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DEBT POLICY RULE, PRODUCTIVE GOVERNMENT SPENDING, AND MULTIPLE GROWTH PATHS: A NOTE

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In a very interesting endogenous growth model, Futagami, Iwaisako, and Ohdoi [*Macroeconomic Dynamics* 12 (2008), 445–462] study the long-run growth effect of borrowing for public investment. Their model exhibits (i) the multiplicity of balanced growth paths (BGPs) in the long run (two steady states) and (ii) a possible indeterminacy of the transition path to the high-growth BGP. The goal of this note is to show that their results depend on a sharp assumption, namely the definition of the public debt target as a ratio to private capital. If the target is defined in terms of public debt–to–GDP ratio, both results vanish: the model exhibits a unique BGP (no multiplicity) and the adjustment path to this unique equilibrium is determinate (no indeterminacy).

Keywords: Deficit, Public Investment, Public Debt, Endogenous Growth, Multiplicity

1. INTRODUCTION

In a very interesting paper, Futagami et al. (2008; hereafter FIO) propose an endogenous growth model with productive government spending. FIO assume that the government adjusts the public debt (B_t)–to–private capital (K_t) ratio $b_t \equiv B_t/K_t$ gradually, so that it equals a target level $\bar{b}$ in the long run. In particular, they assume the following adjustment rule (with $\phi > 0$ the adjustment speed and $\dot{b}_t \equiv db_t/dt$):

$$\dot{b}_t = -\phi(b_t - \bar{b}). \qquad \textbf{(1)}$$

FIO show that (i) this rule leads to the multiplicity of BGPs—namely there are two steady states, one associated with high economic growth and the other with low economic growth—and that (ii) the high-growth BGP may be associated with indeterminacy of the transition path. This note shows that, if the government defines a ratio of bonds *to GDP* (and not a ratio of bonds to *private capital*)

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as the target, the situation changes dramatically. First, multiplicity vanishes: the low-growth BGP disappears, and the model exhibits a unique BGP in the long run, corresponding to the high-growth BGP of FIO. Second, indeterminacy is removed, and the transition path to the unique BGP is determinate.

Section 2 discusses multiplicity, Section 3 deals with indeterminacy, and Section 4 concludes.

2. MULTIPLICITY OR UNIQUENESS OF BALANCED GROWTH PATHS IN THE LONG RUN?

The multiplicity of BGPs in FIO directly comes from the government budget constraint in the long run. In the following, we adopt FIO notations;[1] thus the government budget constraint is

$$\dot{B}_t = (1-\tau)\, r_t B_t - (\tau Q_t - G_t). \tag{2}$$

We define $y_t \equiv G_t/K_t$, and using the production function $Q_t/K_t = Ay_t^{1-\alpha}$ we divide (2) by K_t to find

$$\frac{\dot{B}_t}{K_t} = \dot{b}_t + b_t\frac{\dot{K}_t}{K_t} = (1-\tau)\, r_t b_t - \left(\tau A y_t^{1-\alpha} - y_t\right).$$

Because, in the steady state, $\dot{b}_t = 0$ and $\dot{K}_t/K_t = \gamma$, where γ is a BGP such that $\dot{K}_t/K_t = \dot{B}_t/B_t = \dot{C}_t/C_t = \gamma$, we obtain, in the long run,

$$\tau A y_t^{1-\alpha} - y_t = [(1-\tau)r_t - \gamma]\, \bar{b}. \tag{3}$$

The solvency constraint for the government implies that the public debt growth rate must be inferior to the (net) real interest rate in the long run (the no–Ponzi game condition). Consequently, the cost (i.e., the debt burden) of a permanent borrowing policy, namely $(1-\tau)r_t b_t$, overpasses the flow of permanent resources provided by the deficit financing, namely $\dot{b}_t = \gamma b_t$. Any increase in the long-term public debt target ($\bar{b}$) will need to be financed by an increase in primary budget surpluses ($\zeta(y_t) \equiv \tau A y_t^{1-\alpha} - y_t$).[2] For example, using a simple logarithmic utility function, as in FIO, the Keynes–Ramsey rule leads to $\gamma \equiv \dot{C}_t/C_t = (1-\tau)r_t - \rho$, and relation (3) becomes

$$\zeta\,(y_t) \equiv \tau A y_t^{1-\alpha} - y_t = \rho \bar{b}. \tag{4}$$

According to Fig. 1a, reproduced from FIO (p. 450, Fig. 1), multiplicity occurs, because the primary fiscal balance $\zeta(y_t)$ is a nonlinear function of y_t, with a maximum at $y^{\max} \equiv [(1-\alpha)\, A\, \tau]^{1/\alpha}$. Any increase in the debt target ($\bar{b}$) improves the low-growth BGP ($\bar{y}_L$ moves upward) and reduce the high-growth BGP ($\bar{y}_H$ moves downward). However, notice that the former effect is a very special case. Effectively, in this case, any increase in productive public spending yields a budget surplus (because the productivity of public spending is such that tax collection increases more than public spending). If the government could generate budget

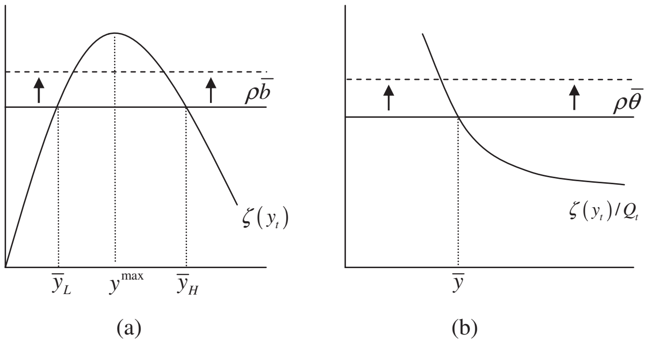

FIGURE 1. (a) Multiplicity of BGPs; (b) uniqueness of BGP.

surpluses simply by raising public spending, it is difficult to justify why it would not (to put it differently, why it would not increase public spending up to $y^{\max}$). Furthermore, it is difficult to justify why it would need to resort to borrowing.

Let us suppose now that the government adopts a debt target defined as a ratio of bonds *to GDP*, and not to private capital. We assume that the government adjusts $\theta_t \equiv B_t/Q_t$ gradually, so that it equals a target level $\bar{\theta}$ in the long run. In particular, let us consider the adjustment rule

$$\dot{\theta}_t = -\phi(\theta_t - \bar{\theta}). \tag{5}$$

Because $b_t = Ay_t^{1-\alpha}\theta_t$, the government budget constraint (4) becomes $\zeta(y_t) \equiv \tau Ay_t^{1-\alpha} - y_t = \rho Ay_t^{1-\alpha}\bar{\theta}$, namely

$$\zeta(y_t)/Q_t = \tau - \frac{y_t^{\alpha}}{A} = \rho\bar{\theta}. \tag{6}$$

By so doing, it removes multiplicity because the budget surplus (expressed as a share of GDP) is now unambiguously negatively linked to public investment. In particular, any increase in the debt target ($\bar{\theta}$) now reduces the amount of productive public spending that the government can finance ($\bar{y}$ moves downward in Fig. 1b) and unambiguously exerts a detrimental effect on the BGP.

3. DETERMINACY OR INDETERMINACY OF THE TRANSITION PATH?

FIO (p. 453, Corollary) state that the high steady state may be associated with indeterminacy of the transition path when $\bar{y}_H < \tilde{y}(\bar{b})$, with

$$\tilde{y}(\bar{b}) \equiv \left[\frac{(1-\alpha)\,A\,[\tau + [1-\alpha\,(1-\tau)]\,b]}{b+1}\right]^{1/\alpha}.$$

The goal of this section is to show that under the debt rule (5) indeterminacy vanishes, and the transition path to the (unique) BGP is determinate.

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Under the adjustment rule (5), the reduced form of the model consists of a three-variable system (θ, x, y), where $x_t \equiv C_t/K_t$. The first relation of the reduced form is the debt law of motion (5). The second equation comes from the Keynes–Ramsey relation $\dot{C}_t/C_t = (1-\tau)r_t - \rho$ (with the real interest rate equalizing the marginal productivity of private capital $r_t = \alpha A y_t^{1-\alpha}$) and the IS equilibrium $(\dot{K}_t/K_t = A y_t^{1-\alpha} - x_t - y_t)$, namely

$$\frac{\dot{x}_t}{x_t} = \frac{\dot{C}_t}{C_t} - \frac{\dot{K}_t}{K_t} = [\alpha\,(1-\tau) - 1]\, A y_t^{1-\alpha} - \rho + x_t + y_t. \tag{7}$$

Finally, the third relation of the reduced form can be obtained from (5) and the log deviation of the production function

$$\left(\frac{\dot{Q}_t}{Q_t} = (1-\alpha)\,\frac{\dot{y}_t}{y_t} + \frac{\dot{K}_t}{K_t}\right): \quad \dot{\theta}_t = \frac{\dot{B}_t}{Q_t} - \theta_t \frac{\dot{Q}_t}{Q_t} = -\phi(\theta_t - \bar{\theta}),$$

so that

$$\frac{\dot{B}_t}{Q_t} = \theta_t \frac{\dot{Q}_t}{Q_t} - \phi(\theta_t - \bar{\theta}) = (1-\alpha)\,\theta_t \frac{\dot{y}_t}{y_t} + \theta_t \frac{\dot{K}_t}{K_t} - \phi(\theta_t - \bar{\theta}). \tag{8}$$

From the government budget constraint (2) we obtain

$$\frac{\dot{B}_t}{Q_t} = \alpha\,(1-\tau)\,\theta_t A y_t^{1-\alpha} - \left(\tau - \frac{y_t^{\alpha}}{A}\right)$$

and, using the IS equilibrium $\dot{K}_t/K_t = A y_t^{1-\alpha} - x_t - y_t$, we extract from (8) the law of motion of y_t,

$$\frac{\dot{y}_t}{y_t} = \frac{1}{1-\alpha}\left\{[\alpha\,(1-\tau) - 1]\, A y_t^{1-\alpha} + y_t + x_t - \frac{1}{\theta_t}\left(\tau - \frac{y_t^{\alpha}}{A}\right) + \phi\left(1 - \frac{\bar{\theta}}{\theta_t}\right)\right\}. \tag{9}$$

System (5)–(7)–(9) is a three-variable three-equation dynamic system. As in FIO, the dynamics of this system depends fundamentally on the dynamics of x_t and y_t, because the dynamics of θ_t is autonomous, and one eigenvalue of the system is simply $-\phi < 0$.

To study transitional dynamics in a more accurate way, let us define the steady state of the model $(\bar{\theta}, \bar{x}, \bar{y})$, where $\bar{\theta}$ is exogenous, and the steady state values $(\bar{x}, \bar{y})$ are the solution of the two-variable system $\dot{x}_t = \dot{y}_t = 0$ (from Equations (7)–(9)). We linearize the system (5)–(7)–(9) in the neighborhood of the steady state, namely

$$\begin{pmatrix} \dot{\theta}_t \\ \dot{x}_t \\ \dot{y}_t \end{pmatrix} = J \begin{pmatrix} \theta_t - \bar{\theta} \\ x_t - \bar{x} \\ y_t - \bar{y} \end{pmatrix}, \tag{10}$$

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with the *Jacobian* matrix defined by[3]

$$J = \begin{bmatrix} -\phi & 0 & 0 \\ 0 & \bar{x} & \rho\bar{x} \\ f(\bar{y},\bar{\theta}) & \dfrac{\bar{y}}{1-\alpha} & \left(\dfrac{\bar{y}}{1-\alpha}\right)\left[\rho + \dfrac{\alpha}{\bar{\theta}A\bar{y}^{1-\alpha}}\right] \end{bmatrix},$$

where

$$f(\bar{y},\bar{\theta}) \equiv \left(\frac{\bar{y}}{1-\alpha}\right)\left(\tau + \phi\bar{\theta} - \frac{\bar{y}^{\alpha}}{A}\right)\frac{1}{\bar{\theta}^2}.$$

We can easily compute

$$\mathrm{Det}(J) = -\frac{\alpha\phi}{\bar{\theta}A\bar{y}^{1-\alpha}}\left(\frac{\bar{x}\bar{y}}{1-\alpha}\right) < 0.$$

Because

$$\mathrm{Tr}(J) = -\phi + \bar{x} + \left(\frac{\bar{y}}{1-\alpha}\right)\left[\rho + \frac{\alpha}{\bar{\theta}A\bar{y}^{1-\alpha}}\right] > -\phi,$$

we obtain only one negative (stable) eigenvalue ($\lambda_1 = -\phi$) and two (unstable) positive eigenvalues (say, $\lambda_2 > 0$ and $\lambda_3 > 0$). The dynamic system (10) is made up of three jumpable variables, but the possible jumps of θ_0 and y_0 at time $t = 0$ are not independent, because the ratio $b_0 \equiv B_0/K_0 = (B_0/Q_0)(Q_0/K_0) = \theta_0 A y_0^{1-\alpha}$ cannot jump (the initial stocks of public debt B_0 and capital K_0 are both predetermined). Therefore, there are only two variables that can jump freely in system (10), and the Blanchard–Kahn conditions ensure that the steady state is well determined (two jumpable variables for two unstable roots).

The Appendix explicitly computes the initial jump of the variables. In Fig. 2, we depict some simulations illustrating the behavior of the variables following a jump in the target $\bar{\theta}$. The initial steady state ($\bar{\theta} = 80\%$) is illustrated by dotted horizontal lines for the four displayed variables. Following an increase in the debt target, the productive public spending ratio (y) initially jumps upward, because borrowing is devoted to productive public spending. As explained before, because $b_0 \equiv B_0/K_0 = A\theta_0 y_0^{1-\alpha}$ is predetermined, this upward jump in y_0 requires a downward jump in the initial public debt ratio θ_0. After this initial jump, the debt ratio converges progressively toward its new steady state level ($\bar{\theta} = 100\%$). Furthermore, the initial jump of y_0 increases economic growth at the impact. Therefore, productive public spending and economic growth increase in the short run, as does the consumption ratio in (7), following an increase in the public debt target. However, in the long run a higher debt ratio generates a crowding-out effect on productive public spending, and thus economic growth, which are lower compared to their initial values. To summarize, raising the long-run public debt target $\bar{\theta}$ increases productive spending and economic growth in the short run, but decreases them in the medium to long run.[4]

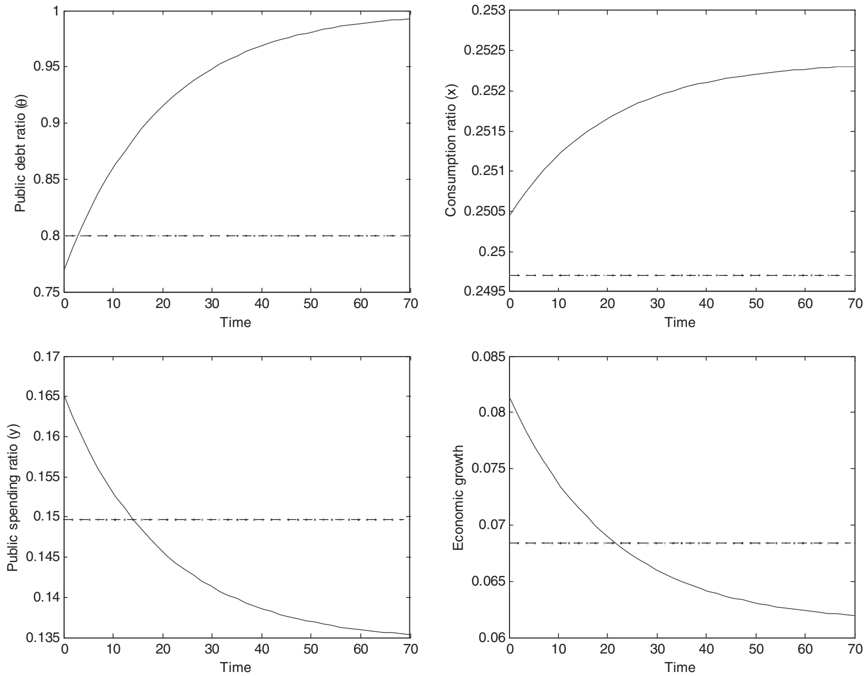

FIGURE 2. Transitional dynamics following an increase in the debt target $\bar{\theta}$ (initial state $\bar{\theta} = 80\%$, final state $\bar{\theta} = 100\%$) for parameter values $A = 1$, $\rho = 0.1$, $\tau = 0.4$, $\alpha = 0.6$, $\phi = 0.05$.

4. CONCLUSION

The goal of this note is to show that some of the main results of FIO (2008) depend on a sharp assumption, namely the definition of the public debt target as a ratio of private capital. If one defines the target in terms of public debt–to–GDP ratio, the model exhibits a unique steady state (multiplicity vanishes), and the transition path to this BGP is well determined (indeterminacy vanishes).

Consequently, there are many arguments for considering a public debt rule as a ratio to GDP rather than as a ratio to private capital. First, from a theoretical standpoint, such a rule avoids multiplicity and possible indeterminacy of the BGP, thus allowing derivation of economic policy implications of the model (such as impulse response functions to changes in parameters). Second, from a practical standpoint, the private capital stock is very difficult to measure (as shown by controversies regarding depreciation and the measurement of net investment), whereas national account systems provide reliable estimations of the GDP. Thus, a GDP-based public debt rule would be not only easier to implement, but also more credible, as the variables involved could be verifiable. Third, from an empirical standpoint, there exist many cases in which governments draw on GDP-based rules to characterize their fiscal stances (such as the upper target of 60% for the public debt–to–GDP ratio in the European Monetary Union, or the Code of Fiscal Stability that limits the public debt–to–GDP ratio to 30% in the United Kingdom).

NOTES

1. τ is the flat-rate tax on the income Q_t, and the interest on the public debt is $r_t B_t$ (r_t is the real interest rate). G_t stands for public investment and C_t stands for consumption.
2. For a detailed analysis, see Minea and Villieu (2010).
3. Note that $[(1-\tau)\alpha - 1] A\bar{y}^{1-\alpha} + \bar{x} + \bar{y} = \rho$ in the steady state.
4. Minea and Villieu (in press) find similar results for a deficit-to-GDP target (instead of a public debt–to–GDP target).

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APPENDIX

Formally, the solution of system (10) can be written as

$$\begin{cases} \theta_t - \bar{\theta} = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + c_3 \exp(\lambda_3 t) \\ x_t - \bar{x} = c_1 v_{12} \exp(\lambda_1 t) + c_2 v_{22} \exp(\lambda_2 t) + c_3 v_{32} \exp(\lambda_3 t), \\ y_t - \bar{y} = c_1 v_{13} \exp(\lambda_1 t) + c_2 v_{23} \exp(\lambda_2 t) + c_3 v_{33} \exp(\lambda_3 t) \end{cases} \tag{A.1}$$

where

$$\vec{V}_i = \begin{pmatrix} 1 \\ v_{i2} \\ v_{i3} \end{pmatrix}$$

is the eigenvector associated with the eigenvalue λ_i. Because $\lambda_2 > 0$ and $\lambda_3 > 0$, a standard transversality condition ensures that $c_2 = c_3 = 0$, so that

$$\begin{cases} \dot{\theta}_t = \lambda_1(\theta_t - \bar{\theta}) \\ \dot{x}_t = \lambda_1 (x_t - \bar{x}), \\ \dot{y}_t = \lambda_1 (y_t - \bar{y}) \end{cases} \tag{A.2}$$

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and the initial jump of the variables can easily been computed from (10) as

$$\begin{cases} x_0 - \bar{x} = \left(\dfrac{\rho \bar{x}}{\bar{x} + \phi}\right) W(\theta_0 - \bar{\theta}) \\ y_0 - \bar{y} = -W(\theta_0 - \bar{\theta}) \\ b_0 = \theta_0 A y_0^{1-\alpha} \end{cases}, \tag{A.3}$$

with predetermined $b_0 \equiv B_0 / K_0$ and where

$$W \equiv \frac{f(\bar{y}, \bar{\theta})}{\left(\dfrac{\bar{y}}{1-\alpha}\right)\left[\rho + \dfrac{\alpha}{\bar{\theta} A \bar{y}^{1-\alpha}} - \left(\dfrac{\rho \bar{x}}{\bar{x} + \phi}\right)\right] + \phi}.$$

Once the initial conditions are known from system (A.3), the transition path of the variables toward the long-run BGP may easily be computed from system (A.2).

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