+\eta]$ does not commute with ∂_t, this will be overcome with the same method as for the zero-order estimate. The commutator $[\partial_t, L] = J\partial_x\delta^3 H(\,\cdot\,, \partial_t(S+\eta)) := \delta L(\,\cdot\,, \partial_t(S+\eta))$ is not zero (for the definition of $\delta^3 H$, see (4.22) in the proof of lemma 4.8), thus to get higher order estimates it is more natural to use the

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

operator $\widetilde{L} := L + \sum_{k=1}^n c_k\chi_k^2\partial_x$. Indeed using

$$[L, \widetilde{L}] = \sum_{k=1}^n [L,\ c_k\chi_k^2\partial_x] = \sum_{k=1}^n c_k[L, \chi_k^2]\partial_x - c_k\chi_k^2\delta L(\cdot, \partial_x(S+\eta)),$$

we find for any V

$$\begin{aligned}[\partial_t - L, \widetilde{L}]V = \sum_{k=1}^n \chi_k^2\delta L(V, (\partial_t + c_k\partial_x)(S+\eta)) \\ + 2c_k\chi_k(\partial_t\chi_k)\partial_x V - [L, c_k\chi_k^2]\partial_x V,\end{aligned} \tag{4.33}$$

therefore recalling $\partial_x V_j = \partial_x U_j$ for $j \geqslant 2$

$$\begin{aligned}\partial_t(\widetilde{L}^n U) &= L(\widetilde{L}^n U) + \sum_{q=0}^{n-1} \widetilde{L}^q[\partial_t - L, \widetilde{L}]L^{n-q-1}U \\ &= L(\widetilde{L}^n U) + \sum_{q=0}^{n-1} \widetilde{L}^q \left(\sum_{j\neq k} \chi_k^2\delta L(\cdot, (c_k - c_j)\partial_x V_j)\right. \\ &\quad \left. + \chi_k^2\delta L(\cdot, (\partial_t + c_k\partial_x)\eta)\right) \widetilde{L}^{n-q-1}U \\ &\quad + \sum_{q=0}^{n-1} \widetilde{L}^q \left(\sum_{k=1}^n 2c_k\chi_k(\partial_t\chi_k)\partial_x - [L, c_k\chi_k^2]\partial_x\right) \widetilde{L}^{n-q-1}U \\ &= L(\widetilde{L}^n U) + C_1 + C_2.\end{aligned} \tag{4.34}$$

Let us recall that if $S + \eta = \begin{pmatrix}\rho_1 \\ v_1\end{pmatrix}$, for any $U = (r, u)^t$, we have

$$LV = -\partial_x \begin{pmatrix} u\rho_1 + rv_1 \\ \left(g' - \frac{K''(\partial_x\rho_1)^2}{2} - K'\partial_x^2\rho_1\right) r - \partial_x(K\partial_x r) + uv_1 \end{pmatrix},$$

the first coefficient contains derivatives of r, u, ρ_1, v_1 of order at most 1, the second contains derivatives of v_1, v of order at most 1, and derivatives of ρ_1, l of order at most 3 so using the rules of § 2.2,

$$\forall\, N \geqslant 0,\ \|LV\|_{\mathcal{H}^N} \leqslant C_N(\|\eta\|_{\mathcal{H}^{N+2}})\|V\|_{\mathcal{H}^{N+2}}, \tag{4.35}$$

$$\Rightarrow \|\widetilde{L}V\|_{\mathcal{H}^N} \leqslant M_N(\|\eta\|_{\mathcal{H}^{N+2}})\|V\|_{\mathcal{H}^{N+2}}, \tag{4.36}$$

with C_N, M_N positive locally bounded functions (we recall that S is smooth and is unimportant in the estimates). With this observation, estimate (4.15) and

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

computations similar to (4.20) we deduce

$$\|C_2(t)\|_{\mathcal{H}^0} \lesssim \frac{\|U(t)\|_{\mathcal{H}^{2n}}}{A} \leqslant \varepsilon \|U(t)\|_{\mathcal{H}^{2n}}.$$

Similarly, $C_1(t)$ is estimated thanks to (4.16) as for (4.24),

$$\|C_1(t)\|_{\mathcal{H}^0} \lesssim \left(\frac{\mathrm{e}^{-\alpha c_0 t}}{A} + \|\eta\|_{X^{2n}}\right) \|U(t)\|_{\mathcal{H}^{2n}} \leqslant \varepsilon \|U(t)\|_{\mathcal{H}^{2n}}.$$

Conversely thanks to the interpolation estimate $\|\partial_x^p \varphi\|_{L^2} \leqslant \|\varphi\|_{L^2}^{1-p/q} \|\partial_x^q \varphi\|_{L^2}^{p/q}$, and Young's inequality, $\|\partial_x^p \varphi\|_{L^2} \leqslant C(\varepsilon)\|\varphi\|_{L^2} + \varepsilon \|\partial_x^q \varphi\|_{L^2}$ for any $\varepsilon > 0$, $p < q$, therefore using once more the explicit formula for $\widetilde{L}V$, we obtain

$$\begin{aligned} &\|\widetilde{L}V\|_{\mathcal{H}^0} \gtrsim \|V\|_{\mathcal{H}^2} - C\|V\|_{\mathcal{H}^0}, \quad \text{and by induction} \\ &\|\widetilde{L}^n U\|_{\mathcal{H}^0} \gtrsim \|U\|_{\mathcal{H}^{2n}} - C_n \|U\|_{\mathcal{H}^0}. \end{aligned} \tag{4.37}$$

Equation (4.34) is of the form $\partial_t V = LV + f$. We apply Duhamel's formula and corollary 4.10

$$\begin{aligned} \|\widetilde{L}^n U(t)\|_{\mathcal{H}^0} &\lesssim (1 + |t-s|)\|U_0\|_{\mathcal{H}^{2n}} \mathrm{e}^{C\varepsilon^{1/4}|t-s|} \\ &\quad + \varepsilon \int_s^t (1 + |t-\tau|)\, \mathrm{e}^{C\varepsilon^{1/4}|t-\tau|} \|U(\tau)\|_{\mathcal{H}^{2n}} \,\mathrm{d}s \qquad (4.38) \\ &\lesssim (1 + |t-s|)\|U_0\|_{\mathcal{H}^{2n}}\, \mathrm{e}^{C\varepsilon^{1/4}|t-s|} + \varepsilon^{3/4} \int_s^t \mathrm{e}^{2C\varepsilon^{1/4}|t-\tau|} \|U(\tau)\|_{\mathcal{H}^{2n}} \,\mathrm{d}\tau, \qquad (4.39) \end{aligned}$$

where the last estimate simply follows from $(1+s)\,\mathrm{e}^s \lesssim \mathrm{e}^{2s}$.

We use (4.39), the bound $\|U(t)\|_{\mathcal{H}^0} \leqslant C(1 + |t-s|)\mathrm{e}^{C\varepsilon^{1/4}|t-s|}\|U_0\|_{\mathcal{H}^0}$ of lemma 4.8 and the lower bound (4.37) with Gronwall's lemma

$$\begin{aligned} \|U(t)\|_{\mathcal{H}^{2n}} &\lesssim (1 + |t-s|)\, \mathrm{e}^{C\varepsilon^{1/4}|t-s|} \|U_0\|_{\mathcal{H}^{2n}} \\ &\quad + \varepsilon^{3/4} \int_s^t \mathrm{e}^{2C\varepsilon^{1/4}(t-\tau)} \|U(s)\|_{\mathcal{H}^{2n}} \,\mathrm{d}s, \\ \Rightarrow \|U(t)\|_{\mathcal{H}^{2n}} &\lesssim (1 + |t-s|)\|U_0\|_{\mathcal{H}^{2n}}\, \mathrm{e}^{C(\varepsilon^{1/4} + \varepsilon^{3/4})|t-s|}. \end{aligned}$$

For ε small $\varepsilon^{3/4} + \varepsilon^{1/4} = O(\varepsilon^{1/4})$ and the proof is complete. □

5. Construction of an approximate solution

We construct here an approximate solution V^a close to the multi-soliton and such that the error $\partial_t V^a - J\partial_x \delta H[V^a]$ is rapidly decaying in t. It is done with Newton's algorithm, initialized with S as the first approximate solution.

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

THEOREM 5.1. *For any $n \in \mathbb{N}$, $\varepsilon > 0, C_e > 0$, there exists A_0 such that for $A \geqslant A_0$, there exists $U^a \in L^\infty_t \mathcal{H}^n$, $\alpha > 0$ such that*

$$\forall\, t \geqslant 0,\ \|\partial_t(S+U^a) - J\partial_x \delta H[S+U^a]\|_{\mathcal{H}^n} \leqslant \varepsilon \mathrm{e}^{-C_e t} \quad \textit{and}\ \|U^a(t)\|_{\mathcal{H}^n} \lesssim \frac{\mathrm{e}^{-\alpha c_0 t}}{A}.$$

We recall Newton's algorithm: $S^0 = S,\ f^0 = \partial_t S^0 - J\partial_x \delta H[S^0]$, $\eta^0 = 0$, and recursively

$$\eta^{j+1} \text{ is the solution of } \begin{cases} \partial_t \eta^{j+1} = J\partial_x \delta^2 H[S^j]\eta^{j+1} - f^j, \\ \lim\limits_{t\to\infty} \eta^{j+1} = 0, \end{cases}$$
$$S^{j+1} = S^j + \eta^{j+1},\ f^{j+1} = \partial_t S^{j+1} - J\partial_x \delta H[S^{j+1}].$$

Of course since $\partial_t S^{j+1} - J\partial_x \delta H[S^{j+1}] = J\partial_x \delta H[S^j] + J\partial_x \delta^2 H[S^j]\eta^{j+1} - J\partial_x \delta H[S^{j+1}]$ we need some Taylor expansion estimate :

LEMMA 5.2. *For $V := S + U$, $(U,\eta) \in (\mathcal{H}^{n+2})^2$ such that $\|V\|_{L^\infty} + \|\eta\|_{L^\infty} \leqslant \inf \rho_S/2$ (non vacuum condition), then*

$$\begin{aligned} &\|J\partial_x \delta H[V+\eta] - J\partial_x \delta H[V] - J\partial_x \delta^2 H[V]\eta\|_{\mathcal{H}^n} \\ &\quad \leqslant C(\|\eta\|_{\mathcal{H}^{n+2}} + \|U\|_{\mathcal{H}^{n+2}})\|\eta\|^2_{\mathcal{H}^{n+2}}, \end{aligned}$$

with C continuous.

Proof. We set $V = \begin{pmatrix} \rho \\ v \end{pmatrix}$, $U = \begin{pmatrix} r \\ u \end{pmatrix}$, $\eta = \begin{pmatrix} \gamma \\ \omega \end{pmatrix}$.

Elementary computations lead to

$$\begin{aligned} &J\partial_x \delta H[V+\eta] - J\partial_x \delta H[V] - J\partial_x \delta^2 H[V]\eta \\ &= -\partial_x \begin{pmatrix} \gamma\omega \\ \frac{\omega^2}{2} + (g(\rho+\gamma) - g(\rho) - \gamma g'(\rho)) - (K(\rho+\gamma) - K(\rho))\partial_x^2 \gamma \\ +(K(\rho) + \gamma K'(\rho) - K(\rho+\gamma))\partial_x^2 \rho - (K'(\rho+\gamma) - K'(\rho))\partial_x \rho \partial_x \gamma \\ -\frac{1}{2} K'(\rho+\gamma)(\partial_x \gamma)^2 \end{pmatrix}. \end{aligned}$$

We have using the rules in 2.2

$$\|\partial_x(\gamma\omega)\|_{H^{n+1}} \lesssim \|\gamma\|_{H^{n+2}}\|\omega\|_{H^{n+2}} \leqslant \|\eta\|^2_{\mathcal{H}^{n+2}}.$$

For the second coordinate, we only treat a few terms. Thanks to the smallness assumption in L^∞, $\inf \rho, \inf \rho + \gamma \geqslant \inf \rho_S/2$ so that we can use the composition

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

rules again

$$\|\partial_x((K(\rho+\gamma)-K(\rho))\partial_x^2\gamma)\|_{H^n}$$
$$\lesssim \|K(\rho+\gamma)-K(\rho)\|_{H^{n+1}}\|\gamma\|_{H^{n+3}}\lesssim \|\gamma\|_{H^{n+3}}^2,$$
$$\|\partial_x(K(\rho)+\gamma K'(\rho)-K(\rho+\gamma))\partial_x^2\rho)\|_{H^n}$$
$$\lesssim \|K(\rho)+\gamma K'(\rho)-K(\rho+\gamma)\|_{H^{n+1}}\|r\|_{H^{n+3}}$$
$$\lesssim \|\gamma\|_{H^{n+2}}^2.$$

After similar estimates for the other terms, we end up with

$$\|J\partial_x\delta H[U+\eta]-J\partial_x\delta H[U]-J\partial_x\delta^2H[U]\eta\|_{\mathcal{H}^n}$$
$$\lesssim \|\omega\|_{H^{n+2}}^2+\|\gamma\|_{H^{n+3}}^2=\|\eta\|_{\mathcal{H}^{n+2}}^2. \qquad \square$$

Proof of theorem 5.1. We fix some $k\in\mathbb{N}$ large, and iterate Newton's algorithm k times.

The proof is divided in the following steps : (1) Control of f^0, (2) Control of η^1, (3) Iteration argument.

Step 1: Control of f^0. We use the partition of unity (4.14):

$$\partial_t S-J\partial_x\delta H[S]=\sum_{j,k,\ j\neq k}\chi_j^2J\partial_x\delta H[V_k]+\sum_j\chi_j^2J\partial_x(\delta H[V_j]-\delta H[S]).$$

We set $V_j=\begin{pmatrix}\rho_j\\ v_j\end{pmatrix}$, $S=\begin{pmatrix}\rho_S\\ v_S\end{pmatrix}$, and explicit the second term

$$\delta H[V_j]-\delta H[S]=\begin{pmatrix}-(\rho_S-\rho_j)v_j-\rho_S(v_S-v_j)\\ (u_S-u_j)(u_j+u_S)+(K(\rho_j)-K(S))\partial_x^2\rho_j\\ +K(S)\partial_x^2(\rho_j-S)\\ +\frac12K'(\rho_j)(\partial_x\rho_j)^2-\frac12K'(S)(\partial_xS)^2\end{pmatrix}.$$

Since all terms are smooth and spatially decorrelated, they are small and exponentially decaying in t for any Sobolev norms, for example,

$$\|\chi_j\partial_x((K(\rho_j)-K(\rho_S))\partial_x^2\rho_j)\|_{H^{2n+2k}}$$
$$\lesssim \|\partial_x\rho_j\|_{H^{2n+2(k+1)}}\|K(\rho_j)-K(S)\|_{H^{2n+2k+1}(\mathrm{supp}(\chi_j))}$$
$$\lesssim \frac{\mathrm{e}^{-\alpha c_0t}}{A}.$$

With similar computations, we find the existence of C_0 only depending on S,n,k such that

$$\|f^0\|_{\mathcal{H}^{2n+2k}}\leqslant C_0\frac{\mathrm{e}^{-\alpha c_0t}}{A}.$$

Step 2: Control of η^1 *and* η^2. At this point some care with the constants is required. We denote R_η the resolvent operator associated to $J\partial_x\delta^2H[S+\eta]$. Following the

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

notation and result of theorem 4.11, we set $C_e = \max_{n\leqslant p\leqslant n+k} C_{e,p}$. There exists a constant C_R such that for $1 \leqslant l \leqslant k$, under the conditions

$$C_e(1/A + \|\eta\|_{X^{2n+2l}})^{1/4} \leqslant \alpha c_0/4,\ \ 1/A + \|\eta\|_{X^{2n+2l}} \leqslant \min_{1\leqslant j\leqslant k} \varepsilon_{n+j}, \tag{5.1}$$

and using $te^t \leqslant e^{2t}$ then

$$\|R_\eta(t,s)\|_{\mathcal{L}(\mathcal{H}^{2n+2l})} \leqslant C(1+|t-s|)\, \mathrm{e}^{\alpha c_0|t-s|/4} \leqslant C_R e^{\alpha c_0|t-s|/2}. \tag{5.2}$$

According to lemma 5.2, there exists C_{Tayl} such that for

$$j \leqslant n+k,\ \ \eta \in \mathcal{H}^{2j+2}, U \in \mathcal{H}^{2j+2},\ \ V = S+U,\ \ \|\eta\|_{L^\infty} + \|U\|_{L^\infty} \leqslant \inf S/2, \tag{5.3}$$

$$\Rightarrow \|J\partial_x \delta H[V+\eta] - J\partial_x \delta H[V] - J\partial_x \delta^2 H[V]\eta\|_{\mathcal{H}^{2j}} \leqslant C_{Tayl}\|\eta\|^2_{\mathcal{H}^{2j+2}}. \tag{5.4}$$

With these notations we bound η^1 using the Duhamel formula,

$$\eta^1 = \int_t^\infty R_0(t,s) f^0(s)\,\mathrm{d}s,$$

$$\Rightarrow \|\eta^1\|_{\mathcal{H}^{2n+2k}} \leqslant \int_t^\infty C_0 C_{\mathrm{Re}}^{\alpha c_0/2|t-s|} \frac{\mathrm{e}^{-\alpha c_0 s}}{A}\,\mathrm{d}s = \frac{2C_0C_R}{A\alpha c_0}\,\mathrm{e}^{-\alpha c_0 t}.$$

We define $\delta = 2C_0C_R/(A\alpha c_0)$, which can be as small as needed.

To bound $f^1 = J\partial_x\delta H[S] + J\partial_x\delta H[S]\eta^1 - J\partial_x\delta H[S+\eta^1]$ we can use estimate (5.4) (note that up to decreasing $\min_{1\leqslant j\leqslant k}\varepsilon_{n+j}$, condition (5.1) is stronger than (5.3))

$$\|f^1\|_{\mathcal{H}^{2n+2(k-1)}} \leqslant C_{Tayl}\delta^2\,\mathrm{e}^{-2\alpha c_0 t}.$$

Next to use the resolvent estimate (5.2), we need to bound $\|\eta^1\|_{X^{2n+2(k-1)}}$. This is done thanks to a general estimate : if $\partial_t\eta = J\partial_x\delta H[S+U]\eta + f$, then using (4.35)

$$\forall\, N \geqslant 2, \|\partial_t\eta\|_{\mathcal{H}^{N-2}} \leqslant C_{X,N}(\|U\|_{\mathcal{H}^N})\|\eta\|_{\mathcal{H}^N} + \|f\|_{\mathcal{H}^{N-2}}. \tag{5.5}$$

We define $C_X = \max_{1\leqslant j\leqslant k} C_{X,2n+2j}(\varepsilon_{n+j})$. Since $\partial_t\eta^1 = J\partial_x\delta^2 H[S]\eta^1 - f^0$, (5.5) gives

$$\|\partial_t\eta^1\|_{\mathcal{H}^{2n+2(k-1)}} \leqslant C_X\|\eta^1\|_{\mathcal{H}^{2n+2k}} + \frac{C_0\,\mathrm{e}^{-\alpha c_0 t}}{A} \leqslant C_X\delta + \frac{C_0}{A}.$$

In particular, $\|\eta^1\|_{X^{2n+2(k-1)}} \leqslant (C_X+1)\delta + C_0/A$. Therefore (up to increasing A) condition (5.1) is satisfied (with $l = k$), Duhamel's formula gives again

$$\|\eta^2\|_{\mathcal{H}^{2n+2(k-1)}} \leqslant \int_t^\infty C_R C_{\mathrm{Tayl}}\delta^2 \mathrm{e}^{\alpha c_0|t-s|/2}\,\mathrm{e}^{-2\alpha c_0 s}\,\mathrm{d}s$$
$$\leqslant \frac{C_R C_{\mathrm{Tayl}}\delta}{\alpha c_0}\delta\mathrm{e}^{-2\alpha c_0 t} := q\delta\mathrm{e}^{-2\alpha c_0 t}.$$

We note for later use that for A large enough, we have $q \leqslant 1/2$ so that $\sum_{j\geqslant 0} q^j \leqslant 1$.

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

Step 3. Induction. Assume we have constructed $(\eta^i)_{1\leqslant i\leqslant j}$ for some $j<k$, with

$$\|\eta^1\|_{\mathcal{H}^{2n+2k}} \leqslant \delta \mathrm{e}^{-\alpha c_0 t},\ \forall\, i\geqslant 2,\ \|\eta^i\|_{\mathcal{H}^{2n+2(k-i+1)}} \leqslant q^{2^{i-1}}\delta \mathrm{e}^{-2^{i-1}\alpha c_0 t}.$$

In particular, $\|\sum_1^j \eta^i\|_{\mathcal{H}^{2n+2(k-i+1)}} \leqslant 2\delta$ and from (5.4)

$$\forall\, 1\leqslant i\leqslant j,\ \|f^i(t)\|_{\mathcal{H}^{2n+2(k-i)}} \leqslant C_{\mathrm{Tayl}} q^{2^i}\delta^2 \mathrm{e}^{-2^i\alpha c_0 t}.$$

Estimating $\partial_t \eta^i$ as for $\partial_t\eta^1$ we have

$$\begin{aligned}\left\|\partial_t\left(\sum_{i=1}^j \eta^i\right)\right\|_{\mathcal{H}^{2n+2(k-j)}} &\leqslant \sum_{i=1}^j C_X\|\eta^i\|_{\mathcal{H}^{2n+2(k-i+1)}} + \|f^{i-1}\|_{\mathcal{H}^{2n+2(k-i)}}\\ &\leqslant 2C_X\delta + 2C_{\mathrm{Tayl}}\delta^2 + \frac{C_0}{A},\end{aligned}$$

therefore $\|\sum_1^j \eta^i\|_{X^{2n+2(k-j)}} \leqslant 2(C_X+1)\delta + 2C_{\mathrm{Tayl}}\delta^2 + C_0/A$, and the smallness conditions (5.1) are satisfied (with $l=k-j$) for A large enough independent of j.

We can use the uniform resolvent estimate (5.2) and Taylor estimate (5.4) as for the construction of η^2

$$\|\eta^{j+1}\|_{\mathcal{H}^{2n+2(k-j)}} \leqslant \int_t^\infty C_R C_{\mathrm{Tayl}} q^{2^j}\delta^2 \mathrm{e}^{\alpha c_0|t-s|/2}\,\mathrm{e}^{-2^j\alpha c_0 s}\,\mathrm{d}s \leqslant q^{2^j}\delta \mathrm{e}^{-2^j\alpha c_0 t}.$$

By induction, we obtain $(\eta^j)_{1\leqslant j\leqslant k}$, the function $U^a \sum_1^k \eta^j$ is sufficient to end the proof since by construction and estimate (5.4)

$$\partial_t\left(S+\sum_1^k \eta^j\right) = J\partial_x\delta H\left[S+\sum_1^k\eta^j\right] + f^k,\quad \text{with } \|f^k\|_{\mathcal{H}^{2n}} \leqslant C_{\mathrm{Tayl}} q^{2^k}\delta \mathrm{e}^{-2^k t},$$

so that the remainder f^k is as small and rapidly decreasing as required for k large enough. □

6. Proof of the main result

This section is a compactness argument. Let $V^a = \begin{pmatrix}\rho^a\\ v^a\end{pmatrix} = S + U^a$ be an approximate solution given by theorem 5.1 with $U^a\in\mathcal{H}^{2n+2}$ and $n,\varepsilon,\ C_e$ to choose later. Define V^k the solution of (1.1) with Cauchy data $V^k(k)=V^a(k)$. According to the energy estimate of proposition 3.1, we have for $\Delta U^k := V^k - V^a$

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\|}\Delta U^k\|_{\mathcal{H}^{2n}}\right| &\leqslant C(\|\Delta U^k\|_{\mathcal{H}^{2n}} + \|U^a\|_{\mathcal{H}^{2n+2}} + \|1/\rho^k + 1/\rho^a\|_{L^\infty})\\ &\quad\times(\widetilde{\|}\Delta U^k\|_{\mathcal{H}^{2n}} + \varepsilon \mathrm{e}^{-C_e t}).\end{aligned}$$

Let $m=\inf_{(x,t)\in\mathbb{R}\times\mathbb{R}^+}\rho^a$, we pick ε such that $\widetilde{\|}\Delta U^k\|_{\mathcal{H}^{2n}} < 2\varepsilon \Rightarrow \inf\rho^k \geqslant m/2$, and $\|\Delta U^k\|_{\mathcal{H}^{2n}} \leqslant C_1\widetilde{\|}\Delta U^k\|_{\mathcal{H}^{2n}}$. Set $C = C(2C_1\varepsilon + \|U^a\|_{\mathcal{H}^{2n+2}} + 3/m)$, and fix $C_e \geqslant$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

$2C$. We can assume $C \geqslant 1$. Since $\Delta U^k(k) = 0$, the energy estimate backwards in time gives (as long as $\|U(t)\|_{\mathcal{H}^{2n}} \leqslant 2\varepsilon$)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{Ct}\widetilde{\|}\Delta U^k\|_{\mathcal{H}^{2n}}) \geqslant -\varepsilon \mathrm{e}^{-(C_e-C)t} \Rightarrow \widetilde{\|}\Delta U^k(t)\|^2_{\mathcal{H}^{2n}} \leqslant \frac{\varepsilon}{C_e - C}\mathrm{e}^{-C_e t} \leqslant \varepsilon \mathrm{e}^{-C_e t}.$$

From a (backwards) continuation argument, the solution is well defined on $[0, k]$ for ε small enough, and independently of k

$$\forall\, 0 \leqslant t \leqslant k,\ \|\Delta U^k(t)\|_{\mathcal{H}^{2n}} \leqslant C_1 \varepsilon \mathrm{e}^{-C_e t}.$$

For n large (actually $n = 2$ is enough), we have from the equation $\partial_t V^k \in \mathcal{C}_b(\mathbb{R}^+, \mathcal{H}^{2n-2})$, indeed we recall that $\partial_x S$ is smooth and rapidly decaying:

$$\begin{aligned} \partial_t \rho^k &= -\partial_x(\rho^k u^k), \quad \text{with } \rho^k \in (\rho_S + H^{2n+1}), \\ & \qquad u^k \in (u_S + H^{2n}), \quad \text{thus } \partial_t \rho^k \in H^{2n-1}, \\ \partial_t u^k &= -\partial_x \left(g(\rho^k) + \underbrace{\frac{(u^k)^2}{2}}_{(u_S)^2 + H^{2n}} - \underbrace{K\partial_x^2 \rho^k}_{H^{2n-1}} - \underbrace{\frac{1}{2}K'(\partial_x \rho^k)^2}_{H^{2n}} \right) \in H^{2n-2}. \end{aligned}$$

Similarly, $\partial_t U^a \in C_b(\mathbb{R}^+, \mathcal{H}^{2n})$. We deduce that $V^k - S = U^a + \Delta U^k$ is bounded in $C_b([0, k], \mathcal{H}^{2n})$ and $C_b^1([0, k], \mathcal{H}^{2n-2}$. By weak* compactness, up to an extraction $V^k - S$ converges weakly to some $U \in L^\infty(\mathbb{R}^+, \mathcal{H}^{2n})$. Moreover for any bounded interval J, we have the compact embedding $\mathcal{H}^{2n-2}(J) \subset \mathcal{H}^{2n-3}(J)$, so using the Ascoli-Arzela theorem, up to another extraction $V^k - S$ converges to U in $C_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{H}^{2n-2}_{\mathrm{loc}})$. For $2n - 3 \geqslant 2$ it is not hard to check that $S + U$ is a solution of the Euler–Korteweg system (1.1).

Now due to the uniform estimate $\|V^k(t) - V^a(t)\|_{\mathcal{H}^{2n}} \leqslant \varepsilon \mathrm{e}^{-C_e t}$ $(t \leqslant k)$, passing to the (weak) limit

$$\|(S + U)(t) - V^a(t)\|_{\mathcal{H}^{2n}} \leqslant \varepsilon \mathrm{e}^{-C_e t} \text{ (a.e.)}.$$

From theorem 5.1, we also know $\|V^a - S\|_{\mathcal{H}^{2n}} \lesssim \mathrm{e}^{-\alpha c_0 t}/A$, therefore we can conclude

$$\lim_{t\to\infty} \|U(t) - S(t)\|_{\mathcal{H}^{2n}} = 0.$$

Remark 6.1. A priori, the pointwise $\mathcal{H}^{2n}$ convergence holds only almost everywhere in t, however using the well-posedness theorem 1.1 in [4], one can prove that U coincides with the $C(\mathbb{R}^+, \mathcal{H}^{2n})$ solution, and by continuity the convergence holds for all t.

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

Appendix A. Complements on travelling waves

Existence of kinks A travelling wave satisfies

$$\begin{cases} -c\partial_x\rho + \partial_x(\rho v) &= 0, \\ -c\partial_x v + \partial_x(v^2/2) + \partial_x g(\rho) &= \partial_x\Big(K\partial_x^2\rho + \frac{1}{2}K'(\partial_x\rho)^2\Big). \end{cases}$$

A first integration gives

$$\begin{cases} \rho(v-c) &= j, \\ \dfrac{(v-c)^2}{2} + g(\rho) - K\partial_x^2\rho - \dfrac{1}{2}K'(\partial_x\rho)^2 &= q. \end{cases}$$

Assuming $\lim_{\pm\infty}\rho = \rho_\pm,\ \lim_{\pm\infty} v = v_\pm$, we have

$$j = \rho(v-c) = \rho_+(v_+ - c) = \rho_-(v_- - c), \tag{A.1}$$

$$q = \frac{(v-c)^2}{2} + g(\rho) - K\partial_x^2\rho - \frac{1}{2}K'(\partial_x\rho)^2 \tag{A.2}$$

$$= \frac{(v_+-c)^2}{2} + g(\rho_+) \tag{A.3}$$

$$= \frac{(v_--c)^2}{2} + g(\rho_-). \tag{A.4}$$

This implies

$$q = \frac{j^2}{2\rho^2} + g(\rho) - K\partial_x^2\rho - \frac{1}{2}K'(\partial_x\rho)^2 = \frac{j^2}{2\rho_+^2} + g(\rho_+) \tag{A.5}$$

$$= \frac{j^2}{2\rho_-^2} + g(\rho_-). \tag{A.6}$$

Set $f(\rho) = j^2/2\rho^2 - q + g(\rho)$, we get two conditions

$$f(\rho_+) = f(\rho_-) = 0. \tag{A.7}$$

Multiplying (A.5) by $\partial_x\rho$ and integrating from ρ_- to ρ

$$\frac{1}{2}K(\partial_x\rho)^2 = -q(\rho-\rho_-) - \frac{j^2}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_-}\right) + G(\rho) := F(\rho), \tag{A.8}$$

with G the primitive of g such that $G(\rho_-) = 0$. From this integrated momentum equation we get one condition :

$$F(\rho_+) = 0. \tag{A.9}$$

This condition can be written only in term of ρ_-, ρ_+:

$$\frac{G(\rho_+) - G(\rho_-)}{\rho_- - \rho_-} = \frac{g(\rho_+)\rho_+ + g(\rho_-)\rho_-}{\rho_+ + \rho_-}. \tag{A.10}$$

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

Lastly according to (A.2) ρ satisfies the following system of ODE

$$\begin{cases}\sqrt{K}\partial_x\rho &= w\\ \sqrt{K}\partial_x w &= j^2/2\rho^2+g(\rho)-q,\end{cases}$$

up to a change of variable it is Hamiltonian (with energy $F(\rho)$) therefore steady states can only be centres or saddles, and a travelling wave connects two saddle points. So $(\rho_\pm,0)$ should be a saddle point, which leads to a last condition: the characteristic equation at $(\rho_\pm,0)$ is

$$\lambda^2+j^2/\rho_\pm^3-g'(\rho_\pm)=0,$$

and the roots in λ are real with opposite sign under the condition

$$j^2<\rho_\pm^3 g'(\rho_\pm)\Leftrightarrow(v_\pm-c)^2<\rho_\pm g'(\rho_\pm)\Leftrightarrow f'(\rho_\pm)>0. \tag{A.11}$$

(we will see several interpretations of this condition). Conversely, assume (A.7), (A.9), (A.11) are satisfied, and that f only changes sign once on (ρ_-,ρ_+). Due to (A.7), (A.11), $f'(\rho_\pm>0$ thus f must be positive then negative on (ρ_-,ρ_+), and from (A.9), F remains positive on (ρ_-,ρ_+), but vanishes at second order at $\rho_\pm$. The existence of a kink then just follows from the integration of $\pm\sqrt{K}\partial_x\rho/\sqrt{2F(\rho)}=1$ (with a choice of sign adapted to the one of $\rho_--\rho_+$). To summarize, provided this equation is satisfied c is a free parameter, and either ρ_+ or ρ_- is used to fully parametrize the travelling waves. Kinks should thus form locally two-dimensional manifolds.

Remark A.1. As the construction of the profile ρ depends on $(v_+-c)^2$, we can assume $c-v_+>0$.

The speed of kinks, some geometry The momentum equation is

$$\frac{j^2}{2\rho^2}-\frac{j^2}{2\rho_+^2}+g(\rho)=K\partial_x^2\rho+\frac{1}{2}K'(\partial_x\rho)^2,$$

$$\Rightarrow -\frac{j^2(\rho-\rho_+)^2}{2\rho\rho_+^2}+G(\rho)=\frac{1}{2}K(\partial_x\rho)^2\geqslant 0. \tag{A.12}$$

Letting $x\to\pm\infty$, from the sign condition we find again (A.11)

$$\frac{j^2}{\rho_\pm^3}=\frac{(v_\pm-c)^2}{\rho_\pm}\leqslant g'(\rho_\pm). \tag{A.13}$$

This inequality gives a geometric interpretation of (A.7), that we rewrite

$$g(\rho_-)=\frac{-j^2}{2\rho_-^2}+q,\ g(\rho_+)=\frac{-j^2}{2\rho_+^2}+q,$$

meaning that $\rho_\pm$ are intersection points of the curves $g,-j^2/2\rho^2+q$, and conditions (A.11) mean that the curves intersect transversally at $\rho_\pm$. Condition (A.9) means that the total signed area between the two curves from ρ_- to ρ_+ must be zero. See figure A1. When g follows a Van Der Waals law, such conditions can be met we refer to [4] for some relevant examples.

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

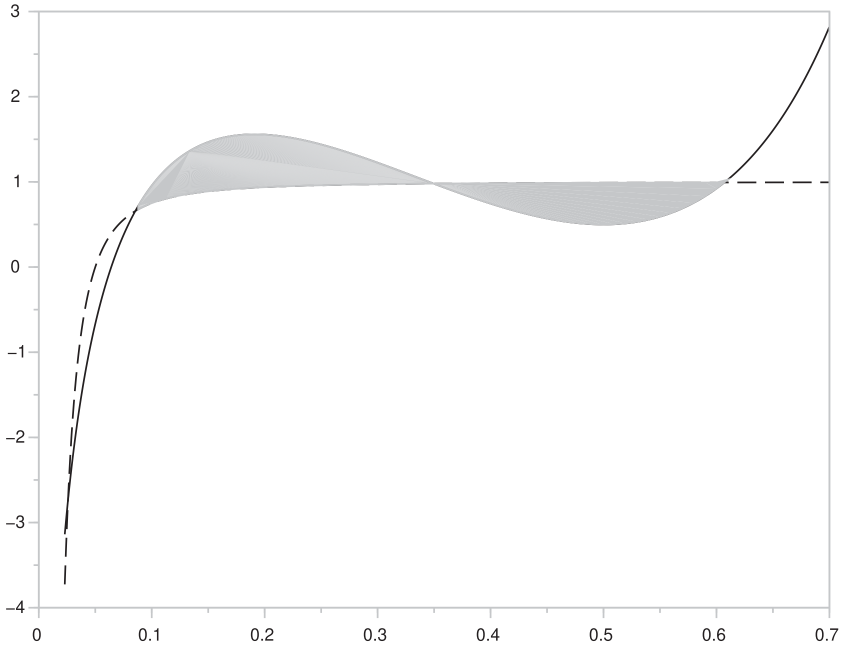

Figure A1. Full line g, dashed line $-j^2/(2\rho^2)+q$, the two shaded areas should be equal.

The dimension of families of kinks. There exists a kink provided equations (A.7), (A.9) are met ((A.11) is open and therefore plays no role for the dimension), namely

$$f(\rho_+) = f(\rho_-) = \int_{\rho_-}^{\rho_+} f(\rho)\,\mathrm{d}\rho = 0, \qquad \text{where } f \text{ depends on} \rho_\pm, c, j, q.$$

Consider the application $\varphi : (\rho_\pm, j, q, c) \to \begin{pmatrix} f(\rho_+) \\ f(\rho_-) \\ \int_{\rho_-}^{\rho_+} f(\rho)\,\mathrm{d}\rho \end{pmatrix}$, we have

$$\begin{aligned} D\varphi &= \begin{pmatrix} -j^2/\rho_-^3 + g'(\rho_-) & 0 & j/\rho_-^2 & -1 & 0 \\ 0 & -j^2/\rho_+^3 + g'(\rho_+) & j/\rho_+^2 & -1 & 0 \\ -f(\rho_-) & f(\rho_+) & \frac{j^2}{2}(1/\rho_- - 1/\rho_+) & \rho_- - \rho_+ & 0 \end{pmatrix} \\ &= \begin{pmatrix} -j^2/\rho_-^3 + g'(\rho_-) & 0 & j/\rho_-^2 & -1 & 0 \\ 0 & -j^2/\rho_+^3 + g'(\rho_+) & j/\rho_+^2 & -1 & 0 \\ 0 & 0 & \frac{j^2}{2}(1/\rho_- - 1/\rho_+) & \rho_- - \rho_+ & 0 \end{pmatrix}. \end{aligned}$$

According to the sign condition (A.11), in the generic case $-j^2/\rho_\pm^3 + g'(\rho_\pm) > 0$, so the rank of the matrix is three and the kinks form a manifold of dimension two.

The case of solitons Kinks can not provide a nontrivial soliton in the limit $\rho_- \to \rho_+$, indeed kinks are monotonous therefore the 'soliton' limit of a kink is actually a constant solution. Nevertheless, the construction of solitons follows the same lines.

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

We denote $\rho_+ = \lim_{\pm\infty} \rho$. Since g is a primitive of g', we can assume $g(\rho_+) = 0$. Equation (A.7) gives

$$f(\rho_+) = \frac{j^2}{2\rho_+^2} - q = 0 \Rightarrow f(\rho) = \frac{j^2}{2\rho^2} - \frac{j^2}{2\rho_+^2} + g(\rho).$$

Then (A.9) is free so

$$\frac{1}{2}K(\rho')^2 = \frac{-j^2}{2\rho\rho_+^2}(\rho_+ - \rho)^2 + G(\rho) = \frac{-(c-v_+)^2}{2\rho}(\rho_+ - \rho)^2 + G(\rho). \tag{A.14}$$

For $j^2 < \rho_+^3 g'(\rho_+) \Leftrightarrow (v_+ - c)^2 < \rho_+ g'(\rho_+)$, we can define

$$\rho_m = \sup\left\{\rho < \rho_+ : \ \frac{-j^2}{2\rho\rho_+^2}(\rho_+ - \rho)^2 + G(\rho) = 0\right\}.$$

From basic ODE arguments, there exists a homoclinic orbit to ρ_+ with minimal value ρ_m; a 'bubble' decreasing from ρ_+ to ρ_m then increasing back to ρ_+.

Remark A.2. We recall that a kink of speed c_k and right endstate (ρ_+, v_+) satisfies $(v_+ - c_k)^2 < \rho_+ g'(\rho_+)$, and since its construction depends on $(v_+ - c_k)^2$ rather than $v_+ - c_k$, we may assume $c_k - v_+ \geqslant 0$. In particular since there exists solitons of speed c_s with $(v_+ - c_s)^2$ arbitrarily close to $\rho_+ g'(\rho_+)$, there always exists solitons faster than the kink and sharing the same endstate.

Existence of kink-stable solitons configuration According to remark A.2, given a kink with right endstate (ρ_+, v_+), there exists solitons with same endstate and larger speed satisfying $c - v_+ > 0$. We are left to check wether such solitons are stable.

We assume here that the asymptotic state (ρ_+, v_+) is fixed, so that solitons only depend on the speed c, and we also assume $g''(\rho_+) \geqslant 0$ (this is true for the Van Der Waals case).

For consistency, we first prove that the stability condition $\mathrm{d}P/\mathrm{d}c < 0$ is indeed equivalent to the stability condition of Benzoni *et al.* [4]. To do so, we recall the definition of momentum of instability from [4]. The equations satisfied by a soliton are

$$\begin{cases} -c(v - v_+) + v^2/2 + g(\rho) - K\partial_x^2\rho - \frac{1}{2}K'(\partial_x\rho)^2 & = & v_+^2/2 + g(\rho_+), \\ -c(\rho - \rho_+) + \rho v & = & \rho_+ v_+. \end{cases}$$

Defining $H = 1/2 \displaystyle\int \rho v^2 - \rho_+ v_+^2 + K(\partial_x\rho)^2 + 2G(\rho)\,\mathrm{d}x$, and recalling $P = \int(\rho - \rho_+)(v - v_+)$, they can be expressed in an abstract way

$$\delta H - c\delta P = (u_+^2/2 + g(\rho_+))\delta P_1 + \rho_+ v_+ \delta P_2 := \lambda_1 \delta P_1 + \lambda_2 \delta P_2, \tag{A.15}$$

where $P_1 = \int \rho - \rho_+ \mathrm{d}x,\ P_2 = \int v - v_+ \mathrm{d}x$. The momentum of instability is then

$$m(c) = H - cP - \lambda_1 P_1 - \lambda_2 P_2, \tag{A.16}$$

and the stability condition of [4] is $m''(c) > 0$.

https://doi.org/10.1017/prm.2019.51 Published online by Cambridge University Press

Lemma A.3. *The condition $m''(c) > 0$ is equivalent to*

$$\frac{\mathrm{d}P}{\mathrm{d}c} = \frac{\mathrm{d}}{\mathrm{d}c}\int_{\mathbb{R}} \frac{(\rho-\rho_+)^2}{\rho}(c-v_+)\,\mathrm{d}x < 0. \tag{A.17}$$

Proof. Denote $'$ the derivative with respect to c, using (A.15) we have

$$m'(c) = H' - cP' - P - \lambda_1 P_1' - \lambda_2 P_2' = -P, \tag{A.18}$$

We differentiate again and use the identity $\rho(v-c) = \rho_+(v_+ - c)$

$$\begin{aligned} m''(c) = -P' &= -\frac{\mathrm{d}}{\mathrm{d}c}\int_{\mathbb{R}} (\rho-\rho_+)(v-v_+)\,\mathrm{d}x \\ &= -\frac{\mathrm{d}}{\mathrm{d}c}\int_{\mathbb{R}} \frac{(\rho-\rho_+)^2}{\rho}(c-v_+)\,\mathrm{d}x. \end{aligned}$$

The condition $m'' > 0$ gives the expected result. □

The so-called transonic limit corresponds to $j^2/\rho_+^2 = (v_+ - c)^2 \to \rho_+ g'(\rho_+)$, so we set $j^2 = \rho_+^3 g'(\rho_+)(1-\varepsilon)$. From numerical computations it was conjectured in [4] that solitons are stable in the transonic limit, and this is rigorously proved with the following result. As it gives the existence of stable solitons with speed arbitrarily close to $\sqrt{\rho_+ g'(\rho_+)}$, it also provides the existence of kink-stable soliton configurations.

Lemma A.4. *For ε small enough and $g''(\rho_+) > 0$, 'bubble' solitons of speed $\sqrt{\rho_+ g'(\rho_+)(1-\varepsilon)}$ are stable.*

Proof. The condition $\mathrm{d}P/\mathrm{d}c < 0$ is equivalent to $\mathrm{d}P/\mathrm{d}\varepsilon > 0$ and equation (A.14) reads

$$\begin{aligned} &\frac{1}{2}K(\partial_x\rho)^2 \\ &\quad = (\rho-\rho_+^2)\left(\frac{g'(\rho_+)}{2} + \frac{g''(\rho_+)(\rho-\rho_+)}{6} - \frac{\rho_+ g'(\rho_+)(1-\varepsilon)}{2\rho} + O(\rho-\rho_+)^2\right) \\ &\quad = (\rho-\rho_+)^2\left(\frac{\varepsilon\rho_+ g'(\rho_+)}{2\rho} + \left(\frac{g''(\rho_+)}{6} + \frac{g'(\rho_+)}{2\rho}\right)(\rho-\rho_+) + O(\rho-\rho_+)^2\right) \\ &\quad := \frac{\rho_+ g'(\rho_+)}{2\rho}(\rho-\rho_+)^2(\varepsilon + \alpha(\rho-\rho_+)) + O(\rho-\rho_+)^4. \end{aligned}$$

Note that $\alpha > 0$, in the limit $\varepsilon \to 0+$, solitons have an amplitude $\rho_+ - \rho_m \sim \varepsilon/\alpha \to 0$, where $\rho_m(\varepsilon)$ is the minimum of ρ, and in this regime $\rho_m'(\varepsilon) < 0$. Up to translation, we can assume that the minimum of ρ is reached at $x = 0$, and ρ is strictly decreasing on $(-\infty, 0)$. Using on $x \in (0,\infty)$ the change of variable $\rho(x) = \rho$, $\mathrm{d}x = \sqrt{K/2F}\,\mathrm{d}\rho$ we find

$$P = 2\int_{\rho_m}^{\rho_+} \frac{(\rho-\rho_+)^2(c-v_+)}{\rho}\sqrt{\frac{K}{2F}}\,\mathrm{d}\rho. \tag{A.19}$$

As is expectable, the situation is somewhat degenerate at $\varepsilon = 0$, as one can check that $P(c) = P(\sqrt{\rho_+ g'(\rho_+)(1-\varepsilon)}) = O(\varepsilon^{3/2})$. This is handled by a factorization of

F (see (A.8)):

$$F = \frac{\rho_+ g'(\rho_+)(\rho-\rho_+)^2}{2\rho}\left(\varepsilon + \frac{2\rho G}{\rho_+ g'(\rho_+)(\rho-\rho_+)^2} - 1\right)$$
$$:= \frac{\rho_+ g'(\rho_+)(\rho-\rho_+)^2}{2\rho}(\varepsilon + H(\rho)).$$

Here $H(\rho_+) = 0$ and by construction $H(\rho_m(\varepsilon)) + \varepsilon = 0$. The condition $\alpha > 0$ implies $H'(\rho_+) > 0$, so $\varphi(\rho,\varepsilon) := (H+\varepsilon)/(\rho-\rho_m)$ is well defined near $(\rho,\varepsilon) = (\rho_+, 0)$, smooth and does not cancel. To summarize, $F = \rho_+ g'(\rho_+)(\rho-\rho_+)^2(\rho-\rho_m)/2\rho\varphi(\rho,\varepsilon)$. Denoting $\delta(\varepsilon) = \rho_+ - \rho_m$, we use the change of variables $\rho = \rho_+ - \delta r$:

$$P = 2\int_0^1 \frac{\delta^2 r^2(c-v_+)}{\rho}\sqrt{\frac{\rho K}{\rho_+ g'(\rho_+)\delta^2 r^2 \delta(1-r)\varphi(r,\varepsilon)}}\,\delta\,\mathrm{d}r$$
$$= 2\int_0^1 \frac{\delta^{3/2} r(c-v_+)}{\rho}\sqrt{\frac{\rho K}{\rho_+ g'(\rho_+)(1-r)\varphi(r,\varepsilon)}}\,\mathrm{d}r.$$

From $\rho_m'(\varepsilon) < 0$, $\varepsilon \to \delta(\varepsilon)$ is locally invertible and the stability condition is equivalent to $\mathrm{d}P/\mathrm{d}\delta > 0$, but it is clear from the formula that

$$\mathrm{d}P/\mathrm{d}\delta = \frac{3\delta^{1/2}}{2}\int_0^1 \frac{r(c-v_+)}{\rho}\sqrt{\frac{\rho K}{\rho_+ g'(\rho_+)(1-r)\varphi(r,\varepsilon)}}\,\mathrm{d}r + O(\delta^{3/2}),$$

which is positive for δ small enough. □

Acknowledgements

The author was supported by the French National Research Agency project NABUCO, grant ANR-17-CE40-0025. The author thanks Stefan Le Coz for pointing out some useful references.

References

1 C. Audiard. Small energy traveling waves for the Euler–Korteweg system. *Nonlinearity* **30** (2017), 3362–3399.

2 C. Audiard and B. Haspot. Global well-posedness of the Euler–Korteweg system for small irrotational data. *Comm. Math. Phys.* **351** (2017), 201–247.

3 I. V. Barashenkov and V. G. Makhankov. Soliton-like ‘bubbles’ in a system of interacting bosons. *Phys. Lett. A* **128** (1988), 52–56.

4 S. Benzoni–Gavage, R. Danchin, S. Descombes and D. Jamet. Structure of Korteweg models and stability of diffuse interfaces. *Interfaces Free Bound.* **7** (2005), 371–414.

5 S. Benzoni–Gavage, R. Danchin and S. Descombes. Well-posedness of one-dimensional Korteweg models. *Electron. J. Differ. Equ.* (2006), 59, 35 pp. (electronic).

6 S. Benzoni–Gavage, R. Danchin and S. Descombes. On the well-posedness for the Euler–Korteweg model in several space dimensions. *Indiana Univ. Math. J.* **56** (2007), 1499–1579.

7 F. Béthuel, P. Gravejat and D. Smets. Stability in the energy space for chains of solitons of the one-dimensional Gross–Pitaevskii equation. *Ann. Inst. Fourier (Grenoble)* **64** (2014), 19–70.

8 R. Carles, R. Danchin and J.-C. Saut. Madelung, Gross–Pitaevskii and Korteweg. *Nonlinearity* **25** (2012), 2843–2873.

9 V. Combet. Multi-existence of multi-solitons for the supercritical nonlinear Schrödinger equation in one dimension. *Discrete Contin. Dyn. Syst.* **34** (2014), 1961–1993.

10 R. Côte, Y. Martel and F. Merle. Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations. *Rev. Mat. Iberoam.* **27** (2011), 273–302.

11 J. Giesselmann, C. Lattanzio and A. E. Tzavaras. Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics. *Arch. Ration. Mech. Anal.* **223** (2017), 1427–1484.

12 M. Grillakis, J. Shatah and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.* **74** (1987), 160–197.

13 S. Le Coz and T.-P. Tsai. Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations. *Nonlinearity* **27** (2014), 2689–2709.

14 L. Lin and T.-P. Tsai. Mixed dimensional infinite soliton trains for nonlinear Schrödinger equations. *Discrete Contin. Dyn. Syst.* **37** (2017), 295–336.

15 Y. Martel and F. Merle. Multi solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23** (2006), 849–864.

16 M. Ming, F. Rousset and N. Tzvetkov. Multi-solitons and related solutions for the water-waves system. *SIAM J. Math. Anal.* **47** (2015), 897–954.

17 R. L. Pego and M. I. Weinstein. Eigenvalues, and instabilities of solitary waves. *Philos. Trans. Roy. Soc. London Ser. A* **340** (1992), 47–94.

18 V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.* **61** (1971), 118–134.