Ergod. Th. & Dynam. Sys. (2012), **32**, 919–940
doi:10.1017/S0143385711000083

Co-induction in dynamical systems

ANTHONY H. DOOLEY† and GUOHUA ZHANG†‡§

† *School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia*
(e-mail: a.dooley@unsw.edu.au)
‡ *School of Mathematical Sciences and LMNS, Fudan University, Shanghai 200433, PR China*
(e-mail: chiaths.zhang@gmail.com)

(*Received* 6 *September* 2010 *and accepted in revised form* 27 *January* 2011)

Abstract. If a countable amenable group G contains an infinite subgroup Γ, one may define, from a measurable action of Γ, the so-called co-induced measurable action of G. These actions were defined and studied by Dooley, Golodets, Rudolph and Sinelsh'chikov. In this paper, starting from a topological action of Γ, we define the co-induced topological action of G. We establish a number of properties of this construction, notably, that the G-action has the topological entropy of the Γ-action and has uniformly positive entropy (completely positive entropy, respectively) if and only if the Γ-action has uniformly positive entropy (completely positive entropy, respectively). We also study the Pinsker algebra of the co-induced action.

1. *Introduction*

A well-known result of Ornstein [**20, 22**] states that there is a completely positive entropy (c.p.e.) non-Bernoulli $\mathbb{Z}$-action of any given entropy. This result was subsequently generalized by many authors; for example, Ornstein and Shields showed that there is an uncountable family of pairwise non-isomorphic c.p.e. non-Bernoulli $\mathbb{Z}$-actions with the same entropy [**23**], and Feldman [**8**] found a c.p.e. non-loosely Bernoulli $\mathbb{Z}$-action (each Bernoulli action is loosely Bernoulli). Kalikow [**18**] subsequently gave a very simple example of a c.p.e. non-loosely Bernoulli $\mathbb{Z}$-action. For further results see [**12, 17, 21, 25**].

A natural extension of a $\mathbb{Z}$-action is the action of an amenable group. A classical result is that each $\mathbb{Z}$-action on a compact metric space admits an invariant Borel probability measure on the space, and it is now well known that the amenability of the group suffices for the existence of an invariant Borel probability measure. The class of amenable groups includes finite groups, solvable groups and compact groups. However, non-amenable groups such as the free group on two generators can admit actions on a compact space with no invariant Borel probability measures.

§ Corresponding author.

The foundations of the theory of amenable group actions were laid by Ornstein and Weiss [**24**]. Rudolph and Weiss [**26**] developed the notion of c.p.e. in this setting and solved a long-standing open problem by proving that, for any infinite countable discrete amenable group action, c.p.e. implies mixing of all orders. Inspired by this, Dooley and Golodets [**6**] proved that every free ergodic c.p.e. action of an infinite countable discrete amenable group has a countable Lebesgue spectrum. Recently, Dooley *et al* [**7**] considered the problem of the existence of a c.p.e. non-Bernoulli action for infinite countable discrete amenable groups. They proved that if the group contains an element of infinite order then there exists an uncountable family of pairwise non-isomorphic c.p.e. actions with any given entropy. In fact, for an infinite countable discrete amenable group G containing an element of infinite order, one starts from a c.p.e. non-Bernoulli action of a subgroup Γ isomorphic to $\mathbb{Z}$ (by Ornstein's result and by the assumptions on G and Γ, such an action must exist); [**7**] constructed the co-induced action of G, which was shown to be non-Bernoulli and c.p.e.

In general, if Γ is a subgroup of G, [**7**] used co-induction to construct a G-action from a Γ-action, and proved that co-induction preserves measure-theoretic entropy and that the co-induced action has c.p.e. (is Bernoulli, respectively) if and only if the original Γ-action has c.p.e. (is Bernoulli, respectively).

In this paper, we consider co-induction for topological group actions. Following [**7**], for any infinite countable discrete amenable group G containing an infinite subgroup Γ, we show how to co-induce a continuous Γ-action to obtain a continuous G-action. We shall show that, as in the measure-theoretic case, co-induction preserves topological entropy. We further investigate the relationship of the original action and the co-induced action with respect to the properties of (topological) c.p.e. and u.p.e. (uniformly positive entropy).

To explain this a little more, let us recall some results on local entropy from [**16**], generalizing [**1–4, 9, 11, 13–15**]. If an infinite countable discrete amenable group acts on a compact metric space, its local entropy theory is established by proving a local variational principle for a given finite open cover; we introduce entropy tuples in both the topological and the measure-theoretic settings, and prove variational relations between these two kinds of entropy tuples. It is then straightforward to define u.p.e. and c.p.e. topological actions. (See §6 for details.)

We shall show that the induced topological action has c.p.e. (respectively u.p.e.) if and only if the original action has c.p.e. (respectively u.p.e.). In order to do so, we define and study the relative Pinsker algebra of a measurable dynamical system and relative c.p.e. actions in a measure-theoretic setting, proving that the Pinsker algebra of the co-induced G-action is the product of the Pinsker algebra of the Γ-action. Hence, we characterize entropy tuples of the G-action by entropy tuples of the Γ-action.

The paper is organized as follows. Section 2 is devoted to preliminaries. In §3, we study the relative Pinsker algebra of a measurable dynamical system and relative c.p.e. actions in the measure-theoretic setting. In §4, we recall from [**7**] the definition and basic properties of measure-theoretic co-induction, and introduce and discuss it in the topological setting. We prove that co-induction preserves topological entropy. In §5, we describe the Pinsker algebra of the co-induced action. Then, using the results of the previous sections, we prove in §6 that the co-induced (topological) action has u.p.e. (respectively c.p.e.) if and only if the original action has u.p.e. (respectively c.p.e.). In §7, we discuss the situation for more general groups.

2. *Preliminaries*

In this section, we shall give some definitions and theorems, which we shall use without further comment. Further details of these notions may be found in [**10, 16, 24, 26, 28, 29**].

Let G be an infinite countable discrete group. Denote by $\mathcal{F}_G$ the set of all non-empty finite subsets of G. G is called *amenable* if for each $K \in \mathcal{F}_G$ and any $\delta > 0$, there exists $F \in \mathcal{F}_G$ such that $|F \Delta KF| < \delta |F|$, where $|\bullet|$ is the counting measure, $KF = \{kf : k \in K, f \in F\}$ and $F \Delta KF = (F \setminus KF) \cup (KF \setminus F)$. Let $K \in \mathcal{F}_G$ and $\delta > 0$. Set $K^{-1} = \{k^{-1} : k \in K\}$. $A \in \mathcal{F}_G$ is called (K, δ)*-invariant* if $|K^{-1}A \cap K^{-1}(G \setminus A)| < \delta |A|$. A sequence $\{F_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}_G$ is called a *Følner sequence for* G if for each $K \in \mathcal{F}_G$ and $\delta > 0$, F_n is (K, δ)-invariant whenever $n \in \mathbb{N}$ is sufficiently large. It is not hard to see that G is amenable if and only if G has a Følner sequence.

Throughout the paper, let G denote an infinite countable discrete amenable group and Γ an infinite subgroup of G (and so Γ is also amenable). Let e_G denote the unit of the group G. Throughout this section, we take $\{F_n\}_{n \in \mathbb{N}}$ to be a Følner sequence for G.

2.1. *Topological dynamical systems.*

Let X be a compact metric space. By a *TDS (topological dynamical G-system)* (X, G) we mean that G is a group of homeomorphisms of X with e_G acting as the identity map. When X is not a singleton, we say that the TDS (X, G) is *non-trivial*.

For $\emptyset \neq W \subseteq X$, denote the diameter of W by $\mathrm{diam}(W)$. Given a family of non-empty subsets $\mathcal{W}$ of X, we set $\mathrm{diam}(\mathcal{W}) = \sup\{\mathrm{diam}(W) : W \in \mathcal{W}\}$, and denote by $\mathcal{B}_X$ the Borel σ-algebra of X. Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X, $\mathcal{M}(X, G)$ the set of all G-invariant elements in $\mathcal{M}(X)$ and $\mathcal{M}^e(X, G)$ the set of all ergodic elements in $\mathcal{M}(X, G)$, respectively. Note that the amenability of G ensures $\mathcal{M}(X, G) \supseteq \mathcal{M}^e(X, G) \neq \emptyset$. A *cover of* X is a finite family of Borel subsets of X whose union is X, a *partition of* X is a cover of X whose elements are pairwise disjoint, and a *finite open cover of* X is a cover of X whose elements are all open subsets of X. Denote by $\mathcal{C}_X$, $\mathcal{P}_X$ and $\mathcal{C}_X^o$ the set of all covers, partitions and finite open covers of X, respectively. Let $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$. $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (denoted by $\mathcal{U} \succeq \mathcal{V}$ or $\mathcal{V} \preceq \mathcal{U}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. We set $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and $\mathcal{U}_F = \bigvee_{g \in F} g^{-1}\mathcal{U}$ for each $F \in \mathcal{F}_G$.

Let (X, G) be a TDS, $\mathcal{U} \in \mathcal{C}_X^o$, and take $N(\mathcal{U})$ to be the minimum among cardinalities of all subfamilies of $\mathcal{U}$ covering X. We may define the *topological entropy of* $\mathcal{U}$ by

$$h_{\mathrm{top}}(G, \mathcal{U}) = \lim_{n \to +\infty} \frac{1}{|F_n|} \log N(\mathcal{U}_{F_n}) \quad (\leq \log N(\mathcal{U})).$$

(Observe that the limit in the above expression always exists and the value of the limit is independent of the choice of the Følner sequence $\{F_n\}_{n \in \mathbb{N}}$.) The *topological entropy of* (X, G) may be defined as

$$h_{\mathrm{top}}(G, X) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_{\mathrm{top}}(G, \mathcal{U}).$$

The following basic facts are easy to obtain.

PROPOSITION 2.1. *Let* (X, G) *be a TDS,* $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}_X^o$ *and* $F \in \mathcal{F}_G$. *Then:*

(1) $h_{\text{top}}(G, (\mathcal{U}_1)_F) = h_{\text{top}}(G, \mathcal{U}_1)$;
(2) $h_{\text{top}}(G, \mathcal{U}_1 \vee \mathcal{U}_2) \leq h_{\text{top}}(G, \mathcal{U}_1) + h_{\text{top}}(G, \mathcal{U}_2)$;
(3) $h_{\text{top}}(G, \mathcal{U}_1) \leq h_{\text{top}}(G, \mathcal{U}_2)$ *if* $\mathcal{U}_1 \preceq \mathcal{U}_2$; *and*
(4) *if* $\{\mathcal{V}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}_X^o$ *satisfies* $\lim_{n\to+\infty} \text{diam}(\mathcal{V}_n) = 0$ *then* $\lim_{n\to+\infty} h_{\text{top}}(G, \mathcal{V}_n) = h_{\text{top}}(G, X)$.

Let (X, G) be a TDS. For non-empty subsets U, V of X, we introduce $N_G(U, V) = \{g \in G : gU \cap V \neq \emptyset\}$. We say that (X, G) is *minimal* if $Gx \doteq \{gx : g \in G\}$ is dense in the space X for each $x \in X$; *transitive* if $N_G(U, V) \neq \emptyset$ whenever U and V are both non-empty open subsets of X; *weakly mixing* if (X^n, G) is transitive for each $n \in \mathbb{N}$, here $X^n = X \times \cdots \times X$ (n times); *mildly mixing* if $(X \times Y, G)$ is transitive whenever (Y, G) is a transitive TDS containing no isolated points; *strongly mixing* if $G \setminus N_G(U, V)$ is a finite set whenever U and V are both non-empty open subsets of X.

LEMMA 2.1. *Suppose that* (X, G) *is a transitive TDS without isolated points and that* U, V *are non-empty open subsets of* X. *Then* $N_G(U, V)$ *is an infinite subset of* G.

Proof. Assume on the contrary that $N_G(U, V)$ is a finite subset of G; let $N_G(U, V) = \{g_1, \ldots, g_n\}$, $n \in \mathbb{N}$. As X contains no isolated points, there exists a non-empty open subset $V^* \subseteq V$ such that $U \setminus \bigcup_{i=1}^n g_i^{-1} V^*$ has a non-empty interior. Since (X, G) is transitive and $V^* \subseteq V$, we may choose $g \in N_G(U \setminus \bigcup_{i=1}^n g_i^{-1} V^*, V^*) \subseteq N_G(U, V)$. However, by definition, $g \notin \{g_1, \ldots, g_n\}$, a contradiction. Thus, $N_G(U, V) \subseteq G$ is an infinite set. □

Remark 1. Note that there exists an infinite transitive TDS (X, G) containing an isolated point such that $N_G(U, V)$ is a finite subset of G for some non-empty open subsets U, V of X. In fact, let G^* be the one point compactification of G; then G has a natural action on G^* and it is easy to check that (G^*, G) is such a TDS.

It is easy to deduce the following basic facts from the definitions:

(1) strong mixing $\Longrightarrow$ weak mixing $\Longrightarrow$ transitivity;
(2) mild mixing $\Longrightarrow$ transitivity (as there exists a transitive TDS containing no isolated points, for example, take Y to be the compact metric space $\{0, 1\}^G$ and let G act on Y naturally, then (Y, G) is a transitive TDS containing no isolated points (note that G is an infinite group));
(3) for a space containing no isolated points, mild mixing $\Longrightarrow$ weak mixing; and
(4) strong mixing $\Longrightarrow$ mild mixing (a direct corollary of Lemma 2.1).

Let (X_1, G) and (X_2, G) be TDSs. A *factor map* $\pi : (X_2, G) \to (X_1, G)$ is a continuous surjective map satisfying $\pi g = g\pi$ for each $g \in G$. In this case, we say that (X_1, G) is a *factor of* (X_2, G) and (X_2, G) is an *extension of* (X_1, G).

2.2. *Measurable dynamical systems.* Let $(X, \mathcal{B}, \mu)$ be a standard Lebesgue space. By an *MDS (measurable dynamical G-system)* $(X, \mathcal{B}, \mu, G)$ we mean that G is a group of invertible measure-preserving transformations of $(X, \mathcal{B}, \mu)$ with e_G acting as the identity transformation.

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Let $(X, \mathcal{B}, \mu)$ be a standard Lebesgue space. As above in the case of a TDS, we introduce $\mathcal{C}_X$ and $\mathcal{P}_X$. Let $\alpha \in \mathcal{P}_X$ and $\mathcal{A} \subseteq \mathcal{B}$ a sub-σ-algebra. Set

$$H_\mu(\alpha|\mathcal{A}) = \sum_{A \in \alpha} \int_X -\mathbb{E}_\mu(1_A|\mathcal{A}) \log \mathbb{E}_\mu(1_A|\mathcal{A})\, d\mu,$$

where $\mathbb{E}_\mu(1_A|\mathcal{A})$ is the μ-expectation of the characteristic function 1_A with respect to $\mathcal{A}$. It is a standard fact that $H_\mu(\alpha|\mathcal{A})$ increases with α and decreases as $\mathcal{A}$ increases. Note that $\beta \in \mathcal{P}_X$ naturally generates a sub-σ-algebra $\mathcal{F}(\beta)$ of $\mathcal{B}$; where there is no ambiguity, we write $\mathcal{F}(\beta)$ as β. Observe that $H_\mu(\alpha|\mathcal{A}) = 0$ if and only if $\alpha \subseteq \mathcal{A}$ (up to μ-null sets). Set $\mathcal{N}_X = \{\emptyset, X\}$, $H_\mu(\alpha) = H_\mu(\alpha|\mathcal{N}_X) = \sum_{A \in \alpha} -\mu(A) \log \mu(A)$. It is easy to check, for $\alpha, \beta \in \mathcal{P}_X$, that $H_\mu(\alpha|\beta)$ (i.e. $H_\mu(\alpha|\mathcal{F}(\beta))) = H_\mu(\alpha \vee \beta) - H_\mu(\beta)$. More generally, for a sub-σ-algebra $\mathcal{A} \subseteq \mathcal{B}$, we have

$$H_\mu(\alpha \vee \beta|\mathcal{A}) = H_\mu(\beta|\mathcal{A}) + H_\mu(\alpha|\beta \vee \mathcal{A}). \tag{1}$$

Now let $(X, \mathcal{B}, \mu, G)$ be an MDS with $\alpha \in \mathcal{P}_X$ and $\mathcal{A} \subseteq \mathcal{B}$ a sub-σ-algebra. If $\mathcal{A}$ is invariant, i.e. $g^{-1}\mathcal{A} = \mathcal{A}$ (up to μ-null sets) for each $g \in G$, then we may define the *measure-theoretic μ-entropy of α with respect to $\mathcal{A}$* by

$$h_\mu(G, \alpha|\mathcal{A}) = \lim_{n \to +\infty} \frac{1}{|F_n|} H_\mu(\alpha_{F_n}|\mathcal{A}) \left(= \inf_{F \in \mathcal{F}_G} \frac{1}{|F|} H_\mu(\alpha_F|\mathcal{A}) \le H_\mu(\alpha|\mathcal{A}) \right), \tag{2}$$

where the limit always exists; the second identity is to be proved later. (In particular, the limit is independent of the choice of the Følner sequence $\{F_n\}_{n \in \mathbb{N}}$.) Then the *measure-theoretic μ-entropy of (X, G) with respect to $\mathcal{A}$* is defined by

$$h_\mu(G, X|\mathcal{A}) = \sup_{\alpha \in \mathcal{P}_X} h_\mu(G, \alpha|\mathcal{A}).$$

To simplify the notation, when $\mathcal{A} = \mathcal{N}_X$, we shall omit the qualification 'with respect to $\mathcal{A}$' or '$|\mathcal{A}$'. For example, we shall write $h_\mu(G, \alpha) = h_\mu(G, \alpha|\mathcal{N}_X)$, $h_\mu(G, X) = h_\mu(G, X|\mathcal{N}_X)$.

By the proof of [**16**, Lemma 3.1], we obtain the following lemma.

LEMMA 2.2. *Let $(X, \mathcal{B}, \mu, G)$ be an MDS, $\alpha \in \mathcal{P}_X$, $\mathcal{A} \subseteq \mathcal{B}$ a sub-σ-algebra of $\mathcal{B}$ and $E, F \in \mathcal{F}_G$. Then*

$$H_\mu(\alpha_F|\mathcal{A}) \le \sum_{g \in F} \frac{1}{|E|} H_\mu(\alpha_{Eg}|\mathcal{A}) + |F \setminus \{g \in G : E^{-1}g \subseteq F\}| \log |\alpha|.$$

Here, $|\alpha|$ denotes the cardinality of α.

Proof of the second identity in (2). Recall that if $\{F_n\}_{n \in \mathbb{N}}$ is a Følner sequence of G, $g\mu = \mu$ and $g^{-1}\mathcal{A} = \mathcal{A}$ (up to μ-null sets) for each $g \in G$; here, $(g\mu)(B) = \mu(g^{-1}B)$ for each $B \in \mathcal{B}$. Now let $E \in \mathcal{F}_G$ and $n \in \mathbb{N}$. By Lemma 2.2, one has

$$\begin{aligned} \frac{1}{|F_n|} H_\mu(\alpha_{F_n}|\mathcal{A}) &\le \frac{1}{|F_n|} \sum_{g \in F_n} \frac{1}{|E|} H_\mu(\alpha_{Eg}|\mathcal{A}) + \frac{1}{|F_n|} |F_n \setminus \{g \in G : E^{-1}g \subseteq F_n\}| \log |\alpha| \\ &= \frac{1}{|E|} H_\mu(\alpha_E|\mathcal{A}) + \frac{1}{|F_n|} |F_n \setminus \{g \in G : E^{-1}g \subseteq F_n\}| \log |\alpha|. \end{aligned} \tag{3}$$

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Note that $F_n \backslash \{g \in G : E^{-1}g \subseteq F_n\} = F_n \cap E(G \backslash F_n) \subseteq K^{-1}F_n \cap K^{-1}(G \backslash F_n)$, where $K = E^{-1} \cup \{e_G\}$. Thus, if $n \in \mathbb{N}$ is sufficiently large then

$$\frac{1}{|F_n|}|F_n \backslash \{g \in G : E^{-1}g \subseteq F_n\}|$$

is arbitrarily small, which implies $h_\mu(G, \alpha|\mathcal{A}) \leq (1/|E|)H_\mu(\alpha_E|\mathcal{A})$ (using (3)). Now the conclusion follows since $E \in \mathcal{F}_G$ was chosen arbitrarily. □

Remark 2. The above proof is a re-working of the proof of [**16**, Lemma 3.1(4)]. We include it here for completeness.

The following basic facts are easy to see.

PROPOSITION 2.2. *Let $(X, \mathcal{B}, \mu, G)$ be an MDS, $\alpha, \beta \in \mathcal{P}_X$, $F \in \mathcal{F}_G$ and $\mathcal{A} \subseteq \mathcal{B}$ an invariant sub-σ-algebra. Then:*
(1) $h_\mu(G, \alpha_F|\mathcal{A}) = h_\mu(G, \alpha|\mathcal{A}) \leq h_\mu(\Gamma, \alpha|\mathcal{A}) \leq H_\mu(\alpha|\mathcal{A}) \leq \log|\alpha|$;
(2) $h_\mu(G, \alpha|\mathcal{A}) \leq h_\mu(G, \beta|\mathcal{A}) + H_\mu(\alpha|\beta \vee \mathcal{A})$;
(3) $h_\mu(G, \alpha \vee \beta|\mathcal{A}) \leq h_\mu(G, \alpha|\mathcal{A}) + h_\mu(G, \beta|\mathcal{A})$*; and*
(4) $h_\mu(G, \alpha|\mathcal{A}) \leq h_\mu(G, \beta|\mathcal{A})$ *if* $\alpha \preceq \beta$.

Let $(X_1, \mathcal{B}_1, \mu_1, G)$ and $(X_2, \mathcal{B}_2, \mu_2, G)$ be MDSs. A *factor map* $\pi : (X_2, \mathcal{B}_2, \mu_2, G) \to (X_1, \mathcal{B}_1, \mu_1, G)$ is a measurable map satisfying $\pi\mu_2 = \mu_1$ and $\pi g = g\pi$ (up to μ_2-null sets) for each $g \in G$. In this case, we say that $(X_1, \mathcal{B}_1, \mu_1, G)$ is a *factor of* $(X_2, \mathcal{B}_2, \mu_2, G)$ and $(X_2, \mathcal{B}_2, \mu_2, G)$ is an *extension of* $(X_1, \mathcal{B}_1, \mu_1, G)$.

2.3. *A variational principle for entropy.*

Observe that if (X, G) is a TDS and there is a measure $\mu \in \mathcal{M}(X, G)$ then $(X, \mathcal{B}_X^\mu, \mu, G)$ is an MDS, where $\mathcal{B}_X^\mu$ is the μ-completion of $\mathcal{B}_X$. For simplicity, we also denote this by $\mathcal{B}_X$ if there is no ambiguity.

The following variational relationship for topological and measure-theoretic entropy is established in [**16, 19, 27**].

THEOREM 2.1. *Let (X, G) be a TDS. Then*

$$h_{\text{top}}(G, X) = \sup_{\mu \in \mathcal{M}(X,G)} h_\mu(G, X) = \sup_{\mu \in \mathcal{M}^e(X,G)} h_\mu(G, X).$$

We also have (see [**16**], for example) the following proposition.

PROPOSITION 2.3. *Let (X, G) be a TDS, $\alpha \in \mathcal{P}_X$ and $\mu \in \mathcal{M}(X, G)$. Then $\bullet \mapsto h_\bullet(G, \alpha)$ and $\bullet \mapsto h_\bullet(G, X)$ are both affine functions on $\mathcal{M}(X, G)$. Moreover, if the ergodic decomposition of μ is $\mu = \int_{\mathcal{M}^e(X,G)} \theta \, d\lambda(\theta)$ then*

$$h_\mu(G, \alpha) = \int_{\mathcal{M}^e(X,G)} h_\theta(G, \alpha)\, d\lambda(\theta) \quad \text{and} \quad h_\mu(G, X) = \int_{\mathcal{M}^e(X,G)} h_\theta(G, X)\, d\lambda(\theta).$$

3. *Relative c.p.e. for measurable dynamical systems*

In this section, we shall introduce and discuss the Pinsker algebra of a given MDS and the property of relative c.p.e. These will both be important in later sections.

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Let $(X, \mathcal{B}, \mu, G)$ be an MDS and $\mathcal{A} \subseteq \mathcal{B}$ an invariant sub-σ-algebra of $\mathcal{B}$. Define $\mathcal{P}^{\mathcal{A}}(X, \mathcal{B}, \mu, G)$ to be the sub-σ-algebra of $\mathcal{B}$ generated by $\{\alpha \in \mathcal{P}_X : h_\mu(G, \alpha|\mathcal{A}) = 0\}$. This is called the *Pinsker algebra of* $(X, \mathcal{B}, \mu, G)$ *with respect to* $\mathcal{A}$. In the case where $\mathcal{A}$ is the algebra $\mathcal{N}_X$ of μ-null sets, we write $\mathcal{P}(X, \mathcal{B}, \mu, G) = \mathcal{P}^{\mathcal{N}_X}(X, \mathcal{B}, \mu, G)$ and call it the *Pinsker algebra of* $(X, \mathcal{B}, \mu, G)$.

It is easy to check that $\mathcal{P}^{\mathcal{A}}(X, \mathcal{B}, \mu, G)$ is invariant, $\mathcal{A} \vee \mathcal{P}(X, \mathcal{B}, \mu, G) \subseteq \mathcal{P}^{\mathcal{A}}(X, \mathcal{B}, \mu, G)$ and, for $\alpha \in \mathcal{P}_X$,

$$h_\mu(G, \alpha) = h_\mu(G, \alpha|\mathcal{P}(X, \mathcal{B}, \mu, G)). \tag{4}$$

We say that $(X, \mathcal{B}, \mu, G)$ has $\mathcal{A}$*-relative c.p.e.* if $\mathcal{P}^{\mathcal{A}}(X, \mathcal{B}, \mu, G) = \mathcal{A}$ (up to μ-null sets), and *c.p.e.* if it has $\mathcal{N}_X$-relative c.p.e.

The main result of this section is the following theorem.

THEOREM 3.1. *Let* $(X, \mathcal{B}, \mu, G)$ *be an MDS and* $\mathcal{A} \subseteq \mathcal{B}$ *an invariant sub-*σ*-algebra of* $\mathcal{B}$*. Then* $(X, \mathcal{B}, \mu, G)$ *has* $\mathcal{A}$*-relative c.p.e. if and only if for each* $\alpha \in \mathcal{P}_X$*, and for any* $\epsilon > 0$*, there exists* $K \in \mathcal{F}_G$ *such that if* $F \in \mathcal{F}_G$ *satisfies* $FF^{-1} \cap (K \setminus \{e_G\}) = \emptyset$ *then*

$$\left|\frac{1}{|F|} H_\mu(\alpha_F|\mathcal{A}) - H_\mu(\alpha|\mathcal{A})\right| < \epsilon. \tag{5}$$

Proof. The implication $\Longrightarrow$ is just [**5**, Theorem 0.1]. We shall prove $\Longleftarrow$ using some ideas from the proof of [**7**, Theorem 4.2].

Choose $\alpha \in \mathcal{P}_X$ with $H_\mu(\alpha|\mathcal{A}) > 0$. By assumption, for $\epsilon \doteq \frac{1}{2} H_\mu(\alpha|\mathcal{A}) > 0$, there exists $K \in \mathcal{F}_G$ such that (5) holds for each $F \in \mathcal{F}_G$ satisfying $FF^{-1} \cap (K \setminus \{e_G\}) = \emptyset$. Now let $E \in \mathcal{F}_G$ and $g \in E$. Obviously, there exists $S \in \mathcal{F}_G$ such that $g \in S \subseteq E$, $SS^{-1} \cap (K \setminus \{e_G\}) = \emptyset$ and $(S \cup \{g'\})(S \cup \{g'\})^{-1} \cap (K \setminus \{e_G\}) \neq \emptyset$ for any $g' \in E \setminus S$. It is not hard to check that $E \setminus S \subseteq (K \setminus \{e_G\})S \cup (K \setminus \{e_G\})^{-1} S = (K \cup K^{-1} \setminus \{e_G\})S$, hence $S \subseteq E \subseteq (K \cup K^{-1} \cup \{e_G\})S$, and one has $|E| \le (2|K| + 1)|S|$. It follows that

$$\frac{1}{|E|} H_\mu(\alpha_E|\mathcal{A}) \ge \frac{1}{(2|K|+1)|S|} H_\mu(\alpha_S|\mathcal{A}) \ge \frac{1}{2(2|K|+1)} H_\mu(\alpha|\mathcal{A}) \quad \text{(using (5))} > 0.$$

In the above inequality, let E vary over all elements in $\mathcal{F}_G$ and then combine with (2) to obtain $h_\mu(G, \alpha|\mathcal{A}) > 0$. We have shown that $(X, \mathcal{B}, \mu, G)$ has $\mathcal{A}$-relative c.p.e. □

Remark 3. Let $(X, \mathcal{B}, \mu, G)$ be an MDS. Note that if $\mathcal{A} \subseteq \mathcal{B}$ is an invariant sub-σ-algebra such that $(X, \mathcal{B}, \mu, G)$ has $\mathcal{A}$-relative c.p.e. then $\mathcal{P}(X, \mathcal{B}, \mu, G) \subseteq \mathcal{A}$ (in the sense of μ), and $(X, \mathcal{B}, \mu, G)$ has $\mathcal{P}(X, \mathcal{B}, \mu, G)$-relative c.p.e.

As an application of Theorem 3.1, we have the following corollary.

COROLLARY 3.1. *For each* $i \in I$*, let* $(X_i, \mathcal{B}_i, \mu_i, G)$ *be an MDS and* $\mathcal{A}_i \subseteq \mathcal{B}_i$ *an invariant sub-*σ*-algebra of* $\mathcal{B}_i$ *such that* $(X_i, \mathcal{B}_i, \mu_i, G)$ *has* $\mathcal{A}_i$*-relative c.p.e., where* I *is an index set which is at most countable. Then* $(\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{B}_i, \bigotimes_{i \in I} \mu_i, G)$ *has* $\prod_{i \in I} \mathcal{A}_i$*-relative c.p.e. In particular,*

$$\mathcal{P}\left(\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{B}_i, \bigotimes_{i \in I} \mu_i, G\right) = \bigotimes_{i \in I} \mathcal{P}(X_i, \mathcal{B}_i, \mu_i, G). \tag{6}$$

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Proof. First, we prove that $(\prod_{i\in I} X_i, \prod_{i\in I} \mathcal{B}_i, \bigotimes_{i\in I} \mu_i, G)$ has $\prod_{i\in I} \mathcal{A}_i$-relative c.p.e.

Let $\alpha \in \mathcal{P}_{\prod_{i\in I} X_i}$ and $\epsilon > 0$. It is not hard to choose a finite subset $\emptyset \neq J \subseteq I$ and $\beta \in \mathcal{P}_{\prod_{i\in I} X_i}$, $\alpha_j \in \mathcal{P}_{X_j}$, $j \in J$ such that, for $\gamma \doteq \prod_{j\in J} \alpha_j \times \prod_{i\in I\setminus J} \mathcal{N}_{X_i}$,

$$\gamma \succeq \beta \quad \text{and} \quad H_{\bigotimes_{i\in I}\mu_i}(\alpha|\beta) + H_{\bigotimes_{i\in I}\mu_i}(\beta|\alpha) < \frac{\epsilon}{2}. \tag{7}$$

Note that for each $j \in J$, $(X_j, \mathcal{B}_j, \mu_j, G)$ has $\mathcal{A}_j$-relative c.p.e., by Theorem 3.1, hence there exists $K_j \in \mathcal{F}_G$ such that if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (K_j \setminus \{e_G\}) = \emptyset$ then

$$\left|\frac{1}{|F|} H_{\mu_j}((\alpha_j)_F|\mathcal{A}_j) - H_{\mu_j}(\alpha_j|\mathcal{A}_j)\right| < \frac{\epsilon}{2|J|}. \tag{8}$$

Thus, if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (\bigcup_{j\in J} K_j \setminus \{e_G\}) = \emptyset$ then, for each $j \in J$, (8) holds for F. Summing over all $j \in J$, we obtain the following estimate:

$$\left|\frac{1}{|F|} H_{\bigotimes_{j\in J}\mu_j}\left(\left(\prod_{j\in J}\alpha_j\right)_F \middle| \prod_{j\in J}\mathcal{A}_j\right) - H_{\bigotimes_{j\in J}\mu_j}\left(\prod_{j\in J}\alpha_j \middle| \prod_{j\in J}\mathcal{A}_j\right)\right| < \frac{\epsilon}{2}. \tag{9}$$

Now recall that

$$H_{\bigotimes_{i\in I}\mu_i}\left(\gamma_F \middle| \prod_{i\in I}\mathcal{A}_i\right) = H_{\bigotimes_{j\in J}\mu_j}\left(\left(\prod_{j\in J}\alpha_j\right)_F \middle| \prod_{j\in J}\mathcal{A}_j\right) \tag{10}$$

and

$$H_{\bigotimes_{i\in I}\mu_i}\left(\gamma \middle| \prod_{i\in I}\mathcal{A}_i\right) = H_{\bigotimes_{j\in J}\mu_j}\left(\prod_{j\in J}\alpha_j \middle| \prod_{j\in J}\mathcal{A}_j\right). \tag{11}$$

Combining (9) with (10) and (11), one has

$$\left|\frac{1}{|F|} H_{\bigotimes_{i\in I}\mu_i}\left(\gamma_F \middle| \prod_{i\in I}\mathcal{A}_i\right) - H_{\bigotimes_{i\in I}\mu_i}\left(\gamma \middle| \prod_{i\in I}\mathcal{A}_i\right)\right| < \frac{\epsilon}{2}, \tag{12}$$

and this implies

$$\begin{aligned}
&H_{\bigotimes_{i\in I}\mu_i}\left(\beta \middle| \prod_{i\in I}\mathcal{A}_i\right) - \frac{1}{|F|} H_{\bigotimes_{i\in I}\mu_i}\left(\beta_F \middle| \prod_{i\in I}\mathcal{A}_i\right)\\
&\quad = H_{\bigotimes_{i\in I}\mu_i}\left(\gamma \middle| \prod_{i\in I}\mathcal{A}_i\right) - \frac{1}{|F|} H_{\bigotimes_{i\in I}\mu_i}\left(\gamma_F \middle| \prod_{i\in I}\mathcal{A}_i\right) - H_{\bigotimes_{i\in I}\mu_i}\left(\gamma \middle| \prod_{i\in I}\mathcal{A}_i \vee \beta\right)\\
&\qquad + \frac{1}{|F|} H_{\bigotimes_{i\in I}\mu_i}\left(\gamma_F \middle| \prod_{i\in I}\mathcal{A}_i\right) - \frac{1}{|F|} H_{\bigotimes_{i\in I}\mu_i}\left(\beta_F \middle| \prod_{i\in I}\mathcal{A}_i\right) \quad \text{(using (1), as } \gamma \succeq \beta)\\
&\quad \le H_{\bigotimes_{i\in I}\mu_i}\left(\gamma \middle| \prod_{i\in I}\mathcal{A}_i\right) - \frac{1}{|F|} H_{\bigotimes_{i\in I}\mu_i}\left(\gamma_F \middle| \prod_{i\in I}\mathcal{A}_i\right) \quad \text{(using (1) again).}
\end{aligned} \tag{13}$$

We deduce that

$$\begin{aligned}
&H_{\bigotimes_{i\in I}\mu_i}\left(\alpha \middle| \prod_{i\in I}\mathcal{A}_i\right) - \frac{1}{|F|} H_{\bigotimes_{i\in I}\mu_i}\left(\alpha_F \middle| \prod_{i\in I}\mathcal{A}_i\right)\\
&\quad \le H_{\bigotimes_{i\in I}\mu_i}\left(\beta \middle| \prod_{i\in I}\mathcal{A}_i\right) - \frac{1}{|F|} H_{\bigotimes_{i\in I}\mu_i}\left(\beta_F \middle| \prod_{i\in I}\mathcal{A}_i\right) + H_{\bigotimes_{i\in I}\mu_i}(\alpha|\beta)\\
&\qquad + H_{\bigotimes_{i\in I}\mu_i}(\beta|\alpha) \quad \text{(using (1))}\\
&\quad < \epsilon \quad \text{(using (7), (12) and (13)).}
\end{aligned}$$

This implies that $(\prod_{i\in I} X_i, \prod_{i\in I} \mathcal{B}_i, \bigotimes_{i\in I} \mu_i, G)$ has $\prod_{i\in I} \mathcal{A}_i$-relative c.p.e.

To finish the proof, it remains to prove (6). Using Remark 3, we obtain the containment $\subseteq$. As for $\supseteq$, it is easy to check that

$$h_{\bigotimes_{i\in I}\mu_i}\Big(G, \prod_{j\in J}\alpha_j \times \prod_{i\in I\setminus J}\mathcal{N}_{X_i}\Big) = \sum_{j\in J} h_{\mu_j}(G, \alpha_j),$$

which implies

$$\mathcal{P}\Big(\prod_{i\in I}X_i, \prod_{i\in I}\mathcal{B}_i, \bigotimes_{i\in I}\mu_i, G\Big) \supseteq \prod_{j\in J}\mathcal{P}(X_j, \mathcal{B}_j, \mu_j, G) \times \prod_{i\in I\setminus J}\mathcal{N}_{X_i}$$

whenever $\emptyset \neq J \subseteq I$ is finite and $\alpha_j \in \mathcal{P}_{X_j}$, $j \in J$, and hence the opposite containment holds. □

Let $(X, \mathcal{B}, \mu, G)$ be an MDS and $\mathcal{A} \subseteq \mathcal{B}$ an invariant sub-σ-algebra. We say that $(X, \mathcal{B}, \mu, G)$ is *$\mathcal{A}$-relatively Bernoulli* if there exists a sub-σ-algebra $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{B}$ such that $\{g^{-1}\mathcal{E} : g \in G\}$ generate $\mathcal{B}$ and are independent relative to $\mathcal{A}$ (that is

$$\mathbb{E}_\mu\Big(1_{\bigcap_{i=1}^n g_i^{-1}E_i}\Big|\mathcal{A}\Big) = \prod_{i=1}^n \mathbb{E}_\mu(1_{E_i}|\mathcal{A}) \quad \text{(in the sense of } \mu) \tag{14}$$

whenever $E_1, \ldots, E_n \in \mathcal{E}$, $\{g_1, \ldots, g_n\} \subseteq G$, $n \in \mathbb{N}$) and *Bernoulli* if it is $\mathcal{N}_X$-relatively Bernoulli. This coincides with the standard definition of a Bernoulli MDS.

The next result is a relative version of the well-known fact that each Bernoulli MDS has c.p.e.

PROPOSITION 3.1. *Let $(X, \mathcal{B}, \mu, G)$ be an MDS and $\mathcal{A} \subseteq \mathcal{B}$ an invariant sub-σ-algebra of $\mathcal{B}$ such that $(X, \mathcal{B}, \mu, G)$ is $\mathcal{A}$-relatively Bernoulli. Then $(X, \mathcal{B}, \mu, G)$ has $\mathcal{A}$-relative c.p.e.*

Proof. Suppose that the MDS $(X, \mathcal{B}, G, \mu)$ is $\mathcal{A}$-relatively Bernoulli. Suppose further that $\mathcal{E} \subseteq \mathcal{B}$ is a sub-σ-algebra of $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{E}$ and $\{g^{-1}\mathcal{E} : g \in G\}$ generate $\mathcal{B}$ and are independent relative to $\mathcal{A}$. Let $\beta \in \mathcal{P}_X$ and $\epsilon > 0$. By the assumptions on $\mathcal{E}$, there exist $S \in \mathcal{F}_G$ and $\alpha, \gamma \in \mathcal{P}_X$, $\gamma \subseteq \mathcal{E}$ such that

$$\gamma_S \succeq \alpha \quad \text{and} \quad H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha) < \epsilon. \tag{15}$$

Using (14), it is not hard to check that $H_\mu(\gamma_S|\mathcal{A}) = |S|H_\mu(\gamma|\mathcal{A})$, and if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (S^{-1}S\setminus\{e_G\})$ then $s_1 f_1 \neq s_2 f_2$ whenever $s_1, s_2 \in S$, $s_1 \neq s_2$ and $f_1, f_2 \in F$, $f_1 \neq f_2$, and so $H_\mu((\gamma_S)_F|\mathcal{A}) = |S||F|H_\mu(\gamma|\mathcal{A})$; thus,

$$H_\mu(\gamma_S|\mathcal{A}) = \frac{1}{|F|}H_\mu((\gamma_S)_F|\mathcal{A}). \tag{16}$$

It follows that if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (S^{-1}S\setminus\{e_G\})$ then

$$\begin{aligned}
&H_\mu(\beta|\mathcal{A}) - \frac{1}{|F|}H_\mu(\beta_F|\mathcal{A}) \\
&\leq H_\mu(\alpha|\mathcal{A}) - \frac{1}{|F|}H_\mu(\alpha_F|\mathcal{A}) + (H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha)) \quad \text{(using (1))} \\
&\leq H_\mu(\gamma_S|\mathcal{A}) - \frac{1}{|F|}H_\mu((\gamma_S)_F|\mathcal{A}) + (H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha)) \\
&\quad \text{(recall } \gamma_S \succeq \alpha\text{, by a reasoning similar to (13))} \\
&< \epsilon \quad \text{(using (15) and (16))},
\end{aligned} \tag{17}$$

that is $(X, \mathcal{B}, \mu, G)$ has $\mathcal{A}$-relative c.p.e. (using Theorem 3.1). □

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

4. *Co-induction for topological dynamical systems*

Let H be a subgroup of G and $\pi_{H\backslash G} : G \to H\backslash G$, $g \mapsto Hg$ the natural projection. By an *H-section* we mean a map $s : H\backslash G \to G$ such that $\pi_{H\backslash G} \circ s$ is the identity map.

From now on, let s denote a Γ-section sending Γe_G to e_G. In this section, we shall define co-induction for both measurable and topological dynamical systems.

4.1. *Co-induction for measurable dynamical systems.*

Let $(X, \mathcal{B}, \mu, \Gamma)$ be an MDS and s a Γ-section in G. By the *co-induced action from* $(X, \mathcal{B}, \mu, \Gamma)$ we mean the G-action defined by the MDS $(Y, \mathcal{D}, \nu, G)$, where $(Y, \mathcal{D}, \nu) = \prod_{\Gamma\backslash G}(X, \mathcal{B}, \mu)$ and $(gy)_\theta = s(\theta)gs(\theta g)^{-1}y_{\theta g}$ whenever $g \in G$, $y = (y_\zeta)_{\zeta\in\Gamma\backslash G} \in Y$, $\theta \in \Gamma\backslash G$. It is not hard to check that this is well defined and independent of the particular choice of Γ-sections sending Γe_G to e_G.

Let $(X, \mathcal{B}, \mu, \Gamma)$ be an MDS and $(Y, \mathcal{D}, \nu, G)$ the co-induced action as above. For convenience, we set

$$V_{A_i,\theta_i;1\le i\le n} = \{(x_\theta)_{\theta\in\Gamma\backslash G} : x_{\theta_i} \in A_i,\ 1 \le i \le n\} \in \mathcal{D}$$

and

$$P_{\alpha_i,\theta_i;1\le i\le n} = \{V_{B_i,\theta_i;1\le i\le n} : B_i \in \alpha_i,\ 1 \le i \le n\} \in \mathcal{P}_Y$$

whenever $\{\theta_1, \ldots, \theta_n\} \subseteq \Gamma\backslash G$, $A_1, \ldots, A_n \in \mathcal{B}$, $\alpha_1, \ldots, \alpha_n \in \mathcal{P}_X$, $n \in \mathbb{N}$. It has been proved that $(Y, \mathcal{D}, \nu, G)$ is Bernoulli if and only if $(X, \mathcal{B}, \mu, \Gamma)$ is Bernoulli [**7**, Corollary 3.3], and $(Y, \mathcal{D}, \nu, G)$ has c.p.e. if and only if $(X, \mathcal{B}, \mu, \Gamma)$ has c.p.e. [**7**, Theorem 5.2]. Moreover, the following theorem has been shown [**7**, Proposition 3.4].

THEOREM 4.1. *Let $(X, \mathcal{B}, \mu, \Gamma)$ be an MDS and $(Y, \mathcal{D}, \nu, G)$ the co-induced action. Then $h_\nu(G, Y) = h_\mu(\Gamma, X)$.*

It is now easy to see the following lemma.

LEMMA 4.1. *Let $(X_1, \mathcal{B}_1, \mu_1, \Gamma)$ and $(X_2, \mathcal{B}_2, \mu_2, \Gamma)$ be MDSs and*

$$\pi_X : (X_2, \mathcal{B}_2, \mu_2, \Gamma) \to (X_1, \mathcal{B}_1, \mu_1, \Gamma)$$

a factor map. Denote by $(Y_i, \mathcal{D}_i, \nu_i, G)$ the co-induced actions from $(X_i, \mathcal{B}_i, \mu_i, \Gamma)$, $i = 1, 2$, respectively. Then $(Y_1, \mathcal{D}_1, \nu_1, G)$ is a factor of $(Y_2, \mathcal{D}_2, \nu_2, G)$ via the factor map

$$\pi_Y : ((x_2)_\theta)_{\theta\in\Gamma\backslash G} \mapsto (\pi_X((x_2)_\theta))_{\theta\in\Gamma\backslash G}.$$

Proof. It is easy to check that $\pi_Y : (Y_2, \mathcal{D}_2, \nu_2) \to (Y_1, \mathcal{D}_1, \nu_1)$ is a measurable map and $\pi_Y\nu_2 = \nu_1$. It remains to show that π_Y commutes with the actions of G over $(Y_2, \mathcal{D}_2, \nu_2)$ and $(Y_1, \mathcal{D}_1, \nu_1)$. Let $g \in G$, $((x_2)_\theta)_{\theta\in\Gamma\backslash G} \in Y_2$ and s a Γ-section sending Γe_G to e_G.

$$\begin{aligned} g\pi_Y(((x_2)_\theta)_{\theta\in\Gamma\backslash G}) &= g((\pi_X((x_2)_\theta))_{\theta\in\Gamma\backslash G}) \\ &= (s(\theta)gs(\theta g)^{-1}\pi_X((x_2)_{\theta g}))_{\theta\in\Gamma\backslash G} \\ &= (\pi_X(s(\theta)gs(\theta g)^{-1}(x_2)_{\theta g}))_{\theta\in\Gamma\backslash G} \quad (\text{as } s(\theta)gs(\theta g)^{-1} \in \Gamma) \\ &= \pi_Y(s(\theta)gs(\theta g)^{-1}(x_2)_{\theta g})_{\theta\in\Gamma\backslash G} = \pi_Y g(((x_2)_\theta)_{\theta\in\Gamma\backslash G}). \end{aligned}$$

This finishes our proof. □

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

4.2. *Co-induction for topological dynamical systems.* Let (X, Γ) be a TDS and s a Γ-section. By the *co-induced action of* (X, Γ) we mean the TDS (Y, G), where $Y = \prod_{\Gamma\backslash G} X$ and $(gy)_\theta = s(\theta)gs(\theta g)^{-1}y_{\theta g}$ whenever $g \in G$, $y = (y_\zeta)_{\zeta\in\Gamma\backslash G} \in Y$, $\theta \in \Gamma\backslash G$. Arguing as in [**7**], it is not hard to check that this is well defined and independent of the particular choice of Γ-sections sending Γe_G to e_G. For convenience, set

$$U_{B_i,\theta_i;1\le i\le n} = \{(x_\theta)_{\theta\in\Gamma\backslash G} : x_{\theta_i} \in B_i,\ 1 \le i \le n\} \in \mathcal{B}_Y$$

and

$$\mathcal{U}_{\mathcal{U}_i,\theta_i;1\le i\le n} = \{U_{U_i,\theta_i;1\le i\le n} : U_i \in \mathcal{U}_i,\ 1 \le i \le n\}$$

whenever $\{\theta_1, \ldots, \theta_n\} \subseteq \Gamma\backslash G$, $B_1, \ldots, B_n \in \mathcal{B}_X$, $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathcal{C}_X^o$, $n \in \mathbb{N}$.

By the arguments of Lemma 4.1, one easily sees the following lemma.

LEMMA 4.2. *Let (X_1, Γ) and (X_2, Γ) be TDSs and $\pi_X : (X_2, \Gamma) \to (X_1, \Gamma)$ a factor map. Denote by (Y_i, G) the co-induced actions from (X_i, Γ), $i = 1, 2$, respectively. Then (Y_1, G) is a factor of (Y_2, G) via the factor map $\pi_Y : ((x_2)_\theta)_{\theta\in\Gamma\backslash G} \mapsto (\pi_X((x_2)_\theta))_{\theta\in\Gamma\backslash G}$.*

We can obtain a counterpart of Theorem 4.1 in the topological setting.

THEOREM 4.2. *Let (X, Γ) be a TDS and (Y, G) the co-induced action. Then $h_{\text{top}}(G, Y) = h_{\text{top}}(\Gamma, X)$.*

Proof. The inequality $\ge$ follows from Theorems 2.1 and 4.1. Now we prove $\le$.

Let $\pi_{\Gamma e_G} : Y \to X$, $(x_\theta)_{\theta\in\Gamma\backslash G} \mapsto x_{\Gamma e_G}$. In $\mathcal{P}_X$, we may choose a sequence $\xi_1 \preceq \xi_2 \preceq \cdots$ with $\lim_{n\to+\infty} \operatorname{diam}(\xi_n) = 0$ and set $\eta_n = P_{\xi_n,\Gamma e_G} \in \mathcal{P}_Y$ for each $n \in \mathbb{N}$.

CLAIM 1. *$\mathcal{B}_Y$ is generated by $\{(\eta_n)_F : F \in \mathcal{F}_G, n \in \mathbb{N}\}$.*

Proof of Claim 1. Let $n \in \mathbb{N}$, $\{\theta_1, \ldots, \theta_n\} \subseteq \Gamma\backslash G$, and let $U_1, \ldots, U_n$ be non-empty open subsets of X. We only need to show that there exist $F \in \mathcal{F}_G$ and $m \in \mathbb{N}$ such that some element of $(\eta_m)_F$ is contained in $U_{U_i,\theta_i;1\le i\le n}$. For each $1 \le i \le n$, and $f_i \in \theta_i$, a suitable choice of $\{\xi_n\}_{n\in\mathbb{N}}$ implies that for $m \in \mathbb{N}$ large enough such that each $f_i s(\theta_i)^{-1}U_i$, $1 \le i \le n$ contains some element of ξ_m (say B_i). Obviously, $F \doteq \{f_1, \ldots, f_n\} \in \mathcal{F}_G$ and $\bigcap_{i=1}^n f_i^{-1}V_{B_i,\Gamma e_G}$ is an element of $(\eta_m)_F$. Now we have to check that $\bigcap_{i=1}^n f_i^{-1}V_{B_i,\Gamma e_G} \subseteq U_{U_i,\theta_i;1\le i\le n}$. In fact, if $(x_\theta)_{\theta\in\Gamma\backslash G} \in \bigcap_{i=1}^n f_i^{-1}V_{B_i,\Gamma e_G}$ then, for each $1 \le i \le n$, $f_i(x_\theta)_{\theta\in\Gamma\backslash G} \in V_{B_i,\Gamma e_G}$; equivalently, $s(\Gamma e_G)f_i s(\Gamma f_i)^{-1}x_{\Gamma f_i} \in B_i$, i.e. $f_i s(\theta_i)^{-1}x_{\theta_i} \in B_i$, thus, $x_{\theta_i} \in s(\theta_i)f_i^{-1}B_i \subseteq U_i$ (recall that $B_i \subseteq f_i s(\theta_i)^{-1}U_i$). □

CLAIM 2. *$h_\nu(G, \eta_n) \le h_{\text{top}}(\Gamma, X)$ for each $n \in \mathbb{N}$ and any $\nu \in \mathcal{M}(Y, G)$.*

Proof of Claim 2. Denote by $\mathcal{F}_\Gamma$ the set of all non-empty finite subsets of Γ. Let $n \in \mathbb{N}$ and $\nu \in \mathcal{M}(Y, G)$. Set $\mu = \pi_{\Gamma e_G}\nu$. It is not hard to check that $\mu \in \mathcal{M}(X, \Gamma)$. Then

$$\begin{aligned} h_\nu(G, \eta_n) &\le \inf_{F\in\mathcal{F}_\Gamma} \frac{1}{|F|} H_\nu((\eta_n)_F) \quad (\text{using (2)}) = \inf_{F\in\mathcal{F}_\Gamma} \frac{1}{|F|} H_\nu(P_{(\xi_n)_F,\Gamma e_G}) \\ &= \inf_{F\in\mathcal{F}_\Gamma} \frac{1}{|F|} H_\mu((\xi_n)_F) \quad (\text{as } \mu = \pi_{\Gamma e_G}\nu) \\ &= h_\mu(\Gamma, \xi_n) \quad (\text{using (2)}) \le h_{\text{top}}(\Gamma, X), \end{aligned}$$

where the last inequality follows from Theorem 2.1. □

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Now let $\nu \in \mathcal{M}(Y, G)$ and choose $E_1 \subseteq E_2 \subseteq \cdots$ in $\mathcal{F}_G$ with $\bigcup_{n\in\mathbb{N}} E_n = G$. Then

$$\begin{aligned} h_\nu(G, Y) &= \lim_{n\to+\infty} h_\nu(G, (\eta_n)_{E_n}) \quad (\text{by Claim 1, as } (\eta_1)_{E_1} \preceq (\eta_2)_{E_2} \preceq \cdots) \\ &= \lim_{n\to+\infty} h_\nu(G, \eta_n) \quad (\text{using Proposition 2.2}) \\ &\leq h_{\text{top}}(\Gamma, X) \quad (\text{by Claim 2}). \end{aligned} \tag{18}$$

Letting ν vary over all elements of $\mathcal{M}(Y, G)$ and using Theorem 2.1, we obtain $h_{\text{top}}(G, Y) \leq h_{\text{top}}(\Gamma, X)$. This completes our proof. □

Let (X, Γ) be a TDS and (Y, G) the co-induced action. Generally, we cannot expect $\mathcal{M}(Y, G)$ to be the set of all G-invariant elements in $\bigotimes_{\Gamma\backslash G} \mathcal{M}(X)$. However, the following is a direct corollary of Theorems 2.1, 4.1 and 4.2.

COROLLARY 4.1. *Let (X, Γ) be a TDS and (Y, G) the co-induced action. Then*

$$h_{\text{top}}(G, Y) = \sup_{\mu\in\mathcal{M}(X,\Gamma)} h_{\bigotimes_{\Gamma\backslash G}\mu}(G, Y) = \sup_{\mu\in\mathcal{M}^e(X,\Gamma)} h_{\bigotimes_{\Gamma\backslash G}\mu}(G, Y).$$

Remark 4. Let (X, Γ) be a TDS and (Y, G) the co-induced action. Dooley *et al* [**7**, Proposition 3.6] tell us that $\{\bigotimes_{\Gamma\backslash G}\mu : \mu \in \mathcal{M}^e(X, \Gamma)\} \subseteq \mathcal{M}^e(Y, G)$, and if $[G : \Gamma] = +\infty$ then $\{\bigotimes_{\Gamma\backslash G}\mu : \mu \in \mathcal{M}(X, \Gamma)\} \subseteq \mathcal{M}^e(Y, G)$. Thus, for the co-induced action (Y, G), Corollary 4.1 is a stronger statement than Theorem 2.1.

PROPOSITION 4.1. *Let (X, Γ) be a TDS and (Y, G) the co-induced action.*
(1) *Let $\nu \in \mathcal{M}(Y, G)$. For each $\theta \in \Gamma\backslash G$, define ν_θ by $\nu_\theta(A) = \nu(U_{A,\theta})$ for each $A \in \mathcal{B}_X$. Then $\nu_\theta = s(\theta)gs(\theta g)^{-1}\nu_{\theta g} \in \mathcal{M}(X)$ whenever $\theta \in \Gamma\backslash G$ and $g \in G$.*
(2) *Assume that $\{\nu_\theta : \theta \in \Gamma\backslash G\} \subseteq \mathcal{M}(X)$ satisfies $\nu_\theta = s(\theta)gs(\theta g)^{-1}\nu_{\theta g}$ whenever $\theta \in \Gamma\backslash G$ and $g \in G$. Then:*
 (a) *$\mu \doteq \bigotimes_{\theta\in\Gamma\backslash G} \nu_\theta \in \mathcal{M}(Y, G)$; and*
 (b) *$\nu_\theta \in \mathcal{M}(X, \Gamma)$ for each $\theta \in \Gamma\backslash G$ and is independent of $\theta \in \Gamma\backslash G$.*

Proof. (1) Let $\theta \in \Gamma\backslash G$ and $g \in G$. It is easy to check that $\nu_\theta \in \mathcal{M}(X)$. Let $A \in \mathcal{B}_X$. Then

$$\begin{aligned} \nu_\theta(A) &= \nu(U_{A,\theta}) = \nu(g^{-1}(U_{A,\theta})) \quad (\text{as } \nu \in \mathcal{M}(Y, G)) \\ &= \nu(U_{(s(\theta)gs(\theta g)^{-1})^{-1}A,\theta g}) = \nu_{\theta g}((s(\theta)gs(\theta g)^{-1})^{-1}A) \\ &= (s(\theta)gs(\theta g)^{-1}\nu_{\theta g})(A). \end{aligned}$$

This equality means that $\nu_\theta = s(\theta)gs(\theta g)^{-1}\nu_{\theta g}$.

(2) (a) It is obvious that $\mu \in \mathcal{M}(Y)$. Now let $\emptyset \neq \Theta \subseteq \Gamma\backslash G$ be a finite subset, $\{A_\theta : \theta \in \Theta\} \subseteq \mathcal{B}_X$ and $g \in G$. By the definition, one directly has

$$\begin{aligned} (g\mu)(U_{A_\theta,\theta;\theta\in\Theta}) &= \mu(g^{-1}(U_{A_\theta,\theta;\theta\in\Theta})) = \mu(U_{(s(\theta)gs(\theta g)^{-1})^{-1}A_\theta,\theta g;\theta\in\Theta}) \\ &= \prod_{\theta\in\Theta} \nu_{\theta g}((s(\theta)gs(\theta g)^{-1})^{-1}A_\theta) \\ &= \prod_{\theta\in\Theta} \nu_\theta(A_\theta) \quad (\text{as } \nu_\theta = s(\theta)gs(\theta g)^{-1}\nu_{\theta g}) = \mu(U_{A_\theta,\theta;\theta\in\Theta}). \end{aligned}$$

This gives $g\mu = \mu$ (since $\{A_\theta : \theta \in \Theta\} \subseteq \mathcal{B}_X$ and $\emptyset \neq \Theta \subseteq \Gamma\backslash G$ are arbitrary). Moreover, letting g vary over all elements of G, we obtain $\mu \in \mathcal{M}(Y, G)$.

(b) Let $\theta \in \Gamma\backslash G$. It is easy to check that $G_\theta \doteq \{g \in G : \theta = \theta g\} = s(\theta)^{-1}\Gamma s(\theta)$ and so, for each $g \in G_\theta$, $\nu_\theta = s(\theta) g s(\theta)^{-1}\nu_\theta$, i.e. $\nu_\theta \in \mathcal{M}(X, \Gamma)$. Moreover, observe that, for $\theta \in \Gamma\backslash G$, $s(\theta) g s(\theta g)^{-1} \in \Gamma$ for each $g \in G$ and $\{\theta g : g \in G\} = \Gamma\backslash G$, which implies that ν_θ is independent of $\theta \in \Gamma\backslash G$. This completes the proof. □

Remark 5. Note that in general $\nu \not\approx \bigotimes_{\theta\in\Gamma\backslash G} \nu_\theta$, as not every invariant measure on a product space is an invariant product measure.

5. *The Pinsker algebra of the co-induced measurable dynamical system*

The main result of this section is given in the following theorem.

THEOREM 5.1. *Let $(X, \mathcal{B}, \mu, \Gamma)$ be an MDS and $(Y, \mathcal{D}, \nu, G)$ the co-induced action. Then $\mathcal{P}(Y, \mathcal{D}, \nu, G) = \prod_{\Gamma\backslash G} \mathcal{P}(X, \mathcal{B}, \mu, \Gamma)$.*

One direction of Theorem 5.1 is easy to obtain.

LEMMA 5.1. *Under the assumption of Theorem 5.1,*

$$\mathcal{P}(Y, \mathcal{D}, \nu, G) \supseteq \prod_{\Gamma\backslash G} \mathcal{P}(X, \mathcal{B}, \mu, \Gamma).$$

Proof. Set $\mathcal{A} = \mathcal{P}(X, \mathcal{B}, \mu, \Gamma)$. Since $(X, \mathcal{A}, \mu, \Gamma)$ is a factor of $(X, \mathcal{B}, \mu, \Gamma)$, the co-induced action from $(X, \mathcal{A}, \mu, \Gamma)$ is a factor of $(Y, \mathcal{D}, \nu, G)$ (by Lemma 4.1). In fact, $(Y, \prod_{\Gamma\backslash G} \mathcal{A}, \nu, G)$ is the co-induced action from $(X, \mathcal{A}, \mu, \Gamma)$ and, by Theorem 4.1, the measure-theoretic ν-entropy of $(Y, \prod_{\Gamma\backslash G} \mathcal{A}, \nu, G)$ equals zero, the measure-theoretic μ-entropy of $(X, \mathcal{A}, \mu, \Gamma)$, which implies $\prod_{\Gamma\backslash G} \mathcal{A} \subseteq \mathcal{P}(Y, \mathcal{D}, \nu, G)$, which gives the result. □

The other direction of Theorem 5.1 follows from the following proposition.

PROPOSITION 5.1. *Let $(X, \mathcal{B}, \mu, \Gamma)$ be an MDS, $\mathcal{A} \subseteq \mathcal{B}$ an invariant sub-σ-algebra of $\mathcal{B}$, and $(Y, \mathcal{D}, \nu, G)$ the co-induced action. If $(X, \mathcal{B}, \mu, \Gamma)$ has $\mathcal{A}$-relative c.p.e. then $(Y, \mathcal{D}, \nu, G)$ has $\prod_{\Gamma\backslash G} \mathcal{A}$-relative c.p.e.*

Proof. Set $\mathcal{C} = \prod_{\Gamma\backslash G} \mathcal{A}$. We shall use ideas from the proof of [**7**, Theorem 5.2].

Let $\xi \in \mathcal{P}_Y$ and $\epsilon > 0$. By Theorem 3.1, it suffices to prove that there exists $K \in \mathcal{F}_G$ such that if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (K\backslash\{e_G\}) = \emptyset$ then

$$H_\nu(\xi|\mathcal{C}) - \frac{1}{|F|} H_\nu(\xi_F|\mathcal{C}) < \epsilon. \tag{19}$$

Suppose that $\mathcal{D}_X = \mathcal{B} \times \prod_{(\Gamma\backslash G)\backslash \Gamma e_G} \mathcal{N}_X$. Then $s(\theta)^{-1}\mathcal{D}_X = \mathcal{B} \times \prod_{(\Gamma\backslash G)\backslash\theta} \mathcal{N}_X$ for each $\theta \in \Gamma\backslash G$, and so $\mathcal{D}$ is generated by $s(\theta)^{-1}\mathcal{D}_X$, $\theta \in \Gamma\backslash G$. Moreover, we may choose $\alpha \in \mathcal{P}_X$ and $\{\theta_1, \ldots, \theta_n\} \subseteq \Gamma\backslash G$, $n \in \mathbb{N}$ such that

$$H_\nu(\xi|\eta) + H_\nu(\eta|\xi) < \frac{\epsilon}{2} \tag{20}$$

for some $\eta \preceq \zeta_M$, where $\zeta = P_{\alpha,\Gamma e_G}$ and $M = \{s(\theta_1), \ldots, s(\theta_n)\} \in \mathcal{F}_G$. As $(X, \mathcal{B}, \mu, \Gamma)$ has $\mathcal{A}$-relative c.p.e., by Theorem 3.1, there exists $J \in \mathcal{F}_\Gamma$ such that if $S \in \mathcal{F}_\Gamma$ satisfies $SS^{-1} \cap (J\backslash\{e_G\}) = \emptyset$ then

$$\left|\frac{1}{|S|} H_\mu(\alpha_S|\mathcal{A}) - H_\mu(\alpha|\mathcal{A})\right| < \frac{\epsilon}{2n}, \tag{21}$$

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

where $\mathcal{F}_\Gamma$ denotes the set of all non-empty finite subsets of Γ. Moreover, we may assume without loss of generality that $e_G \in J$. First, note the following.

CLAIM. *Equation (19) holds for* $K \doteq M^{-1}JM \in \mathcal{F}_G$ *when we replace* ϵ *by* $\epsilon/2$ *and* ξ *by* ζ_M.

Proof of claim. Let $F \in \mathcal{F}_G$ with $FF^{-1} \cap (K \backslash \{e_G\}) = \emptyset$. Observe that if there exist $f_i, f_j \in F$, $1 \le i, j \le n$ such that $s(\theta_i) f_i (s(\theta_j) f_j)^{-1} \in J$, or, equivalently, if $f_i f_j^{-1} \in s(\theta_i)^{-1} J s(\theta_j) \subseteq K$, then $f_i f_j^{-1} = e_G$ since $FF^{-1} \cap (K \backslash \{e_G\}) = \emptyset$. This implies that $s(\theta_i)s(\theta_j)^{-1} \in J \subseteq \Gamma$, hence $s(\theta_i) \in \Gamma s(\theta_j)$ and so $\theta_i = \theta_j$. Thus, $s(\theta_i) = s(\theta_j)$. Summing up, and using the fact that $e_G \in J$, we obtain:

$$|MF| = n|F| \quad \text{and} \quad (MF)(MF)^{-1} \cap (J \backslash \{e_G\}) = \emptyset. \tag{22}$$

Now set $\Theta = \{\theta_1, \ldots, \theta_n\}F$. For each $\theta \in \Theta$, note that $\Theta_\theta s(\theta)^{-1} \in \mathcal{F}_\Gamma$ and $(\Theta_\theta s(\theta)^{-1})(\Theta_\theta s(\theta)^{-1})^{-1} \cap (J \backslash \{e_G\}) \subseteq (MF)(MF)^{-1} \cap (J \backslash \{e_G\}) = \emptyset$ (using (22)), where $\Theta_\theta = \{s(\theta_i) g : 1 \le i \le n \text{ and } g \in F \text{ satisfy } \theta_i g = \theta\}$. Thus, by (21),

$$\left| \frac{1}{|\Theta_\theta s(\theta)^{-1}|} H_\mu(\alpha_{\Theta_\theta s(\theta)^{-1}} | \mathcal{A}) - H_\mu(\alpha|\mathcal{A}) \right| < \frac{\epsilon}{2n}. \tag{23}$$

Now we may deduce the following:

$$\begin{aligned}
H_\nu((\zeta_M)_F|\mathcal{C}) &= H_\nu(\zeta_{MF}|\mathcal{C}) \\
&= H_\nu\bigg(\bigvee_{g\in F} \bigvee_{i=1}^{n} P_{(s(\theta_i)gs(\theta_i g)^{-1})^{-1}\alpha, \theta_i g} | \mathcal{C}\bigg) \\
&= H_\nu\bigg(\prod_{\theta\in\Theta} \bigvee_{g\in\Theta_\theta} (gs(\theta)^{-1})^{-1}\alpha \times \prod_{(\Gamma\backslash G)\backslash\Theta} \mathcal{N}_X | \mathcal{C}\bigg) \\
&= \sum_{\theta\in\Theta} H_\mu\bigg(\bigvee_{g\in\Theta_\theta} (gs(\theta)^{-1})^{-1}\alpha | \mathcal{A}\bigg) \quad \bigg(\text{as } \nu = \bigotimes_{\Gamma\backslash G} \mu\bigg) \\
&> \sum_{\theta\in\Theta} |\Theta_\theta| \bigg(H_\mu(\alpha|\mathcal{A}) - \frac{\epsilon}{2n}\bigg) \quad (\text{using (23)}) \\
&= |MF| \bigg(H_\mu(\alpha|\mathcal{A}) - \frac{\epsilon}{2n}\bigg) \\
&= |F| \bigg(nH_\mu(\alpha|\mathcal{A}) - \frac{\epsilon}{2}\bigg) \quad (\text{using (22)}).
\end{aligned} \tag{24}$$

However, since $\nu = \bigotimes_{\Gamma\backslash G} \mu$, we have

$$H_\nu(\zeta_M|\mathcal{C}) = H_\nu\bigg(\bigvee_{i=1}^{n} s(\theta_i)^{-1} P_{\alpha, \Gamma e_G} | \mathcal{C}\bigg) = H_\nu(P_{\alpha, \theta_i; 1\le i\le n}|\mathcal{C}) = nH_\mu(\alpha|\mathcal{A}).$$

Now, using (24), we see that the claim holds. □

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Now, if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (K \setminus \{e_G\}) = \emptyset$, we may argue, as for (17),

$$\begin{aligned} H_\nu(\xi|\mathcal{C}) - \frac{1}{|F|} H_\nu(\xi_F|\mathcal{C}) &\le H_\nu(\eta|\mathcal{C}) - \frac{1}{|F|} H_\nu(\eta_F|\mathcal{C}) + (H_\nu(\xi|\eta) + H_\nu(\eta|\xi)) \\ &\le H_\nu(\zeta_M|\mathcal{C}) - \frac{1}{|F|} H_\nu((\zeta_M)_F|\mathcal{C}) + (H_\nu(\xi|\eta) + H_\nu(\eta|\xi)) \\ &< \epsilon \quad \text{(using (20) and claim).} \end{aligned}$$

This proves (19) and so completes the proof of the theorem. □

Now we give a proof of Theorem 5.1.

Proof of Theorem 5.1. As mentioned in Remark 3, $(X, \mathcal{B}, \mu, \Gamma)$ has $\mathcal{P}(X, \mathcal{B}, \mu, \Gamma)$-relative c.p.e., and so, by Proposition 5.1, $(Y, \mathcal{D}, \nu, G)$ has $\prod_{\Gamma\setminus G} \mathcal{P}(X, \mathcal{B}, \mu, \Gamma)$-relative c.p.e. Thus, using Remark 3 again, $\mathcal{P}(Y, \mathcal{D}, \nu, G) \subseteq \prod_{\Gamma\setminus G} \mathcal{P}(X, \mathcal{B}, \mu, \Gamma)$. Combining with Lemma 5.1, we conclude the proof. □

6. *U.p.e. and c.p.e. for topological co-induction*

In [**16**], local entropy theory for a countable discrete infinite amenable group action was introduced and systematically studied. In particular, the properties of u.p.e. and c.p.e. and the concept of entropy tuples were studied in both the measure-theoretic and the topological settings. In this section, we shall discuss them for co-induced actions of topological dynamical systems.

Let (X, G) be a TDS and $n \in \mathbb{N}\setminus\{1\}$. We say that (X, G) has *u.p.e. of order* n if any cover of X by n non-dense open sets has positive topological entropy; *u.p.e.* if (X, G) has u.p.e. of order two; *u.p.e. of all orders* if (X, G) has u.p.e. of order m for each $m \in \mathbb{N}\setminus\{1\}$; and *c.p.e.* if each non-trivial factor of (X, G) has positive entropy.

As shown in [**16**], these properties can be characterized by entropy tuples.

Let (X, G) be a TDS, $\mu \in \mathcal{M}(X, G)$, $n \in \mathbb{N}\setminus\{1\}$ and $(x_1, \ldots, x_n) \in X^n \setminus \Delta_n(X)$, where $\Delta_n(X) = \{(z_1, \ldots, z_n) \in X^n : z_1 = \cdots = z_n\}$. We say that $\mathcal{U} \in \mathcal{C}_X$ is *admissible with respect to* $(x_1, \ldots, x_n)$ if $\{x_1, \ldots, x_n\}$ is not contained in the closure of U for any $U \in \mathcal{U}$. $(x_1, \ldots, x_n)$ is called a *topological entropy* n*-tuple of* (X, G) if $h_{\text{top}}(G, \mathcal{U}) > 0$ whenever $\mathcal{U} \in \mathcal{C}_X^o$ is admissible with respect to $(x_1, \ldots, x_n)$, and a μ*-entropy* n*-tuple of* (X, G) if $h_\mu(G, \alpha) > 0$ whenever $\alpha \in \mathcal{P}_X$ is admissible with respect to $(x_1, \ldots, x_n)$. Denote by $E_n(X, G)$ and $E_n^\mu(X, G)$ the set of all topological entropy n-tuples and μ-entropy n-tuples of (X, G), respectively. It is easy to check that both $E_n(X, G)$ and $E_n^\mu(X, G)$ are invariant. In fact, $E_n^\mu(X, G)$ can be characterized by $\text{supp}(\lambda_n(\mu))$, the support of an invariant probability measure $\lambda_n(\mu)$ over $(X^n, (\mathcal{B}_X)^n)$, where $\lambda_n(\mu)$ is given by

$$\lambda_n(\mu)\bigg(\prod_{i=1}^n A_i\bigg) = \int_X \prod_{i=1}^n \mathbb{E}_\mu(1_{A_i}|\mathcal{P}(X, \mathcal{B}_X, \mu, G))\, d\mu \tag{25}$$

whenever $A_i \in \mathcal{B}_X$, $i = 1, \ldots, n$ and $(\mathcal{B}_X)^n = \mathcal{B}_X \times \cdots \times \mathcal{B}_X$ (n times).

It is not too hard to obtain [**16**, Proposition 6.3(3)].

PROPOSITION 6.1. *Let* $\pi : (Z, G) \to (X, G)$ *be a factor map between TDSs and* $n \in \mathbb{N}\setminus\{1\}$. *Then*

$$E_n(X, G) \subseteq \{(\pi z_1, \ldots, \pi z_n) : (z_1, \ldots, z_n) \in E_n(Z, G)\} \subseteq E_n(X, G) \cup \Delta_n(X).$$

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Let (X, G) be a TDS. The *support* $\text{supp}(X, G)$ *of* (X, G) is defined as $\cup\{\text{supp}(\mu) : \mu \in \mathcal{M}(X, G)\}$. We say that (X, G) is *fully supported* if $\text{supp}(X, G) = X$.

The most important properties concerning entropy tuples in both measure-theoretic and topological settings are given by the following variational relationship.

PROPOSITION 6.2. *Let (X, G) be a TDS. Then:*

(1) $E_n(X, G) \supseteq E_n^\mu(X, G) = \text{supp}(\lambda_n(\mu))\setminus\Delta_n(X)$ *whenever* $\mu \in \mathcal{M}(X, G)$ *and* $n \in \mathbb{N}\setminus\{1\}$*;*

(2) $(\text{supp}(\mu))^n \supseteq \text{supp}(\lambda_n(\mu)) = E_n^\mu(X, G) \cup \Delta_n^\mu(X)$ *whenever* $\mu \in \mathcal{M}(X, G)$ *and* $n \in \mathbb{N}\setminus\{1\}$*, where* $\Delta_n^\mu(X) = \{(x_1, \ldots, x_n) \in (\text{supp}(\mu))^n : x_1 = \cdots = x_n\}$*;*

(3) *there exists* $\mu \in \mathcal{M}(X, G)$ *such that* $E_n(X, G) = E_n^\mu(X, G)$ *(and hence* $E_n(X, G) \subseteq (\text{supp}(X, G))^n$*) for all* $n \in \mathbb{N}\setminus\{1\}$*;*

(4) *for each* $n \in \mathbb{N}\setminus\{1\}$*,* (X, G) *has u.p.e. of order* n *if and only if* $E_n(X, G) = X^n\setminus\Delta_n(X)$*; and*

(5) (X, G) *has c.p.e. if and only if* X^2 *is the closed invariant equivalence relation generated by* $E_2(X, G)$*. In particular, if* (X, G) *has c.p.e. then it is fully supported.*

Proof. (1) and (3) are [**16**, Theorem 6.16] (except $E_n(X, G) \subseteq (\text{supp}(X, G))^n$); (4) and (5) are proved in [**16**, §7]. Now let $\mu \in \mathcal{M}(X, G)$ and $n \in \mathbb{N}\setminus\{1\}$. It is easy to check that $(\text{supp}(\mu))^n \supseteq \text{supp}(\lambda_n(\mu))$ and $\text{supp}(\lambda_n(\mu)) \cap \Delta_n(X) = \Delta_n^\mu(X)$ from (25), and so (2) follows from (1). Moreover, $E_n(X, G) \subseteq (\text{supp}(X, G))^n$ follows from (2) and the definition of $\text{supp}(X, G)$. □

To proceed with the proof of Theorem 6.1, we need the following lemma.

LEMMA 6.1. *Let (X, G) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. Then*

$$h_{\text{top}}(G, \mathcal{U}) > 0 \Longleftrightarrow \inf_{F\in\mathcal{F}_G} \frac{1}{|F|} \log N(\mathcal{U}_F) > 0. \tag{26}$$

In particular, if $h_{\text{top}}(G, \mathcal{U}) > 0$ *then* $h_{\text{top}}(\Gamma, \mathcal{U}) > 0$*. Here,* $h_{\text{top}}(\Gamma, \mathcal{U})$ *denotes the topological entropy of* $\mathcal{U}$ *under the action of the subgroup* Γ *on* X*.*

Proof. We need only prove (26). The implication $\Longleftarrow$ follows directly from the definition. To prove $\Longrightarrow$ we use some ideas from the proof of [**16**, Lemma 7.9].

By that proof, there exist $\mu \in \mathcal{M}^e(X, G)$, $\alpha \in \mathcal{P}_X$ and $\epsilon > 0$ such that, for $\beta \in \mathcal{P}_X$, if $\beta \succeq \mathcal{U}$,

$$H_\mu(\alpha|\beta \vee \mathcal{P}(X, \mathcal{B}_X, \mu, G)) \le H_\mu(\alpha|\mathcal{P}(X, \mathcal{B}_X, \mu, G)) - \epsilon. \tag{27}$$

Note that there exists $K \in \mathcal{F}_G$ such that if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (K\setminus\{e_G\}) = \emptyset$ then

$$\left|\frac{1}{|F|} H_\mu(\alpha_F|\mathcal{P}(X, \mathcal{B}_X, \mu, G)) - H_\mu(\alpha|\mathcal{P}(X, \mathcal{B}_X, \mu, G))\right| < \frac{\epsilon}{2} \tag{28}$$

(using Theorem 3.1 and Remark 3). For $E \in \mathcal{F}_G$, as in the proof of Theorem 3.1, there exists $S \in \mathcal{F}_G$ such that $SS^{-1} \cap (K\setminus\{e_G\}) = \emptyset$, $S \subseteq E$ and $(2|K| + 1)|S| \ge |E|$. Thus,

$$\left|\frac{1}{|S|} H_\mu(\alpha_S|\mathcal{P}(X, \mathcal{B}_X, \mu, G)) - H_\mu(\alpha|\mathcal{P}(X, \mathcal{B}_X, \mu, G))\right| < \frac{\epsilon}{2} \quad \text{(using (28))}. \tag{29}$$

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Moreover, we can choose $m \doteq N(\mathcal{U}_S)$ elements $V_1, \ldots, V_m$ from $\mathcal{U}_S$ covering the space X and set $\beta = \{V_1, V_2 \setminus V_1, \ldots, V_m \setminus (V_1 \cup \cdots \cup V_{m-1})\} \in \mathcal{P}_X$. Obviously, $\beta \succeq \mathcal{U}_S$ and so $g\beta \succeq \mathcal{U}$ for each $g \in S$. It follows that

$$\begin{aligned} H_\mu(\beta) &\geq H_\mu(\beta \vee \alpha_S | \mathcal{P}(X, \mathcal{B}_X, \mu, G)) - H_\mu(\alpha_S | \beta \vee \mathcal{P}(X, \mathcal{B}_X, \mu, G)) \quad \text{(using (1))} \\ &\geq H_\mu(\alpha_S | \mathcal{P}(X, \mathcal{B}_X, \mu, G)) - \sum_{g \in S} H_\mu(\alpha | g\beta \vee \mathcal{P}(X, \mathcal{B}_X, \mu, G)) \\ &\geq H_\mu(\alpha_S | \mathcal{P}(X, \mathcal{B}_X, \mu, G)) - |S|(H_\mu(\alpha | \mathcal{P}(X, \mathcal{B}_X, \mu, G)) - \epsilon) \quad \text{(using (27))} \\ &\geq \frac{|S|\epsilon}{2} \quad \text{(using (29))}, \end{aligned} \tag{30}$$

which implies

$$\begin{aligned} \frac{1}{|E|} \log N(\mathcal{U}_E) &\geq \frac{1}{|E|} \log N(\mathcal{U}_S) \geq \frac{1}{|E|} H_\mu(\beta) \quad \text{(using Proposition 2.2)} \\ &\geq \frac{|S|\epsilon}{2|E|} \quad \text{(using (30))} \\ &\geq \frac{\epsilon}{2(2|K|+1)} \quad \text{(as } (2|K|+1)|S| \geq |E|) > 0. \end{aligned} \tag{31}$$

Now the conclusion follows by letting E vary over all elements from $\mathcal{F}_G$ in (31). □

Remark 6. In fact, using the theory of [**16**] we can prove a stronger result, viz. $h_{\text{top}}(G, \mathcal{U}) \leq h_{\text{top}}(\Gamma, \mathcal{U})$ for any TDS (X, G) with $\mathcal{U} \in \mathcal{C}_X^o$ (and so $h_{\text{top}}(G, X) \leq h_{\text{top}}(\Gamma, X)$). We shall not use this result, so we just give a sketch of the proof. In the notation of [**16**], for $\mathcal{U} \in \mathcal{C}_X^o$, select $\mu \in \mathcal{M}(X, G) \subseteq \mathcal{M}(X, \Gamma)$ such that

$$h_{\text{top}}(G, \mathcal{U}) = \inf_{\alpha \in \mathcal{P}_X, \alpha \succeq \mathcal{U}} h_\mu(G, \alpha). \tag{32}$$

However, by (2) or Proposition 2.2, one has

$$\inf_{\alpha \in \mathcal{P}_X, \alpha \succeq \mathcal{U}} h_\mu(G, \alpha) \leq \inf_{\alpha \in \mathcal{P}_X, \alpha \succeq \mathcal{U}} h_\mu(\Gamma, \alpha) \leq h_{\text{top}}(\Gamma, \mathcal{U}). \tag{33}$$

The last step uses the fact that $\mu \in \mathcal{M}(X, \Gamma)$, and the theory built in [**16**]. Now, combining (32) with (33), we obtain the desired inequality.

A useful corollary of Lemma 6.1 is the following.

COROLLARY 6.1. *Let* (X, G) *be a TDS,* $\mu \in \mathcal{M}(X, G)$ *and* $n \in \mathbb{N} \setminus \{1\}$*. Then* $E_n(X, G) \subseteq E_n(X, \Gamma)$ *and* $E_n^\mu(X, G) \subseteq E_n^\mu(X, \Gamma)$.

Proof. $E_n(X, G) \subseteq E_n(X, \Gamma)$ and $E_n^\mu(X, G) \subseteq E_n^\mu(X, \Gamma)$ follow from Lemma 6.1 and the fact that $h_\mu(G, \alpha) \leq h_\mu(\Gamma, \alpha)$ for each $\alpha \in \mathcal{P}_X$ (using (2)), respectively. □

We shall also need the following lemma.

LEMMA 6.2. *Let* (X, Γ) *be a TDS and* (Y, G) *the co-induced action. Then* $\text{supp}(Y, G) = \prod_{\Gamma \setminus G} \text{supp}(X, \Gamma)$.

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Proof. Obviously, $\mathrm{supp}(Y, G) \supseteq \prod_{\Gamma\backslash G} \mathrm{supp}(X, \Gamma)$. Let $\nu \in \mathcal{M}(Y, G)$ and $\pi_{\Gamma e_G} : Y \to X$, $(x_\theta)_{\theta\in\Gamma\backslash G} \mapsto x_{\Gamma e_G}$. It is easy to check that $\pi_{\Gamma e_G}\nu \in \mathcal{M}(X, \Gamma)$, which implies that $\mathrm{supp}(\nu) \subseteq U_{\mathrm{supp}(\pi_{\Gamma e_G}\nu),\Gamma e_G} \subseteq U_{\mathrm{supp}(X,\Gamma),\Gamma e_G}$, and so (by the invariance of $\mathrm{supp}(\nu)$),

$$\mathrm{supp}(\nu) \subseteq \bigcap_{g\in G} gU_{\mathrm{supp}(X,\Gamma),\Gamma e_G} = \prod_{\Gamma\backslash G} \mathrm{supp}(X, \Gamma). \tag{34}$$

The result now follows since $\nu \in \mathcal{M}(Y, G)$ is arbitrary. □

Putting these results together, we obtain the following proposition.

PROPOSITION 6.3. *Let (X, Γ) be a TDS, $\mu \in \mathcal{M}(X, \Gamma)$, $n \in \mathbb{N}\backslash\{1\}$ and (Y, G) the co-induced action. Set $\Delta_n^s(X) = \Delta_n(X) \cap (\mathrm{supp}(X, \Gamma))^n$. Then:*
(1) $\mathrm{supp}(\lambda_n(\bigotimes_{\Gamma\backslash G} \mu)) = \bigotimes_{\Gamma\backslash G} \mathrm{supp}(\lambda_n(\mu))$*; and*
(2) $E_n(Y, G) = \prod_{\Gamma\backslash G}(E_n(X, \Gamma) \cup \Delta_n^s(X))\backslash\prod_{\Gamma\backslash G} \Delta_n^s(X)$.

Proof. (1) Observe that, by (25), on $(Y^n, \mathcal{B}_Y^n)$, $\lambda_n(\bigotimes_{\Gamma\backslash G} \mu)$ is given by

$$\begin{aligned}
&\lambda_n\bigg(\bigotimes_{\Gamma\backslash G} \mu\bigg)\bigg(\prod_{i=1}^n U_{A_{i,j},\theta_j;1\le j\le m}\bigg)\\
&\quad= \int_Y \prod_{i=1}^n \mathbb{E}_{\bigotimes_{\Gamma\backslash G}\mu}\bigg(1_{U_{A_{i,j},\theta_j;1\le j\le m}}\bigg|\mathcal{P}\bigg(Y, \mathcal{B}_Y, \bigotimes_{\Gamma\backslash G}\mu, G\bigg)\bigg)\, d\bigotimes_{\Gamma\backslash G}\mu\\
&\quad= \int_{\prod_{j=1}^m X} \prod_{i=1}^n \mathbb{E}_{\prod_{j=1}^m \mu}\bigg(1_{\prod_{j=1}^m A_{i,j}}\bigg|\prod_{j=1}^m \mathcal{P}(X, \mathcal{B}_X, \mu, \Gamma)\bigg)\, d\prod_{j=1}^m \mu\\
&\qquad\text{(using Theorem 5.1)}\\
&\quad= \prod_{j=1}^m \int_X \prod_{i=1}^n \mathbb{E}_\mu(1_{A_{i,j}}|\mathcal{P}(X, \mathcal{B}_X, \mu, \Gamma))\, d\mu = \prod_{j=1}^m \lambda_n(\mu)\bigg(\prod_{i=1}^n A_{i,j}\bigg),
\end{aligned} \tag{35}$$

where $\{\theta_1, \ldots, \theta_m\} \subseteq \Gamma\backslash G$, $m \in \mathbb{N}$ and $A_{i,j} \in \mathcal{B}_X$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. The conclusion follows readily.

(2) By Proposition 2.3 and Proposition 6.2(3), we may choose $\omega \in \mathcal{M}(X, \Gamma)$ with $E_n(X, \Gamma) = E_n^\omega(X, \Gamma)$ and $\Delta_n^s(X) = \Delta_n^\omega(X)$ (hence $\mathrm{supp}(\lambda_n(\omega)) = E_n(X, \Gamma) \cup \Delta_n^s(X)$). Thus,

$$\begin{aligned}
&E_n(Y, G)\\
&\quad\supseteq \mathrm{supp}\bigg(\lambda_n\bigg(\bigotimes_{\Gamma\backslash G}\omega\bigg)\bigg)\bigg\backslash \Delta_n(Y) \quad \text{(using Proposition 6.2)}\\
&\quad= \prod_{\Gamma\backslash G} \mathrm{supp}(\lambda_n(\omega))\bigg\backslash \Delta_n(Y) \quad \text{(using (1))}\\
&\quad= \prod_{\Gamma\backslash G} \mathrm{supp}(\lambda_n(\omega))\bigg\backslash \prod_{\Gamma\backslash G}\Delta_n^s(X) \quad \bigg(\text{as } \prod_{\Gamma\backslash G} \mathrm{supp}(\lambda_n(\omega)) \cap \Delta_n(Y) = \prod_{\Gamma\backslash G}\Delta_n^s(X)\bigg)\\
&\quad= \prod_{\Gamma\backslash G}(E_n(X, \Gamma) \cup \Delta_n^s(X))\bigg\backslash \prod_{\Gamma\backslash G}\Delta_n^s(X) \quad (\text{as } \mathrm{supp}(\lambda_n(\omega)) = E_n(X, \Gamma) \cup \Delta_n^s(X)).
\end{aligned}$$

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

We shall finish the proof by proving an equivalent version of the other direction:

$$E_n(Y, G) \subseteq \prod_{\Gamma\backslash G} (E_n(X, \Gamma) \cup \Delta_n^s(X)). \tag{36}$$

Let $(y_1, \ldots, y_n) \in E_n(Y, G)$. Say $y_i = (x_\theta^i)_{\theta\in\Gamma\backslash G}$, $1 \le i \le n$. Let $\vartheta \in \Gamma\backslash G$. Observe that (X, Γ) is a factor of (Y, Γ) via the factor map $\pi_{\Gamma e_G} : Y \to X$, $(x_\theta)_{\theta\in\Gamma\backslash G} \mapsto x_{\Gamma e_G}$. Furthermore, by Corollary 6.1, $(y_1, \ldots, y_n) \in E_n(Y, \Gamma)$ and hence $(x_\vartheta^1, \ldots, x_\vartheta^n) \in E_n(X, \Gamma) \cup \Delta_n(X)$. It now follows that $(x_\vartheta^1, \ldots, x_\vartheta^n) \in E_n(X, \Gamma) \cup \Delta_n^s(X)$ when $\vartheta = \Gamma e_G$ (by Propositions 6.1, 6.2 and Lemma 6.2). Finally, using an argument similar to (34), the invariance of $E_n(Y, G)$ implies that $(x_\vartheta^1, \ldots, x_\vartheta^n) \in E_n(X, \Gamma) \cup \Delta_n^s(X)$, even if $\vartheta \neq \Gamma e_G$. This proves (36) and completes our proof. □

Now we can prove the following theorem.

THEOREM 6.1. *Let (X, Γ) be a TDS and (Y, G) the co-induced action. Then:*

(1) *(Y, G) has c.p.e. if and only if (X, Γ) has c.p.e.;*

(2) *for each $n \in \mathbb{N}\backslash\{1\}$, (Y, G) has u.p.e. of order n if and only if (X, Γ) has u.p.e. of order n; and*

(3) *(Y, G) has u.p.e. of all orders if and only if (X, Γ) has u.p.e. of all orders.*

Proof. Observe that (3) follows directly from (2). Thus, it suffices to prove (1) and (2).

(1) If (X, Γ) does not have c.p.e., it is easy to check that (Y, G) does not have c.p.e. (using Lemma 4.2). That is, if (Y, G) has c.p.e. then (X, Γ) has c.p.e. Now, if (X, Γ) has c.p.e. then X^2 is the closed Γ-invariant equivalence relation generated by $E_2(X, \Gamma)$ (using Proposition 6.2), and so Y^2 is the closed G-invariant equivalence relation generated by $E_2(Y, G)$ (using Proposition 6.3), thus (Y, G) has c.p.e. (using Proposition 6.2 again).

(2) Let $n \in \mathbb{N}\backslash\{1\}$. First, assume that (X, Γ) has u.p.e. of order n. It follows that $E_n(X, \Gamma) = X^n\backslash\Delta_n(X)$ and so $E_n(Y, G) = Y^n\backslash\Delta_n(Y)$. This now implies that (Y, G) has u.p.e. of order n (using Propositions 6.2 and 6.3). Now, if (Y, G) has u.p.e. of order n, for each $(x_1, \ldots, x_n) \in X^n\backslash\Delta_n(X)$, we can select $(y_1, \ldots, y_n) \in Y^n\backslash\Delta_n(Y)$ with $(y_i)_{\Gamma e_G} = x_i$ for each $1 \le i \le n$. Note that if $\mathcal{U} \in \mathcal{C}_X^o$ is admissible with respect to $(x_1, \ldots, x_n)$ then $\mathcal{U}_{\mathcal{U},\Gamma e_G} \in \mathcal{C}_Y^o$ is admissible with respect to $(y_1, \ldots, y_n)$, and so $h_{\text{top}}(G, \mathcal{U}_{\mathcal{U},\Gamma e_G}) > 0$. Thus,

$$\begin{aligned} 0 &< h_{\text{top}}(\Gamma, \mathcal{U}_{\mathcal{U},\Gamma e_G}) \quad \text{(using Lemma 6.1)} \\ &= h_{\text{top}}(\Gamma, \mathcal{U}) \quad \text{(following directly from the definitions).} \end{aligned}$$

Since $\mathcal{U}$ is arbitrary, one has $(x_1, \ldots, x_n) \in E_n(X, \Gamma)$. Hence, $E_n(X, \Gamma) = X^n\backslash\Delta_n(X)$, i.e. (X, Γ) has u.p.e. of order n (using Proposition 6.2). □

7. *Co-induction for more general groups*

The above definitions and some of the main theorems can be formulated for a general countable discrete group G containing a subgroup Γ with a TDS (X, Γ) and a Γ-section s sending Γe_G to e_G. In this section, we shall consider this situation.

LEMMA 7.1. *Let Γ be a possibly finite subgroup of a countable discrete infinite group G, (X, Γ) a TDS and (Y, G) the co-induced action.*

(1) *If* Γ *is infinite and* (Y, G) *is strongly mixing then* (X, Γ) *is strongly mixing.*

(2) *Assume that* $K\Gamma K^{-1} = G$ *for some* $K \in \mathcal{F}_G$.

  (a) *If* (Y, G) *is transitive then* (X, Γ) *is transitive.*

  (b) *If* (Y, G) *is weakly mixing then* (X, Γ) *is weakly mixing.*

  (c) *If* Γ *is infinite and* (X, Γ) *is strongly mixing then* (Y, G) *is strongly mixing.*

(3) *Assume that either one has*

$$\bigcup_{i=1}^{n}\bigcup_{j=1}^{n} s(\theta_i)^{-1}(\Gamma\setminus N_\Gamma(U_j, V_i))s(\theta_j) \subsetneq G \tag{37}$$

or there exists $\emptyset \neq A \subseteq \{1, \ldots, n\}^2$ *such that*

$$\bigcap_{(i,j)\in A} s(\theta_i)^{-1}N_\Gamma(U_j, V_i)s(\theta_j) \Big\backslash \bigcup_{(i,j)\in\{1,\ldots,n\}^2\setminus A} s(\theta_i)^{-1}\Gamma s(\theta_j) \neq \emptyset \tag{38}$$

whenever $\{\theta_1, \ldots, \theta_n\} \subseteq \Gamma\setminus G$, $n \in \mathbb{N}$ *and* $U_1, \ldots, U_n, V_1, \ldots, V_n$ *are non-empty open subsets of* X. *Then:*

  (a) (Y, G) *is transitive; and*

  (b) (X, Γ) *is transitive when* $K\Gamma K^{-1} = G$ *for some* $K \in \mathcal{F}_G$.

(4) *If* $K\Gamma K^{-1} \subsetneq G$ *for each* $K \in \mathcal{F}_G$ *then* (Y, G) *is weakly mixing.*

(5) *If TDS* (X, Γ) *can be extended to be another TDS* (X, G) *such that* (X, G) *is weakly mixing then* (Y, G) *is weakly mixing.*

Proof. Observe that if $g \in G$, $\{\theta_1, \ldots, \theta_n\} \subseteq \Gamma\setminus G$, $n \in \mathbb{N}$ and $U_1, \ldots, U_n, V_1, \ldots, V_n \in \mathcal{B}_X$ then (Y^n, G) is the co-induced action from TDS (X^n, Γ) and

$$g(U_{U_i,\theta_i;1\le i\le n}) = U_{s(\theta_i g^{-1})gs(\theta_i)^{-1}U_i,\theta_i g^{-1};1\le i\le n}. \tag{39}$$

(1) Let U and V be non-empty open subsets of X. The conclusion follows from the assumption that (Y, G) is strongly mixing and the fact that

$$\Gamma\setminus N_\Gamma(U, V) \subseteq G\setminus N_G(U_{U,\Gamma e_G}, U_{V,\Gamma e_G}).$$

(2) By the above observation, (b) follows directly from (a). It remains to prove (a) and (c). As $K\Gamma K^{-1} = G$ for some $K \in \mathcal{F}_G$, there exists $\{g_1, \ldots, g_n\} \subseteq G$, $n \in \mathbb{N}$ such that $\Gamma g_i \neq \Gamma g_j$ if $1 \le i < j \le n$ and $G = \{g_1, \ldots, g_n\}^{-1}\Gamma\{g_1, \ldots, g_n\}$.

(a) Assume that (Y, G) is transitive. Let U and V be non-empty open subsets of X. As (Y, G) is transitive, we can select

$$g \in N_G(U_{U,\Gamma g_i;1\le i\le n}, U_{V,\Gamma g_i;1\le i\le n}). \tag{40}$$

As $G = \{g_1, \ldots, g_n\}^{-1}\Gamma\{g_1, \ldots, g_n\}$, there exist $1 \le i, j \le n$ such that $\Gamma g_j g^{-1} = \Gamma g_i$, which implies (combining (40) with (39)) that

$$s(\Gamma g_j g^{-1})gs(\Gamma g_j)^{-1}U \cap V \neq \emptyset.$$

In particular, $s(\Gamma g_j g^{-1})gs(\Gamma g_j)^{-1} \in N_\Gamma(U, V)$. Thus, (X, Γ) is transitive.

(c) Let $\{\theta_1, \ldots, \theta_m\} \subseteq \Gamma\setminus G$, $m \in \mathbb{N}$ with $\{\Gamma g_1, \ldots, \Gamma g_n\} \subseteq \{\theta_1, \ldots, \theta_m\}$ and non-empty open subsets $U_1, \ldots, U_m, V_1, \ldots, V_m$ of X. It suffices to prove that

$$N \doteq G\setminus N_G(U_{U_i,\theta_i;1\le i\le m}, U_{V_i,\theta_i;1\le i\le m})$$

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

is a finite subset of G. Note that, as in (a) above, if $g \in G$ then $\theta_j g^{-1} = \theta_i$ for some $1 \le i, j \le m$, and $g \in N$ if and only if there exist $1 \le i, j \le m$ such that $\theta_j g^{-1} = \theta_i$ and $s(\theta_j g^{-1}) g s(\theta_j)^{-1} U_j \cap V_i = \emptyset$ (i.e. $g \in s(\theta_i)^{-1}(\Gamma \backslash N_\Gamma(U_j, V_i)) s(\theta_j)$). Thus,

$$N \subseteq \bigcup_{i=1}^{m} \bigcup_{j=1}^{m} s(\theta_i)^{-1}(\Gamma \backslash N_\Gamma(U_j, V_i)) s(\theta_j).$$

In particular, N is a finite subset of G, as (X, Γ) is strongly mixing, which implies that $\Gamma \backslash N_\Gamma(U_j, V_i)$ is a finite subset of Γ whenever $1 \le i, j \le m$. Hence, (Y, G) is strongly mixing.

(3) We only need to prove (a), as (b) follows from (a) together with (2).

Let us first show that (Y, G) is transitive. If $\{\theta_1, \ldots, \theta_n\} \subseteq \Gamma \backslash G$, $n \in \mathbb{N}$ and $U_1, \ldots, U_n, V_1, \ldots, V_n$ are non-empty open subsets of X, it is not hard to see, similarly to the proof of (2), that

$$\begin{aligned} \emptyset \neq G \Big\backslash & \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} s(\theta_i)^{-1}(\Gamma \backslash N_\Gamma(U_j, V_i)) s(\theta_j) \\ & \cup \bigcup_{\emptyset \neq A \subseteq \{1,\ldots,n\}^2} \bigcap_{(i,j) \in A} s(\theta_i)^{-1} N_\Gamma(U_j, V_i) s(\theta_j) \Big\backslash \bigcup_{(i,j) \in \{1,\ldots,n\}^2 \backslash A} s(\theta_i)^{-1} \Gamma s(\theta_j) \\ & \subseteq N_G(U_{U_i,\theta_i;1 \le i \le n}, U_{V_i,\theta_i;1 \le i \le n}). \end{aligned}$$

This implies that (Y, G) is transitive.

(4) Clearly, if $K \Gamma K^{-1} \subsetneq G$ for each $K \in \mathcal{F}_G$ then (37) always holds for (X^m, Γ), for each $m \in \mathbb{N}$. Hence, by (3), (Y^m, G) is transitive and so (Y, G) is weakly mixing.

(5) Let $\{\theta_1, \ldots, \theta_n\} \subseteq \Gamma \backslash G$, $n \in \mathbb{N}$ and $U_1, \ldots, U_n, V_1, \ldots, V_n$ be non-empty open subsets of X. As (X, G) is weakly mixing, it is not hard to verify that

$$\emptyset \neq \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} N_G(s(\theta_j)^{-1} U_j, s(\theta_i)^{-1} V_i) \subseteq N_G(U_{U_i,\theta_i;1 \le i \le n}, U_{V_i,\theta_i;1 \le i \le n}).$$

It follows that (Y, G) is transitive, and, as above, weakly mixing. □

LEMMA 7.2. *Let Γ be a (possibly finite) subgroup of the countable discrete infinite group G and (X, Γ) a TDS with (Y, G) the co-induced action. Assume that $G = H \otimes K$ for some groups H and K, and $\Gamma = H \otimes \{e_K\} \subsetneq G$. Then (Y, G) is minimal if and only if (X, Γ) is trivial.*

Proof. Obviously, if (X, Γ) is trivial then (Y, G) is also trivial and so minimal. Thus, it remains to prove that if (X, Γ) is not trivial then (Y, G) is not minimal.

We should note that the introduction of co-induction is independent of the particular choice of Γ-sections sending Γe_G to e_G. Suppose that $s : \Gamma \backslash G \to G$ is the Γ-section given by $\Gamma(e_H, k) \mapsto (e_H, k)$ for each $k \in K$. Then, whenever $h \in H$ and $k, \bar{k} \in K$,

$$\begin{aligned} &((h, k)(x_\theta)_{\theta \in \Gamma \backslash G})_{\Gamma(e_H, \bar{k})} \\ &\quad = s(\Gamma(e_H, \bar{k}))(h, k) s(\Gamma(e_H, \bar{k})(h, k))^{-1} x_{\Gamma(e_H, \bar{k})(h,k)} \\ &\quad = (e_H, \bar{k})(h, k)(e_H, \bar{k}k)^{-1} x_{\Gamma(e_H, \bar{k}k)} = (h, e_K) x_{\Gamma(e_H, \bar{k}k)}, \end{aligned}$$

which implies that $G(x^*)_{\theta \in \Gamma \backslash G} = \{(x)_{\theta \in \Gamma \backslash G} : x \in \Gamma x^*\}$ for each $x^* \in X$. Thus, (Y, G) is not minimal if and only if (X, Γ) is not trivial. □

https://doi.org/10.1017/S0143385711000083 Published online by Cambridge University Press

Acknowledgements. We gratefully acknowledge the support of the Australian Research Council. The second author was supported by FANEDD (201018), NSFC (10801035) and a grant from the Chinese Ministry of Education (200802461004).

REFERENCES

[1] F. Blanchard. Fully positive topological entropy and topological mixing. *Symbolic Dynamics and its Applications (Contemporary Mathematics, 135)*. American Mathematical Society, Providence, RI, 1992, pp. 95–105.

[2] F. Blanchard. A disjointness theorem involving topological entropy. *Bull. Soc. Math. France* **121**(4) (1993), 465–478.

[3] F. Blanchard, E. Glasner and B. Host. A variation on the variational principle and applications to entropy pairs. *Ergod. Th. & Dynam. Sys.* **17**(1) (1997), 29–43.

[4] F. Blanchard, B. Host, A. Maass, S. Martinez and D. J. Rudolph. Entropy pairs for a measure. *Ergod. Th. & Dynam. Sys.* **15**(4) (1995), 621–632.

[5] A. I. Danilenko. Entropy theory from the orbital point of view. *Monatsh. Math.* **134**(2) (2001), 121–141.

[6] A. H. Dooley and V. Ya. Golodets. The spectrum of completely positive entropy actions of countable amenable groups. *J. Funct. Anal.* **196**(1) (2002), 1–18.

[7] A. H. Dooley, V. Ya. Golodets, D. J. Rudolph and S. D. Sineĺshchikov. Non-Bernoulli systems with completely positive entropy. *Ergod. Th. & Dynam. Sys.* **28**(1) (2008), 87–124.

[8] J. Feldman. New K-automorphisms and a problem of Kakutani. *Israel J. Math.* **24**(1) (1976), 16–38.

[9] E. Glasner. A simple characterization of the set of μ-entropy pairs and applications. *Israel J. Math.* **102** (1997), 13–27.

[10] E. Glasner, J.-P. Thouvenot and B. Weiss. Entropy theory without a past. *Ergod. Th. & Dynam. Sys.* **20**(5) (2000), 1355–1370.

[11] E. Glasner and X. Ye. Local entropy theory. *Ergod. Th. & Dynam. Sys.* **29**(2) (2009), 321–356.

[12] C. Hoffman. A K counterexample machine. *Trans. Amer. Math. Soc.* **351**(10) (1999), 4263–4280.

[13] W. Huang and X. Ye. A local variational relation and applications. *Israel J. Math.* **151** (2006), 237–279.

[14] W. Huang, X. Ye and G. H. Zhang. A local variational principle for conditional entropy. *Ergod. Th. & Dynam. Sys.* **26**(1) (2006), 219–245.

[15] W. Huang, X. Ye and G. H. Zhang. Relative entropy tuples, relative U.P.E. and C.P.E. extensions. *Israel J. Math.* **158** (2007), 249–283.

[16] W. Huang, X. Ye and G. H. Zhang. Local entropy theory for a countable discrete amenable group action. *Preprint*, 2006, arXiv:1005.1335v1.

[17] A. del Junco. Bernoulli shifts induced by K-automorphisms. *Adv. Math.* **25**(1) (1977), 35–41.

[18] S. A. Kalikow. T, T^{-1} transformation is not loosely Bernoulli. *Ann. of Math.* (2) **115**(2) (1982), 393–409.

[19] J. M. Ollagnier and D. Pinchon. The variational principle. *Studia Math.* **72**(2) (1982), 151–159.

[20] D. S. Ornstein. An example of a Kolmogorov automorphism that is not a Bernoulli shift. *Adv. Math.* **10** (1973), 49–62.

[21] D. S. Ornstein. A K automorphism with no square root and Pinsker's conjecture. *Adv. Math.* **10** (1973), 89–102.

[22] D. S. Ornstein. *Ergodic Theory, Randomness, and Dynamical Systems (Yale Mathematical Monographs, 5)*. Yale University Press, New Haven, CT–London, 1974.

[23] D. S. Ornstein and P. C. Shields. An uncountable family of K-automorphisms. *Adv. Math.* **10** (1973), 63–88.

[24] D. S. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Anal. Math.* **48** (1987), 1–141.

[25] D. J. Rudolph. Two non-isomorphic K-automorphisms all of whose powers beyond one are isomorphic. *Israel J. Math.* **27**(3–4) (1977), 277–298.

[26] D. J. Rudolph and B. Weiss. Entropy and mixing for amenable group actions. *Ann. of Math.* (2) **151**(3) (2000), 1119–1150.

[27] A. M. Stepin and A. T. Tagi-Zade. Variational characterization of topological pressure of the amenable groups of transformations. *Dokl. Akad. Nauk SSSR* **254**(3) (1980), 545–549 (in Russian).

[28] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, New York–Berlin, 1982.

[29] T. Ward and Q. Zhang. The Abramov–Rokhlin entropy addition formula for amenable group actions. *Monatsh. Math.* **114**(3–4) (1992), 317–329.