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Octahedrality in Lipschitz-free Banach spaces

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The aim of this note is to study octahedrality in vector-valued Lipschitz-free Banach spaces on a metric space, under topological hypotheses on it, by analysing the weak-star strong diameter 2 property in Lipschitz function spaces. Also, we show an example that proves that our results are optimal and that octahedrality in vector-valued Lipschitz-free Banach spaces actually relies on the underlying metric space as well as on the Banach one.

Keywords: diameter 2 properties; octahedral norms; slices

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1. Introduction

Lipschitz function spaces (denoted by $\mathrm{Lip}(M)$) and their preduals [23], Lipschitz-free Banach spaces (denoted by $\mathcal{F}(M)$), have recently been studied from a topological point of view (see, for example, [9, 13, 17]). Geometrical properties in such spaces have also been considered, such as the Daugavet property. Indeed, the Daugavet property in Lipschitz functions spaces has been characterized in [15] in terms of 'locality' in the compact case, and provides examples of metric spaces whose Lipschitz-free Banach space has an octahedral norm. On the other hand, in [8] it was recently proved that given an infinite metric space M, the free space $\mathcal{F}(M)$ contains a complemented copy of ℓ_1 and, consequently, $\mathcal{F}(M)$ has an equivalent norm that is an octahedral norm [12].

Motivated by these kinds of results, the aim of this note is to go further and analyse octahedrality in vector-valued Lipschitz-free Banach spaces. Indeed, we introduce the Banach space of vector-valued Lipschitz-free Banach spaces, which, to the best of our knowledge, has not been previously considered, and we prove (see theorem 2.4) that such spaces have an octahedral norm whenever their underlying metric space satisfies some topological assumptions, such as being unbounded or not being uniformly discrete, and a condition of existence of extension of vector-valued Lipschitz functions (see definition 2.3). Consequently, such Banach spaces cannot have any point of Fréchet differentiability. Moreover, we will exhibit an example of a metric space such that, depending on the underlying Banach space, the geometry of the vector-valued Lipschitz-free Banach space changes its behaviour from having a point of Fréchet differentiability to having an octahedral norm. This will have two important consequences: on the one hand, as there are vector-valued Lipschitz-free Banach spaces that contain points of Fréchet differentiability, we prove that our results on octahedrality are optimal; on the other hand, this proves that the geometry of the vector-valued Lipschitz-free Banach spaces is determined by the underlying metric space as well as by the target Banach space. We will end by exhibiting some consequences of theorem 2.4 and open problems in § 3

We shall now introduce some notation. We consider only real Banach spaces. Given a Banach space X, B_X (respectively S_X) stands for the closed unit ball (respectively, the unit sphere) of X. Given a Banach space X, we will mean by a slice of B_X a subset of the form

$$S(B_X, f, \alpha) := \{x \in B_X \colon f(x) > 1 - \alpha\},$$

where $f \in S_{X^*}$ and $\alpha > 0$. If X is a dual space, say $X = Y^*$, by a weak-star slice of B_{X^*} we will mean a slice $S(B_X, y, \alpha)$, where $y \in Y$.

We recall that the *projective tensor product of two Banach spaces X and Y*, denoted by $X\hat{\otimes}_\pi Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\| := \inf \left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| \;\middle|\; n \in \mathbb{N},\ x_i \in X,\ y_i \in Y\ \forall i \in \{1, \ldots, n\},\ u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

for every $u \in X \otimes Y$.

We recall that the space $L(X, Y^*)$ of bounded and linear Y^*-valued operators on X is linearly isometric to the topological dual of $X\hat{\otimes}_\pi Y$.

Given a metric space M with a designated origin 0 and a Banach space X, we will denote by $\mathrm{Lip}(M, X)$ the Banach space of all X-valued Lipschitz functions on M that vanish at 0 under the standard Lipschitz norm

$$\|f\| := \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} \;\middle|\; x, y \in M,\ x \neq y \right\}.$$

First of all, notice that we can consider every point of M as an origin with no loss of generality. Indeed, given $x, y \in M$, let $\mathrm{Lip}_x(M, X)$ ($\mathrm{Lip}_y(M, X)$) be the space of X-valued Lipschitz functions that vanish at x (respectively, at y). Then the map

$$\begin{aligned} \mathrm{Lip}_x(M, X) &\to \mathrm{Lip}_y(M, X) \\ f &\mapsto f - f(y), \end{aligned}$$

defines an onto linear isometry. So the designated origin will be freely chosen.

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From a straightforward application of the Ascoli–Arzelà theorem it can be checked that $B_{\mathrm{Lip}(M,X^*)}$ is a compact set for the pointwise topology. Hence $\mathrm{Lip}(M,X^*)$ is itself a dual Banach space. In fact, the map

$$\begin{aligned} \delta_{m,x}\colon \mathrm{Lip}(M,X^*) &\to \mathbb{R} \\ f &\mapsto f(m)(x) \end{aligned}$$

defines a linear and bounded map for each $m \in M$ and $x \in X$. In other words, $\delta_{m,x} \in \mathrm{Lip}(M,X^*)^*$. So if we define

$$\mathcal{F}(M,X) := \overline{\mathrm{span}}(\{\delta_{m,x} \mid m \in M,\ x \in X\}),$$

then we have that $\mathcal{F}(M,X)^* = \mathrm{Lip}(M,X^*)$. Furthermore, a bounded net $\{f_s\}$ in $\mathrm{Lip}(M,X^*)$ converges in the weak-star topology to a function $f \in \mathrm{Lip}(M,X^*)$ if and only if $\{f_s(m)\} \to f(m)$ for each $m \in M$, where the last convergence is in the weak-star topology of X^*. Now we have the following identification.

PROPOSITION 1.1. *$\mathcal{F}(M,X)$ and $\mathcal{F}(M)\hat{\otimes}_\pi X$ are isometric Banach spaces for every metric space M and for every Banach space X.*

Proof. Note that given a Lipschitz map $f\colon M \to X^*$, there exists a linear operator $T_f\colon \mathcal{F}(M) \to X^*$ such that $T_f \circ \delta_m = f(m)$ for each $m \in M$. This map

$$\begin{aligned} \Phi\colon \mathrm{Lip}(M,X^*) &\to L(\mathcal{F}(M),X^*) \\ f &\mapsto T_f \end{aligned}$$

is an isometric isomorphism (see, for example, [16]). Then it is enough to prove that Φ is $w^* - w^*$ continuous, where the weak-star topologies are respectively induced by $\mathcal{F}(M,X)$ on $\mathrm{Lip}(M,X^*)$ and by $\mathcal{F}(M)\hat{\otimes}_\pi X$ on $L(\mathcal{F}(M),X^*)$.

Note that Φ is $w^* - w^*$ continuous if and only if for every $z \in \mathcal{F}(M)\hat{\otimes}_\pi X$ one has that $z \circ \Phi$ is a weak-star continuous functional. By [11, corollary 3.94] it is enough to prove that, given $z \in \mathcal{F}(M)\hat{\otimes}_\pi X$, we have that $\mathrm{Ker}(z \circ \Phi) \cap B_{\mathrm{Lip}(M,X^*)}$ is weak-star closed. So, pick $z \in \mathcal{F}(M)\hat{\otimes}_\pi X$ and consider a net $\{f_s\}$ in $\mathrm{Ker}(z \circ \Phi) \cap B_{\mathrm{Lip}(M,X^*)}$ that is weak-star convergent to f, and let us prove that $(z \circ \Phi)(f) = 0$. To this end, pick a positive number $\varepsilon > 0$. Note that z can be expressed as

$$z := \sum_{n=1}^{\infty} \gamma_n \otimes x_n,$$

where $\gamma_n \in \mathcal{F}(M)$ and $x_n \in X$ verify that $\sum_{n=1}^{\infty} \|\gamma_n\|\|x_n\| < \infty$ [21, proposition 2.8]. Now, consider a sequence $\{\varepsilon_n\}$ in $\mathbb{R}^+$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon/3$ and consider, for each $n \in \mathbb{N}$, an element $\psi_n \in \mathrm{span}\{\delta_m\colon m \in M\}$ verifying $\|\gamma_n - \psi_n\|\|x_n\| < \varepsilon_n/2$ for each $n \in \mathbb{N}$. As it is clear that $\sum_{n=1}^{\infty} \|\psi_n\|\|x_n\| < \infty$, consider $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} \|\psi_n\|\|x_n\| < \varepsilon/6$. Finally, in view of the weak-star topology of $\mathrm{Lip}(M,X^*)$, it is obvious that $\{f_s(\psi_n)(x_n)\} \to f(\psi_n)(x_n)$ for each $n \in \mathbb{N}$, and hence we can find s such that $|(f - f_s)(\psi_n)(x_n)| < \varepsilon/3k$ for each $n \in \{1, \ldots, k\}$.

Now, keeping in mind that $\|f - f_s\| \leqslant 2$, one has

$$\begin{aligned}
|(z \circ \Phi)(f)| &= |(z \circ \Phi)(f - f_s)| \\
&= \left| \sum_{n=1}^{\infty} T_{f-f_s}(\gamma_n)(x_n) \right| \\
&\leqslant \left| \sum_{n=1}^{\infty} T_{f-f_s}(\psi_n)(x_n) \right| + \|f - f_s\| \sum_{n=1}^{\infty} \|\gamma_n - \psi_n\| \|x_n\| \\
&\leqslant \sum_{n=1}^{k} |(f - f_s)(\psi_n)(x_n)| + \|f - f_s\| \sum_{n=k+1}^{\infty} \|\psi_n\| \|x_n\| + \frac{\varepsilon}{3} \\
&< \sum_{n=1}^{k} \frac{\varepsilon}{3k} + \frac{2\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we conclude that $(z \circ \Phi)(f) = 0$, so we are done. □

The norm on a Banach space X is said to be octahedral if for every $\varepsilon > 0$ and for every finite-dimensional subspace M of X there is some y in the unit sphere of X such that

$$\|x + \lambda y\| \geqslant (1 - \varepsilon)(\|x\| + |\lambda|)$$

holds for every $x \in M$ and for every scalar λ (see [10]).

We recall that a Banach space X satisfies the slice diameter 2 property (respectively, diameter 2 property, strong diameter 2 property) if every slice (respectively, non-empty weakly open subset, convex combination of slices) of the closed unit ball has diameter 2. If X is itself a dual Banach space, then the weak-star slice diameter 2 property (respectively, weak-star diameter 2 property and weak-star strong diameter 2 property) can be defined as usual, invoking weak-star slices (respectively, non-empty weakly-star open subset, convex combination of weak-star slices) of the unit ball of X. These properties, which are extremely opposite to the Radon–Nikodým property, have been deeply studied over the last few years. For instance, it was recently proved [3, 4] that each one of the above properties is different from the rest in an extreme way.

Several Banach spaces that satisfy some of the diameter 2 properties are infinite-dimensional uniform algebras [20], Banach spaces satisfying the Daugavet property [22], non-reflexive M-embedded spaces [19], among others.

It is known that the norm on a Banach space X is octahedral if and only if X^* satisfies the weak-star strong diameter 2 property [2, theorem 2.1]. It is also known that if a Banach space X has an octahedral norm, then X contains an isomorphic copy of ℓ_1 [12].

Finally, given a Banach space X and a point $x \in X$, we say that x is a point of Fréchet differentiability if, for each $h \in X$, we have that

$$f'(x)(h) := \lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t}$$

uniformly for $h \in S_X$, and $f'(x)\colon X \to \mathbb{R}$ is a continuous and linear functional (see [12]).

It is clear that a Banach space that has an octahedral norm does not have any point of Fréchet differentiability.

2. Main results

Let M be a metric space and let X be a Banach space. Notice that we have a useful description of $\mathcal{F}(M,X)$ because we know a dense subspace of it. This fact will play an important role in the following because diameter 2 properties actually rely on dense subspaces in the following sense.

PROPOSITION 2.1. *Let X be a Banach space. Let $Y \subseteq X^*$ a norm dense subspace. Then the following hold.*

(1) *If for each $f \in S_Y$ and $\alpha \in \mathbb{R}^+$ the slice $S(B_X, f, \alpha)$ has diameter 2, then X has the slice diameter 2 property.*

(2) *If for each $f_1, \ldots, f_n \in S_Y$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ such that*

$$W := \bigcap_{i=1}^{n} S(B_X, f_i, \alpha_i) \neq \emptyset$$

it follows that W has diameter 2, then X has the diameter 2 property.

(3) *If for each $f_1, \ldots, f_n \in S_Y, \alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ and $\lambda_1, \ldots, \lambda_n \in]0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ the convex combination of slices $\sum_{i=1}^{n} \lambda_i S(B_X, f_i, \alpha_i)$ has diameter 2, then X satisfies the strong diameter 2 property*

Proof. We will prove statement (1), the proof of (2) and (3) being analogous.

Pick a slice $S := S(B_X, f, \alpha)$ of B_X. As Y is norm dense in X^* we can find $\varphi \in S_Y$ such that $\|f - \varphi\| < \frac{1}{2}\alpha$.

By hypothesis, given an arbitrary $\delta \in \mathbb{R}^+$ we can find $x, y \in S(B_X, \varphi, \frac{1}{2}\alpha)$ such that $\|x - y\| > 2 - \delta$. Let us prove that $x \in S$, the proof of $y \in S$ being similar. Bearing in mind that $\varphi(x) > 1 - \frac{1}{2}\alpha$ and that $\|f - \varphi\| < \frac{1}{2}\alpha$, we deduce that

$$f(x) = \varphi(x) + (f - \varphi)(x) \geqslant \varphi(x) - \|f - \varphi\| > 1 - \alpha.$$

On the other hand, as $x, y \in S$, we conclude that

$$2 - \delta < \|x - y\| \leqslant \operatorname{diam}(S).$$

As $\delta \in \mathbb{R}^+$ was arbitrary we conclude that X has the slice diameter 2 property, as desired. □

We now we consider the weak-star version of proposition above.

PROPOSITION 2.2. *Let X be a Banach space and let $Y \subseteq X$ be a dense subspace. Then the following hold.*

(1) *If for each $y \in S_Y$ and $\alpha \in \mathbb{R}^+$ the slice $S(B_{X^*}, y, \alpha)$ has diameter 2, then X has the weak-star slice diameter 2 property.*

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(2) *If for each* $y_1, \ldots, y_n \in S_Y$ *and* $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ *such that*

$$W := \bigcap_{i=1}^{n} S(B_{X^*}, y_i, \alpha_i) \neq \emptyset$$

one has that W *has diameter* 2*, then* X *has the weak-star diameter* 2 *property.*

(3) *If for* $y_1, \ldots, y_n \in S_Y, \alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ *and* $\lambda_1, \ldots, \lambda_n \in]0, 1]$ *with* $\sum_{i=1}^{n} \lambda_i = 1$ *the convex combination of weak-star slices* $\sum_{i=1}^{n} \lambda_i S(B_{X^*}, y_i, \alpha_i)$ *has diameter* 2*, then* X *satisfies the strong diameter* 2 *property*

Now we need the following definition.

Definition 2.3. Let M be a metric space and let X be a Banach space.

We will say that the pair (M, X) satisfies the contraction–extension property (CEP) if given $N \subseteq M$ and a Lipschitz function $f\colon N \to X$ there exists a Lipschitz function $F\colon M \to X$ that extends to f such that

$$\|F\|_{\mathrm{Lip}(M,X)} = \|f\|_{\mathrm{Lip}(N,X)}.$$

On the one hand note that, in the particular case of M being a Banach space, the definition given above agrees with the one given in [6].

On the other hand, let us give some examples of pairs that have the CEP. First of all, given M a metric space, the pair $(M, \mathbb{R})$ has the CEP [23, theorem 1.5.6]. In addition, in [6, ch. 2] we can find some examples of Banach spaces X such that the pair (X, X) satisfies the CEP, such as Hilbert spaces and ℓ_∞^n. Finally, if Y is a strictly convex Banach space such that there exists a Banach space X with $\dim(X) \geqslant 2$ and verifying that the pair (X, Y) has the CEP, then Y is a Hilbert space [6, theorem 2.11].

Let us explain roughly the key idea of the main result, which proves, for every unbounded or not uniformly discrete metric space M, that the norm on $\mathcal{F}(M, X)$ is octahedral whenever the pair (M, X^*) has the CEP, where X is a Banach space. For this, it is enough to show that every convex combination of w^*-slices C in the unit ball of $\mathrm{Lip}(M, X^*)$ has diameter exactly 2. What is done first is to observe that it is enough to consider w^*-slices given by elements in $\mathrm{span}\{\delta_{m,x} \mid m \in M,\ x \in X\}$, which is based on proposition 2.2. Now, depending on the kind of metric space considered, we construct a pair of Lipschitz functions for every w^*-slice of C. Different pairs are defined on different finite metric subspaces so that each pair of these functions have norm-preserving extensions to Lipschitz functions in a w^*-slice of C from the CEP assumption, and we control the norm of each pair only on a finite metric space so that the difference between the elements of every pair is also controlled. Now the estimates for the extensions are possible and in this way we get that C has diameter 2. Of course, there are details that depend on the kind of metric space considered, but the existence of the above unified idea motivated to us to show the following result in a joint way.

Theorem 2.4. *Let* M *be an infinite pointed metric space and let* X *be a Banach space. Assume that the pair* (M, X^*) *has the CEP. If* M *is unbounded or is not uniformly discrete, then the norm on* $\mathcal{F}(M, X)$ *is octahedral. Consequently, the unit ball of* $\mathcal{F}(M, X)$ *cannot have any point of Fréchet differentiability.*

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Proof. We will prove that $\mathrm{Lip}(M, X^*)$ has the weak-star strong diameter 2 property, which is equivalent to the thesis of the theorem. Let

$$C = \sum_{i=1}^{k} \lambda_i S(B_{\mathrm{Lip}(M,X^*)}, \varphi_i, \alpha)$$

be a convex combination of weak-star slices in $\mathrm{Lip}(M, X^*)$ and let us prove that C has diameter exactly 2. From proposition 2.1 we can assume that $\varphi_i \in \mathrm{span}\{\delta_{m,x} \mid m \in M,\ x \in X\}$ for each $i \in \{1, \dots, k\}$. So assume that

$$\varphi_i = \sum_{j=1}^{n_i} \lambda_j^i \delta_{m_{i,j}, x_{i,j}}$$

for suitable $n_i \in \mathbb{N}$, $m_{i,j} \in M \setminus \{0\}$, $x_{i,j} \in X$, $\lambda_j^i \in \mathbb{R}$ for $i \in \{1, \dots, k\}$, $j \in \{1, \dots, n_i\}$.

Pick $g_i \in S(B_{\mathrm{Lip}(M,X^*)}, \varphi_i, \alpha)$ and $\delta_0 \in \mathbb{R}^+$ verifying

$$0 < \delta < \delta_0 \implies \frac{\varphi_i(g_i)}{1+\delta} > 1 - \alpha \quad \forall i \in \{1, \dots, k\}.$$

Fix $0 < \delta < \delta_0$.

Now, as a first step we will define for every $i \in \{1, \dots, k\}$ a subspace $M_i \subset M$ and functions F_i and G_i in $\mathrm{Lip}(M_i, X^*)$.

We will do this depending on following cases: M is unbounded, M is bounded, discrete but not uniformly discrete, or M is bounded and $0 \in M'$. It is clear that when M is unbounded or not uniformly discrete, it is enough to study each of these three cases.

Assume that M is unbounded. Then there exists $\{m_n\} \subseteq M$ verifying

$$\{d(m_n, 0)\} \to \infty.$$

Hence,

$$\{d(m_n, m)\} \to \infty$$

for each $m \in M$ in view of the triangle inequality. Now pick an positive integer N so that

$$\frac{d(m_{i,j}, 0)}{d(m_N, m_{i,j})} + \frac{\|g_i(m_{i,j})\|}{d(m_N, m_{i,j})} < \delta \quad \forall i \in \{1, \dots, k\},\ j \in \{1, \dots, n_i\}. \tag{2.1}$$

Choose $x^* \in S_{X^*}$ and define

$$M_i := F := \{0\} \cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} \{m_{i,j}\} \cup \{m_N\}$$

for every $1 \leqslant i \leqslant k$. (In this case M_i does not depend on i.) Also, we define $F_i, G_i \colon M_i \to X^*$ given by

$$F_i(m_{i,j}) = G_i(m_{i,j}) = g_i(m_{i,j}), \quad i \in \{1, \dots, k\},\ j \in \{1, \dots, n_i\},$$
$$F_i(0) = G_i(0) = 0, \qquad F_i(m_N) = -G_i(m_N) = d(m_N, 0)x^*.$$

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Assume now that M is bounded and discrete, but not uniformly discrete. As M is discrete we can find $r > 0$ such that

$$B(0,r) = \{0\}, \quad B(m_{i,j}, r) = \{m_{i,j}\} \quad \forall i \in \{1,\dots,k\},\ j \in \{1,\dots,n_i\}.$$

Also, as M is not uniformly discrete we can find a pair of sequences $\{x_n\}$, $\{y_n\}$ in M such that $0 < d(x_n, y_n) \to 0$. Pick $n \in \mathbb{N}$ satisfying $d(x_n, y_n) < \delta$ and so that

$$\frac{1 + d(x_n,y_n)/d(x_n,v)}{1 - d(x_n,y_n)/d(x_n,v)} < 1+\delta \quad \forall v \in \{m_{i,j} \colon 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n_i\} \cup \{0\}. \tag{2.2}$$

Note that such an n exists since $\{d(x_n, v)^{-1}\}$ is a well-defined bounded sequence because M is discrete and bounded in this case. Given $i \in \{1,\dots,k\}$ and $x^* \in S_{X^*}$ define $M_i := \{0\} \cup \bigcup_{j=1}^{n_i}\{m_{i,j}\} \cup \{x_n, y_n\}$ and $F_i, G_i := M_i \to \mathbb{R}$ given by

$$F_i(0) = g_i(0) = 0, \quad F_i(m_{i,j}) = G_i(m_{i,j}) = g_i(m_{i,j}) \quad \forall j \in \{1,\dots,n_i\},$$

and

$$F_i(x_n) = G_i(x_n) = g_i(x_n), \qquad F_i(y_n) = g_i(x_n) + d(y_n, x_n)x^*,$$
$$G_i(y_n) = g_i(x_n) - d(y_n, x_n)x^*.$$

Finally, we assume that M is bounded and $0 \in M'$. Then we can find a sequence $\{m_n\}$ in $M \setminus \{0\}$ such that $\{m_n\} \to 0$. So there exists a positive integer m such that $m_n \notin \{m_{i,j} \colon i \in \{1,\dots,k\},\ j \in \{1,\dots,n_i\}\}$ for every $n \geqslant m$. Now pick $x^* \in S_{X^*}$ and, for each $i \in \{1,\dots,k\}$, we define $M_i := \{0, m_n\} \cup \bigcup_{j=1}^{n_i}\{m_{i,j}\}$ and $F_i, G_i \colon M_i \to X^*$ by the equations

$$F_i(m_{i,j}) = G_i(m_{i,j}) = g_i(m_{i,j}), \quad i \in \{1,\dots,k\},\ j \in \{1,\dots,N_i\}$$

and

$$F_i(m_n) = -G_i(m_n) = d(m_n, 0)x^*, \quad F_i(0) = G_i(0) = 0.$$

Now, for each unbounded or not uniformly discrete metric space M we have defined the desired subspaces M_i and functions F_i and G_i in $\mathrm{Lip}(M_i, X^*)$ for every $1 \leqslant i \leqslant k$.

For the second step we claim that $\|F_i\|_{\mathrm{Lip}(M_i,X^*)} \leqslant 1+\delta$ for all $i \in \{1,\dots,k\}$. For this we have three cases again: M is unbounded, M is bounded, discrete but not uniformly discrete, or M is bounded and $0 \in M'$.

Assume that M is unbounded. Given $u, v \in F$, $u \neq v$, and $i \in \{1,\dots,k\}$ we have two different possibilities.

(a) If $u, v \notin \{m_N\}$, then

$$\frac{\|F_i(u) - F_i(v)\|}{d(u,v)} = \frac{\|g_i(u) - g_i(v)\|}{d(u,v)} \leqslant \|g_i\| \leqslant 1.$$

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(b) If $u = m_N$, then

$$\begin{aligned}\frac{\|F_i(u)-F_i(v)\|}{d(u,v)} &= \frac{\|d(m_N,0)x^*-F_i(v)\|}{d(m_N,v)}\\ &\leqslant \frac{d(m_N,0)}{d(m_N,v)}+\frac{\|g_i(v)\|}{d(m_N,v)}\\ &\leqslant 1+\frac{d(v,0)}{d(m_N,v)}+\frac{\|g_i(v)\|}{d(m_N,v)}\\ &\overset{(2.1)}{<} 1+\delta.\end{aligned}$$

Now, taking the supremum in u and v, one has

$$\|F_i\|_{\mathrm{Lip}(M_i,X^*)} \leqslant 1+\delta.$$

Assume now that M is bounded, discrete but not uniformly discrete. Again, given $u, v \in M_i$, $u \neq v$, and $i \in \{1,\dots,k\}$ we have several different possibilities.

(a) If $u \neq y_n$ and $v \neq y_n$, then we have

$$\frac{\|F_i(u)-F_i(v)\|}{d(u,v)} = \frac{\|g_i(u)-g_i(v)\|}{d(u,v)} \leqslant \|g_i\| \leqslant 1.$$

(b) If $u = y_n$, $v \neq x_n$, then

$$\begin{aligned}\frac{\|F_i(u)-F_i(v)\|}{d(u,v)} &= \frac{\|g_i(x_n)+d(x_n,y_n)x^*-g_i(v)\|}{d(y_n,v)}\\ &\leqslant \frac{\|g_i(x_n)-g_i(v)\|+d(x_n,y_n)}{d(y_n,v)}\\ &< \frac{d(x_n,v)+d(x_n,y_n)}{d(x_n,v)-d(y_n,x_n)}\\ &= \frac{1+d(x_n,y_n)/d(x_n,v)}{1-d(x_n,y_n)/d(x_n,v)}\\ &< 1+\delta.\end{aligned}$$

(c) If $u = y_n$ and $v = x_n$, then

$$\frac{\|F_i(u)-G_i(v)\|}{d(u,v)} = \frac{d(x_n,y_n)\|x^*\|}{d(x_n,y_n)} = 1.$$

Then, taking the supremum in u and v, one has

$$\|F_i\|_{\mathrm{Lip}(M_i,X^*)} \leqslant 1+\delta.$$

If M is bounded and $0 \in M'$, we can get also that

$$\|F_i\|_{\mathrm{Lip}(M_i,X^*)} \leqslant 1+\delta$$

using similar arguments to the ones of the above cases, taking n large enough.

Similar computations also yield

$$\|G_i\|_{\mathrm{Lip}(M_i,X^*)} \leqslant 1+\delta \quad \forall i \in \{1,\dots,k\}.$$

Now, we have defined subspaces $M_i \subset M$ and functions $F_i, G_i \in \mathrm{Lip}(M_i, X^*)$ such that

$$\max_{1\leqslant i\leqslant k}\{\|F_i\|_{\mathrm{Lip}(M_i,X^*)}, \|G_i\|_{\mathrm{Lip}(M_i,X^*)}\} \leqslant 1+\delta.$$

As the pair (M, X^*) has the CEP, for each $i \in \{1,\dots,k\}$ we can find extensions of F_i and G_i to the whole of M, which we will again call F_i and G_i, respectively, such that

$$\|F_i\|_{\mathrm{Lip}(M,X^*)} \leqslant 1+\delta, \qquad \|G_i\|_{\mathrm{Lip}(M,X^*)} \leqslant 1+\delta.$$

So $F_i/(1+\delta), G_i/(1+\delta) \in B_{\mathrm{Lip}(M,X^*)}$ for each $i \in \{1,\dots,k\}$.

The final step of the proof is to see that

$$\sum_{i=1}^k \frac{F_i}{1+\delta} \in C, \qquad \sum_{i=1}^k \frac{G_i}{1+\delta} \in C$$

and to conclude from here that C has diameter 2. We prove this fact for the case in which M is unbounded. For the other cases, the arguments and estimates are similar. So, assume that M is unbounded. Given $i \in \{1,\dots,k\}$ one has

$$\varphi_i\left(\frac{F_i}{1+\delta}\right) = \frac{\sum_{j=1}^{n_i}\lambda_j^i F_i(m_{i,j})(x_{i,j})}{1+\delta} = \frac{\sum_{j=1}^{n_i}\lambda_j^i g_i(m_{i,j})(x_{i,j})}{1+\delta} = \frac{g_i(\varphi_i)}{1+\delta} > 1-\alpha.$$

So $\sum_{i=1}^k \lambda_i F_i/(1+\delta) \in C$. Similarly one has $\sum_{i=1}^k \lambda_i G_i/(1+\delta) \in C$. Hence,

$$\begin{aligned}
\mathrm{diam}(C) &\geqslant \left\|\sum_{i=1}^k \lambda_i \frac{F_i}{1+\delta} - \sum_{i=1}^k \lambda_i \frac{G_i}{1+\delta}\right\| \\
&\geqslant \frac{\|\sum_{i=1}^k \lambda_i F_i(m_N)/(1+\delta) - \sum_{i=1}^k \lambda_i G_i(m_N)/(1+\delta)\|}{d(m_N,0)} \\
&= \frac{\|\sum_{i=1}^k 2\lambda_i d(m_N,0)x^*/(1+\delta)\|}{d(m_N,0)} \\
&= \frac{2}{1+\delta}.
\end{aligned}$$

From the estimate above we deduce that $\mathrm{diam}(C) = 2$ from the arbitrariness of $0<\delta<\delta_0$. □

Now let us end the section by analysing the vector-valued Lipschitz-free Banach space over a concrete metric space. From here, we will get two interesting consequences: on the one hand, we will get several examples of vector-valued Lipschitz-free Banach spaces that not only fail to have an octahedral norm but also whose unit ball contains points of Fréchet differentiability. On the other hand, we will prove that such a construction depends strongly on the underlying target Banach space. So, octahedrality in vector-valued Lipschitz-free Banach spaces relies on the underlying metric spaces as well as on the target Banach one.

For the construction of such a metric space, consider Γ to be an infinite set. Define $M := \Gamma \cup \{0\} \cup \{z\}$. Consider on M the following distance:

$$d(x,y) := \begin{cases} 1 & \text{if } x,y \in \Gamma \cup \{0\},\ x \neq y, \\ 1 & \text{if } x = z,\ y \in \Gamma \text{ or } x \in \Gamma,\ y = z, \\ 2 & \text{if } x = z,\ y = 0 \text{ or } x = 0,\ y = z, \\ 0 & \text{otherwise.} \end{cases}$$

This is obviously an infinite, bounded and uniformly discrete metric space. Moreover, it is not difficult to prove that the pair (M, X) has the CEP for every Banach space X. Consider a Banach space X, pick $y \in S_X$ and notice that $\delta_{z,y}$ is a 2-norm functional, so define $\varphi := \frac{1}{2}\delta_{z,y} \in S_{\mathcal{F}(M,X)}$. Given $\alpha \in \mathbb{R}^+$ consider

$$S_\alpha := S(B_{\mathrm{Lip}(M,X^*)}, \varphi, \tfrac{1}{2}\alpha) = \{f \in B_{\mathrm{Lip}(M,X^*)} \mid f(z)(y) > 2 - \alpha\}.$$

Consider $x \in \Gamma$ and $f \in S_\alpha$. We claim that

$$f(x)(y) > 1 - \alpha.$$

Indeed, assume by contradiction that $f(x)(y) \leqslant 1 - \alpha$. Then

$$1 < f(z)(y) - f(x)(y) = (f(z) - f(x))(y) \leqslant \|f(z) - f(x)\| \leqslant d(z,x) = 1,$$

which is a contradiction.

We will prove that $\inf_\alpha \operatorname{diam}(S_\alpha)$ depends on the target space X^*.

Proposition 2.5. *If y is a point of Fréchet differentiability of B_X, then*

$$\inf_\alpha S_\alpha = 0.$$

Proof. Notice that, as y is a point of Fréchet differentiability, there exists (by Smulian's lemma) $\delta\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\delta(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$ and such that

$$\left.\begin{aligned} x^*, y^* &\in B_{X^*}, \\ x^*(y) &> 1 - \alpha, \\ y^*(y) &> 1 - \alpha, \end{aligned}\right\} \implies \|x^* - y^*\| < \delta(\alpha). \tag{2.3}$$

Pick $f, g \in S(B_{\mathrm{Lip}(M,X^*)}, \varphi, \frac{1}{2}\alpha)$ and $u, v \in M \setminus \{0\}$, $u \neq v$. Our aim is to estimate

$$\begin{aligned} \frac{\|f(u) - g(u) - (f(v) - g(v))\|}{d(u,v)} &\leqslant \|f(u) - g(u) - (f(v) - g(v))\| \\ &\leqslant \|f(u) - g(u)\| + \|f(v) - g(v)\| \\ &=: K. \end{aligned}$$

If $u = z$, then we have

$$\tfrac{1}{2}f(u)(y) > 1 - \tfrac{1}{2}\alpha, \quad \tfrac{1}{2}g(u)(y) > 1 - \tfrac{1}{2}\alpha \quad \overset{(2.3)}{\Longrightarrow} \quad \|f(u) - g(u)\| \leqslant 2\delta(\tfrac{1}{2}\alpha).$$

Similarly, if $u \in \Gamma$, then

$$f(u)(y) > 1 - \alpha, \qquad g(u)(y) > 1 - \alpha \quad \overset{(2.3)}{\Longrightarrow} \quad \|f(u) - g(u)\| \leqslant \delta(\alpha).$$

Hence, $K \leqslant \delta(\alpha) + \max\{\delta(\alpha), 2\delta(\frac{1}{2}\alpha)\}$.

From the arbitrariness of $f, g \in S(B_{\mathrm{Lip}(M)}, \varphi, \frac{1}{2}\alpha)$ we conclude that

$$\mathrm{diam}(S(B_{\mathrm{Lip}(M)}, \varphi, \tfrac{1}{2}\alpha)) \leqslant \delta(\alpha) + \max\{\delta(\alpha), 2\delta(\tfrac{1}{2}\alpha)\}.$$

Finally, taking the infimum in $\alpha \in \mathbb{R}^+$, from the hypothesis on δ and the continuity of the map max we conclude the desired result. □

Despite proposition 2.5, we will prove that $\mathcal{F}(M, X)$ has a dramatically different behaviour whenever X^* has the weak-star slice diameter 2 property.

Proposition 2.6. *If X^* has the weak-star slice diameter 2 property, then*

$$\inf_{\alpha} S_\alpha = 2.$$

Proof. Pick two arbitrary numbers $\alpha > 0$ and $\varepsilon > 0$. As X^* has the weak-star slice diameter 2 property, we can find $x^*, y^* \in S(B_{X^*}, x, \frac{1}{2}\alpha)$ such that $\|x^* - y^*\| > 2 - \varepsilon$. Now define $f, g\colon M \to X^*$ by the equations

$$f(t) := d(t, 0)x^*, \quad g(t) := d(t, 0)y^* \quad \forall t \in M.$$

Now, f, g are clearly norm-one Lipschitz functions. Moreover,

$$\varphi(f) = \tfrac{1}{2} f(z)(x) = x^*(x) > 1 - \tfrac{1}{2}\alpha.$$

So $f \in S_\alpha$. Analogously, $g \in S_\alpha$. Consequently,

$$\mathrm{diam}(S_\alpha) \geqslant \|f - g\| \geqslant \tfrac{1}{2}\|f(z) - g(z)\| = \|x^* - y^*\| > 2 - \varepsilon.$$

As ε and α were arbitrary we conclude that $\mathrm{diam}(S_\alpha) = 2$, so we are done. □

From the two propositions above we can get the desired consequences. From proposition 2.5 we get vector-valued Lipschitz-free Banach spaces with points of Fréchet differentiability, which, keeping in mind that the pair (M, X^*) has the CEP for every Banach space X, proves that theorem 2.4 is optimal. However, from proposition 2.6 we conclude that the existence of such Fréchet differentiability points depends on the target space. Indeed, we can even get octahedrality for suitable choices of X in the example above. For instance, $\mathcal{F}(M, \ell_1) = \mathcal{F}(M)\hat{\otimes}_\pi \ell_1 = \ell_1(\mathcal{F}(M))$ has an octahedral norm.

3. Consequences and open questions

Under the assumptions of theorem 2.4 we have that $\mathrm{Lip}(M, X^*)$ has the weak-star strong diameter 2 property. This gives rise to a natural question.

Question 3.1. If M and X satisfy the hypotheses of theorem 2.4, does $\mathrm{Lip}(M, X^*)$ satisfy the strong diameter 2 property?

Note that in [15] we have a partial answer for the scalar case in terms of the Daugavet property. Also, in the scalar case, when M is a compact metric space such that $\mathrm{lip}(M)$ separates the points in M, it is known that $\mathrm{lip}(M)^* = \mathcal{F}(M)$ and $\mathrm{lip}(M)$ is an M-embedded space (see the remark after theorem 6.6 in [17]), that is, $\mathrm{lip}(M)$ is an M-ideal in $\mathrm{Lip}(M)$ (see [7] for the case in which $M = [0, 1]$). Then we get from [1] that $\text{lip}(M)$ and $\text{Lip}(M)$ satisfy the strong diameter 2 property. Recall that $\text{lip}(M)$ stands for the space of scalar Lipschitz functions on M such that

$$\lim_{\varepsilon\to 0} \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)},\ x\neq y\in M,\ d(x,y)<\varepsilon\right\}=0.$$

Moreover, in [14, theorems 1 and 2] Ivakhno obtained in the scalar case that $\text{Lip}(M)$ has the slice diameter 2 property whenever M satisfies the same assumptions as those of theorem 2.4.

Propositions 2.5 and 2.6 show that the geometry of vector-valued Lipschitz-free Banach spaces does not only depend on the underlying scalar Lipschitz-free space but also on the target Banach space. However, two natural questions arise.

Question 3.2. Let M be a pointed metric space and let X be a non-zero Banach space.

(1) Does theorem 2.4 hold without assuming that the pair (M, X^*) has the CEP?

(2) Does $\mathcal{F}(M, X)$ have an octahedral norm whenever $\mathcal{F}(M)$ does?

Bearing in mind the identification $\mathcal{F}(M, X) = \mathcal{F}(M)\hat{\otimes}_\pi X$, question 3.2(2) is related to the problem of how octahedrality is preserved by projective tensor products. However, question 3.2(2) is an open problem recently posed in [18].

Finally, we have analysed octahedrality in $\mathcal{F}(M, X)$ whenever M is a metric space and X is a Banach space. However, we did not get any result about the dual properties (i.e. diameter 2 properties). More precisely, we have the following question.

Question 3.3. Given a metric space M and a non-zero Banach space X, which assumptions do we need over M and X in order to ensure that $\mathcal{F}(M, X)$ has the slice diameter 2 property (respectively, diameter 2 property, strong diameter 2 property)?

Again, not only do we get a partial answer in the scalar case but also in the vector-valued one. Indeed, again by [15] we know that $\mathcal{F}(M)$ has the strong diameter 2 property whenever M is a metrically convex metric space. Keeping in mind that $\mathcal{F}(M, X) = \mathcal{F}(M)\hat{\otimes}_\pi X$, the next proposition is an immediate application of [5, corollary 3.6].

Proposition 3.4. *Let M be a metric space with a designated origin* 0 *and let X be a Banach space. If M is metrically convex and X has the strong diameter* 2 *property, then $\mathcal{F}(M, X)$ has the strong diameter* 2 *property.*

Despite the above proposition, there are metric spaces whose free-Lipschitz Banach space fails to have any diameter 2 property. Indeed, it is well known that $\mathcal{F}(M)$ has the Radon–Nikodým property whenever M is a totally discrete metric space [17]. Related to the strong diameter 2 property we can even get vector-valued free Lipschitz Banach spaces that fail to have such a property. Indeed, if we consider a Banach space X failing to have the strong diameter 2 property and a totally discrete metric space M, then $\mathcal{F}(M, X) = \mathcal{F}(M)\hat{\otimes}_\pi X$ does not have the strong diameter 2 property [5, corollary 3.13].

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