NON-LINEAR OSCILLATIONS AND BEATS IN THE BETA CANIS MAJORIS STARS

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Notwithstanding a great number of hypotheses, suggested for explaining superpositions of the light- and of the velocity variations of the β Canis Majoris stars, no one of these does it satisfactorily. Possibly it is due to an inadequate elaboration of the non-linearly oscillation theory. Analysis and critical evaluation of the existing hypotheses are given by Mel'nikov and Popov (1970). Our explanation consists in existence of close frequencies corresponding to various oscillation modes which are non-linearly interacting.

Equations of motion of an ideal incompressible fluid under condition of preserving the equilibrium figure symmetry with respect to the equatorial plane (lateral oscillations) have the form (Baranov 1988):

$$\dot{q}_{1,2} = \frac{\partial H}{\partial p_{1,2}}, \quad \dot{p}_{1,2} = -\frac{\partial H}{\partial q_{1,2}}, \tag{1}$$

where variable q_1 and q_2 are in the following way connected with semi-axes a and b of ellipsoid (the variable connected with the remaining semi-axis c of an ellipsoid has been already eliminated from the equation of motion): $\exp(q_{1,2}) = a \pm b$, the Hamiltonian $H = H(q_1, q_2, p_1, p_2, t)$, where t is time. We remind that the variables describing orientation of the figure don't enter into the Hamiltonian since impulses conjugated by it are combinations of invariants of motion: $p_{3,4} = \pm(C \mp L)/2$, where C and L are values of circulation and the moment of motion quantity. Aside from the indicated integrals of motion, the equations (1) allow the energy integral. It is also necessary to take into account the condition of the mass constancy.

Although the system of equations has a canonical form this circumstance is, in fact, not utilized. Nevertheless theorems of existence and uniqueness are almost automatically extended on our problem.

Now substitute into equation (1) expansions of coordinates and impulses from equilibrium state for which are assumed ellipsoids of revolution. The detailed calculations are omitted since they are given by Baranov (1988).

In the case of a stationary state (pure rotation) the equations (1) lead to the known Maclaurin formula.

In linearized approximation the equations (1) are again easily solved.

Finally, we compare the terms, containing ε^2 – the squares of deviations from equilibrium state in both parts of equations (1). Here we have to consider the difference of the true oscillations period from that of the

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linearized oscillations. To this we introduce the new variable τ so that $t = (1 + \varepsilon^2 h + \ldots)\tau$, where quantity h is introduced so that in terms of τ functions describing oscillations have a constant period. Such refinement of the calculation is not influencing all previous calculations.

The equations of motion (1) after some transformations are reduced to the form:

$$Q'' = -\sigma_p^2 Q + \lambda_1 A^2 + \lambda_2, \tag{2}$$

where $A^2 = \tilde{C}(1 + e\sin T)/J$, σ and σ_p are frequencies of lateral and pulsating oscillations, $\tilde{C}$ is an amplitude of lateral oscillations, $T = \sigma\tau$, e is eccentricity of the meridian cross-section of an ellipsoid, $J = 2B_{11} - \Omega^2/2$, in this connection we used in accordance with Chandrasekhar (1969) the notation

$$B_{ijk\ldots} = \alpha_1\alpha_2\alpha_3 \int_0^\infty \frac{ds}{\Delta(\alpha_i^2 + s)(\alpha_j^2 + s)(\alpha_k^2 + s)\ldots},$$

$$(\Delta^2 = (\alpha_1^2 + s)(\alpha_2^2 + s)(\alpha_3^2 + s), \quad \alpha_1 = a, \; \alpha_2 = b, \; \alpha_3 = c).$$

Ω is the angular velocity of a figure rotation. Values of the constants λ_1 and λ_2 are presented by Baranov (1988).

The equation (2) may be solved in a general form

$$Q = \lambda_2/\ \sigma_p^2 + u(\tau) + Q_1 \cos\sigma_p\tau + Q_2 \sin\sigma_p\tau,$$

where $u(\tau)$ is solution of the equation

$$Q'' + \sigma_p^2 Q = \lambda_1 A^2, \tag{3}$$

possessing the same period as the quantity A. Since we consider purely lateral oscillations then assume $Q_1 = Q_2 = 0$. The function $u(\tau)$ is sought in the form

$$u(\tau) = \alpha_1 + \alpha_2 \sin\sigma\tau \tag{4}$$

where α_1 and α_2 are still unknown functions. Substituting expression (4) into equation (3) after transformations which are omitted here, find:

$$\alpha_1 = \frac{\lambda_1\tilde{C}}{\sigma_p^2 J}, \quad \alpha_2 = \frac{\lambda_1\tilde{C}e}{J(\sigma_p^2 - \sigma^2)}.$$

The case $\sigma_p \neq \sigma$ was studied (Baranov 1988).

Phenomenon of various modes oscillations resonance for an ellipsoids of revolution explains a superposition of the light variation harmonics and the radial velocity of the β Canis Majoris stars. Equality of frequencies in the linearized problem is due to common properties of spherically symmetric problems (Bhagavantam and Venkatarayudu 1951, Vilenkin 1965) having as a consequence $2n+1$ - multiple degeneration of frequencies (n is a principal index of a spherical harmonics as an angular part of the equation of oscillations). Various modes originate from one another by turns and a linear superposition. If the linearity is disturbed a superposition is no longer a lawful operation in physical sense. On the other hand, various modes influence each other and loose independence. Therefore, in non-linear analysis there appear typical resonance terms. The situation is in a certain degree analogous to the case of the asteroidal commensurability 1:1 if the asteroid orbit is a quasi-circular but is inclined to Jupiter's orbit. Let it be emphasized that in a non-linear case frequencies of oscillations slightly deviate from predicted values of linear theory at the expense of the final amplitude. If there exist synchronously some oscillation modes then the above mentioned displacement is, generally speaking, different for each of the modes.

Considerations of a qualitative pattern described is connected with one of the interesting and important properties of self-oscillating systems – the phenomenon of a forced synchronization which is sometimes called a caputure. At sufficiently small difference between a proper frequency of the system (frequency of lateral oscillations in this paper) and a frequency of an external force (whose role belongs here to pulsating oscillations) a stable periodic motion acquires the frequency of the latter. If in such manner, the difference $\sigma_p - \sigma$ is sufficiently small, then there takes place a synchronization of frequencies. The main problem of the theory is finding the value of the capture interval i.e. the value of that largest difference of frequencies at which capture still occurs, where as by further increase of difference between frequencies the capture does not take place and there appears a special regime related to presence in the system of quasiperiodic motion with two main frequencies which the one is the frequency of pulsating oscillations and the other – more or less changed frequency of lateral oscillations (regime of beats).

It is clear that possible is the superposition of the greater number of frequencies which in absence of the simple resonance correlations between these leads to the oscillations having still less regularities. Multifrequency oscillations in many aspects remind the stochastic processes in the sense that prediction of the further course of evolution encounters if not with principal then with substantial difficulties. In the above aspect the synchronization phenomenon and that of stochasticity are contrary. Emergence of synchronism leads to suppression of stochasticity and on the contrary development of stochasticity implies the lesser degree of the oscillations synchronism of separate parts of the system.

Emphasize the principal difference of our hypothesis from the point of view expressed elsewhere on an independent origin of various frequencies met in the model of oscillations of the β Canis Majoris stars (Chandrasekhar and Lebovitz 1962). On the contrary in our scheme one considers the oscillation frequencies in the linearized problem which differ in orientation only and therefore synchronous by their character.

We remind that we have considered the homogeneous case only but it clarifies many features of more general and complex structural models since degeneration with respect to symmetry does not depend upon the concrete law of a density change.

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