Proceedings of the Edinburgh Mathematical Society (2002) **45**, 449–465 ©
DOI:10.1017/S0013091500001176 Printed in the United Kingdom

EXPLICIT CAPELLI IDENTITIES FOR SKEW SYMMETRIC MATRICES

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(Received 27 November 2000)

Dedicated to Professor Mitsuo Sugiura on the occasion of his 70th birthday

Abstract The main purpose of this paper is to obtain an explicit Capelli identity relating skew-symmetric matrices under the action of the general linear group GL_N. In particular, we give an explicit formula for the *skew Capelli element* in terms of the trace of powers of a matrix defined by the standard infinitesimal generators of GL_N.

Keywords: Capelli identity; skew Capelli element; Pfaffian; universal enveloping algebra; invariant differential operator; Young diagram

AMS 2000 *Mathematics subject classification:* Primary 17B35
Secondary 17B10; 15A15; 20G05

1. Introduction

Various (explicit and/or abstract) Capelli identities corresponding to multiplicity-free actions were established in [**4**]. Among the explicit identities given there, only those relating skew-symmetric matrices under the action of GL_N have not yet been completely understood explicitly, although their general nature was determined. What is desired here is to obtain an explicit formula for the *skew Capelli element* (see p. 592, Remark (a) in [**4**] and see also [**8**]). Since in [**4**] the skew Capelli elements are introduced through their eigenvalues in terms of the irreducible summands of the decomposition of the ring of polynomials on the space of skew-symmetric matrices, it is difficult to obtain explicit formulae directly from the definition. The main purpose of the present paper is to provide such an explicit Capelli identity by using the minor summation formula of Pfaffians established in [**5**]. In particular, we give an explicit formula for the skew Capelli element, which in fact belongs to the centre $\mathcal{ZU}(\mathfrak{gl}_N)$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_N)$, in terms of the trace of powers of a matrix $E = E_N$. Here, the (i, j)th element of the matrix E is given by the (i, j)th standard basis elements E_{ij} of $\mathfrak{gl}_N$. In this context, the expression 'the Capelli identity' refers to a description of the invariant (polynomial coefficients) differential operators in terms of some suitable elements in $\mathcal{ZU}(\mathfrak{gl}_N)$ under the corresponding infinitesimal action of GL_N.

To find the skew Capelli elements, we look at the following identity as our starting point (see p. 296, Remark 2 in [**5**]):

$$\sum_{t=0}^{[n/2]} \lambda^t \sum_{\boldsymbol{i}\in I_{2t}^n} \mathrm{pf}(X_{\boldsymbol{i}})\,\mathrm{pf}(\partial_{\boldsymbol{i}}) = (-1)^{(n(n-1))/2}\,\mathrm{pf}\begin{pmatrix} -X & I_n \\ -I_n & \lambda\partial \end{pmatrix}. \tag{1.1}$$

Here, ∂ represents a dual matrix of $X=(x_{ij})$ obtained by replacing x_{ij} with $\partial/\partial x_{ij}$, and $X_{\boldsymbol{i}}$ and $\partial_{\boldsymbol{i}}$, respectively, represent the sub-matrix of X and the sub-matrix of ∂ determined by the subscript $\boldsymbol{i}$ (see § 2 below for the precise meaning of the notation). The important fact here is that the coefficients of the λ^t on the left-hand side give rise to the generators of the ring of invariant differential operators with polynomial coefficients. Actually, it is easy to see from this expression that the operators are GL_N-invariant. Therefore, the problem is to determine how we can express the right-hand side in terms of suitable polynomials in E_{ij} under the action in question. In fact, these polynomials turn out to be elements of $\mathcal{ZU}(\mathfrak{gl}_N)[\lambda]$.

Our strategy is to use the idea described above to obtain the desired identities. For motivation and guidance, we start by considering the much simpler commutative case. In § 2 we recall the classical Newton formula for describing the expression of elementary symmetric polynomials in terms of power-sum symmetric polynomials and vice versa. In other words, this is a relation between the trace of powers and the sum of minor-determinants of a given square matrix. If we replace ∂ appearing in (1.1) by a matrix Y whose elements commute elementwise with those of X, then we may state a certain variant of Newton's formula in determinant form for a skew-symmetric matrix. This variant describes a relation between the trace of powers and the sum of minor-Pfaffians. This enables us to establish a 'commutative' counterpart (or, rather, the 'leading term' of a differential operator) of a formula for the target Capelli identity (see Theorem 2.7). From this identity we may guess the explicit form of the skew Capelli elements and the identities, respectively. In particular, it seems quite reasonable to expect that the skew Capelli elements can be written nicely using the trace of the matrix powers E^k of E. Of course, if one could compute eigenvalues of $\mathrm{tr}(E^k)$ in their entirety, then one might naturally expect that the final answer could be found by employing the representation/invariant theoretic characterization of the operators. However, this is not the case. Fortunately, though, one can determine constant multiples which relate the generators (of the invariant differential operators with polynomial coefficients) and the desired central elements defined in terms of the various $\mathrm{tr}(E^k)$. Combining the information about such constant multiples which can be obtained from Theorem 2.7 regarding the inductive properties which our expected elements possess, we will finally obtain the sought-after Capelli identity in Theorem 3.2. It is worth noting that a similar procedure also works when one tries to find a certain variant (using the trace of the matrix powers E^k) of the Capelli identity for the most standard $GL_N \times GL_N$ action on the full matrix algebra (see Theorem 3.15).

Moreover, in the very final part of the paper, we devote ourselves to computing the explicit eigenvalues of our skew Capelli elements $\tilde{\mathbb{T}}_k(E)$. In order to do this, we first evaluate the eigenvalues of these elements on some class of representations (parametrized by partitions of the box-type shape) which consists of a part of the above-mentioned irreducible summands. Indeed, comparing this result with the very definition of the aforementioned skew Capelli elements C_k^Λ via the eigenvalues introduced in [**4**] we finish the computation. This means that our skew Capelli elements $\tilde{\mathbb{T}}_k(E_N)$ coincide with C_k^Λ.

2. Newton's formula for Pfaffians

In this section we give a Pfaffian version of Newton's formula for a skew-symmetric matrix.

We first recall the definition of the Pfaffian $\mathrm{pf}(X)$ for an $2m \times 2m$ skew-symmetric matrix X:

$$\mathrm{pf}(X) = \sum_{\sigma \in \mathfrak{S}_{2m}/\mathfrak{B}_m} \mathrm{sgn}(\sigma) x_{\sigma(1)\sigma(2)} x_{\sigma(3)\sigma(4)} \cdots x_{\sigma(2m-1)\sigma(2m)}.$$

Here $\mathfrak{B}_m$ is the subgroup of the symmetric group $\mathfrak{S}_{2m}$ consisting of elements which preserve the collection of pairs $\{\{1,2\},\{3,4\},\dots,\{2m-1,2m\}\}$. It is also convenient to note that $\mathrm{pf}(X)$ can be expressed as

$$\mathrm{pf}(X) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{E}_m} \mathrm{sgn}(\sigma) x_{\sigma(1)\sigma=(2)} x_{\sigma(3)\sigma(4)} \cdots x_{\sigma(2m-1)\sigma(2m)},$$

where we define $\mathfrak{E}_m = \{\sigma = (\sigma(1),\dots,\sigma(2m)) \in \mathfrak{S}_{2m};\ \sigma(2k-1) < \sigma(2k)\ (1 \leqslant k \leqslant m)\}$.

We now define

$$I_k^n = \{\boldsymbol{i} = (i_1,\dots,i_k);\ 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$$

for $n, k \in \mathbb{N}$ satisfying $k \leqslant n$. Let $X = (x_{ij})$ be an $n \times n$ matrix. For each $\boldsymbol{i} = (i_1,\dots,i_k) \in I_k^n$, we denote by $X_{\boldsymbol{i}}$ the sub-matrix of X defined by

$$X_{\boldsymbol{i}} = \begin{pmatrix} x_{i_1 i_1} & \dots & x_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ x_{i_k i_1} & \dots & x_{i_k i_k} \end{pmatrix}.$$

We now recall the minor summation formula of Pfaffians from [**5**].

Theorem 2.1 (the minor summation formula of Pfaffians). *Let T be an $n \times n$ matrix and let X and Y be $n \times n$ skew-symmetric matrices. Put $m = [n/2]$, the integer part of $n/2$. If we define $Z = TY\,{}^{\mathrm{t}}T$, then*

$$\sum_{\ell=0}^{m} \lambda^\ell \sum_{\boldsymbol{i} \in I_{2\ell}^n} \mathrm{pf}(X_{\boldsymbol{i}})\, \mathrm{pf}(Y_{\boldsymbol{i}}) \det(T_{\boldsymbol{i}}) = (-1)^{(n(n-1))/2}\, \mathrm{pf} \begin{pmatrix} -X & I_n \\ -I_n & \lambda Z \end{pmatrix}, \tag{2.1}$$

where λ is a parameter, I_n is the $n \times n$ identity matrix and ${}^{\mathrm{t}}T$ denotes the matrix transposition of T.

To simplify the following discussion we assume that n is even throughout this section.

Corollary 2.2. *Suppose $n = 2m$ is even. Let X and Y be $n \times n$ skew-symmetric matrices. Then the multiplicity of each eigenvalue of the product XY is even.*

Proof. Taking the square of both sides of (2.1) when $T = I_n$, we have

$$\begin{aligned}\left\{\sum_{t=0}^{m}\lambda^t \sum_{\boldsymbol{i}\in I_{2t}^n} \operatorname{pf}(X_{\boldsymbol{i}}) \operatorname{pf}(Y_{\boldsymbol{i}})\right\}^2 &= \left\{(-1)^{(n(n-1))/2} \operatorname{pf}\begin{pmatrix} -X & I_n \\ -I_n & \lambda Y\end{pmatrix}\right\}^2 = \det\begin{pmatrix} -X & I_n \\ -I_n & \lambda Y\end{pmatrix} \\ &= \det(-X)\det(-X^{-1}+\lambda Y) = \det(I_n - \lambda XY) \\ &= \lambda^n \det(\lambda^{-1} I_n - XY).\end{aligned}$$

Put $s = \lambda^{-1}$. Then the formula above turns out to be the characteristic polynomial of the matrix XY. Namely, we have

$$\det(sI_n - XY) = \left\{\sum_{t=0}^{m} s^{m-t} \sum_{\boldsymbol{i}\in I_{2t}^n} \operatorname{pf}(X_{\boldsymbol{i}}) \operatorname{pf}(Y_{\boldsymbol{i}})\right\}^2. \tag{2.2}$$

It hence follows that the multiplicities of the eigenvalues of XY are all even. □

We now recall the classical Newton formula.

Lemma 2.3 (the Newton formula). *Let*

$$e_k = \sum_{i_1<i_2<\cdots<i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

be the kth elementary symmetric polynomials and $p_k = \sum x_i^k$ be the kth power-sum symmetric polynomials. Among these symmetric polynomials the following relations hold:

$$k \cdot e_k = \sum_{i=1}^{k} (-1)^{i-1} p_i \cdot e_{k-i} \quad (k = 1, 2, \dots). \tag{2.3}$$

This formula can also be expressed in the following determinant form, which is useful for subsequent analysis (see, for example, p. 28 in [**7**]):

$$e_k = \frac{1}{k!}\det\begin{pmatrix} p_1 & 1 & 0 & \dots & 0 & 0 \\ p_2 & p_1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{k-2} & p_{k-3} & p_{k-4} & \dots & k-2 & 0 \\ p_{k-1} & p_{k-2} & p_{k-3} & \dots & p_1 & k-1 \\ p_k & p_{k-1} & p_{k-2} & \dots & p_2 & p_1 \end{pmatrix}. \tag{2.4}$$

https://doi.org/10.1017/S0013091500001176 Published online by Cambridge University Press

Definition 2.4. For an arbitrary $n \times n$ matrix T and $k \in \mathbb{Z}_{\geqslant 0}$, we define $\mathbb{T}_k(T)$ by

$$\mathbb{T}_k(T) = \frac{1}{2^k k!} \det \begin{pmatrix} \mathrm{tr}(T) & 2 & 0 & \dots & 0 & 0 \\ \mathrm{tr}(T^2) & \mathrm{tr}(T) & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{tr}(T^{k-2}) & \mathrm{tr}(T^{k-3}) & \mathrm{tr}(T^{k-4}) & \dots & 2(k-2) & 0 \\ \mathrm{tr}(T^{k-1}) & \mathrm{tr}(T^{k-2}) & \mathrm{tr}(T^{k-3}) & \dots & \mathrm{tr}(T) & 2(k-1) \\ \mathrm{tr}(T^k) & \mathrm{tr}(T^{k-1}) & \mathrm{tr}(T^{k-2}) & \dots & \mathrm{tr}(T^2) & \mathrm{tr}(T) \end{pmatrix}$$

for $k \geqslant 1$ and $\mathbb{T}_0(T) = 1$.

Using Lemma 2.3 we have the following lemma.

Lemma 2.5. *Let $n, m \in \mathbb{N}$ be such that $n = 2m$ and let X and Y be $n \times n$ skew-symmetric matrices. Then*

$$\mathrm{pf}(X)\,\mathrm{pf}(Y) = \mathbb{T}_m({}^{\mathrm{t}}XY). \tag{2.5}$$

Proof. Let $\alpha_1 \cdots \alpha_m$ be the eigenvalues of ${}^{\mathrm{t}}XY$. If we put $s = 0$ in equation (2.2), then $\{\mathrm{pf}(X)\,\mathrm{pf}(Y)\}^2 = \alpha_1^2 \cdots \alpha_m^2$. Take the square root of each side of this relation. Then substitute an appropriate matrix to determine the sign. This yields

$$e_m = \mathrm{pf}(X)\,\mathrm{pf}(Y) = \alpha_1 \cdots \alpha_m. \tag{2.6}$$

Thus, since

$$\mathrm{tr}\{({}^{\mathrm{t}}XY)^k\} = 2(\alpha_1^k + \cdots + \alpha_m^k) \tag{2.7}$$

for any $k \in \mathbb{N}$, the assertion follows immediately from expression (2.4). □

The proof of the following lemma is the same as that of Corollary 2.2.

Lemma 2.6. *We retain the notation and the assumptions of Lemma 2.5. We then have*

$$\mathrm{pf}\begin{pmatrix} -X & I_n \\ -I_n & \lambda Y \end{pmatrix} = \mathrm{pf}(-X)\,\mathrm{pf}(\lambda Y - X^{-1}). \tag{2.8}$$

In view of the minor summation formula (2.1), using Lemmas 2.5 and 2.6 we see that

$$\sum_{\ell=0}^{m} \lambda^\ell \sum_{\boldsymbol{i} \in I_{2\ell}^n} \mathrm{pf}(X_{\boldsymbol{i}})\,\mathrm{pf}(Y_{\boldsymbol{i}})\det(T_{\boldsymbol{i}}) = (-1)^{(n(n-1))/2}\mathbb{T}_m(\lambda XZ - I_n). \tag{2.9}$$

We are now in a position to state the following theorem. This theorem asserts that the minor summation of the product of Pfaffians and/or determinants can be represented by just one determinant whose entries are given by the trace of the related matrix powers.

Theorem 2.7. *Let T be an $n \times n$ matrix and let X and Y be $n \times n$ skew-symmetric matrices. Put $Z = TY\,{}^{\mathrm{t}}T$. For $0 \leqslant \ell \leqslant [n/2]$, the following identity holds:*

$$\sum_{\boldsymbol{i} \in I_{2\ell}^n} \mathrm{pf}(X_{\boldsymbol{i}})\,\mathrm{pf}(Y_{\boldsymbol{i}})\det(T_{\boldsymbol{i}}) = \mathbb{T}_\ell({}^{\mathrm{t}}XZ). \tag{2.10}$$

For simplicity, we shall give the proof of Theorem 2.7 only for the case of even n. We need the following expansion formula for $\mathbb{T}_k(\lambda A + I_n)$ with respect to the parameter λ.

Proposition 2.8. *Let $n = 2m$ be an even integer and A be an $n \times n$ matrix. Suppose $0 \leqslant k \leqslant m$. Then we have*

$$\mathbb{T}_k(\lambda A + I_n) = \sum_{i=0}^{k} \binom{m-i}{k-i} \mathbb{T}_i(A)\lambda^i. \tag{2.11}$$

Proof. We prove the lemma by induction on k. It clearly holds when $k = 0$. Now, suppose that the formula holds for any l less than k. By the definition of $\mathbb{T}_k$, if we expand the determinant with respect to the last row, then we obtain

$$\mathbb{T}_k(A) = \frac{1}{2k} \sum_{i=1}^{k} (-1)^{i-1} \operatorname{tr}(A^i)\mathbb{T}_{k-i}(A). \tag{2.12}$$

Using this relation, by the induction hypothesis and the binomial theorem we observe

$$\begin{aligned}
\mathbb{T}_k(\lambda A + I_n) &= \sum_{i=1}^{k} (-1)^{i-1} \frac{1}{2k} \operatorname{tr}\{(\lambda A + I_n)^i\} \mathbb{T}_{k-i}(\lambda A + I_n) \\
&= \sum_{i=1}^{k} (-1)^{i-1} \frac{1}{2k} \left\{ \sum_{j=0}^{i} \binom{i}{j} \operatorname{tr}(A^j)\lambda^j \right\} \left\{ \sum_{l=0}^{k-i} \binom{m-l}{k-i-l} \mathbb{T}_l(A)\lambda^l \right\} \\
&= \sum_{i=1}^{k} \sum_{j=0}^{i} \sum_{l=0}^{k-i} (-1)^{i-1} \frac{1}{2k} \binom{i}{j} \binom{m-l}{k-i-l} \operatorname{tr}(A^j)\mathbb{T}_l(A)\lambda^{j+l}.
\end{aligned}$$

We now divide the sum in $\mathbb{T}_k(\lambda A + I_n)$ into two parts. That is, we set

$$\mathbb{T}_k(\lambda A + I_n) = \chi_1 + \chi_2,$$

where

$$\begin{aligned}
\chi_1 &= \sum_{i=1}^{k} \sum_{j=1}^{i} \sum_{l=0}^{k-i} (-1)^{i-1} \frac{1}{2k} \binom{i}{j} \binom{m-l}{k-i-l} \operatorname{tr}(A^j)\mathbb{T}_l(A)\lambda^{j+l}, \\
\chi_2 &= \sum_{i=1}^{k} \sum_{l=0}^{k-i} (-1)^{i-1} \frac{m}{k} \binom{m-l}{k-i-l} \mathbb{T}_l(A)\lambda^l.
\end{aligned}$$

Now we put $t = j + l$, $s = j$ and $t - s = l$ and change the order of the summations appearing in χ_1. Then we have

$$\chi_1 = \sum_{t=1}^{k} \sum_{s=1}^{t} \sum_{i=0}^{k-t} (-1)^{i+s-1} \frac{1}{2k} \binom{i+s}{s} \binom{m-t+s}{k-t-i} \operatorname{tr}(A^s)\mathbb{T}_{t-s}(A)\lambda^t.$$

We now recall two identities relating binomial coefficients. These identities can also be easily verified by induction (see, for example, [**2**]). For any $l, m, n \in \mathbb{Z}_{\geqslant 0}$ and $r \in \mathbb{R}$, we have

$$\sum_{i=0}^{l}(-1)^i\binom{r}{i} = (-1)^l\binom{r-1}{l}, \tag{2.13}$$

$$\sum_{i=0}^{l}(-1)^i\binom{l-i}{m}\binom{r}{i-n} = (-1)^{l+m}\binom{r-m-1}{l-m-n}. \tag{2.14}$$

Using (2.14), χ_1 turns out to be

$$\chi_1 = \sum_{t=1}^{k}\sum_{s=1}^{t}(-1)^{s-1}\frac{1}{2k}\binom{m-t-1}{k-t}\operatorname{tr}(A^s)\mathbb{T}_{t-s}(A)\lambda^t.$$

Making use of (2.12) again, this expression becomes

$$\chi_1 = \sum_{t=1}^{k}\frac{t}{k}\binom{m-t-1}{k-t}\mathbb{T}_t(A)\lambda^t.$$

Similar manipulation yields

$$\chi_2 = \sum_{t=0}^{k-1}\frac{m}{k}\binom{m-t-1}{k-t-1}\mathbb{T}_t(A)\lambda^t.$$

Adding χ_1 and χ_2, we thus obtain

$$\begin{aligned}\mathbb{T}_k(\lambda A + I_n) &= \sum_{t=1}^{k}\frac{t}{k}\binom{m-t-1}{k-t}\mathbb{T}_t(A)\lambda^t + \sum_{t=0}^{k-1}\frac{m}{k}\binom{m-t-1}{k-t-1}\mathbb{T}_t(A)\lambda^t \\ &= \sum_{t=0}^{k}\binom{m-t}{k-t}\mathbb{T}_t(A)\lambda^t.\end{aligned}$$

This completes the proof of the proposition. □

Proof of Theorem 2.7. If we consider the case $k = m$ in (2.11), then we see that

$$\mathbb{T}_m(\lambda XZ - I_n) = (-1)^m\mathbb{T}_m(-\lambda XZ + I_n) = (-1)^m\sum_{\ell=0}^{k}\mathbb{T}_\ell({}^{\mathrm{t}}XZ)\lambda^\ell. \tag{2.15}$$

Since $(n(n-1))/2 \equiv m \mod 2$, if we combine (2.9) with (2.15), then the theorem follows immediately. □

3. Explicit Capelli identities

Let $\Lambda^2 GL_N$ denote the action of GL_N on skew-symmetric two tensors. This is naturally thought of as the action $\lambda(g) : X \to gX\,{}^{\mathrm{t}}g$ ($g \in GL_N$) on the space of skew-symmetric matrices $\Lambda^2(\mathbb{C}^N)$. We use E_{ij} to represent a standard basis of the Lie algebra $\mathfrak{gl}_N$ of GL_N. Then, let $\mathcal{U}(\mathfrak{gl}_N)$ be the universal enveloping algebra of $\mathfrak{gl}_N$ and $\mathcal{ZU}(\mathfrak{gl}_N)$ be the centre of $\mathcal{U}(\mathfrak{gl}_N)$. We also denote the derived action of $\mathcal{U}(\mathfrak{gl}_N)$ by λ. Then the standard basis elements E_{ij} of $\mathfrak{gl}_N$ are represented by the linear vector fields as follows:

$$\lambda(E_{ij}) = \sum_{t=1}^{N}(x_{it}\partial_{jt} + x_{ti}\partial_{tj}). \tag{3.1}$$

Remark 3.1. In applying this formula x_{ij} and x_{ji} should be treated as independent variables. In other words, if we regard x_{ij} ($i < j$) as the standard coordinate functions on $\Lambda^2(\mathbb{C}^N)$, then we use the convention on notation that $x_{ji} = -x_{ij}$ for $i < j$ and also use the convention $\partial_{ji} = -\partial_{ij}$. Consequently, the infinitesimal action λ is interpreted as $\lambda(E_{ij}) = \sum_{t=1}^{N} x_{it}\partial_{jt}$.

Let $\mathcal{P}(\Lambda^2(\mathbb{C}^N))$ be the ring of polynomials on $\Lambda^2(\mathbb{C}^N)$. It is well known that the natural action of GL_N on $\mathcal{P}(\Lambda^2(\mathbb{C}^N))$ is multiplicity-free and that, in fact, the irreducible decomposition of $\mathcal{P}(\Lambda^2(\mathbb{C}^N))$ under the action of $\mathfrak{gl}_N$ is described by

$$\mathcal{P}(\Lambda^2(\mathbb{C}^N)) \cong \bigoplus_{D} \varrho_N^D,$$

where $D = (b_1, b_1, b_2, b_2, \dots)$ runs over Young diagrams with columns of even length and with $\operatorname{depth}(D) \leqslant N$, and ϱ_N^D is the corresponding polynomial representation of GL_N (see, for example, [**3**]). The fundamental generators associated with $\Lambda^2(\mathbb{C}^N)$ are given by

$$D_{2k} = \underbrace{(1, 1, 1, 1, \dots, 1, 1)}_{2k\text{-terms}},$$

and the corresponding fundamental highest-weight vectors are expressed in terms of (the principal minor) Pfaffians $\varphi_k = \operatorname{pf}(X_{(1,2,\dots,2k-1,2k)})$ (see Lemma 3.8 below). Note that $\varrho_N^{D_{2k}} \cong \Lambda^{2k}(\mathbb{C}^N)$. Moreover, the highest-weight vector φ_N^D of ϱ_N^D for $D = (b_1, b_1, b_2, b_2, \dots)$ is given by $\prod_{k=1}^{[N/2]} \varphi_k^{b_k - b_{k+1}}$. Let $\mathcal{PD}(\Lambda^2(\mathbb{C}^N))^{GL_N}$ be the algebra of GL_N-invariant differential operators with polynomial coefficients on $\Lambda^2(\mathbb{C}^N)$. We now denote a typical element of this space by X; that is, X is an $N \times N$ skew-symmetric matrix, and ∂ is a dual matrix obtained by replacing x_{ij} with $\partial_{ij} = (\partial/\partial x_{ij})$. Then it is known that the canonical generators of $\mathcal{PD}(\Lambda^2(\mathbb{C}^N))^{GL_N}$ are given by

$$\Gamma_k^{\Lambda} = \sum_{\boldsymbol{i} \in I_{2k}^N} \operatorname{pf}(X_{\boldsymbol{i}}) \operatorname{pf}(\partial_{\boldsymbol{i}}) \quad (1 \leqslant k \leqslant [N/2]).$$

The operator Γ_k^{Λ} is characterized as an element of $\mathcal{PD}(\Lambda^2(\mathbb{C}^N))^{GL_N}$ up to constant multiples by the following properties [**4**].

(i) Γ_k^{Λ} has degree k as a differential operator.

(ii) Γ_k^{Λ} annihilates all ϱ_N^D occurring in $\mathcal{P}(\Lambda^2(\mathbb{C}^N))$ with depth $(D) < 2k$.

We now come to our main theorem, which gives Capelli identities for skew-symmetric matrices in explicit form.

Theorem 3.2 (the skew Capelli identity). *Define the (central) elements $T_{i,j} \in \mathcal{ZU}(\mathfrak{gl}_N)$ by*

$$T_{i,j}(E) = \operatorname{tr}\Biggl\{\prod_{t=j}^{i}(E-(1-\delta_{it})(N-2t))\Biggr\} \quad (i \geqslant j).$$

Also define the element $\tilde{\mathbb{T}}_k(E) \in \mathcal{ZU}(\mathfrak{gl}_N)$ by putting

$$\tilde{\mathbb{T}}_k(E) = \frac{1}{2^k k!}\det\begin{pmatrix} T_{1,1}(E) & 2 & 0 & \dots & 0 & 0 \\ T_{2,1}(E) & T_{2,2}(E) & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{k-2,1}(E) & T_{k-2,2}(E) & T_{k-2,3}(E) & \dots & 2(k-2) & 0 \\ T_{k-1,1}(E) & T_{k-1,2}(E) & T_{k-1,3}(E) & \dots & T_{k-1,k-1}(E) & 2(k-1) \\ T_{k,1}(E) & T_{k,2}(E) & T_{k,3}(E) & \dots & T_{k,k-1}(E) & T_{k,k}(E) \end{pmatrix}$$

for each $k \in \mathbb{Z}_{\geqslant 1}$ and $\tilde{\mathbb{T}}_0(E) = 1$. Then the following (skew Capelli) identities hold:

$$\lambda(\tilde{\mathbb{T}}_k(E)) = \Gamma_k^{\Lambda} \quad (0 \leqslant k \leqslant N/2), \tag{3.2 a}$$

$$\lambda(\tilde{\mathbb{T}}_k(E)) = 0 \quad (k > N/2). \tag{3.2 b}$$

Remark 3.3. The statement (3.2 b) is equivalent to saying that the elements $\tilde{\mathbb{T}}_k(E)$, for $k > N/2$, vanish on ϱ_N^D for all Young diagrams with columns of even length at most N.

We shall prove the theorem by induction on N. Thus, it is necessary to have precise information about the properties and relations for the $\tilde{\mathbb{T}}_k(E)$ (and hence the $T_{i,j}(E)$) between two cases $\mathfrak{gl}_N$ and $\mathfrak{gl}_{N-1}$.

Let $\mathcal{I}_N = \langle E_{1N}, E_{2N}, \dots, E_{NN}\rangle$ be a left ideal of $\mathcal{U}(\mathfrak{gl}_N)$ generated by the elements $\{E_{jN}\}_{j=1,\dots,N}$ of the Lie algebra $\mathfrak{gl}_N$. We now regard $\mathfrak{gl}_{N-1}$ as a subalgebra of $\mathfrak{gl}_N$ in an obvious way, that is, $\mathfrak{gl}_{N-1}$ is a subalgebra generated by $\{E_{jk}\}_{1\leqslant j,\ k\leqslant N-1}$. When it is necessary to specify the degree N of $\mathfrak{gl}_N$, we denote the matrix $E = (E_{jk})_{1\leqslant j,\ k\leqslant N} \in \mathrm{Mat}_N(\mathfrak{gl}_N)$ by E_N. Then we have the following proposition.

Proposition 3.4. *For $k \geqslant j$ $(k, j \in \mathbb{Z}_{\geqslant 0})$, the following relation holds:*

$$T_{k,j}(E_N) \equiv T_{k,j}(E_{N-1}) \mod \mathcal{I}_N.$$

From this proposition, we have in particular the following corollary.

Corollary 3.5.

$$\tilde{\mathbb{T}}_k(E_N) \equiv \tilde{\mathbb{T}}_k(E_{N-1}) \mod \mathcal{I}_N.$$

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In order to give a proof of the proposition above we need to prepare some lemmas. First we observe the following lemma.

Lemma 3.6. *For any $k \geqslant 0$, we have*

$$\operatorname{tr} E_N^k \equiv \sum_{j=1}^{k} \binom{k-1}{k-j} \operatorname{tr} E_{N-1}^j \mod \mathcal{I}_N.$$

Proof. It is obvious to see that

$$\operatorname{tr} E_N^k \equiv \sum_{i=1}^{N-1} \sum_{\ell=0}^{k-1} \sum_{\substack{1\leqslant i_1,\dots,i_{k-1}\leqslant N,\\ \#\{j\in(1,\dots,k-1);\ i_j=N\}=\ell}} E_{ii_1}E_{i_1i_2}\cdots E_{i_{k-1}i} \mod \mathcal{I}_N.$$

Take now a pair of indices $(j_1,\dots,j_\ell)$ such that $i_{j_t}=N$ $(1\leqslant t\leqslant \ell)$ and fix it. Then one easily proves

$$\sum_{i=1}^{N-1} \sum_{\substack{1\leqslant i_j\leqslant N-1\ (j\neq j_t \text{ for all } 1\leqslant t\leqslant \ell),\\ i_j=N\ (j=j_t \text{ for some } t=1,\dots,\ell)}} E_{ii_1}E_{i_1i_2}\cdots E_{i_{k-1}i} \equiv \operatorname{tr} E_{N-1}^{k-\ell} \mod \mathcal{I}_N, \tag{3.3}$$

by induction on k. Since the number of ways for choosing a pair of i_j from the set $(i_1,\dots,i_{k-1})$ such that $\#\{j\in(1,\dots,k-1);\ i_j=N\}=\ell$ is obviously equal to $\binom{k-1}{\ell}$, the desired formula follows from (3.3) immediately. □

Lemma 3.7. *Define the numbers $a_\ell^N(k)$ by the following equation:*

$$\{x-(N-2)\}\{x-(N-4)\}\cdots\{x-(N-2(k-1))\} = \sum_{\ell=1}^{k} a_\ell^N(k) x^{\ell-1}.$$

Then we have

$$a_\ell^{N-1}(k) = \sum_{\ell=j}^{k} \binom{\ell-1}{\ell-j} a_\ell^N(k). \tag{3.4}$$

Proof. By definition it is clear that

$$\sum_{\ell=1}^{k} a_\ell^{N-1}(k) x^{\ell-1} = \sum_{\ell=1}^{k} a_\ell^N(k)(x+1)^{\ell-1},$$

whence the binomial theorem implies (3.4). □

Proof of Proposition 3.4. We first remark that the equations

$$\begin{aligned} T_{k,j}(E_N) &= T_{k-1,j-1}(E_N) + 2(k-j)T_{k-1,j}(E_N),\\ T_{k,k}(E_N) &= T_{1,1}(E_N) = \operatorname{tr} E_N \end{aligned}$$

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hold, simply because of the definition of $T_{k,j}(E_N)$. By means of these equations, it is sufficient to prove the statement in the case of $k > 1$ and $j = 1$. Now we note that

$$\operatorname{tr} T_{k,1}(E_N) = \sum_{\ell=1}^{k} a_\ell^N(k) \operatorname{tr} E_N^\ell.$$

Substituting the relation described in Lemma 3.6 we see

$$\begin{aligned}\operatorname{tr} T_{k,1}(E_N) &\equiv \sum_{\ell=1}^{k} a_\ell^N(k) \sum_{j=1}^{\ell} \binom{\ell-1}{\ell-j} \operatorname{tr} E_{N-1}^j \mod \mathcal{I}_N \\ &\equiv \sum_{j=1}^{k} \biggl\{ \sum_{\ell=j}^{k} \binom{\ell-1}{\ell-j} a_\ell^N(k) \biggr\} \operatorname{tr} E_{N-1}^j \mod \mathcal{I}_N.\end{aligned}$$

Hence the result follows from Lemma 3.7. This completes the proof of the proposition. □

Before going on to prove the theorem, we need the following lemma.

Lemma 3.8. *Suppose $0 \leqslant n \leqslant N$. Let $x = (x_{ij})_{1 \leqslant i,\ j \leqslant n}$ be an $n \times n$ principal submatrix of the skew-symmetric matrix $X = (x_{ij})_{1 \leqslant i,\ j \leqslant N}$. Then we have*

$$\lambda(E_{ij}) \operatorname{pf}(x) = \begin{cases} 0 & (n < j), \\ 0 & (1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant n,\ i \neq j), \\ \operatorname{pf}(x) & (1 \leqslant j \leqslant n,\ i = j). \end{cases}$$

In particular, if we put $m = [N/2]$, then we have

$$\lambda(E_{ij})(\varphi_1^{\alpha_1} \cdots \varphi_m^{\alpha_m}) = \delta_{ij} \biggl(\sum_{k=[(i+1)/2]}^{m} \alpha_k \biggr) \varphi_1^{\alpha_1} \cdots \varphi_m^{\alpha_m} \quad (i \leqslant j).$$

Proof. The first line follows immediately from the expression (3.1). Since $\operatorname{pf}({}^{\mathrm{t}}gxg) = \det(g) \operatorname{pf}(x)$, we see that the second and third lines are also obviously true. □

Remark 3.9. If $b \in GL_N$ is an upper triangular matrix, then the formula

$$\operatorname{pf}(({}^{\mathrm{t}}bXb)_{(1,2,\dots,2k-1,2k)}) = \delta_{2k}(b) \operatorname{pf}(X_{(1,2,\dots,2k-1,2k)})$$

holds for any skew-symmetric matrix X. Here $\delta_{2k}(b)$ denotes the principal minor determinant of b of order $2k$ (see, for example, [**1**]). This formula also implies the statement of Lemma 3.8.

Proof of Theorem 3.2. By the definition of $\tilde{\mathbb{T}}_k(E_N)$, it is clear that $\lambda(\tilde{\mathbb{T}}_k(E_N))$ is a degree k differential operator. Indeed, we know that $\lambda(\tilde{\mathbb{T}}_k(E_N))$ has degree k as GL_N-invariant differential operator with polynomial coefficients. (Note also that the abstract Capelli identity $\lambda(\mathcal{ZU}(\mathfrak{gl}_N)) = \mathcal{PD}(\Lambda^2(\mathbb{C}^N))^{GL_N}$ holds (see [**4**]).) Comparing the definition (determinant expression) of $\tilde{\mathbb{T}}_k(E_N)$ and Theorem 2.7, we see that the highest-order term (the leading term as a differential operator) of $\lambda(\tilde{\mathbb{T}}_k(E_N))$ is exactly given by Γ_k^Λ. Actually, this fact is easily seen from the determinant expression of $\tilde{\mathbb{T}}_k(E_N)$ by comparing with the determinant $\mathbb{T}_k(X\partial)$. This implies that the difference $\lambda(\tilde{\mathbb{T}}_k(E_N)) - \Gamma_k^\Lambda$ is of degree less than or equal to $k-1$. This means that if we show $\lambda(\tilde{\mathbb{T}}_k(E_N))$ satisfies condition (ii) of the characterization of Γ_k^Λ, then we conclude that (3.2 a) becomes true. We show the desired identities (3.2 a), (3.2 b) by induction on N. First, it is clear that (3.2 a), (3.2 b) hold for $N = 1$.

Suppose that the identities (3.2 a), (3.2 b) hold for N. Recall the highest-weight vector

$$\varphi_N^D = \prod_{k=1}^{[N/2]} \varphi_k^{b_k - b_{k+1}}$$

of ϱ_N^D, where $\varphi_k = \mathrm{pf}(X_{(1,2,\dots,2k-1,2k)})$ for a Young diagram $D = (b_1, b_1, b_2, b_2, \dots)$ of even length. Note that $\lambda(E_{iN+1})\varphi_k^\ell = 0$ for any ℓ less than or equal to $[N/2]$ by Lemma 3.8. Note also that if $\mathrm{depth}(D) \leqslant 2k$ ($\leqslant 2[N/2]$), then we have $\varphi_{N+1}^D = \varphi_N^D$. By Corollary 3.5, for each k ($0 \leqslant k \leqslant (N+1)/2$), if $\mathrm{depth}(D) \leqslant 2k$ ($\leqslant 2[N/2]$), then we have

$$\tilde{\mathbb{T}}_k(E_{N+1})\varphi_{N+1}^D = \tilde{\mathbb{T}}_k(E_N)\varphi_{N+1}^D = \tilde{\mathbb{T}}_k(E_N)\varphi_N^D.$$

Therefore, if D satisfies $\mathrm{depth}(D) < 2k$ ($\leqslant 2[N/2]$), then $\tilde{\mathbb{T}}_k(E_{N+1})\varphi_{N+1}^D = 0$ holds for $N+1$ by the induction hypothesis (3.2 a) for N. The remaining case that we have to show holds is only the case when $k = (N+1)/2$ and $\mathrm{depth}(D) = N - 1 = 2[N/2]$ for odd N, i.e. $\tilde{\mathbb{T}}_{(N+1)/2}(E_{N+1})\varphi_{N+1}^D = 0$. But this in fact follows from the induction hypothesis (3.2 b) for N. Thus the operator $\tilde{\mathbb{T}}_k(E_{N+1})$ ($0 \leqslant k \leqslant (N+1)/2$) satisfies the characterization (i), (ii) of Γ_k^Λ, whence the identity (3.2 a) for $N+1$ follows. A similar argument actually works when proving (3.2 b) for $N+1$. This completes the proof of the theorem. □

Our next task is to compute the explicit eigenvalues of the Capelli elements $\tilde{\mathbb{T}}_k(E)$. In other words, by Theorem 3.2 this is equivalent to determining the explicit value of the constant s_k which appeared as a ratio of Γ_k^Λ to $\lambda(C_k^\Lambda)$, where C_k^Λ is the skew Capelli element introduced in [**4**] (see p. 592). We shall prove below that $s_k = 1$, that is, C_k^Λ coincides with $\tilde{\mathbb{T}}_k(E)$ for each $1 \leqslant k \leqslant N/2$.

To prove this it is necessary and sufficient to have precise information regarding the eigenvalues of $T_{i,j}(E) \in \mathcal{ZU}(\mathfrak{gl}_N)$ for some special representations. In fact, we shall directly compute the eigenvalues for each representation corresponding to the Young diagram of the box-type shape with columns of even length.

Using Lemma 3.8 we have the following lemma.

Lemma 3.10. *Suppose $0 \leqslant n \leqslant N$. Let $k, b \in \mathbb{N}$. Let $x = (x_{ij})_{1\leqslant i,\ j\leqslant n}$ be an $n \times n$ principal submatrix of the skew-symmetric matrix $X = (x_{ij})_{1\leqslant i,\ j\leqslant N}$.*

(a) *If $n < i_1 \leqslant N$, then*

$$\lambda(E_{i_1 i_k} E_{i_k i_{k-1}} \cdots E_{i_2 i_1})\, \mathrm{pf}(x)^b = 0. \tag{3.5}$$

(b) *If* $1 \leqslant i_1 \leqslant n$ *and* $i_j \in \{i_1\} \cup \{n+1, n+2, \ldots, N\}$ *for all* $j \in \{2, 3, \ldots, n\}$, *then*

$$\lambda(E_{i_1 i_k} E_{i_k i_{k-1}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b = b^{\#\{j \in \{2,\ldots,k\}: i_j = i_1\}+1} \operatorname{pf}(x)^b. \tag{3.6}$$

(c) *If* $1 \leqslant i_1 \leqslant n$ *and there exists* $j \in \{2, 3, \ldots, n\}$ *such that* $i_j \notin \{i_1\} \cup \{n+1, n+2, \ldots, N\}$, *then*

$$\lambda(E_{i_1 i_k} E_{i_k i_{k-1}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b = 0. \tag{3.7}$$

Proof. By Lemma 3.8, statement (a) is clear. To prove statement (b), we first assume $i_j \in \{n+1, n+2, \ldots, N\}$ for all $j \in \{2, 3, \ldots, N\}$. We carry out this proof by induction on k. If $k = 2$, then by Lemma 3.8 we have

$$\begin{aligned}\lambda(E_{i_1 i_2} E_{i_2 i_1}) \operatorname{pf}(x)^b &= \lambda([E_{i_1 i_2}, E_{i_2 i_1}] + E_{i_2 i_1} E_{i_1 i_2}) \operatorname{pf}(x)^b \\ &= \lambda(E_{i_1 i_1} - E_{i_2 i_2} + E_{i_2 i_1} E_{i_1 i_2}) \operatorname{pf}(x)^b = b \operatorname{pf}(x)^b.\end{aligned}$$

Assume that the statement holds for any j less than k. Then, since $i_2 > n$ we have

$$\begin{aligned}&\lambda(E_{i_1 i_k} \cdots E_{i_3 i_2} E_{i_2 i_1}) \operatorname{pf}(x)^b \\ &\quad = \lambda(E_{i_1 i_k} \cdots E_{i_4 i_3}([E_{i_3 i_2}, E_{i_2 i_1}] + E_{i_2 i_1} E_{i_3 i_2})) \operatorname{pf}(x)^b \\ &\quad = \lambda(E_{i_1 i_k} \cdots E_{i_5 i_4} E_{i_4 i_3} E_{i_3 i_1} + E_{i_1 i_k} \cdots E_{i_4 i_3} E_{i_2 i_1} E_{i_3 i_2}) \operatorname{pf}(x)^b = b \operatorname{pf}(x)^b\end{aligned}$$

provided $i_j \neq i_1$ for all $j = 2, \ldots, k$. We next suppose $i_l = i_1$ for some l (> 1). We may assume that l is the smallest among such integers. Then by the same reasoning it is clear that

$$\lambda(E_{i_l i_{l-1}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b = b \operatorname{pf}(x)^b.$$

If there are other values of j such that $i_j = i_1$, then by repeating the same procedure we obtain (b).

We prove (c) by induction on l. Let l be the smallest integer such that $i_l \notin \{i_1\} \cup \{n+1, n+2, \ldots, N\}$. If $l = 2$, statement (c) follows immediately from Lemma 3.8. Assume that (3.7) holds for any j less than l. Then we have

$$\begin{aligned}&\lambda(E_{i_l i_{l-1}} E_{i_{l-1} i_{l-2}} E_{i_{l-2} i_{l-3}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b \\ &\quad = \lambda(([E_{i_l i_{l-1}}, E_{i_{l-1} i_{l-2}}] + E_{i_{l-1} i_{l-2}} E_{i_l i_{l-1}}) E_{i_{l-2} i_{l-3}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b \\ &\quad = \lambda(E_{i_l i_{l-2}} E_{i_{l-2} i_{l-3}} E_{i_{l-3} i_{l-4}} \cdots E_{i_2 i_1} + E_{i_{l-1} i_{l-2}} E_{i_l i_{l-1}} E_{i_{l-2} i_{l-3}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b.\end{aligned}$$

Hence, the induction hypothesis yields

$$\begin{aligned}&\lambda(E_{i_l i_{l-1}} E_{i_{l-1} i_{l-2}} E_{i_{l-2} i_{l-3}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b \\ &\quad = \lambda(E_{i_{l-1} i_{l-2}} E_{i_l i_{l-1}} E_{i_{l-2} i_{l-3}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b \\ &\quad = \lambda(E_{i_{l-1} i_{l-2}}([E_{i_l i_{l-1}}, E_{i_{l-2} i_{l-3}}] + E_{i_{l-2} i_{l-3}} E_{i_l i_{l-1}}) E_{i_{l-3} i_{l-4}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b \\ &\quad = \lambda(\delta_{i_{l-1} i_{l-2}} E_{i_{l-1} i_{l-2}} E_{i_l i_{l-3}} E_{i_{l-3} i_{l-4}} \cdots E_{i_2 i_1} \\ &\qquad\qquad + E_{i_{l-1} i_{l-2}} E_{i_{l-2} i_{l-3}} E_{i_l i_{l-1}} E_{i_{l-3} i_{l-4}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b.\end{aligned}$$

Again by the induction hypothesis, we find

$$\begin{aligned}\lambda(E_{i_l i_{l-1}} E_{i_{l-1} i_{l-2}} E_{i_{l-2} i_{l-3}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b \\ = \lambda(E_{i_{l-1} i_{l-2}} E_{i_{l-2} i_{l-3}} E_{i_l i_{l-1}} E_{i_{l-3} i_{l-4}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b.\end{aligned}$$

Repeating the same procedure, we have $\lambda(E_{i_1 i_k} E_{i_k i_{k-1}} \cdots E_{i_2 i_1}) \operatorname{pf}(x)^b = 0$. This proves the lemma. $\square$

Corollary 3.11. *Let $0 \leqslant n \leqslant N$ and let $k, b \in \mathbb{N}$. Retain the notation in Lemma 3.10. Then we have*

$$\lambda(\operatorname{tr}(E^k)) \operatorname{pf}(x)^b = nb(N-n+b)^{k-1} \operatorname{pf}(x)^b. \tag{3.8}$$

Proof. The definition of $\operatorname{tr}(E^k)$ and statements (a) and (c) of Lemma 3.10 imply

$$\begin{aligned}\lambda(\operatorname{tr}(E^k)) \operatorname{pf}(x)^b &= \sum_{1 \leqslant i_1, \ldots, i_k \leqslant N} \lambda(E_{i_1 i_k} \cdots E_{i_3 i_2} E_{i_2 i_1}) \operatorname{pf}(x)^b \\ &= \sum_{i_1=1}^{n} \sum_{i_2, \ldots i_k \in \{i_1\} \cup \{n+1, \ldots, N\}} \lambda(E_{i_1 i_k} \cdots E_{i_3 i_2} E_{i_2 i_1}) \operatorname{pf}(x)^b \\ &= \sum_{i_1=1}^{n} \sum_{t=0}^{k-1} \sum_{\substack{i_2, \ldots, i_k; \\ \#\{j \in (2,3,\ldots,k);\ i_j = i_1\} = t}} \lambda(E_{i_1 i_k} \cdots E_{i_3 i_2} E_{i_2 i_1}) \operatorname{pf}(x)^b.\end{aligned}$$

Since the number of ways to choose a pair of i_j from $(i_2, \ldots, i_k)$ such that $\#\{j \in (2,3,\ldots,k);\ i_j = i_1\} = t$ is obviously equal to $\binom{k-1}{t}$, by virtue of (b) in Lemma 3.10 we see that

$$\lambda(\operatorname{tr}(E^k)) \operatorname{pf}(x)^b = \sum_{i_1=1}^{n} \sum_{t=0}^{k-1} \binom{k-1}{t} b^{t+1} (N-n)^{k-t-1} \operatorname{pf}(x)^b.$$

The last expression gives (3.8) immediately. $\square$

We now compute the eigenvalue of the operator $\lambda(\tilde{\mathbb{T}}_k(E))$ on the box-type representation $\varrho_N^{bD_{2l}}$ ($bD_{2l} = (b, b, \ldots, b, b)$). By Corollary 3.11, for any $0 \leqslant l \leqslant [N/2]$ and $b \in \mathbb{N}$, it is easy to verify that

$$\begin{aligned}\lambda(T_{i,j}(E)) \varphi_l^b &= \lambda\left(\operatorname{tr}\left\{\prod_{t=j}^{i} (E - (1-\delta_{it})(N-2t))\right\}\right) \varphi_l^b \\ &= \begin{cases} 2lb\varphi_l^b & (i = j), \\ 2lb \displaystyle\prod_{t=j}^{i-1} (2t + b - 2l) \varphi_l^b & (i > j). \end{cases}\end{aligned}$$

The following relation can be easily seen from the expression for the eigenvalue of $\lambda(T_{i,j}(E))$:

$$\lambda(T_{i,j}(E)) \varphi_l^b = \{2(i-1) + b - 2l\} \lambda(T_{i-1,j}(E)) \varphi_l^b \quad (i > j). \tag{3.9}$$

Then, expanding the determinant $\lambda(\tilde{\mathbb{T}}_k(E))$ with respect to the last column and using (3.9), we have

$$\begin{aligned}\lambda(\tilde{\mathbb{T}}_k(E))\varphi_l^b &= \frac{1}{2k}[T_{k,k}(E) - 2(k-1)\{2(k-1)+b-2l\}]\lambda(\tilde{\mathbb{T}}_{k-1}(E))\varphi_l^b \\ &= \frac{(2k-2+b)(l+1-k)}{k}\lambda(\tilde{\mathbb{T}}_{k-1}(E))\varphi_l^b.\end{aligned}$$

Hence, by repeating this procedure k times we get

$$\begin{aligned}\lambda(\tilde{\mathbb{T}}_k(E))\varphi_l^b(= \varrho_N^{bD_{2l}}(\tilde{\mathbb{T}}_k(E))\varphi_l^b) &= \frac{1}{k!}\prod_{t=1}^{k}\{2(k-t)+b\}\prod_{t=1}^{k}(l+1-t)\varphi_l^b \\ &= \frac{\Gamma(l+1)}{\Gamma(l+1-k)\Gamma(k+1)}\prod_{t=0}^{k-1}(2t+b)\varphi_l^b, \qquad (3.10)\end{aligned}$$

where $\Gamma(z)$ is the gamma function.

Theorem 3.1 and Corollary (11.3.19) in [**4**] imply that the operators $\tilde{\mathbb{T}}_k(E)$ actually realize the skew Capelli elements C_k^{Λ} up to constant multiples. By comparing eqn (11.3.6) in [**4**] and the actual value (3.10) for the case $D = bD_{2l}$, it is easy to check that the constant should be equal to 1. Specifically, this implies that the general eigenvalue of the operator $\tilde{\mathbb{T}}_k(E)$ on ϱ_N^D is given as follows.

Proposition 3.12. *We have*

$$C_k^{\Lambda} = \tilde{\mathbb{T}}_k(E) \quad (1 \leqslant k \leqslant N/2).$$

In particular, for any Young diagram $D = (b_1, b_1, b_2, b_2, \dots)$ with columns of even length we have

$$\varrho_N^D(\tilde{\mathbb{T}}_k(E)) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant[N/2]}\prod_{t=1}^{k}\{2(k-t)+b_{i_t}\}. \qquad (3.11)$$

As an application of this formula or the special case of (3.10), we note, for example, the following simple evaluation of the (simplest) Cayley-type formula (b-function) attached to $\Lambda^2\mathbb{C}^{2m}$.

Corollary 3.13. *Let $X = (x_{ij})$ be an $N \times N$ skew-symmetric matrix and $\partial = (\partial_{ij})$ be the dual matrix of X. Then we have*

$$\operatorname{pf}(\partial)\operatorname{pf}(X)^s = \prod_{t=0}^{[N/2]-1}(s+2t)\operatorname{pf}(X)^{s-1}. \qquad (3.12)$$

We finish the paper by making one remark concerning the Capelli identities for the standard $GL_N \times GL_N$ action on the space of $N \times N$ matrices.

Remark 3.14. The procedure developed in this paper may also be used to obtain the Capelli-type identity for the $N \times N$ full matrix algebra under the $GL_N \times GL_N$-action.

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Actually, we may replace the ordinary Capelli elements (see, for example, [**10**]) by the traces of powers of E as central elements of $\mathcal{U}(\mathfrak{gl}_N)$. Moreover, since the action is faithful, this new expression for the Capelli identity provides a new type of the Newton formula which is a variant of the formula discussed in [**6**] and [**9**]. Indeed, under the notation in [**4**], we have the following theorem.

Theorem 3.15. *We have the following Capelli identities attached to the $GL_N \times GL_N$ action on the space of all $N \times N$ matrices:*

$$\Gamma_k = L(\tilde{\mathbb{T}}'_k(E)) = L(C_k) \quad (k \leqslant N),$$

where we define the element $\tilde{\mathbb{T}}'_k(E) \in \mathcal{ZU}(\mathfrak{gl}_N)$ by putting

$$\tilde{\mathbb{T}}'_k(E) = \frac{1}{k!}\det\begin{pmatrix} T'_{1,1}(E) & 1 & 0 & \dots & 0 & 0 \\ T'_{2,1}(E) & T'_{2,2}(E) & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ T'_{k-2,1}(E) & T'_{k-2,2}(E) & T'_{k-2,3}(E) & \dots & k-2 & 0 \\ T'_{k-1,1}(E) & T'_{k-1,2}(E) & T'_{k-1,3}(E) & \dots & T'_{k-1,k-1}(E) & k-1 \\ T'_{k,1}(E) & T'_{k,2}(E) & T'_{k,3}(E) & \dots & T'_{k,k-1}(E) & T'_{k,k}(E) \end{pmatrix}$$

for each $k \in \mathbb{Z}_{\geqslant 1}$ and $\tilde{\mathbb{T}}'_0(E) = 1$. Here the matrix coefficients $T'_{i,j}(E) \in \mathcal{ZU}(\mathfrak{gl}_N)$ are given by

$$T'_{i,j}(E) = \operatorname{tr}\left\{\prod_{t=j}^{i}(E - (1-\delta_{it})(N-t))\right\} \quad (i \geqslant j).$$

Acknowledgements. The authors express their thanks to Hiroyuki Ochiai for making them aware of a gap in the proof of the main theorem in the first draft of the paper and Roger Howe for his helpful comments. M.W. was partly supported by grant-in-aid for Scientific Research (B) no. 09440022, from the Ministry of Education, Science and Culture of Japan.

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