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FROG MODELS ON TREES THROUGH RENEWAL THEORY

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Abstract

We study a class of growing systems of random walks on regular trees, known as *frog models with geometric lifetime* in the literature. With the help of results from renewal theory, we derive new bounds for their critical parameters. Our approach also improves the existing bounds for the critical parameter of a percolation model on trees known as *cone percolation*.

Keywords: Frog model; critical probability; cone percolation; homogeneous tree; renewal theory

2010 Mathematics Subject Classification: Primary 60K35
Secondary 60K05; 82C41

1. Introduction

In this work we explore a new approach for studying the localization of the critical parameter of a growing system of random walks on regular trees, known as *frog models with lifetime* in the literature. This approach is based on a link between undelayed renewal sequences and a frog model on directed regular trees.

On general infinite connected graphs, the original frog model with geometric lifetime is inspired by a process of information transmission on a moving population and may be informally described as follows. Assume that at time 0 each vertex of the graph has one particle which may be in one of two states: active or inactive. Each active particle performs, independently of the others, a discrete-time random walk through the vertices of the graph during a random number of steps, geometrically distributed with parameter $1-p$ for some $p \in (0, 1)$. This time is called the lifetime of the active particle, and once it is reached we assume that the particle is removed from the system. Removed particles play no role in the spreading procedure. On the other hand, if an active particle jumps to a vertex containing an inactive one, then the inactive particle becomes active and starts an independent random walk on the graph. Usually, it is assumed that the process starts with one active particle at a fixed vertex, and inactive particles everywhere else. We refer the reader to [1] for a formal definition of the model.

One of the main questions of interest is the survival, or not, of a particular realization of the process, that is, whether or not there is, at any time, at least one active particle on the graph. Alves *et al.* [1] and Lebensztayn *et al.* [16] addressed this question with regard to homogeneous trees and other infinite graphs such as hypercubic lattices. On infinite trees, a simple coupling argument can be used to show that the probability of survival of the frog model

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on $\mathbb{T}_d$, the homogeneous tree of degree $d+1$ (that is, any vertex that has $d+1$ neighbors), which we denote by $\theta(d, p)$, is a nondecreasing function of p and, therefore, we may define the critical parameter as $p_c(d) := \inf\{p : \theta(d, p) > 0\}$. Alves *et al.* [1] presented a necessary and sufficient condition for the existence of a phase transition on $\mathbb{T}_d$, that is, for $0 < p_c(d) < 1$, and also stated the bounds $(d+1)/(2d+1) \leq p_c(d) \leq (d+1)/(2d-2)$ for the critical parameter. Their upper bound was later improved by Lebensztayn *et al.* [16], who proved that $p_c(d) \leq (d+1)/(2d)$. Similar results were obtained by Lebensztayn *et al.* [14] for the modified version of the model in which each active particle performs a self-avoiding discrete-time random walk on the tree. In this paper we follow the approach in these papers. We refer the reader to [1] and [9], and the references therein, for results related to the behavior of the frog model when the lifetimes of the particles are not restricted. More precisely, Alves *et al.* [1] obtained a shape theorem for the model on the d-dimensional hypercubic lattice, while Hoffman *et al.* [9] studied recurrence/transience for the model on trees. The latter is a topic of current research.

The paper is organized as follows. In Section 2 we define a frog model on directed trees and obtain tight bounds for the critical parameter as a function of d. In Section 3 we compare our model to three models (the original frog model, the self-avoiding frog model, and the cone percolation with geometric radius), and improve several previously known bounds. Section 4 is an interlude on renewal processes, containing results that we will use for our proofs. Section 5 contains the proofs, where our strategy will be to use the link, originally showed in [7], between some long-range information propagation models on $\mathbb{N}$ and undelayed renewal sequences.

2. Frog model on directed trees

2.1. Definition of the model

We consider the directed rooted tree $\overrightarrow{\mathbb{T}}_d = (\mathcal{V}, \overrightarrow{\mathcal{E}})$, defined by making all the edges of the d-regular tree $\mathbb{T}_d = (\mathcal{V}, \mathcal{E})$ point away from the root. We define the distance between $u, v \in \mathcal{V}$, denoted by $d(u, v)$, as the number of edges in the unique path connecting them. We write $u < v$ if $u \neq v$ and u is one of the vertices of the path connecting the root to v. In this paper we consider a frog model on $\overrightarrow{\mathbb{T}}_d$, that is, the active particles try to activate other particles localized away from the root, as shown in Figure 1.

In order to define the model, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where the following random objects are independent and well defined for any $v \in \mathcal{V}$: $(S_n^v)_{n\geq 0}$ is a discrete-time symmetric random walk on $\overrightarrow{\mathbb{T}}_d$ starting from v, and $\mathcal{T}_v$ is an $\mathbb{N}$-valued random variable satisfying $\mathbb{P}(\mathcal{T}_v \geq n) = c(dq)^n$ for some $c \in (0, 1]$ and $q \in (0, 1)$ such that $c(dq)^n < 1$ for any $n \geq 1$. The random walk $(S_n^v)_{n\geq 0}$ represents the trajectory of the particle starting at v and $\mathcal{T}_v$ represents its lifetime. We now define the truncated random walk starting at v, $(R_n^v)_{n\geq 0}$, by

$$R_n^v := \begin{cases} S_n^v, & n < \mathcal{T}_v, \\ S_{\mathcal{T}_v - 1}^v, & n \geq \mathcal{T}_v. \end{cases}$$

Observe that, by symmetry, for any $n \geq 1$ and any $u \in \partial T_v^n := \{u \in \mathcal{V} : u > v, \operatorname{dist}(v, u) = n\}$,

$$\mathbb{P}(R_n^v = u) = cq^n. \tag{1}$$

Remark 1. A number of authors have considered models in which the process is started with a random number of particles at each vertex of the tree. We can also do this, but in order to simplify the presentation, we only present the results in the one-per-site case. We refer the

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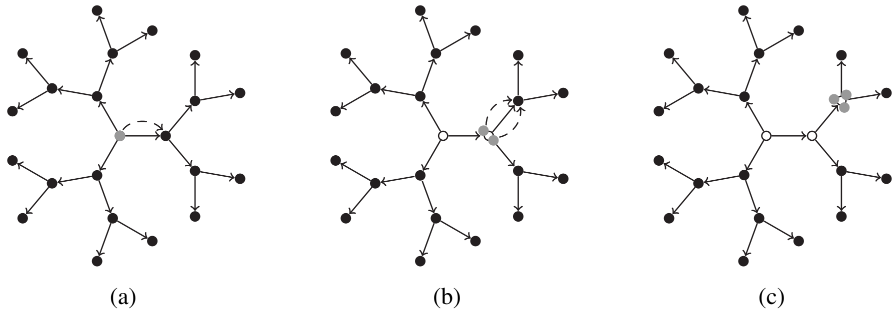

FIGURE 1: Realization of the frog model on T_2. Active particles are represented as shaded circles, and inactive particles as solid circles. Open circles indicate vertices of the graph, which may be unoccupied or occupied by one or more particles. Solid arrows indicate the directions that particles may move on the graph. Dashed arrows indicate the paths taken by activate particles in this example. (a) At $t = 0$, an inactive particle located at the root becomes active. (b) At $t = 1$, the active particle has activated one of the three available inactive particles. (c) At $t = 2$, the two active particles have activated a third particle.

reader interested in an extension to the random initial configuration to [16, Section 5] where the changes required in the proofs are explained.

2.2. Results concerning frogs on directed trees

Let $\theta(d, c, q)$ denote the probability of survival of this model. Furthermore, for this model a coupling argument can be used to show that for $d \geq 2$, $\theta(d, c, q)$ is monotone nondecreasing in q and, thus, there exists a critical parameter

$$q_{\mathrm{c}}(d, c) := \inf\{q > 0 \colon \theta(d, c, q) > 0\}.$$

Our first main result is an equality that the critical parameter q_{c} must satisfy.

Theorem 1. *For any fixed $d \geq 2$ and $c \in (0, 1]$, the critical value $q_{\mathrm{c}} = q_{\mathrm{c}}(c, d)$ is the solution to*

$$\sum_{k\geq 1} c(dq)^k \prod_{i=1}^{k-1}(1 - cq^i) = 1. \tag{2}$$

In particular, $q_{\mathrm{c}} \in (0, 1/d)$.

Remark 2. We point out that there are different ways to present condition (2). The q-Pochhammer symbol is defined as $(a; x)_k := \prod_{i=0}^{k-1}(1 - ax^i)$ with the convention that $(a; x)_0 = 1$. Then, Theorem 1 states that q_{c} is the solution of $\sum_{k\geq 1} c(dq)^k (cq; q)_k = 1$. An alternate way of stating (2) is by considering $f_k^{(q)} := cq^k \prod_{i=1}^{k-1}(1 - cq^i)$, $k \geq 1$, a probability distribution indexed by q (recall that c and d are fixed). Thus, letting N_q be a random variable with distribution $f_k^{(q)}$, $k \geq 1$, q_{c} is such that $\mathbb{E}d^{N_{q_{\mathrm{c}}}} = 1$.

We obtain the following bounds relating q_{c}, d, and c.

Corollary 1. *We have*

$$\frac{(c+1)(1 - \sqrt{1 - 4q_{\mathrm{c}}(c^2/(c+1)^2)})}{2q_{\mathrm{c}}^2 c^2} \leq d \leq \frac{(c+1)(1 - \sqrt{1 - 4(c^2/(c+1)^2)q_{\mathrm{c}}(q_{\mathrm{c}}+1)}}{2c^2 q_{\mathrm{c}}^2 (q_{\mathrm{c}}+1)}.$$

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In order to identify the bounds for q_c as a function of c and d, we have to revert the above bounds in order to isolate q_c. This is carried out using MATHEMATICA®. Although the obtained bounds are explicit functions of c and d, the expressions are not very friendly, nor really informative, so for the sake of simplicity we do not include them here; we refer he interested reader to [6, Equation (5)], which is a previous version of this paper. We use these expressions to obtain our tightest numerical bounds, which are presented in the columns 'Corollary 1' of Tables 1 and 2. For instance, in Table 1 we present numerical bounds for our frog model in the $c = 1$ case.

In the next corollary we present more explicit expressions.

Corollary 2. *We have, for any* $d \geq 3$,

$$\frac{1}{d(c+1) - (c/(c+1))^2} \leq q_c(d) \leq \frac{F(c,d) - \sqrt{F^2(c,d) - 224c^2(c+1)^2}}{16c^2},$$

where $F(c, d) := 7d(c+1)^3 - 8c^2$. *The lower bound also holds for* $d = 2$.

3. Applications

Now we discuss the application of our results to some models in the literature. While the applications that we will present in Subsections 3.2 and 3.3 are direct consequences of the above results, the case of the *cone percolation*, that we first investigate, is an application of the method of proof.

3.1. Cone percolation with geometric radius

The first application we discuss is an improvement of both the upper and lower bounds for the critical probability of a long-range percolation model on $\overrightarrow{\mathbb{T}}_d$ called the cone percolation model; see [11]. There is a random variable X_v associated to each vertex $v \in \mathcal{V} := \mathcal{V}(\mathbb{T}_d)$, representing a radius of propagation of a piece of information. The X_v are independent copies of $X \sim \text{geo}(1-p)$ for some $p \in (0, 1)$, and it is assumed that the information propagates through the graph as follows. At time 0, only the root has the information. At time 1, all the vertices at a distance of at most $X_{\mathbf{0}}$ from the root of the tree are informed. At each step, each newly informed vertex v will inform all noninformed vertices $v' > v$ such that $d(v, v') \leq X_v$. Junior *et al.* [11] proved that there exists a critical value p_c^{cp} (where 'cp' denotes cone percolation) above which infinitely many vertices are informed (that is, the model percolates) with positive probability.

Our interest is to compare this information propagation with the frog model on the directed tree, making $c = 1$ and $q = p$ in (1). A quick look to the first step in the proof of Theorem 1 shows that (8) and (9) are also valid using the dynamics of the cone percolation in place of our frog model. The remainder of the proof relies only on the dynamics along one single branch, which is the same in both models with our choice of parameters. Thus, making $c = 1$ and $q = p$, the results of our frog model are valid for the geometric cone percolation, and this leads to the next proposition.

Proposition 1. *The critical parameter* p_c^{cp} *is the solution in* q *of* (2) *with* $c = 1$. *More explicitly, we have* $0.266\,667 \leq p_c^{\text{cp}}(2) \leq 0.277\,206$ *and, for* $d \geq 3$,

$$\frac{1}{2d - 1/4} \leq p_c^{\text{cp}}(d) \leq \frac{(7d-1)\left(1 - \sqrt{1 - 14/(7d-1)^2}\right)}{2}.$$

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TABLE 1: Numerical values obtained by reverting Corollary 1 (with MATHEMATICA when $c = 1$) and from Proposition 1 as well as the existing bounds from the literature.

d	Lower bounds			Upper bounds		
	Corollary 1	Proposition 1	[11, Example 5.2]	Corollary 1	Proposition 1	[11, Example 5.2]
2	0.269 594	0.266 667	0.250 000	0.277 206	0.277 206	0.292 893
3	0.174 659	0.173 913	0.166 667	0.176 343	0.176 559	0.183 503
4	0.129 326	0.129 032	0.125 000	0.129 961	0.130 258	0.133 975
5	0.102 709	0.102 564	0.100 000	0.103 015	0.103 255	0.105 573
6	0.085 188	0.085 106	0.083 333	0.085 358	0.085 544	0.087 129
7	0.072 777	0.072 727	0.071 428	0.072 882	0.073 027	0.074 179
8	0.063 525	0.063 492	0.062 500	0.063 594	0.063 710	0.064 585
9	0.056 361	0.056 338	0.055 556	0.056 408	0.056 503	0.057 191
10	0.050 649	0.050 632	0.050 000	0.050 684	0.050 762	0.051 316
15	0.033 618	0.033 613	0.033 333	0.033 628	0.033 664	0.033 908
20	0.025 159	0.025 157	0.025 000	0.025 163	0.025 184	0.025 320
30	0.016 737	0.016 736	0.016 667	0.016 738	0.016 748	0.016 807
50	0.010 025	0.010 025	0.010 000	0.010 025	0.010 028	0.010 050
100	0.005 006	0.005 006	0.005 000	0.005 006	0.005 007	0.005 012

These bounds improve the bounds presented in [11]; namely,

$$\frac{1}{2d} \leq p_{\rm c}^{\rm cp}(d) \leq 1 - \sqrt{1 - \frac{1}{d}}. \tag{3}$$

See Table 1 for a comparison between (3) and Proposition 1.

3.2. Improved upper bounds for the original frog model and its self-avoiding version

Now we present an improvement of the known upper bounds for the critical parameter of the original frog model, as well as the self-avoiding frog model, on $\mathbb{T}_d$. We start with the frog model with one-per-site configuration, independent and identically distributed (i.i.d.) geometric lifetimes of parameter $1 - p$ for some $p \in (0, 1)$, and denote by $p_c^{\rm o}(d)$ its critical parameter for $d \geq 2$ (where 'o' denotes original, to point out that it refers to the original model). A useful result in order to obtain an upper bound for $p_c^{\rm o}(d)$ is [16, Lemma 2.1]. It states that, for any two vertices u and v such that $u < v$ and $d(u, v) = n \geq 1$, vertex v will be visited by the active particle starting at u with probability r^n, where

$$r = r(p) := \frac{d + 1 - \sqrt{(d + 1)^2 - 4dp^2}}{2dp}.$$

Our frog model on $\overrightarrow{\mathbb{T}}_d$ with $c = 1$ and $q = r(p)$ can be coupled to the original frog model in such a way that our model is *below* (in the sense that if our model survives, the original frog model survives also). To this end, we start with the one-per-site configuration for both models (our on $\overrightarrow{\mathbb{T}}_d$ and the original on $\mathbb{T}_d$) and we realize both processes in such a way that an active particle hits a given vertex in $\overrightarrow{\mathbb{T}}_d$ only if the corresponding particle on $\mathbb{T}_d$ also hits this vertex. Thus, if our frog model on $\overrightarrow{\mathbb{T}}_d$ with $q = r(p)$ and $c = 1$ survives, the original frog model with p also survives. So we can use the upper bound of Corollary 2 (and revert Corollary 1 for $d = 2$) to obtain our next result.

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TABLE 2: Numerical values of the upper bounds for the original frog model and its self-avoiding version from Corollary 1 and Proposition 2, and existing values from the literature.

d	Upper bound—original frog model			Upper bound —self-avoiding frog model		
	Corollary 1	Proposition 2	[16, Theorem 4.1]	Corollary 1	Proposition 3	[14, Theorem 2.1]
2	0.720 836	0.720 836	0.750 000	0.648 045	0.648 045	0.697 224
3	0.645 182	0.645 837	0.666 667	0.599 063	0.600 229	0.627 719
4	0.608 681	0.609 897	0.625 000	0.574 870	0.576 225	0.594 875
5	0.586 944	0.588 174	0.600 000	0.560 271	0.561 544	0.575 571
6	0.572 482	0.573 624	0.583 333	0.550 468	0.551 621	0.562 829
7	0.562 156	0.563 197	0.571 429	0.543 421	0.544 461	0.553 778
8	0.554 410	0.555 358	0.562 500	0.538 107	0.539 048	0.547 013
9	0.548 384	0.549 249	0.555 556	0.533 955	0.534 812	0.541 764
10	0.543 561	0.544 355	0.550 000	0.530 620	0.531 406	0.537 571
15	0.529 076	0.529 632	0.533 333	0.520 543	0.521 093	0.525 021
20	0.521 822	0.522 248	0.525 000	0.515 458	0.515 881	0.518 759
30	0.514 559	0.514 848	0.516 667	0.510 341	0.510 628	0.512 503
50	0.508 741	0.508 917	0.510 000	0.506 222	0.506 397	0.507 501
100	0.504 373	0.504 461	0.505 000	0.503 118	0.503 206	0.503 750

Proposition 2. *We have* $p_c^o(2) \le 0.720\,836$ *and, for* $d \ge 3$,

$$p_c^o(d) \le \frac{(d+1)[(7d-1) - \sqrt{(7d-1)^2 - 14}]}{d(7d-1)^2 - 7d + 2 - d(7d-1)\sqrt{(7d-1)^2 - 14}}.$$

This upper bound improves the bound presented in Lebensztayn [16], that is,

$$p_c^o(d) \le \frac{d+1}{2d}.$$

In the left panel of Table 2 we present a numerical comparison between the upper bounds. We point out that the bound obtained in [16] was an improvement over that of [5], in which the authors obtained $p_c^o(d) \le (d+1)/(2d-2)$.

Now we consider the self-avoiding version of the frog model on $\mathbb{T}_d$ introduced in [14]. The only difference with respect to the preceding model is that each particle performs a self-avoiding random walk on $\mathbb{T}_d$ when it is activated. Again, a coupling argument allows us to compare this model with the frog model on $\vec{\mathbb{T}}_d$, but now taking $c = d/(d+1)$ and $q = p/d$ in (1). In the coupled versions, we remove the activated particles of $\vec{\mathbb{T}}_d$ for which the corresponding particle on $\mathbb{T}_d$ jumps in the direction of the root. In any other case both particles follow the same trajectory away from the root with the same geometric lifetimes. We see that survival for our model implies survival for the self-avoiding frog model. So here, denoting by $p_c^{sa}(d)$ (where 'sa' denotes self avoiding) the critical parameter of the self-avoiding model, we can also use the upper bound of Corollary 2 (and revert Corollary 1 for $d = 2$) to obtain the next proposition.

Proposition 3. *We have* $p_c^{sa}(2) \le 0.648\,046$ *and, for* $d \ge 3$,

$$p_c^{sa}(d) \le (d+1)^2 \frac{F(d/(d+1), d) - \sqrt{F^2(d/(d+1), d) - 224c^2(d/(d+1)+1)^2}}{16d},$$

where we recall that $F(c, d) := 7d(c+1)^3 - 8c^2$.

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This upper bound improves the bound obtained in [14], that is,

$$p_c^{sa}(d) \le \tfrac{1}{2}\big(2d + 1 - \sqrt{4d^2 - 3}\big). \tag{4}$$

In Table 2 we present a numerical comparison between (4) and Proposition 3.

3.3. Frog model with removal at visited vertices

In the previous subsection we presented some improvements for the original frog model and its self-avoiding version on $\mathbb{T}_d$. As a final application of our results, we discuss another version of the frog model which has not been explored on infinite graphs. It is a frog model with i.i.d. geometric lifetimes of parameter $1 - p$, in which we remove any particle which did not activate any other particles for L number of times, where $L \ge 1$. This modification was suggested by Popov [18] and to the best of the authors' knowledge, the only rigorous results on infinite graphs are proved in $\mathbb{Z}$ for some related models; see, for example, [15] and the references therein. On the other hand, the model has been well studied on some finite graphs. In such a case the issue of interest is the study of the final proportion of visited vertices at the end of the process. In this direction, Alves *et al.* [2] obtained the first results for the model defined on a complete graph with $L = 1$ and $p = 1$ by means of a mean-field approximation analysis and computational simulations. Their work was later generalized by Kurtz *et al.* [12] for $L \ge 1$ and $p = 1$ in the form of limit theorems obtained for the proportion of visited vertices at the absorption time in the process as the size of the graph goes to ∞. More recently, Lebensztayn and Rodriguez [13] stated the connection between this model and the well-known Maki–Thompson rumor model; see [17]. In view of the connection obtained in [13], one may consider the model presented here as a rumor process on a moving population.

If we consider this model on $\mathbb{T}_d$, starting from a one-per-site configuration, then a realization of the resulting process when $L = 1$ coincides with a realization of our general frog model on $\overrightarrow{\mathbb{T}}_d$, making $c = 1$ and $q = p/(d + 1)$ in (1). Therefore, denoting by $p_c^r(d)$ the critical parameter of this model (where 'r' denotes removal) with $L = 1$, we directly obtain the next proposition.

Proposition 4. *The critical parameter* p_c^r *is the solution in* p *of (2) with* $c = 1$ *and* $q = p/(d + 1)$. *More explicitly, we have* $0.8 \le p_c^r(2) \le 0.831\,619$, *and, for* $d \ge 3$,

$$\frac{d+1}{2d - 1/4} \le p_c^r(d) \le \frac{(d+1)(7d-1)\big(1 - \sqrt{1 - 14/(7d-1)^2}\big)}{2}.$$

4. Interlude: renewal convergence rates

In our proofs we use a parallel between information propagation on $\mathbb{N}$ and undelayed renewal sequences. For this reason we dedicate this section to the description of some aspects of renewal theory. Let $\boldsymbol{T} = (T_n)_{n \ge 1}$ be an i.i.d. sequence of random variables, taking values in $\{1, 2, \ldots\} \cup \{\infty\}$ with common distribution $(f_k)_{k \in \{1,2,\ldots\} \cup \{\infty\}}$. The undelayed renewal sequence is the $\{0, 1\}$-valued stochastic chain $\boldsymbol{Y} = (Y_n)_{n \ge 0}$ defined through $Y_0 = 1$ and, for any $n \ge 1$, $Y_n = \mathbf{1}(T_1 + \cdots + T_i = n \text{ for some } i)$. The distribution $(f_k)_{k \in \{1,2,\ldots\} \cup \{\infty\}}$ is called the interarrival distribution. The well-known renewal theorem states that

$$\mathbb{P}(Y_n = 1) \to \frac{1}{\mathbb{E}T}$$

with the convention that $1/\infty = 0$. The question of identifying the rate at which this convergence holds (renewal convergence rate), based on the interarrival distribution is a very classical one; see the introduction of [8] for a rapid survey.

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For our purposes, we first observe that the renewal property implies that

$$\begin{aligned}\mathbb{P}(Y_n = 1)\mathbb{P}(Y_m = 1) &= \mathbb{P}(Y_{n+m} = 1 \mid Y_{n=1})\mathbb{P}(Y_n = 1)\\ &= \mathbb{P}(Y_n = 1, Y_{n+m} = 1)\\ &\le \mathbb{P}(Y_{n+m} = 1).\end{aligned}$$

This means that, in particular, $\log \mathbb{P}(Y_n = 1)$ is super-additive; thus, by Fekete's lemma, it follows that $\lim_n (1/n) \log \mathbb{P}(Y_n = 1)$ exists. We introduce the renewal convergence rate of the process, γ, defined through

$$\log \gamma := -\lim_n \frac{1}{n} \log \mathbb{P}(Y_n = 1).$$

Naturally, the value of $\log \gamma$ depends on the interarrival distribution. Here we focus (having in mind a future application to the frog model described in Section 2) on the interarrival distribution having the property that the hazard rate

$$h_k := \frac{f_k}{\sum_{i \ge k} f_i} = cq^k, \qquad k \ge 1,$$

for some $c > 0$ and $q \in (0, 1)$. Conversely, we have the following expression for f_k:

$$f_k = cq^k \prod_{i=1}^{k-1}(1 - cq^i), \qquad k \ge 1, \tag{5}$$

with the convention $\prod_{i=1}^{0}(1 - cq^i) = 1$. This is a defective probability distribution since

$$\mathbb{P}(T \ge n) = \sum_{k \ge n} f_k = \prod_{i=1}^{n-1}(1 - cq^i) \to \prod_{i \ge 1}(1 - cq^i) > 0. \tag{6}$$

So we have $f_\infty := \mathbb{P}(T = \infty) = \prod_{i \ge 1}(1 - cq^i) > 0$.

In the proofs of our results we make use of the next two lemmas.

Lemma 1. *For a renewal process with interarrival distribution given by (5), we have*

$$\sum_{k \ge 1} \gamma^k cq^k \prod_{i=1}^{k-1}(1 - cq^i) = 1.$$

Proof. From [3, Theorem 3.5], if for some defective distribution $(f_k)_{k \in \{1,2,\dots\} \cup \{\infty\}}$ there exists some $\alpha > 1$ such that

$$F(\alpha) := \sum_{n \ge 1} \alpha^n f_n = 1, \tag{7}$$

then the limit $\lim_n \alpha^n \mathbb{P}(Y_n = 1) = (\sum_n n\alpha^n f_n)^{-1}$.

In our case (recall that f_n, $n \ge 1$ satisfies (5)), we can prove that (7) actually holds for some α. This fact was proved, for example, in [4, Proof of Proposition 2(iii)]. For completeness, we include the argument here. Observe first that $F(1) = \mathbb{P}(T < \infty) < 1$. Moreover, by (5) and (6), we have $f_n/(cq^n) \to \prod_{i \ge 1}(1 - cq^i) > 0$ meaning that, in particular, the radius of convergence of F, $\lim_n f_n^{-1/n} = \lim_n (cq^n)^{-1/n} = q^{-1}$, which is larger than 1. Thus, $F(1) < 1$ and we can find $\delta \in (1, q^{-1})$ such that $1 < F(\delta) < +\infty$. By continuity of F, there exists α such that $F(\alpha) = 1$.

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So (7) holds for some $1 < \alpha < q^{-1}$, meaning, moreover, that $(\sum_n n\alpha^n f_n)^{-1} \in (0, \infty)$. Using [3, Theorem 3.5], we have

$$\begin{aligned} 0 &= -\lim_n \frac{\log \alpha^n \mathbb{P}(Y_n = 1)}{n} \\ &= -\lim_n \frac{\log \mathbb{P}(Y_n = 1)}{n} - \lim_n \frac{\log \alpha^n}{n} \\ &= -\lim_n \frac{\log \mathbb{P}(Y_n = 1)}{n} - \log \alpha; \end{aligned}$$

therefore, $-\lim_n \log \mathbb{P}(Y_n = 1)/n = \log \alpha > 0$, and α is indeed the renewal convergence rate γ of the process. □

In the next lemma, we consider that c is fixed, and thus γ can be seen as a function of q.

Lemma 2. *It holds that γ is a continuous function of q on $q \in (0, 1/cd)$.*

Proof. Fix $q \in (0, 1/cd)$ and consider any sequence $q_\varepsilon \to q$ as $\varepsilon \to 0$, where $q_\varepsilon \in (0, 1/cd)$. Define naturally $f_{k,\varepsilon} := cq_\varepsilon^k \prod_{i=1}^{k-1}(1 - cq_\varepsilon^i)$, $k \geq 1$, and observe that $f_{k,\varepsilon} \to f_k$ for any $k \geq 1$. For any ε, we can use the proof of Lemma 1 to show that there exists a unique solution $\alpha > 1$ of

$$F_\varepsilon(\alpha) := \sum_{n \geq 1} \alpha^n f_{n,\varepsilon} = 1.$$

We naturally denote this solution by $\gamma(q_\varepsilon)$. Observe, moreover, that we can find $\delta \in (1, q^{-1})$ such that $F(\delta) > 1$ and $F_\varepsilon(\delta) < \infty$ for any sufficiently small ε. In these conditions, it was proved in [19] that $\gamma(q_\varepsilon) \to \gamma(q)$, showing that γ is a continuous function of q on $(0, 1/cd)$. □

5. Proofs of the main results

Proof of Theorem 1. The proof comprises three main steps. First, we transform the problem of survival of the frog model on the tree into that of controlling the propagation along one single branch (see(8) and (9) below). Second, we use a result of [7] which implies that the probability that the dynamics along one single branch reaches distance n is equal to the probability that a specific renewal process renews at time n. This allows us to relate to the preceding section, and specifically to prove that $\gamma_c := \gamma(q_c) = d$. Finally, we conclude the proof using Lemma 1 of the preceding section.

Step 1. We use a simple union bound for the lower bound on the critical parameter, and a classical coupling with branching processes for the upper bound. We introduce A_n and A_∞ to respectively denote {a frog at distance n of the root is activated} and {infinitely many frogs are activated}. Fix $d \geq 2$ and $c > 0$, and consider our frog model parametrized by q. Naturally, we have $\theta(q) = \mathbb{P}_q(A_\infty) = \lim_n \mathbb{P}_q(A_n)$. For any $v \in \mathcal{V}$, let $A^v :=$ {the frog of vertex v is activated}.

To find a lower bound for q_c, observe that $\mathbb{P}_q(A_n) = \mathbb{P}_q(\bigcup_{v:\, d(0,v)=n} A^v) \leq d^n p_{q,n}$, where $p_{q,n}$ denotes the common value (by symmetry) of the $\mathbb{P}_q(A^v)$ for any v at distance n of the root. Thus,

$$d^n p_{q,n} \to 0 \quad \Longrightarrow \quad q < q_c. \tag{8}$$

In other words, to find a nontrivial lower bound for q_c, it is sufficient to find a value $q > 0$ such that the left-hand side of (8) holds.

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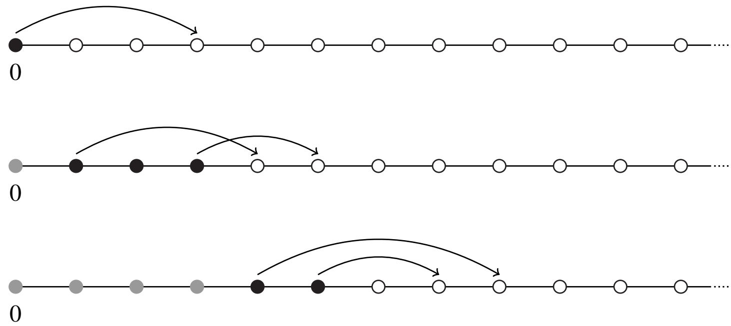

FIGURE 2: First three steps of the firework process. The open, shaded, and solid circles are sites that represent ignorants, spreaders from earlier stages, and current spreaders, respectively.

In order to find an upper bound for q_c, we couple our frog model, rescaled by some length $n \geq 1$, with a branching process. The coupling is the same as the one used in [14, Section 3.1] and [16, Section 3], so we only describe it informally here. The individuals of the branching process are identified with vertices of the tree recursively as follows. Originally, there is one individual which is the root. Its offspring is identified with the set of vertices at distance n from the root which are visited by active frogs at some time of the process. That is, a vertex v is identified with an individual of the offspring of the root provided $d(0, v) = n$ and the event A^v occurs. Recursively, we start one active frog from each vertex v of the offspring of the $(k-1)$th generation, and identify its offspring (of the kth generation) with the set of vertices visited by active frogs and which are at distance n from v.

The resulting branching process is below our frog model in the sense that if it survives, our model survives also. Note that the offspring distribution of the first generation (children of the root) is different from the others, since on $\overrightarrow{\mathbb{T}}_d$, there are $(d+1)d^{n-1}$ vertices at distance n from the root (whereas there are d^n at distance n from any other vertex). This has no effect on the fact that the branching process survives if there exists $N \geq 1$ such that $\mathbb{E}_p(\sum_{v\colon d(0,v)=N} \mathbf{1}(A^v)) = d^N p_{q,N} > 1$. This is also a sufficient condition for our frog model, so

$$\text{there exists } N\colon d^N p_{q,N} > 1 \quad \Longrightarrow \quad q \geq q_c. \tag{9}$$

Step 2. With (8) and (9) in hand, we obtain information about $p_{q,n}$, $n \geq 1$, the probability that a given vertex at distance n from the root is activated. In other words, we investigate how the process propagates along one single branch of the tree, since this is the only way to reach this vertex. At this stage we make a comparison with another process in the literature, originally introduced in [10] under the name of a 'firework process' to model information spreading on $\mathbb{N}$. We briefly describe this model. We start with one spreader at site 0 and ignorants at all the other sites of $\mathbb{N}$. The spreaders transmit the information within a random distance, which are independent copies of an $\mathbb{N}$-valued random variable D, on their right; see Figure 2.

Gallo *et al.* [7, Lemma 1] stated that the probability the firework process on $\mathbb{N}$ reaches site n is equal to the probability that an undelayed renewal sequence (see Section 4 above) $\boldsymbol{Y}$ with hazard rate

$$h_k := \frac{f_k}{\sum_{i \geq k} f_i} = \mathbb{P}(D \geq k)$$

renews at time n.

Returning to the dynamics of the frogs along one single branch, we identify the branch with $\mathbb{N}$ and our active/inactive particles with spreaders/ignorants. The i.i.d. radius of transmission corresponds to the reach of the random walks of the activated frogs along the corresponding branch in $\overrightarrow{\mathbb{T}}_d$. From this parallel we have $p_{q,n} = \mathbb{P}(Y_n = 1)$ if we take $\mathbb{P}(D \geq k) = cq^k$.

As explained in Section 2, there exists $\gamma > 0$ such that $\log \gamma = -\lim_n (1/n) \log \mathbb{P}(Y_n = 1)$. Suppose that $d/\gamma > 1$ (respectively, $d/\gamma < 1$). There exists $\varepsilon = \varepsilon(d, \gamma) > 0$ such that $(d/\gamma)\mathrm{e}^{-\varepsilon} > 1$ (respectively, $(d/\gamma)\mathrm{e}^{\varepsilon} < 1$). On the other hand, by the definition of the limit, we know that for any $\varepsilon > 0$, there exist N such that for any $n \geq N$, we have $\mathrm{e}^{-n(\log\gamma+\varepsilon)} \leq \mathbb{P}(Y_n = 1) \leq \mathrm{e}^{-n(\log\gamma-\varepsilon)}$; thus, in particular,

$$\left(\frac{d}{\gamma}\mathrm{e}^{-\varepsilon}\right)^n \leq d^n \mathbb{P}(Y_n = 1) \leq \left(\frac{d}{\gamma}\mathrm{e}^{\varepsilon}\right)^n.$$

We therefore have the following sequence of implications:

$$\frac{d}{\gamma} > 1 \quad \Longrightarrow \quad \text{there exists } \varepsilon\colon \frac{d}{\gamma}\mathrm{e}^{-\varepsilon} > 1 \quad \Longrightarrow \quad \text{there exists } N\colon d^N \mathbb{P}(Y_N = 1) > 1.$$

Conversely,

$$\begin{aligned} \frac{d}{\gamma} < 1 \quad &\Longrightarrow \quad \text{there exists } \varepsilon\colon \frac{d}{\gamma}\mathrm{e}^{\varepsilon} < 1 \\ &\Longrightarrow \quad d^n \mathbb{P}(Y_n = 1) \leq \left(\frac{d}{\gamma}\mathrm{e}^{\varepsilon}\right)^n \to 0 \quad \text{for all } n \geq N. \end{aligned}$$

Thus, using (8) and (9), and recalling that $\gamma = \gamma(q)$ is a function of q, we have proved that

$$\gamma(q) > d \quad \Longrightarrow \quad q < q_\mathrm{c} \quad \text{and} \quad \gamma(q) < d \quad \Longrightarrow \quad q \geq q_\mathrm{c}.$$

Owing to Lemma 2, we necessarily have $\gamma(q_\mathrm{c}) = d$. Indeed, from the previous relations, observe that $d \leq \lim_{q \nearrow q_\mathrm{c}} \gamma(q) = \gamma(q_\mathrm{c}) = \lim_{q \searrow q_\mathrm{c}} \gamma(q) \leq d$.

Step 3. To conclude the proof, we use Lemma 1, that is,

$$\sum_{k\geq 1} \gamma^k(q) c q^k \prod_{i=1}^{k-1} (1 - cq^i) = 1.$$

Thus, we have

$$\sum_{k\geq 1} d^k c q_\mathrm{c}^k \prod_{i=1}^{k-1} (1 - cq_\mathrm{c}^i) = 1. \qquad \square$$

Proof of Corollary 1. We have to find bounds for $\sum_{k\geq 1} d^k c q_\mathrm{c}^k \prod_{i=1}^{k-1} (1 - cq_\mathrm{c}^i) = 1$. For $n \geq 2$, we can show that

$$1 - cq - cq^2 \leq \prod_{i=1}^{n-1} (1 - cq^i) \leq (1 - cq),$$

where the right-hand side is trivial, and the left-hand side follows easily by recursion. Now we have

$$cdq_\mathrm{c} + (1 - cq_\mathrm{c} - cq_\mathrm{c}^2) c \sum_{n\geq 2} (dq_\mathrm{c})^n \leq 1 \leq cdq_\mathrm{c} + (1 - cq_\mathrm{c}) c \sum_{n\geq 2} (dq_\mathrm{c})^n. \tag{10}$$

The first inequality of (10) yields

$$d \leq \frac{(c+1)\big(1-\sqrt{1-4(c^2/(c+1)^2)q(q+1)}\big)}{2c^2q^2(q+1)},$$

while the second inequality of (10) yields

$$d \geq \frac{(c+1)\big(1-\sqrt{1-4q(c^2/(c+1)^2)}\big)}{2q^2c^2}. \qquad \square$$

Proof of Corollary 2. Using the fact that $\sqrt{1-x} \leq 1-\frac{1}{2}x-\frac{1}{8}x^2$ for $x \in [0,1]$, we obtain

$$d \geq \frac{1+q(c^2/(c+1)^2)}{(c+1)q}.$$

Reverting this inequality yields the lower bound of the corollary. On the other hand, using the fact that $\sqrt{1-x} \geq 1-\frac{1}{2}x-\frac{1}{7}x^2$ for $x \in [0, 0.24]$, we obtain

$$d \leq \frac{1+(8c^2/7(c+1)^2)q(q+1)}{(c+1)q} \tag{11}$$

when $4c^2(c+1)^{-2}q(q+1) \leq 0.24$. Recalling that $c \leq 1$, it is enough to prove that this inequality holds with $c = 1$. From Table 1, we see that the inequality $q_c(q_c+1) \leq 0.06$ holds for $d \geq 3$. Reverting (11) yields the upper bound of the corollary. $\square$

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