

UNIVERSAL MINIMAL FLOWS OF GENERALIZED WAŻEWSKI DENDRITES

ALEKSANDRA KWIATKOWSKA

Abstract. We study universal minimal flows of the homeomorphism groups of generalized Ważewski dendrites W_P , $P \subseteq \{3, 4, \dots, \omega\}$. If P is finite, we prove that the universal minimal flow of the homeomorphism group $H(W_P)$ is metrizable and we compute it explicitly. This answers a question of Duchesne. If P is infinite, we show that the universal minimal flow of $H(W_P)$ is not metrizable. This provides examples of topological groups which are Roelcke precompact and have a nonmetrizable universal minimal flow with a comeager orbit.

§1. Introduction. The *order* of a point x in a topological space is the number of connected components we obtain after removing x . A *ramification point* is a point which has order at least 3. An *endpoint* is a point of order 1. A *continuum* is a compact connected topological space. A *dendrite* is a locally connected continuum that contains no simple closed curve. All dendrites we consider in this article will be metrizable. A *Ważewski dendrite* W_ω is a dendrite such that each ramification point of W_ω is of order ω and each arc I contained in W_ω contains a ramification point.

Moreover, for every $P \subseteq \{3, 4, \dots, \omega\}$, there exists a *generalized Ważewski dendrite* W_P , that is, a dendrite such that each ramification point of W_P is of order that belongs to P and for every $p \in P$ and an arc I contained in W_P , I contains a ramification point of order p . For every $P \subseteq \{3, 4, \dots, \omega\}$, a generalized Ważewski dendrite is unique up to homeomorphism, see Charatonik-Dilks [6, Theorem 6.2]. Duchesne-Monod [9] studied structural properties of homeomorphism groups of generalized Ważewski dendrites, in particular, they showed that these groups are simple.

The homeomorphism group of a generalized Ważewski dendrite is isomorphic (as a topological group) to the automorphism group of a certain Fraïssé-HP structure (i.e., the Fraïssé limit of a family of finite first-order structures, which has the joint embedding and the amalgamation properties, but not necessarily the hereditary property), which we now describe. Let P be fixed and consider W_P . Let M_P be the set of all ramification points of W_P . Let \mathcal{L}_P be the first-order language that consists of a 4-ary relation symbol D and of unary relation symbols K_p for every $p \in P$. We let M_P to be the structure with universe M_P , $D^{M_P}(a, b, c, d)$ iff the path in W_P

Received December 17, 2017.

2010 *Mathematics Subject Classification.* 05D10, 37B05, 54F15, 03C98.

Key words and phrases. universal minimal flows, homeomorphism groups of Ważewski dendrites, Fraïssé limits.

© 2018, Association for Symbolic Logic
0022-4812/18/8304-0017
DOI:10.1017/jsl.2018.26

connecting a and b and the path connecting c and d do not intersect (we emphasize that we allow here trivial paths, i.e., we allow $a = b$ or $c = d$), and let $K_p^{M_p}(a)$ iff a is a ramification point of the order equal to p .

Instead of coding the tree structure using the D relation, we could use the ternary betweenness relation B , where $B(a, b, c)$ iff b belongs to the path ac . Indeed, $B(a, b, c)$ holds iff $D(a, c, b, b)$ does not hold, and $D(a, b, c, d)$ does not hold iff there exists e such that $B(a, e, b)$ and $B(c, e, d)$. Later on, we will also work with a C relation, which will be defined using the D relation, moreover, the D relation is used to describe boron trees, see [13], therefore we decided to work in this article with the D relation rather than with the B relation.

Propositions 2.4 and 6.1 in [9] imply:

PROPOSITION 1.1. *The homeomorphism group of the generalized Ważewski dendrite W_p , equipped with the uniform metric, is isomorphic (as a topological group) to the automorphism group of M_p , equipped with the pointwise convergence metric.*

A *tree* is an acyclic connected undirected graph. For a tree T we denote by $V(T)$ the set of vertices and by $E(T)$ the set of edges of T . A *degree* of a vertex in a graph is the number of edges that come out of that vertex. An *endpoint* is a vertex of degree 1. A *path* is a tree such that each vertex either is an endpoint or it has degree 2. Note that for any two vertices in a tree there is exactly one path joining them. A path joining vertices a and b we will often denote by ab . A *rooted tree* is a tree with a distinguished point, which we call the *root*. On a rooted tree T with the root r we consider the tree order \leq_T letting $x \leq_T y$ iff x belongs to the path ry . A *branch* in a rooted tree is a path ra , where r is the root and a is an endpoint. The *meet* of $a, b \in T$ is the greatest lower bound of a and b with respect to \leq_T . In a rooted tree we can talk about the *height* of each vertex. The root has the height equal to 0 and the height of $x \in T$ is taken to be the maximum plus 1 of heights of $\{v \in T : v <_T x\}$. The *height* of a rooted tree T is the maximum of the heights of all of its vertices, we denote it by $ht(T)$. Note that the height of $x \in T$ is equal to the length of the path rx , where the *length* of a path is defined to be the number of edges in the path. A *successor* of a vertex x is any point $y \neq x$ such that $x \leq_T y$. A vertex y is an *immediate successor* of a vertex x if it is a successor of x and there is no successor $w \neq y$ of x such that $x \leq_T w \leq_T y$.

Let \mathcal{F}_p be the family of all finite structures in the language \mathcal{L}_p such that the universe is a finite tree and the degree of each vertex is different from 2. If $A \in \mathcal{F}_p$, we let $D^A(a, b, c, d)$ iff the path ab and the path cd , do not intersect. Take K_p^A such that for every $a \in A$ there is exactly one $p \in P$ such that $K_p^A(a)$, and if $K_p^A(a)$ then the degree of a is not greater than p .

A first-order structure M is *ultrahomogeneous* with respect to a family of finite substructures \mathcal{F} if for any finite substructures $A, B \subseteq M$, $A, B \in \mathcal{F}$, and an isomorphism $p: A \rightarrow B$, there is an automorphism of M extending p .

Proposition 6.1 in [9] together with Proposition 1.1 imply:

PROPOSITION 1.2. *For every $P \subseteq \{3, 4, \dots, \omega\}$, the structure M_P is ultrahomogeneous with respect to \mathcal{F}_P .*

The proposition above implies that \mathcal{F}_P has the joint embedding property and the amalgamation property. Note that \mathcal{F}_P does not have the hereditary property.

Moreover, as additionally for every finite subset $X \subseteq M_P$ there is $A \in \mathcal{F}_P$ such that $X \subseteq A \subseteq M_P$, we have that M_P is the Fraïssé limit of \mathcal{F}_P .

REMARK 1.3. Let $i: (S, D^S) \rightarrow (T, D^T)$ be an embedding of trees S and T in which each vertex has degree $\neq 2$. Then every edge in S is mapped to a path in T and T is obtained from S in a sequence of the following simple steps:

1. Start with a tree T' . Pick an edge $[a, b]$ in T' . Let c and d be points not in T' . Get S' by removing edge $[a, b]$, and by adding points c and d , and edges $[a, c]$, $[c, b]$ and $[c, d]$.
2. Start with a tree T' . Pick an endpoint e in T' . Let c and d be points not in T' . Get S' by adding points c and d , and edges $[e, c]$ and $[e, d]$.
3. Start with a tree T' . Pick a vertex v in T' which is not an endpoint. Let c be a point not in T' . Get S' by adding the point c and the edge $[v, c]$.

REMARK 1.4. Note that the relation D remembers which pairs of vertices are joined by an edge. Given a tree T and $a, b \in T$, $a \neq b$. Then there is an edge between a and b iff for every $c \in T$, $c \neq a, b$ we have that c does not belong to the path ab iff for every $c \in T$, $c \neq a, b$, $D^T(a, b, c, c)$ holds.

REMARK 1.5. Let T be a tree and let E be the set of endpoints of T . Then $D^T \upharpoonright E$ on the set E is an example of a D -relation, as defined in [1, Section 22]. Moreover, D^T on the tree T satisfies (D1)–(D3) in the definition of a D -relation, but not (D4).

§2. The universal minimal flow—preliminaries. Our goal is to compute universal minimal flows of the homeomorphism groups $H(W_P)$, equivalently, of the automorphism groups $\text{Aut}(M_P)$.

We will work in the framework provided by Kechris–Pestov–Todorćević. Let us recall relevant definitions and theorems. The presentation below is essentially copied from [2], Section 3.6. Lemma 2.3, Theorem 2.6, and Corollary 2.7 are proved there.

A topological group G is *extremely amenable* if every G -flow has a fixed point. A *coloring* of a set X is any function $c: X \rightarrow \{1, 2, \dots, r\}$, for some $r \geq 2$; we say that $Y \subseteq X$ is *c-monochromatic* (or just *monochromatic*) if $r \upharpoonright Y$ is constant.

Let \mathcal{G} be a family of finite structures in a language \mathcal{L} . For $A, B \in \mathcal{G}$ write $A \leq B$ if A embeds into B . For A, B in \mathcal{G} , let $\binom{B}{A}$ denotes the set of all embeddings of A into B . We say that $A \in \mathcal{G}$ is a *Ramsey object* if for every $B \in \mathcal{G}$ with $A \leq B$ and every integer $r \geq 2$ there exists $C \in \mathcal{G}$ such that for every coloring $c: \binom{C}{A} \rightarrow \{1, 2, \dots, r\}$ there exists $h \in \binom{C}{B}$ such that $\{h \circ f: f \in \binom{B}{A}\}$ is monochromatic. Note that to check that A is a Ramsey object it suffices to check it only for $r = 2$. We say that \mathcal{G} is a *Ramsey class* (or that it has *Ramsey property*) if every structure in \mathcal{G} is a Ramsey object.

A structure $A \in \mathcal{G}$ is *rigid* if it has trivial automorphism group.

Kechris–Pestov–Todorćević [14] worked with Fraïssé families and their ordered Fraïssé expansions, their work was generalized by Nguyen Van Thé [17] to Fraïssé families and to arbitrary relational Fraïssé expansions. The Kechris–Pestov–Todorćević correspondence remains true for Fraïssé–HP families, which was checked by several people, and it appears in [22], see also [2].

THEOREM 2.1 (Kechris–Pestov–Todorcevic [14], see Theorem 5.1 in [22]). *Let \mathcal{G} be a Fraïssé-HP family, let \mathbb{G} be its Fraïssé limit, and let $G = \text{Aut}(\mathbb{G})$. Then the following are equivalent:*

- (1) *The group G is extremely amenable.*
- (2) *The family \mathcal{G} is a Ramsey class and it consists of rigid structures.*

Let \mathcal{G} be a Fraïssé-HP family in a language \mathcal{L} , let \mathbb{G} be its Fraïssé limit, and let $G = \text{Aut}(\mathbb{G})$. Let \mathcal{G}^* be a Fraïssé-HP family in a language $\mathcal{L}^* \supseteq \mathcal{L}$, $\mathcal{L}^* \setminus \mathcal{L}$ relational, such that the map defined on \mathcal{G}^* and given by $A^* \mapsto A^* \upharpoonright \mathcal{L}$ is onto \mathcal{G} . In that case we say that A^* is an *expansion* of $A^* \upharpoonright \mathcal{L}$ and that $A^* \upharpoonright \mathcal{L}$ is a *reduct* of A^* , and that \mathcal{G}^* is an *expansion* of \mathcal{G} . Let \mathbb{G}^* be the Fraïssé limit of \mathcal{G}^* , and let $G^* = \text{Aut}(\mathbb{G}^*)$.

We say that the expansion \mathcal{G}^* of \mathcal{G} is *reasonable* if for any $A, B \in \mathcal{G}$, an embedding $\alpha: A \rightarrow B$ and an expansion $A^* \in \mathcal{G}^*$ of A , there is an expansion $B^* \in \mathcal{G}^*$ of B such that $\alpha: A^* \rightarrow B^*$ is an embedding. It is *precompact* if for every $A \in \mathcal{G}$ there are only finitely many $A^* \in \mathcal{G}^*$ such that $A^* \upharpoonright \mathcal{L} = A$. We say that \mathcal{G}^* has the *expansion property* relative to \mathcal{G} if for any $A^* \in \mathcal{G}^*$ there is $B \in \mathcal{G}$ such that for any expansion $B^* \in \mathcal{G}^*$, there is an embedding $\alpha: A^* \rightarrow B^*$. The following proposition explains the importance of the notion of reasonability.

PROPOSITION 2.2 ([14], [17], see Proposition 5.3 in [22]). *The expansion \mathcal{G}^* of \mathcal{G} is reasonable if and only if $\mathbb{G}^* \upharpoonright \mathcal{L} = \mathbb{G}$.*

We say that \mathcal{G}^* has the *relative HP* (the relative hereditary property) with respect to \mathcal{G} if for every $A, B \in \mathcal{G}$ such that A is a substructure of B and for $B^* \in \mathcal{G}^*$, an expansion of B , we have $B^* \upharpoonright A \in \mathcal{G}^*$. This is equivalent to saying that for any $A \in \mathcal{G}$ and an embedding $i: A \rightarrow \mathbb{G}$ there is an expansion $A^* \in \mathcal{G}^*$ of A such that $i: A^* \rightarrow \mathbb{G}^*$ is an embedding. The relative HP property is used to show that when an expansion \mathcal{G}^* of \mathcal{G} is precompact, then $\text{Aut}(\mathbb{G})/\text{Aut}(\mathbb{G}^*)$ is precompact in the quotient of the right uniformity, the proof is contained in Section 3.6 in [2].

LEMMA 2.3. *Suppose that \mathcal{G}^* is a reasonable precompact expansion of \mathcal{G} and that the relative HP holds. Then the right uniform space $\text{Aut}(\mathbb{G})/\text{Aut}(\mathbb{G}^*)$ is precompact.*

Below (\mathbb{G}, \vec{R}) denotes an expansion of \mathbb{G} to a structure in \mathcal{L}^* . Instead of (\mathbb{G}, \vec{R}) we will often just write \vec{R} .

Define

$$X_{\mathbb{G}^*} = \{ \vec{R} : \text{for every } A \in \mathcal{G}, \text{ and an embedding } i: A \rightarrow \mathbb{G} \text{ there exists } A^* \in \mathcal{G}^*, \text{ such that } i: A^* \rightarrow (\mathbb{G}, \vec{R}) \text{ is an embedding} \}.$$

The relative HP implies that the space $X_{\mathbb{G}^*}$ contains \mathbb{G}^* .

We make $X_{\mathbb{G}^*}$ a topological space by declaring sets

$$V_{i,A^*} = \{ \vec{R} \in X_{\mathbb{G}^*} : \text{the map } i: A^* \rightarrow (\mathbb{G}, \vec{R}) \text{ is an embedding} \},$$

where $i: A \rightarrow \mathbb{G}$ is an embedding, $A^* \in \mathcal{G}^*$, and $A^* \upharpoonright \mathcal{L} = A$, to be open. The group $\text{Aut}(\mathbb{G}^*)$ acts continuously on $X_{\mathbb{G}^*}$ via

$$g \cdot \vec{R}(\vec{a}) = \vec{R}(g^{-1}(\vec{a})).$$

Reasonability and precompactness of the expansion \mathcal{G}^* of \mathcal{G} imply that the space $X_{\mathbb{G}^*}$ is compact and zero-dimensional.

From now on till the end of this section, we will assume that the expansion \mathcal{G}^ of \mathcal{G} is reasonable, precompact, and satisfies the relative HP.*

THEOREM 2.4 ([14], [17], see Proposition 5.5 in [22]). *The following are equivalent:*

- (1) *The flow $G \curvearrowright X_{\mathbb{G}^*}$ is minimal.*
- (2) *The family \mathcal{G}^* has the expansion property relative to \mathcal{G} .*

THEOREM 2.5 (Kechris–Pestov–Todorćević [14], Nguyen Van Th e [17], see Theorem 5.7 in [22]). *The following are equivalent:*

- (1) *The flow $G \curvearrowright X_{\mathbb{G}^*}$ is the universal minimal flow of G .*
- (2) *The family \mathcal{G}^* is a rigid Ramsey class and has the expansion property relative to \mathcal{G} .*

A proof of Theorem 2.6 is contained in Section 3.6 in [2].

Let $\vec{R}^{\mathbb{G}}$ be such that $\mathbb{G}^* = (\mathbb{G}, \vec{R}^{\mathbb{G}})$.

THEOREM 2.6. *The map $g \text{Aut}(\mathbb{G}^*) \mapsto g \cdot \vec{R}^{\mathbb{G}}$ from $\text{Aut}(\mathbb{G})/\text{Aut}(\mathbb{G}^*)$ to $X_{\mathbb{G}^*}$ is a uniform isomorphism.*

We will say that flows $G \curvearrowright X$ and $G \curvearrowright Y$ are *isomorphic* if there is a homeomorphism from X onto Y which is a G -map.

COROLLARY 2.7. *The flow $G \curvearrowright \overline{\text{Aut}(\mathbb{G})/\text{Aut}(\mathbb{G}^*)}$ is isomorphic to the flow $G \curvearrowright X_{\mathbb{G}^*}$.*

§3. The universal minimal flow—construction. In this section, we show:

THEOREM 3.1. *For any $P \subseteq \{3, \dots, \omega\}$ there is a reasonable Fra iss e-HP expansion \mathcal{F}_P^* of \mathcal{F}_P , which has the relative HP, the expansion, and the Ramsey properties. In the case when P is finite, this expansion \mathcal{F}_P^* is also precompact.*

Then using the Kechris–Pestov–Todorćević correspondence and Proposition 1.1, we obtain a description of the universal minimal flow of the homeomorphism group of the generalized Wa zewski dendrite W_P , for all finite P . In particular, we will obtain that this universal minimal flow is metrizable, when P is finite. This answers a question of Duchesne asked during his talk at the Workshop “Structure and Geometry of Polish groups” in Oaxaca in 06/2017. In the special case, when $P = \{\omega\}$, the universal minimal flow of $H(W_{\{\omega\}})$, independently of our work, was identified by Duchesne in [8].

Given \mathcal{F}_P in the language \mathcal{L}_P , we first construct a family \mathcal{T}_P^* of rooted trees with ordered and labeled branches, and then we construct the required family \mathcal{F}_P^* that is Ramsey and has the expansion property with respect to \mathcal{F}_P .

The family \mathcal{T}_P^ .* Take $\mathcal{L}_{\mathcal{T}_P^*} = \mathcal{L}_P \cup \{C, \prec, G_1, G_2, \dots\}$, where C is a ternary relation symbol, \prec, G_1, G_2, \dots are binary relation symbols. If P is finite, it suffices to take G_1, \dots, G_{m-1} , where $m = \max(P \setminus \{\omega\})$.

Let $A \in \mathcal{F}_P$.

STEP 1. Choosing the root for A .

Let x be an edge of A or a vertex of A such that its degree is strictly less than p satisfying $K_p^A(x)$. In the case when x is a vertex, denote $r = x$ and consider A with the distinguished point r , which we call the root. Denote this rooted tree by $T_{A,r}$. In the case when $x = [a, b]$ is an edge, remove x from A , take a new point r and add edges $[a, r]$ and $[r, b]$. The obtained tree with the distinguished point r , which we call the root, denote by $T_{A,r}$ as before.

For simplicity, write $T = T_{A,r}$. Similarly as before, we let for $a, b, c, d \in T$, $D^T(a, b, c, d)$ iff the paths ab and cd do not intersect. Let for $a, b, c \in T$, $C^T(a, b, c)$ iff $D^T(a, b, c, r)$. (The relation C^T “remembers” that the root r of T is the smallest with respect to \leq_T element of T .)

It is crucial that we are allowed to choose the root both with respect to edges and with respect to vertices. Otherwise, the relative HP would fail, see Remark 3.5.

REMARK 3.2. Let T be a tree and let E be the set of endpoints of T . Then $C^T \upharpoonright E$ is an example of a C -relation on the set E , as defined in [1, Section 10]. Moreover, C^T on the tree T satisfies (C1)–(C3) in the definition of a C -relation, but not (C4).

STEP 2. Labeling the root r .

If $r \in A$ (which is exactly in the case when in Step 1 the x we picked was a vertex) then already there is $p \in P$ such that $K_p^A(r)$, i.e., $K_p^T(r)$. Otherwise, if $r \notin A$, we pick some $p \in P$ and let $K_p^T(r)$.

STEP 3. Ordering and labeling branches of T .

Here we have to do two things: we will introduce a binary relation that induces an order of branches of T , and then for every $a \in T$ such that for a finite $p \in P$ we have $K_p^T(a)$, we will put additional labels on the successors of a .

The binary relation \prec^T : For every $a \in T$ we fix a strict linear order \prec_a^T of its immediate successors. Then we let $c \prec^T d$ iff for some $a \in T$ there are $i < j$ such that $a_i \leq_T c$ and $a_j \leq_T d$, where $a_1 \prec_a^T \dots \prec_a^T a_n$ are immediate successors of a , for some n .

The binary relations G_i^T : If $a \in T$ and $p \in P \setminus \{\omega\}$ are such that $K_p^T(a)$, and $a_1 \prec^T \dots \prec^T a_n$ are the immediate successors of a , fix an increasing injection $k: \{1, \dots, n\} \rightarrow \{1, \dots, p - 1\}$. We let for $b \in T$, $G_{k(i)}^T(a, b)$ iff $a_i \leq_T b$.

Clearly \prec^T induces an ordering of branches of T . Moreover, if $\omega \notin P$, then \prec^T can be recovered from G_1^T, G_2^T, \dots . Note that if $n < p - 1$ in the definition of an injection k , G_1^T, G_2^T, \dots carry more information than just \prec^T . The reason why we include \prec^T rather than just work with G_1^T, G_2^T, \dots is that in the case when $\omega \in P$ and P is finite, we do not want to work with infinitely many G_i 's (otherwise precompactness will fail); in the case $\omega \notin P$, it suffices to work only with G_1^T, G_2^T, \dots and not introduce \prec^T .

Finally, put into \mathcal{T}_p^* any structure obtained from A (in a very nonunique way) in the procedure described in Steps 1–3. Note that every vertex in a $T^* \in \mathcal{T}_p^*$, except possibly the root, has the degree different from 2.

The family \mathcal{F}_p^* . Take $\mathcal{L}_p^* = \mathcal{L}_{\mathcal{T}_p^*} \cup \{R_p\}_{p \in P} \cup \{H_{ij}\}_{1 \leq i < j}$, where each R_p and H_{ij} is a binary relation symbol, and $i, j \in \mathbb{N}$.

Start with $A \in \mathcal{F}_p$ and let $T^* \in \mathcal{T}_p^*$ be any rooted tree obtained from A . The universes of A and of T^* either are equal or there is an extra point, the root r of T^* , which is not in A . All the relations in $\mathcal{L}_{\mathcal{T}_p^*}$ we simply restrict from T^* to A . However, note that in the case $r \notin A$, we “forgot” this way for which $p \in P$ it holds $K_p^{T^*}(r)$ and for which $1 \leq i$ it holds $G_i^{T^*}(r, a)$, whenever $a \in T^*$, $a \neq r$. In order to remember these two pieces of information after removing the root, we set for any two incomparable with respect to \leq^{T^*} elements $a, b \in A$ and c equal to the meet of

a and b in the rooted tree T^* : $R_p^A(a, b)$ iff $K_p^{T^*}(c)$ and we set $H_{ij}^A(a, b)$ iff $G_i^{T^*}(c, a)$ and $G_j^{T^*}(c, b)$.

PROPOSITION 3.3. *Let $f: S \rightarrow T$ be an injection between finite rooted trees S and T with roots r_S and r_T , respectively. Then the following are equivalent:*

- (1) f preserves the relations C (defined with respect to r_S and r_T) and D ;
- (2) f preserves the relation C ;
- (3) f preserves the meet (i.e., for each $a, b \in S$ and their meet c , $f(c)$ is the meet of $f(a)$ and $f(b)$).

PROOF. Clearly (1) implies (2). Assume now (2). First notice that then f preserves \leq_S and \leq_T . Then note that if for some $a, b \in S$ and c , the meet of a and b , we had that $f(c)$ is strictly lower with respect to \leq_T than the meet of $f(a)$ and $f(b)$, then $\neg C^S(a, b, c)$ and $C^T(f(a), f(b), f(c))$, which is impossible. Therefore we get (3). Now if we assume (3), then f also preserves \leq_S and \leq_T . Essentially from the definitions of the relations C and D it follows that if (3) holds then f preserves C and D , and hence we get (1). \dashv

PROPOSITION 3.4. *The category \mathcal{F}_P^* with embeddings and the category \mathcal{T}_P^* with embeddings are equivalent via a covariant functor.*

PROOF. To $A^* \in \mathcal{F}_P^*$ assign $T^* \in \mathcal{T}_P^*$ by adding the root if it is not already in A^* . From the relation C^{A^*} we can recover where the root is. Recover the information needed about the root using the relations R_p and H_{ij} . To $T^* \in \mathcal{T}_P^*$ assign $A^* \in \mathcal{F}_P^*$ by removing the root if it was added (which is exactly when the degree of the root is equal to 2).

To an embedding $f: A^* \rightarrow B^*$ assign an embedding $g: S^* \rightarrow T^*$, where S^* corresponds to A^* and T^* corresponds to B^* in the following way. If S^* contains the root r which is not already in A^* and this root was added with respect to an edge $x = [a, b]$, we take g to be the extension of f in which r is mapped to the meet of $f(a)$ and $f(b)$. Again the relations R_p and H_{ij} remember all the information needed for such a g to be an embedding. On the other hand, having an embedding $g: S^* \rightarrow T^*$, we obtain an embedding $f: A^* \rightarrow B^*$ by simply removing the root from S^* , in case it was added, and restricting g . \dashv

Let \mathcal{T}_P denote the set of reducts of elements in \mathcal{T}_P^* to the language $\mathcal{L}_P \cup \{C\}$. We now prove that the family \mathcal{F}_P^* has all the properties required in Theorem 3.1.

3.1. \mathcal{F}_P^* is reasonable. Let $A, B \in \mathcal{F}_P$ such that A is a substructure of B , be given, and fix an expansion $A^* \in \mathcal{F}_P^*$ of A . We will define $B^* \in \mathcal{F}_P^*$, an expansion of B which when restricted to A is equal to A^* . If the root $r = r_{A^*}$ of A^* is a vertex such that there is no $b \in B \setminus A$ and an edge $[r, b]$ in B , we let r to be the root of B . If r is a vertex such that there is $b_0 \in B \setminus A$ and an edge $[r, b_0]$ in B , we let the vertex of B to be any endpoint e of B such that b_0 belongs to the path er in B . The resulted rooted tree denote by T and note that $T \in \mathcal{T}_P$. If the root of A^* was added with respect to an edge $[x_1, x_2]$ in A , then take any edge $[y_1, y_2] \subseteq [x_1, x_2]$ in B , and add the root to B with respect to $[y_1, y_2]$. Take $T \in \mathcal{T}_P$ equal to B with the root r_T added with respect to $[y_1, y_2]$ and let for every $p \in P$, $K_p^T(r_T)$ iff $K_p^{S^*}(r_{S^*})$, where $S^* \in \mathcal{F}_P^*$ with the root r_{S^*} corresponds to A^* .

View $S \in \mathcal{T}_P$ equal to the reduct of S^* as embedded in T . In the case when the root r_{S^*} is not in A , this embedding takes r_{S^*} to r_T . We still have to define

\prec^T and G_1^T, G_2^T, \dots , which extend \prec^{S^*} and $G_1^{S^*}, G_2^{S^*}, \dots$. For this, for any $b \in T$ and its immediate successors b_1, \dots, b_n , it is enough to define \prec^T on $\{b_1, \dots, b_n\}$ and specify for each i and k whether $G_i^T(b, b_k)$ holds or not. Let $p \in P$ be such that $K_p^B(b)$. In case $b \notin S^*$, we define \prec^T and G_1^T, \dots, G_{p-1}^T in an arbitrary way that Step 3 in the construction allows us. In the case $b \in S^*$, we define \prec^T and G_1^T, \dots, G_{p-1}^T in any way allowed in Step 3 so that additionally if for some $c \in S^*$, $b_k \leq_T c$ and $G_i^{S^*}(b, c)$ then $G_i^T(b, b_k)$ and if for some $c, d \in S^*$, $b_k \leq_T c$, $b_l \leq_T d$ and $c \prec^{S^*} d$ then $b_k \prec^T b_l$. This defines $T^* \in \mathcal{T}_p^*$, which corresponds to $B^* \in \mathcal{F}_p^*$ we are looking for.

3.2. \mathcal{F}_p^* is precompact with respect to \mathcal{F}_p . Clear. P has to be finite.

3.3. \mathcal{F}_p^* has the JEP. For this we can instead work with the family \mathcal{T}_p^* . Take $S^*, T^* \in \mathcal{T}_p^*$. Let r_{S^*} be the root of S^* , and let r_{T^*} be the root of T^* . Pick a new element r , pick $p \in P$, and if $p < \omega$ pick $1 \leq i < j \leq p - 1$. Let $R^* \in \mathcal{T}_p^*$ be obtained as follows. We take the union of S^* and T^* together with the point r and vertices $[r, r_{S^*}]$ and $[r, r_{T^*}]$. We declare r to be the root of R^* , i.e., we define $C^{R^*}(a, b, c)$ iff $D^{R^*}(a, b, c, r)$, and let $K_p^{R^*}(r)$. For any $r_{S^*} \leq_{S^*} a$ and $r_{T^*} \leq_{T^*} b$, let $a \prec^{R^*} b$. If $p < \omega$, then if $r_{S^*} \leq_{S^*} a$, we let $G_i^{R^*}(r, a)$ and if $r_{T^*} \leq_{T^*} a$, we let $G_j^{R^*}(r, a)$. We also make sure that the degrees of r_{S^*} and r_{T^*} in R^* are at least 3 by adding additional edges and extending \prec^{R^*} and $G_i^{R^*}$, if needed. Then this R^* is as required for S^* and T^* . (Note that in this proof we used that the degree of r_{S^*} in S^* is strictly less than $p_0 \in P$ such that $K_{p_0}^{S^*}(r_{S^*})$, and similarly for T^* .)

3.4. \mathcal{F}_p^* has the AP. One can show it directly, but it also follows from the rigidity of each $A \in \mathcal{F}_p^*$ together with the JEP and the Ramsey properties for \mathcal{F}_p^* . The proof of this fact is essentially due to Nešetřil-Rödl (see [16, p. 294, Lemma 1]), the framework in which they work is somewhat different from ours. Their proof is for families of structures which are rigid, hereditary, have the JEP and Ramsey properties, see also [14, p. 129]. Nevertheless, for a Fraïssé-HP family \mathcal{F} , whenever $A \in \mathcal{F}$ then every structure isomorphic to A is also in \mathcal{F} . Therefore the proof presented by Kechris–Pestov–Todorcevic [14] applies to Fraïssé-HP families as well.

3.5. \mathcal{F}_p^* has the relative HP. Fix $A, B \in \mathcal{F}_p$ and $B^* \in \mathcal{F}_p^*$ extending B . Take $T^* \in \mathcal{T}_p^*$ that corresponds to B^* and view A as embedded in the \mathcal{L}_p -reduct of T^* . There are either one or two minimal elements in $A \subseteq (T^*, \leq_{T^*})$. Let r be this minimal element, if there is exactly one, and otherwise let r be the meet of the two minimal elements. Take $S = A \cup \{r\}$, a rooted tree with the root r . Let S^* be the substructure of T^* such that the universe is S . Then $S^* \in \mathcal{T}_p^*$ and hence the corresponding structure $A^* \in \mathcal{F}_p^*$ satisfies $B^* \upharpoonright A \in \mathcal{F}_p^*$.

REMARK 3.5. It is possible to have $A, B \in \mathcal{F}_p$, A embedded into B , the expansion $B^* \in \mathcal{F}_p^*$ of B , such that its root was added with respect a vertex, but the root of A^* , the restriction of B^* to A , has the root added with respect to an edge. Let for example B^* be the rooted tree that consists of 4 vertices: r, a, b, c , where r is the root, and edges $[r, a]$, $[r, b]$, $[r, c]$, and let A be the subtree that consists of 2 vertices a, b and the edge $[a, b]$. Similarly, it is not hard to give an example of B^* and A such that B^* has the root added with respect to an edge and A^* has the root added with respect to a vertex.

3.6. \mathcal{F}_p^* has the expansion property with respect to \mathcal{F}_p . Let $A^* \in \mathcal{F}_p^*$ be given. Without loss of generality, let the root of A^* belong to A^* (as we can always embed the A^* we started with in an element of \mathcal{F}_p^* with such a property). Therefore we can think that $A^* \in \mathcal{T}_p^*$. Take a rooted tree $T \in \mathcal{T}_p$ which has the property that all its expansions to an element in \mathcal{T}_p^* are isomorphic, the degree of the root is ≥ 2 , and A^* embeds in some/every expansion of T . (For this, note that any tree $V \in \mathcal{T}_p$ with the properties: (1) if x and y have the same height, $K_p^V(x)$ and $K_p^V(y)$ hold, then $p = q$; (2) if x is not an endpoint, $p < \omega$, and $K_p^V(x)$, then x has exactly $p - 1$ immediate successors; (3) there is $M \geq 2$ such that every x which is not an endpoint and $K_\omega^V(x)$, has exactly M immediate successors; is such that all its expansions to an element in \mathcal{T}_p^* are isomorphic.) Finally, let B be obtained as follows. Take T' and T'' , two disjoint copies of T . Denote their roots by $r_{T'}$ and $r_{T''}$, respectively. The disjoint union of T' and T'' together with the edge $[r_{T'}, r_{T''}]$ is a required B . This is because whenever we expand B to a $B^* \in \mathcal{F}_p^*$ then we can embed T^* (the unique expansion of T) into B^* . If, say, the vertex or edge with respect to which is added the root of B^* lies in T' , then the unique expansion of T'' embeds into B^* .

3.7. \mathcal{F}_p^* has the Ramsey property. We generalize the Ramsey theorems by Deuber [7] and by Sokić [18] (Theorems 2.2 and 6.1). For related Ramsey theorems, where it is additionally assumed that endpoints of a rooted tree are mapped to endpoints, see [4, 13, 20].

THEOREM 3.6. *For any nonempty $P \subseteq \{3, \dots, \omega\}$, the family \mathcal{T}_P^* , and hence the family \mathcal{F}_P^* , is Ramsey.*

Consider $T \in \mathcal{T}_p^*$ with the root r_T and let q be such that $K_q^T(r_T)$. Let $V \in \mathcal{T}_p^*$ and let M be the maximum of 2 and the number of immediate successors of all vertices in V labeled with ω . We are going to define $V[T] \in \mathcal{T}_p^*$. First consider $V' \in \mathcal{T}_p^*$ defined as follows. For every endpoint $e \in V$ take p_e such that $K_{p_e}^V(e)$ and take new points $x_1^e, \dots, x_{p_e'}^e$, where $p_e' = p_e - 1$ if $p_e < \omega$ and $p_e' = M$ if $p_e = \omega$, and add edges $[e, x_i^e]$, $i = 1, \dots, p_e'$. Then let $V' \in \mathcal{T}_p^*$ be the tree we obtain by letting $K_{q'}^V(x_i^e)$ for each endpoint e , and each i , and (uniquely) choosing $\prec^{V'}$ and $G_i^{V'}$. To obtain $V[T]$, to each endpoint of V' attach the tree T by identifying this endpoint with the root r_T .

EXAMPLE 3.7. Let $V = 2^{\leq 1}$ with $K_5^V(\emptyset)$, $K_3^V(0)$, and $K_\omega^V(1)$. Let $T = 2^{\leq 1}$ with $K_7^T(\emptyset)$, $K_{10}^T(0)$, and $K_6^T(1)$. Then $S = V[T] = 2^{\leq 3}$ with $K_5^S(\emptyset)$, $K_3^S(0)$, $K_\omega^S(1)$, $K_7^S(00)$, $K_7^S(01)$, $K_7^S(10)$, $K_7^S(11)$, $K_{10}^S(000)$, $K_{10}^S(010)$, $K_{10}^S(100)$, $K_{10}^S(110)$, $K_6^S(001)$, $K_6^S(011)$, $K_6^S(101)$, and $K_6^S(111)$.

For a family \mathcal{G} of first-order structures in some language denote by \mathcal{G}_m the family $\{(A_1, \dots, A_m) : A_j \in \mathcal{G}\}$. We say that (A_1, \dots, A_m) embeds into (B_1, \dots, B_m) if for every j , A_j embeds into B_j .

In the inductive step of the proof of Theorem 3.6 we will be using the product Ramsey theorem.

THEOREM 3.8 (Sokić, Theorem 2 in [19]). *Let \mathcal{G} be a family of first-order structures in some language, which is a Ramsey class. For any $(A_1, \dots, A_m), (B_1, \dots, B_m) \in \mathcal{G}_m$ such that (A_1, \dots, A_m) embeds into (B_1, \dots, B_m) there is $C \in \mathcal{G}$ such that for any coloring of embeddings of (A_1, \dots, A_m) in (C, \dots, C) into finitely many colors there*

is a embedding $h = (h_1, \dots, h_m)$ of (B_1, \dots, B_m) in (C, \dots, C) such that the set of all functions $h \circ f$, where $f = (f_1, \dots, f_m)$ is an embedding of (A_1, \dots, A_m) into (B_1, \dots, B_m) , is monochromatic.

Moreover, from the proof of Sokić’s theorem it follows that for every i : If every $A \in \mathcal{G}$ with $ht(A) \leq i$ is a Ramsey object in \mathcal{G} , then every $(A_1, \dots, A_m) \in \mathcal{G}_m$ with each A_j satisfying $ht(A_j) \leq i$, is a Ramsey object in \mathcal{G}_m .

We show that \mathcal{T}_p^* is a Ramsey class, i.e., we show that for every $S, T \in \mathcal{T}_p^*$ with $S \leq T$ there exists $U \in \mathcal{T}_p^*$ such that for every coloring $c : \binom{U}{S} \rightarrow \{\text{blue, red}\}$ there exists $h \in \binom{U}{T}$ such that $\{h \circ f : f \in \binom{T}{S}\}$ is monochromatic.

PROOF OF THEOREM 3.6. We show that every $S \in \mathcal{T}_p^*$ is a Ramsey object by induction on the height of S . First let $S \in \mathcal{T}_p^*$ be a one-element structure. Take $T \in \mathcal{T}_p$ such that S embeds into T . Without loss of generality, for any $x \in T$, which is not an endpoint, and $p \in P$ such that $K_p^T(x)$, if $p < \omega$, then the number of immediate successors of x is equal to $p - 1$. Moreover assume that there is M such that the for any x , which is not an endpoint, such that $K_\omega^T(x)$, the number of immediate successors of x is exactly M . Suppose that $S = \{a\}$ and let $p_S \in P$ be such that $K_{p_S}^S(a)$.

Let $T_0 = T$ and $T_k = T_{k-1}[T]$, $1 \leq k \leq h = ht(T)$, and we claim that $U = T_h$ is as required. Denote the set of copies of T attached to T_{k-1} in the construction of T_k by \mathcal{T}_k , and let $\mathcal{T}_0 = \{T_0\}$. Color embeddings of S into U into two colors: blue and red. If there is k and $T' \in \mathcal{T}_k$ such that all embeddings of S into T' are in the same color, we are done. Otherwise, for each k and $T' \in \mathcal{T}_k$ there is a blue embedding of S into T' .

We construct the required embedding f of T into U by induction. First we construct $f(r_T)$, where r_T denotes the root of T . Let $p_T \in P$ be such that $K_{p_T}^T(r_T)$. If $p_T \neq p_S$, let $f(r_T) = r_U$, where r_U is the root of U . If $p_T = p_S$, let $f(r_T)$ be an image of any blue embedding of S into $T' = T_0 \in \mathcal{T}_0$. Now let $x \in T$ be of height k and suppose that we constructed $f(x)$ and that $f(x) \in T'$ for some $T' \in \mathcal{T}_k$. Let $x_1 \prec^T \dots \prec^T x_m$ be the list of immediate successors of x , and we construct $f(x_1), \dots, f(x_m)$, each will be in a copy of T that lies in \mathcal{T}_{k+1} . Suppose that $p \in P$ is such that $K_p^T(x)$ and that $y_1 \prec^U \dots \prec^U y_{p'}$ is the list of immediate successors of $f(x)$, where $p' = p - 1$ if $p < \omega$ and $p' = M$ when $p = \omega$. By the construction of U , there are $r_1 \prec^U \dots \prec^U r_{p'}$, such that r_l is a successor of y_l in U and r_l is the root of some $T^l \in \mathcal{T}_{k+1}$. For each l , let $p_l \in P$ be such that $K_{p_l}^T(x_l)$, and if $p_l \neq p_S$, let $f(x_l)$ be equal to the point in T_l corresponding to x_l in the obvious isomorphism between T and T^l . Otherwise, if $p_l = p_S$, we let $f(x_l)$ to be the image of a blue embedding of S into $T^l \subseteq U$. This gives a “blue” embedding of T into U and finishes the base step of the induction.

For the inductive step, let $S, T \in \mathcal{T}_p^*$ such that $S \leq T$ be given. We assume that every tree in \mathcal{T}_p^* of the height strictly less than the height of S is a Ramsey object. Let $V \in \mathcal{T}_p^*$ be such that whenever we color embeddings of $\{r_S\}$ into V into two colors, then there exists an embedding $g : T \rightarrow V$ such that $\{g \circ f : f \in \binom{T}{S}\}$ is monochromatic. Without loss of generality, we assume that for any $x \in V$, which is not an endpoint, and $p \in P$ such that $K_p^V(x)$, if $p < \omega$, then the number of immediate successors of x is equal to $p - 1$. Let $a_1 \prec^S \dots \prec^S a_k$ be the list of immediate successors of r_S , and let $S_i = S_{a_i} = \{b \in S : a_i \leq_S b\}$.

Using the well-founded recursion along V , for each $x \in V$ we construct a tree $V^x \in \mathcal{T}_P^*$. The $U = V^{rv}$ will be as needed for S and T and two colors. For an endpoint $x \in V$, let $V^x = \{x\}$ with $K_p^{V^x}(x)$ iff $K_p^V(x)$ for every $p \in P$. Now let $x \in V$ not be an endpoint, let $x_1 \prec^V \dots \prec^V x_n$ be the list of all immediate successors of x , and assume that we already defined V^{x_1}, \dots, V^{x_n} . Let V_0^x be obtained from the disjoint union of $\{x\}$ and V^{x_1}, \dots, V^{x_n} , adding edges $[x, r_{V^{x_i}}]$. For $a \in V^{x_i}$ and $t = 1, \dots, n$ we let $G_t^{V_0^x}(x, a)$ iff $G_t^V(x, x_i)$. Similarly, for $a \in V^{x_i}$ and $b \in V^{x_j}$ we let $a \prec_{V_0^x}^x b$ iff $x_i \prec^V x_j$ and let $K_p^{V_0^x}(x)$ iff $K_p^V(x)$.

If S does not embed into V_0^x in a way that r_S is mapped to $r_{V_0^x}$, the root of V_0^x , let $V^x = V_0^x$. Otherwise, and if p such that $K_p^V(x)$ is finite, apply the product Ramsey theorem to (S_1, \dots, S_k) and $(V^{x_{b(1)}}, \dots, V^{x_{b(k)}})$, where $b: \{1, \dots, k\} \rightarrow \{1, \dots, n\}$ is an increasing injection such that for any t and i , $G_t^S(r_S, a_i)$ iff $G_t^V(x, x_{b(i)})$, and let $U^x \in \mathcal{T}_P^*$ be the structure we obtain from the product Ramsey theorem. For each $j \in \text{rng}(b)$, take any $U^j \in \mathcal{T}_P^*$ such that U^x embeds into it and for any $p \in P$, $K_p^{U^j}(r_{U^j})$ iff $K_p^{V^{x_j}}(r_{V^{x_j}})$. Finally, let V^x be equal to V_0^x with each $V^{x_{b(i)}}$ replaced by $U^{b(i)}$. If $K_\omega^V(x)$, take l such that whenever we color increasing injections of $\{1, \dots, k\}$ to $\{1, \dots, l\}$ into two colors then there is an increasing injection $g: \{1, \dots, n\} \rightarrow \{1, \dots, l\}$ such that all maps $g \circ f$, where $f: \{1, \dots, k\} \rightarrow \{1, \dots, n\}$ is an increasing injection, are in the same color. Let $U_0^x \in \mathcal{T}_P^*$ be any structure that all V^{x_1}, \dots, V^{x_n} embed into it. Enumerate increasing injections of $\{1, \dots, k\}$ to $\{1, \dots, l\}$ into e_1, \dots, e_m . Define recursively U_{i+1}^x , $i = 0, \dots, m - 1$ to be the result of applying the product Ramsey theorem to (S_1, \dots, S_k) and (U_i^x, \dots, U_i^x) . Define V^x to be the disjoint union of $\{x\}$ and l many U_m^x , we add edges $[x, r_{U_m^x}]$, and specify $\prec_{U_m^x}^x$, and let $K_p^{V^x}(x)$ iff $K_p^V(x)$.

Observe that for every $x \in V$ with immediate successors $x_1 \prec^V \dots \prec^V x_n$, which is not an endpoint, we have for every $p \in P$, $K_p^V(x)$ iff $K_p^{V^x}(r_{V^x})$. If p is finite and such that $K_p^V(x)$ or if S does not embed into V^x in a way that r_S is mapped to r_{V^x} , then x has exactly n many immediate successors $x'_1 \prec^{V^x} \dots \prec^{V^x} x'_n$ in V^x and they are such that for any t and i , $G_t^{V^x}(r_{V^x}, x'_i)$ iff $G_t^V(x, x_i)$, and for any p and i , $K_p^{V^x}(x'_i)$ iff $K_p^V(x_i)$. Moreover, for any coloring into two colors of embeddings of S into V^x such that r_S is mapped to r_{V^x} , there is an embedding $g: V_0^x \rightarrow V^x$ taking $r_{V_0^x}$ to r_{V^x} such that $\{g \circ f: f \in \binom{V_0^x}{S}\}$ taking r_S to $r_{V_0^x}$ is monochromatic. If $K_\omega^V(x)$, set $V^i = \{a \in V_0^x: x_i \leq_{V_0^x} a\}$, where $x_1 \prec_{V_0^x}^x \dots \prec_{V_0^x}^x x_n$ are the immediate successors of x in V_0^x . Then for any coloring into two colors of embeddings of S into V^x such that r_S is mapped to r_{V^x} , there are immediate successors $y_1 \prec^{V^x} \dots \prec^{V^x} y_n$ of r_{V^x} and an embedding $g: V_0^x \rightarrow V^x$ taking $r_{V_0^x}$ to r_{V^x} and satisfying $y_i \leq_{V^x} g(V^i)$, $i = 1, \dots, n$, such that $\{g \circ f: f \in \binom{V_0^x}{S}\}$ taking r_S to $r_{V_0^x}$ is monochromatic.

Color embeddings of S into U into two colors. Using the observations above, find an embedding $h: V \rightarrow U$ such that any two embeddings $g_1, g_2: S \rightarrow U$ whose image is contained in $h(V)$ and with $g_1(r_S) = g_2(r_S)$, are in the same color. Finally, by the choice of V , for the induced coloring of embeddings of $\{r_S\}$ into V into two colors, there exists an embedding $g: T \rightarrow V$ such that $\{g \circ f: f \in \binom{T}{S}\}$ is monochromatic. Then $h \circ g$ is as required. \dashv

We finish this section relating Theorem 3.6 to the work of Sokić [18].

A *semilattice* is a poset such that every 2 elements have an infimum. If A is a semilattice, we define a binary operation \circ on A by $a \circ b = \inf(a, b)$ and a partial order \leq_A by $a \leq_A b$ iff $a \circ b = a$. Say that (A, \circ^A) is a *treeable semilattice* if the induced poset is a rooted tree, i.e., it has the minimum, called the root, and for each $a \in A$, the set $\{b \in A: b \leq_A a\}$ is linearly ordered by \leq_A .

Let \mathcal{T} be the family of all finite treeable semilattices in the language $\{\circ\}$. Let $A \in \mathcal{T}$ and say that \preceq^A is a *convex ordering* on A if for every $a, b, c \in A$ with $a \circ b = c$, $a \neq c$ and $b \neq c$, we have $a \preceq^A b$ iff $a' \preceq^A b'$, where a', b' are immediate successors of c , $a' \leq_A a$ and $b' \leq_A b$. Denote the set of all convex ordering on A by $co(A)$ and let

$$CT = \{(A, \circ^A, \preceq^A): (A, \circ^A) \in \mathcal{T}, \preceq^A \in co(A)\}.$$

THEOREM 3.9 (Sokić, Theorem 2.2 in [18]). *CT is a Ramsey class.*

The theorem above is a special case of Theorem 3.6 and is equivalent to the statement that $\mathcal{T}_{\{\omega\}}^*$ is a Ramsey class. Indeed, the categories CT and $\mathcal{T}_{\{\omega\}}^*$ are equivalent via a covariant functor, which follows from Proposition 3.3 and an observation that convex orderings on treeable semilattices correspond to binary relations allowed in Step 3 of the definition of \mathcal{T}_P^* .

Similarly, Theorem 6.1 in [18] is equivalent to the statement that each $\mathcal{T}_{\{k\}}^*$ is a Ramsey class, $k \geq 3$, therefore again it is a special case of Theorem 3.6.

§4. The generalized Ważewski dendrite W_P , for an infinite $P \subseteq \{3, 4, \dots, \omega\}$. In this section, we show that in the case P is infinite, the universal minimal flow of the homeomorphism group of the generalized Ważewski dendrite W_P is nonmetrizable, and we point out two important consequences this fact (see Sections 4.1 and 4.2).

Let \mathcal{G} be a family of finite structures. Say that $A \in \mathcal{G}$ has *Ramsey degree* $\geq t$ iff there exist $A \leq B$ such that for every $B \leq C$ there exists a coloring $c_0: \binom{C}{A} \rightarrow \{1, 2, \dots, t\}$ such that for every $g \in \binom{C}{B}$, $\{g \circ f: f \in \binom{B}{A}\}$ assumes $\geq t$ colors.

The Ramsey degree is infinite if for every t it is $\geq t$.

THEOREM 4.1 (Zucker [22], Theorem 8.7). *Let \mathcal{G} be a Fraïssé-HP family and let \mathbb{G} be its Fraïssé limit. Then some $A \in \mathcal{G}$ has infinite Ramsey degree iff the universal minimal flow of $\text{Aut}(\mathbb{G})$ is nonmetrizable.*

Theorems 4.1 and 4.2 imply that when P is infinite then the universal minimal flow of $H(W_P)$ is nonmetrizable.

THEOREM 4.2. *Suppose that P is infinite. Then there is $A \in \mathcal{F}_P$ which has infinite Ramsey degree.*

PROOF. Let $p_1 < p_2 < \dots$ be the increasing enumeration of $P \setminus \{\omega\}$, let $A = \{a, b\}$ be such that $V(A) = \{a, b\}$, $E(A) = \{[a, b]\}$, $K_{p_1}^A(a)$ and $K_{p_1}^A(b)$, and let $t \geq 2$ be given. Take B constructed as follows. First let B_1 consists of a single point x_0 such that $K_{p_1}^{B_1}(x_0)$. Let B_2 be the tree that consists of vertices x_0, y_1, \dots, y_{p_1} and edges $[x_0, y_i]$, $K_{p_1}^{B_2}(x_0)$, and $K_{p_2}^{B_2}(y_i)$, $i = 1, \dots, p_1$. Having constructed B_k , $k \leq t$, such that for every endpoint e in B_k it holds $K_{p_k}^{B_k}(e)$, we obtain B_{k+1} from B_k in the following way. For every endpoint e in B_k pick new points $y_1^e, \dots, y_{p_k-1}^e$ and add vertices $[e, y_i^e]$. Note that e has degree p_k in B_{k+1} . If $k < t$, we let $K_{p_{k+1}}^{B_{k+1}}(y_i^e)$, and if $k = t$, let $K_{p_1}^{B_{k+1}}(y_i^e)$. Take $B = B_t$.

Now take any $C \in \mathcal{F}_P$ such that $B \leq C$. Pick an endpoint r in C and consider C as a rooted tree with the root r . Let $c_0: \binom{C}{A} \rightarrow \{1, 2, \dots, t\}$ be the following coloring. For an embedding $f: A \rightarrow C$, if $f(a)$ and $f(b)$ do not lie on the same branch in the rooted tree C , let $c_0(f) = i$ iff $K_{p_i}^C(c)$, where c is the meet of $f(a)$ and $f(b)$. Otherwise, if $f(a)$ and $f(b)$ do lie on the same branch, let $c_0(f)$ be an arbitrary color from $\{1, \dots, t\}$.

Let $g: B \rightarrow C$ be an embedding. There are j_1 and j_2 such that $g(x_0) \leq_C g(y_{j_1}), g(y_{j_2}), y_{j_1}, y_{j_2} \in B_2$ (in fact, all $j = 1, \dots, p_1$ except one have this property). Clearly in the rooted tree B' , obtained from B by removing all vertices z such that some $y_j \in B_2, j \neq j_1, j_2$, is on the path connecting z and $x_0 \in B_1$, we have that for any i there are endpoints e_1 and e_2 in B' such that the meet of e_1 and e_2 is a vertex c such that $K_{p_i}^{B'}(c)$. That implies that $g(B')$ and hence $g(B)$ assumes all t colors. \dashv

COROLLARY 4.3. *Suppose that P is infinite. Then the universal minimal flow of $H(W_P)$ is nonmetrizable.*

4.1. $H(W_P)$ has a nonmetrizable universal minimal flow and is Roelcke precompact. A subgroup H of $S(X)$, the group of all permutations of a countable set X with the pointwise convergence topology, is *oligomorphic* when for every n , the diagonal action of H on X^n has only finitely many orbits. Note that we do not assume that H is a closed subgroup of $S(X)$. A topological group H is *Roelcke precompact* if for every open neighbourhood U of $1 \in H$ there exists a finite set $F \subseteq H$ such that $H = UFU$. As shown by Tsankov [21, Theorem 2.4] a subgroup of $S(X)$ is Roelcke precompact if and only if it is an inverse limit of oligomorphic groups.

As observed by Todor Tsankov (private communication in 2013), $\text{Aut}(M_P)$, for each $P \subseteq \{3, 4, \dots, \omega\}$, is a Roelcke precompact group. This is because when we take

$$M_n = \{m \in M_P : K_p(m) \text{ for some } p \in \{3, \dots, n, \omega\}\},$$

$$G_n = \text{Aut}(M_n) = \text{Aut}(M_n, D^{M_n}, (K_p^{M_n})_{p \in P \cap \{3, \dots, n, \omega\}}),$$

and

$$H_n = \{h \in G_n : \text{there exists } f \in \text{Aut}(M_P) \text{ such that } h = f \upharpoonright M_n\},$$

then H_n is an oligomorphic group and the inverse limit of H_n is equal to $\text{Aut}(M_P)$.

Melleray–Nguyen Van Thé–Tsankov [15] asked:

QUESTION 4.4 (Question 1.5 in [15]). *Is the universal minimal flow of every Roelcke precompact Polish group metrizable?*

Moreover, Bodirsky–Pinsker–Tsankov [5] asked if every ω -categorical structure has an ω -categorical expansion which is Ramsey (which by the work of Zucker [22] is equivalent to the question above with “Roelcke precompact Polish” replaced by “oligomorphic”).

Evans–Hubička–Nešetřil [11] answered Question 4.4 in the negative. They provided an example of an oligomorphic group with a nonmetrizable universal minimal flow. Their example is much more involved than ours, it is based on a very nontrivial construction due to Hrushovski [12], see also [10].

4.2. $H(W_P)$ has a nonmetrizable universal minimal flow with a comeager orbit. Ben Yaacov–Melleray–Tsankov [3], generalizing a result of Zucker [22], showed:

THEOREM 4.5 (Theorem 1.2 in [3]). *Let G be a Polish group whose universal minimal flow $M(G)$ is metrizable. Then $M(G)$ has a comeager orbit.*

They asked if the converse holds:

QUESTION 4.6 (Question 1.3 in [3]). *Suppose that G is a Polish group such that $M(G)$ has a comeager orbit. Is it true that $M(G)$ is metrizable?*

After this preprint was posted on arXiv, Zucker [23] showed:

THEOREM 4.7. *Let \mathcal{G} be a Fraïssé–HP family. Suppose that there exists a reasonable Fraïssé–HP expansion \mathcal{G}^* of \mathcal{G} , which has the relative HP, the expansion, and the Ramsey properties, but the precompactness fails. Then the universal minimal flow of $\text{Aut}(\mathbb{G})$, where \mathbb{G} is the Fraïssé limit of \mathcal{G} , has a comeager orbit.*

Theorems 3.1 and 4.7, together with Corollary 4.3, provide the negative answer to Question 4.6.

§5. Acknowledgments. I thank Todor Tsankov for our discussions on generalized Ważewski dendrites in 2013. I also thank Miodrag Sokić, Bruno Duchesne, and the referee, for many valuable remarks that improved the presentation of the article. The author was supported by Narodowe Centrum Nauki grant 2016/23/D/ST1/01097.

REFERENCES

- [1] S. A. ADELEKE and P. M. NEUMANN, *Relations related to betweenness: Their structure and automorphisms. Memoirs of the American Mathematical Society*, vol. 131 (1998), no. 623, viii+125 pp.
- [2] D. BARTOŠOVÁ and A. KWIAKOWSKA, *The universal minimal flow of the homeomorphism group of the Lelek fan. Transactions of the American Mathematical Society*, to appear, arXiv:1706.09154.
- [3] I. BEN YAACOV, J. MELLERAY, and T. TSANKOV, *Metrizable universal minimal flows of Polish groups have a comeager orbit. Geometric and Functional Analysis*, vol. 27 (2017), no. 1, pp. 67–77.
- [4] M. BODIRSKY and D. PIGUET, *Finite trees are Ramsey under topological embeddings*, preprint, 2010, arXiv:1002.1557.
- [5] M. BODIRSKY, M. PINSKER, and T. TSANKOV, *Decidability of definability*, this JOURNAL, vol. 78 (2013), no. 4, pp. 1036–1054.
- [6] W. J. CHARATONIK and A. M. DILKS, *On self-homeomorphic spaces. Topology and its Applications*, vol. 55 (1994), no. 3, pp. 215–238.
- [7] W. DEUBER, *A generalization of Ramsey’s theorem for regular trees. Journal of Combinatorial Theory, Series B*, vol. 18 (1975), pp. 18–23.
- [8] B. DUCHESNE, *Topological properties of Ważewski dendrite groups*, preprint, 2017.
- [9] B. DUCHESNE and N. MONOD, *Structural properties of dendrite groups. Transactions of the American Mathematical Society*, to appear, arXiv:1610.08488.
- [10] D. EVANS, *Homogeneous structures, ω -categoricity and amalgamation constructions*. Notes on a Minicourse given at HIM, Bonn, September 2013. Available at <https://www.him.uni-bonn.de/programs/past-programs/past-trimester-programs/universality-and-homogeneity/mini-courses/>.
- [11] D. EVANS, J. HUBIČKA, and J. NEŠETŘIL, *Automorphism groups and Ramsey properties of sparse graphs*, preprint, 2017, arXiv:1801.01165.
- [12] E. HRUSHOVSKI, *A stable \aleph_0 -categorical pseudoplane*, unpublished notes, 1988.
- [13] J. JASIŃSKI, *Ramsey degrees of boron tree structures. Combinatorica*, vol. 33 (2013), no. 1, pp. 23–44.

- [14] A. S. KECHRIS, V. G. PESTOV, and S. TODORCEVIC, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*. *Geometric and Functional Analysis*, vol. 15 (2005), no. 1, pp. 106–189.
- [15] J. MELLERAY, L. NGUYEN VAN THÉ, and T. TSANKOV, *Polish groups with metrizable universal minimal flows*. *International Mathematics Research Notices*, vol. 2016 (2016), no. 5, pp. 1285–1307.
- [16] J. NEŠETŘIL and V. RÖDL, *Partitions of finite relational and set systems*. *Journal of Combinatorial Theory, Series A*, vol. 22 (1977), no. 3, pp. 289–312.
- [17] L. NGUYEN VAN THÉ, *More on the Kechris-Pestov-Todorcevic correspondence: Precompact expansions*. *Fundamenta Mathematicae*, vol. 222 (2013), no. 1, pp. 19–47.
- [18] M. SOKIĆ, *Semilattices and the Ramsey property*, this JOURNAL, vol. 80 (2015), no. 4, pp. 1236–1259.
- [19] ———, *Ramsey property of finite posets II*. *Order*, vol. 29 (2012), no. 1, pp. 31–47.
- [20] S. SOLECKI, *Abstract approach to Ramsey theory and Ramsey theorems for finite trees*. *Asymptotic Geometric Analysis, Fields Institute Communications* (M. Ludwig, V. D. Milman, W. Pestov, and N. Tomczak-Jaegermann, editors), Springer, New York, 2013, pp. 313–340.
- [21] T. TSANKOV, *Unitary representations of oligomorphic groups*. *Geometric and Functional Analysis*, vol. 22 (2012), no. 2, pp. 528–555.
- [22] A. ZUCKER, *Topological dynamics of automorphism groups, ultrafilter combinatorics, and the Generic Point Problem*. *Transactions of the American Mathematical Society*, vol. 368 (2016), no. 9, pp. 6715–6740.
- [23] ———, *New directions in the abstract topological dynamics of Polish groups*, Ph.D. thesis, Carnegie Mellon University, April 2018.

INSTITUT FÜR MATHEMATISCHE LOGIK UND GRUNDLAGENFORSCHUNG
 UNIVERSITÄT MÜNSTER
 EINSTEINSTRASSE 62
 48149 MÜNSTER, GERMANY

and

INSTYTUT MATEMATYCZNY
 UNIwersytet WROCLAWSKI
 PL. GRUNWALDZKI 2/4
 50-384 WROCLAW, POLAND
E-mail: kwiatkoa@uni-muenster.de