

# A Synthesis of Two Factor Estimation Methods

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## Abstract

Two-pass cross-sectional regression (TPCSR) is frequently used in estimating factor risk premia. Recent papers argue that the common practice of grouping assets into portfolios to reduce the errors-in-variables (EIV) problem leads to loss of efficiency and masks potential deviations from asset pricing models. One solution that allows the use of individual assets while overcoming the EIV problem is iterated TPCSR (ITPCSR). ITPCSR converges to a fixed point regardless of the initial factors chosen. ITPCSR is intimately linked to the asymptotic principal components (APC) method of estimating factors since the ITPCSR estimates are the APC estimates, up to a rotation.

## I. Introduction

Cross-sectional regression has long been an important tool in evaluating asset pricing models, beginning with the classic work of Lintner (1965), Black, Jensen, and Scholes (1972), and Fama and MacBeth (1973). In the standard cross-sectional regression framework, one begins with a postulated factor model for asset returns. In the first step, time-series regressions of asset returns on risk factors are used to estimate the factor loadings, or betas, of the assets. In the second step, cross-sectional regressions of asset returns on asset betas are used to estimate the factor-mimicking returns and the zero-beta return (Fama (1976)). Time-series means of these portfolio returns are often used to estimate unconditional factor risk premia. We call this procedure two-pass cross-sectional regression (TPCSR). TPCSR has been applied by many researchers to the estimation of factor returns and testing of asset pricing models (some examples, in addition to those above, are Fama and French (1992), Lettau and Ludvigson (2001), and Jagannathan and Wang (2007)). Econometric analyses of the properties of the

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TPCSR estimator include Shanken (1982), (1983), (1992), Jagannathan and Wang (1998), Kan and Zhang (1999), Skoulakis (2008), Kleibergen (2009), Jagannathan, Skoulakis, and Wang (2010), and Lewellen, Nagel, and Shanken (2010).

Since the factor loadings from the first-stage regression are estimated with error, the second-stage risk premium estimates suffer from an errors-in-variables (EIV) bias. A common approach to reducing EIV bias is to use portfolios rather than individual assets in the second-stage regression. Diversification inherent in portfolios reduces the error in the estimated factor loadings and, therefore, reduces the bias in the second-stage regression. An ancillary cost to forming portfolios is reduction in the cross-sectional dispersion in factor loadings, which reduces the precision of the risk premium estimates. Ang, Liu, and Schwarz (2010) show that using estimated factor loadings from portfolios reduces the precision of the estimated risk premia.

The EIV problem motivates the use of portfolios for estimating factor loadings. Alternatives that can take advantage of the efficiency gains of using individual stocks without the associated EIV bias include maximum likelihood estimation (MLE) (as in Ang et al. (2010)) or iterated TPCSR (ITPCSR) (as in Kruskal (1978), Brown and Weinstein (1983), and Shanken (1983)).<sup>1</sup>

We analyze ITPCSR in which each iteration uses the last iteration's factor beta estimates as the new inputs for the cross-sectional regression. We show that ITPCSR has a fixed point. That is, iterating the TPCSR until convergence leads to the same factor estimates, *independent of the choice of the initial factors*. Thus, starting with the Fama and French (1993) 3-factor model, or a model using three macroeconomic series, or three nonsensical factors (like sunspot numbers) leads to the same final estimates (up to a  $k$ -dimensional rotation, where  $k$  is the number of factors).

In statistical factor models, such as asymptotic principal components (APC), factor estimates are not dependent on prespecification of the nature of the economic factors but are statistically derived factor returns that explain the observed common movements across assets (see Connor and Korajczyk (1986), (1988)). Given that ITPCSR requires the researcher to take a stance on the nature of the economic factors and APC uses only the returns data to define the factors, the approaches seem to be quite distinct. We show they are not distinct. The ITPCSR factor estimates are identical to the APC factor estimates (again, up to a  $k$ -dimensional rotation).

This paper provides a new look at factor estimation techniques by synthesizing two existing approaches: iterated two-pass cross-sectional regression and asymptotic principal components. This synthesis has interest in its own right, since it unifies two previously disparate methodologies. It also leads to the derivation of several new results about these estimators and some suggestions for estimation and testing strategies. In the simple constrained case (the only case we treat in detail), the two estimators are identical, and therefore the more difficult

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<sup>1</sup>A separate, but related, issue is choosing the set of test assets used to evaluate a set of prospective factors. Lewellen et al. (2010) suggest expanding test assets beyond the often-used size and book-to-market sorted portfolios, since the cross section of mean returns of these portfolios is strongly related to many alternative factor models.

and time-consuming one (ITPCSR) is redundant. Nonetheless, our equivalence result has a range of implications in the use and potential extensions of TPCSR and statistically based factor estimation methods.

In Section II, we consider the balanced-panel case, in which all assets in the sample have observed returns each period. In Section III, we extend the techniques to the case of an unbalanced panel. Section IV presents our empirical and simulation findings, and Section V concludes.

## II. Estimating Factor-Mimicking Portfolio Returns from a Balanced Panel

We assume that the returns on all securities obey a multibeta asset pricing model, as, for example, the capital asset pricing model (CAPM) or an equilibrium version of the arbitrage pricing theory (APT) (Connor (1984)). Let  $e^n$  be an  $n$ -vector of 1s. Let  $B$  be an  $n \times k$  matrix of factor loadings, or betas. Let  $r_{f,t}$  denote the zero-beta return for period  $t$ ,  $f_t$  denote the  $k$ -vector of zero-mean factor shocks at period  $t$ , and  $\mu_t$  denote the  $k$ -vector of factor risk premia at period  $t$ . Let  $\epsilon_t$  be an  $n$ -vector of idiosyncratic returns, and let  $r_t$  denote the  $n$ -vector of asset returns, which we assume are independent and identically distributed (i.i.d.) across time. The equilibrium asset pricing model implies

$$(1) \quad r_t - e^n r_{f,t} = B(\mu_t + f_t) + \epsilon_t,$$

where  $E[f_t] = 0$  and  $E[\epsilon_t] = 0$ . We strengthen the 0 expectation to hold conditional on  $f_t$ ,  $E[\epsilon_t | f_t] = 0$ . We need no additional structure on returns for most of our results.

### A. Estimation Using ITPCSR

Now we will discuss the estimation of  $B$  and  $\mu_t + f_t$  using TPCSR. Let  $R_t$  denote the  $n$ -vector of excess returns,  $r_t - e^n r_{f,t}$ , and  $R$  denote an  $n \times T$  matrix of realized excess returns on the  $n$  securities for  $T$  time periods. We will assume throughout this section that the panel of observed returns is balanced, that is, there are no missing observations. In matrix notation we can write the security returns as

$$(2) \quad R = BF + \phi,$$

where  $F$  is the  $k \times T$  matrix of the realizations of the factors plus risk premia and  $\phi$  is the  $n \times T$  matrix of idiosyncratic returns. Suppose that we begin with initial estimates of  $F$ , which we will call  $F^0$ . Many models prespecify a particular set of portfolio returns or macroeconomic innovations as the underlying factors. For example, for the CAPM,  $F^0$  is the  $T$ -vector of realized excess returns on the market portfolio proxy; for the Chen, Roll, and Ross (1986) implementation of the APT,  $F^0$  is the  $k \times T$  matrix of macroeconomic innovations (in which case  $F^0$  contains only the factor shocks,  $f_t$ , and not the risk premia,  $\mu_t$ ); and for the Fama and French (1993) model,  $F^0$  is the  $3 \times T$  matrix of excess returns on the market and two zero-investment portfolios: high-minus-low book-to-market

ratio (HML) and small-minus-big market capitalization (SMB). Let  $\widehat{B}_{i,\bullet}^0$  be the coefficients from a time-series ordinary least squares (OLS) of  $R_{i,\bullet}$  (where for a matrix  $X$ ,  $X_{i,\bullet}$  denotes the  $i$ th row of  $X$  and  $X_{\bullet,j}$  denotes the  $j$ th column of  $X$ ) on  $F^0$ , and let  $\widehat{B}^0$  be the  $n \times k$  matrix of estimated coefficients from all  $n$  of these time-series regressions,  $i = 1, \dots, n$ :

$$(3) \quad \widehat{B}^0 = RF^{0'}(F^0F^{0'})^{-1}.$$

Given the estimated matrix of factor betas,  $\widehat{B}^0$ , we can compute the excess returns on factor-mimicking portfolios by second-pass, cross-sectional regressions of returns on the matrix of estimated betas (Fama (1976), ch. 9). Let  $\widehat{F}^1$  denote the  $k \times T$  matrix of these TPCSR estimates of the factors plus risk premia from the cross-sectional OLS regression of excess returns on  $\widehat{B}^0$ :

$$(4) \quad \widehat{F}^1 = (\widehat{B}^{0'}\widehat{B}^0)^{-1}\widehat{B}^{0'}R.$$

Equations (3) and (4) are standard and appear in many variations throughout the empirical asset pricing literature.<sup>2</sup> Most analysts include an intercept in the first-pass regressions, and some include an intercept in the second-pass regression. If the factors are portfolio excess returns (as they are after the first iteration in our setting), Shanken (1983) has noted that including an intercept produces a loss of estimation efficiency if equation (1) is the true return model. Yuan and Savickas (2009) argue that the restricted (no-intercept) estimates yield significantly improved estimates, and Lewellen et al. (2010) argue that imposing the asset pricing restrictions provides significant improvement in our ability to evaluate alternative asset pricing models. We impose the asset pricing restrictions and do not include an intercept.

Our iterative TPCSR procedure uses the “output” of equation (4),  $\widehat{F}^j$ , as input into equation (3) to estimate  $\widehat{B}^j$ . Equations (3) and (4) are reestimated in turn until we reach a fixed point. The final estimates at convergence solve

$$(5) \quad \widehat{F} = (\widehat{B}'\widehat{B})^{-1}\widehat{B}'R$$

and

$$(6) \quad \widehat{B} = R\widehat{F}'(\widehat{F}\widehat{F}')^{-1}.$$

Kruskal (1978), Brown and Weinstein (1983), and Shanken (1983) show that iterating until convergence produces an  $n$ -consistent<sup>3</sup> MLE for  $F$ . Lemma 1 in the Appendix proves the existence of a solution to equations (5) and (6) but also

<sup>2</sup>We use a very simple formulation of the TPCSR algorithm. There are nearly as many variations and enhancements to TPCSR as there are implementations. Some authors use rolling estimates of  $B$ , while some use the full sample to estimate the factor loadings, as we do here. Various authors have suggested generalized least squares (GLS) or weighted least squares in the first pass (time-series) or second pass (cross-sectional) regressions. The correct weighting matrix is never known, but various proxies have been suggested. Since we are using the full cross section of assets, full GLS is not feasible, since it requires the inversion of an  $n \times n$  matrix.

<sup>3</sup>By  $n$ -consistency, we mean that the estimate approaches the true value as the number of cross-sectional observations grows large. See Shanken ((1983), pp. 47–50).

shows that the solution is not unique. The nonuniqueness problem is easily remedied. There exists a unique solution to equations (5) and (6) that is optimal in the sense of minimizing the estimated variance of idiosyncratic returns. We eliminate the indeterminacy by adding this condition to equations (5) and (6):

$$(7) \quad \widehat{F} = \arg \min \text{trace}(\widehat{\phi}'\widehat{\phi}),$$

where  $\widehat{\phi} = R - \widehat{B}\widehat{F}$ . We call the unique matrix that solves equations (5), (6), and (7) the iterative TPCSR estimate.

The estimator defined by the fixed point in equations (5) and (6) is an MLE only under the assumption that the covariance matrix of idiosyncratic returns  $V = \sigma^2 I$ . With cross-sectional heteroskedasticity, the estimator is quasi-MLE. Theoretically, estimates of the factors with more precision could be obtained by applying GLS in the estimation (e.g., Lewellen et al. (2010)) if  $V$  is known. For large  $n > T$ , full GLS is not feasible, since it requires the inversion of  $\widehat{V}$ , which is singular. Imposing some additional structure on  $V$  may allow the use of a restricted version of GLS. In unreported results, we simulate returns following a strict factor model with cross-sectional heteroskedasticity (i.e.,  $V$  equal to a diagonal matrix) and estimate factors using OLS and weighted least squares (WLS). The OLS and WLS estimates converge to the same factor estimates.

### B. Estimation Using Asymptotic Principal Components

Connor and Korajczyk (1986) suggest APC as an alternative method of estimating factor portfolio returns. Let  $\Omega$  denote the  $T \times T$  cross-product matrix of excess returns:

$$(8) \quad \Omega = \frac{R'R}{n}.$$

Let  $\widehat{F}$  denote the  $k \times T$  matrix of the  $k$  eigenvectors of  $\Omega$  corresponding to the largest  $k$  eigenvalues of  $\Omega$ . Connor and Korajczyk (1986) show that  $\widehat{F}$  is an  $n$ -consistent estimate of  $F$ , which they call the asymptotic principal components estimator.

### C. The Equivalence of ITPCSR and APC Estimators

Observe that if  $F$  is replaced by  $LF$  in the factor model of equation (2), where  $L$  is any nonsingular  $k \times k$  matrix, the return model is unaltered if  $B$  is replaced by  $BL^{-1}$ . This is referred to as the “rotational indeterminacy” of factor models. Therefore, to show the equivalence of two estimators, we need only to show that they are nonsingular linear transformations of each other. Iterative TPCSR provides estimates that are identical to APC up to a linear transformation regardless of the initial factors chosen,  $F^0$ .

*Theorem 1.*  $\widehat{F}$  is the iterative TPCSR estimate if and only if  $L\widehat{F}$  is the APC estimate for some nonsingular matrix  $L$ . (The proof is in the Appendix.)

Note that Theorem 1 is an algebraic, rather than probabilistic, relationship between estimators. The two estimates are exactly equal for any sample, and this

equality does not require any assumptions about the true return distribution of the assets. It relies on showing that a fixed point solution to equations (5) and (6) must consist (up to a rotation) of exactly  $k$  eigenvectors of  $\Omega$  and, if the solution also obeys equation (7), these eigenvectors must correspond to the  $k$  largest eigenvalues (which we will call the “first  $k$ ” eigenvectors, implicitly ordering from largest eigenvalue to smallest).

We have not analyzed the statistical properties (e.g., unbiasedness,  $n$ -consistency) of iterative TPCSR. Because of Theorem 1, this is not necessary. All of the properties of APC carry over unchanged to iterative TPCSR. Additionally, note that the MLE result for ITPCSR (Kruskal (1978), Brown and Weinstein (1983), and Shanken (1983)) thereby applies to APC. See Connor and Korajczyk (1986), (1988) for a discussion of the statistical properties of APC under a set of assumptions on the return-generating process.

The convergence properties of iterating TPCSR to the fixed point are interesting. It is easily shown (see the proof of Theorem 1) that any set of  $k$  eigenvectors, not just the first  $k$ , will give a fixed point of equations (5) and (6). This means that if we started the iteration using any set of  $k$  eigenvectors as factors, the repeated regressions would remain at that local minimum. However, from a typical arbitrary starting point that is not a set of eigenvectors, the repeated two-step regression minimizations are drawn toward the joint minimum given by equation (7). We have not proven that the iteration is convergent almost everywhere within the vector space of initial factors, but we have found quick convergence in all our test cases.

Theorem 1 shows the equivalence of APC and iterative TPCSR only up to an arbitrary nonsingular transformation. In some applications, this arbitrary transformation is important and cannot be ignored. Consider, for example, the TPCSR application to macroeconomic factor models (Connor, Goldberg, and Korajczyk (2010), ch. 6). One begins by specifying a set of economic variates whose innovations serve as reasonable proxies for shifts in the consumption and investment opportunity sets (e.g., industrial production, the term structure, corporate bond premia, and long- and short-term inflation). These are used as the initial inputs for the factors. Typically, one cycle of TPCSR is estimated (i.e., equations (3) and (4)) to produce estimates of factor returns. Each factor estimate represents the excess return to a portfolio with unit sensitivity to an inputted economic shock. For example, the first factor captures the excess return from holding industrial production risk, the second factor captures the excess return from holding term structure risk, and so on. This relationship between estimated factors and economic variates is not preserved across rotations. Suppose that instead of stopping after a single iteration, one iterates the TPCSR to convergence. The final factor estimates will be identical to APC except that they will differ by a rotation. In the case where we wish to interpret the risk premia associated with particular economic factors, the iterative TPCSR estimates might be preferred to those from APC. However, the APC factor estimates can be rotated to match the economic factors (e.g., see Connor and Korajczyk (1991)).

One of the advantages of APC is that it can be applied to individual asset returns, since it does not require portfolio grouping. In most applications, TPCSR is known to require portfolio grouping in order to eliminate (or at least mitigate)

the EIV problems from using estimated betas in the cross-sectional regressions (see Fama and MacBeth (1973) for a discussion). However, in the restricted form of ITPCSR that we use (no nonfactor characteristics, no estimation of the zero-beta return, pricing restriction imposed), there is no EIV bias from using estimated betas.<sup>4</sup> With nonfactor characteristics included, or with estimation of the zero-beta return or mispricing terms, portfolio grouping is required for  $n$ -consistency.<sup>5</sup>

Our equivalence result does not invalidate the use of TPCSR in all circumstances. It does provide justification for cross-checking or supplementing empirical analysis based on TPCSR with the comparative use of statistically estimated factors. Valid risk premia estimated by TPCSR should correspond, at least approximately, to a linear rotation of statistically estimated factor risk premia. Statistically estimated factor risk premia, which can be rotated as needed, provide an obvious alternative to TPCSR-estimated risk premia. Whether it is preferable to use TPCSR or statistically estimated risk premia may depend on the particular objective of the empirical analysis. This equivalence also has relevance when deciding upon the number of factors to use in a TPCSR-estimated model. The number of TPCSR-estimated factors should not exceed the number of statistical factors; otherwise, at least one of the TPCSR-estimated factors is not truly providing an independent influence on returns.

### III. Estimating Factor-Mimicking Portfolio Returns from an Unbalanced Panel

It is not unusual for empirical analyses of factor models to estimate factor-mimicking portfolios from balanced panels of data (e.g., Roll and Ross (1980), Connor and Korajczyk (1988), Lehmann and Modest (1988), and Jones (2001)). However, requiring a balanced panel induces survivorship bias into the sample used to construct factor-mimicking portfolios. Connor and Korajczyk (1987) suggest a method of factor estimation with missing data. This procedure estimates  $\Omega^u$  over the observed data (the  $u$  superscript denotes an unbalanced panel). Define  $I_{i,t} = 1$  if the  $\{i, t\}$  element of  $R$  is observed, and  $I_{i,t} = 0$  otherwise, and define the  $\{t, \tau\}$  element of  $\Omega$  as

$$(9) \quad \Omega_{t,\tau}^u = \frac{\sum_{i=1}^n I_{i,t} I_{i,\tau} R_{i,t} R_{i,\tau}}{\sum_{i=1}^n I_{i,t} I_{i,\tau}}$$

Factor-mimicking portfolio returns are estimated from the eigenvectors of the redefined  $\Omega^u$ . We will take a slightly different, quasi-MLE, approach here. Under stronger assumptions than are necessary for consistency of the APC estimator

<sup>4</sup>Although Shanken (1983) does not consider this case explicitly, it is easy to derive from his results. See Shanken ((1983), pp. 51–53) and consider the case in which the cross-sectional regressions do not include an intercept.

<sup>5</sup>See Miller and Scholes ((1972), pp. 60–63).



(i.e.,  $\phi_{i,t} \sim \text{i.i.d. } N(0, \sigma^2)$ ) the MLE of  $\{B, F\}$  minimizes the nonlinear least squares objective function (see Stock and Watson (1998)):

$$(10) \quad A = (nT)^{-1} \sum_{i=1}^n \sum_{t=1}^T I_{i,t} (R_{i,t} - B_{i,\bullet} F_{\bullet,t})^2.$$

The first-order conditions are

$$(11) \quad F_{\bullet,t} = \left( \sum_{t=1}^T I_{i,t} B'_{i,\bullet} B_{i,\bullet} \right)^{-1} \left( \sum_{t=1}^T I_{i,t} B'_{i,\bullet} R_{i,t} \right)$$

and

$$(12) \quad B_{i,\bullet} = \left( \sum_{t=1}^T I_{i,t} R_{i,t} F'_{\bullet,t} \right) \left( \sum_{t=1}^T I_{i,t} F_{\bullet,t} F'_{\bullet,t} \right)^{-1},$$

which correspond to the time-series and cross-sectional regressions (5) and (6) applied to the observed data in the unbalanced panel. We can obtain the MLEs of  $F$  and  $B$  by iterating between the first-order conditions, equations (11) and (12) (Stock and Watson (1998)), which is ITPCSR applied to the observed data. An alternative approach to obtaining the MLEs is to minimize  $A$  using the EM algorithm of Dempster, Laird, and Rubin (1977). Let  $A^*$  denote the negative complete data log-likelihood function

$$(13) \quad A^*(B, F) = (nT)^{-1} \sum_{i=1}^n \sum_{t=1}^T (R_{i,t}^* - B_{i,\bullet} F_{\bullet,t})^2,$$

where  $R_{i,t}^*$  is the latent value of  $R_{i,t}$ . The EM algorithm iteratively maximizes the expected value of the complete data likelihood (minimizes the expected value of  $A^*(B, F)$ ), conditional on the estimates from the prior iteration. Under the assumed error structure, this amounts to minimizing, at iteration  $j$ ,

$$(14) \quad (nT)^{-1} \sum_{i=1}^n \sum_{t=1}^T (R_{i,t}^{*,j-1} - B_{i,\bullet}^j F_{\bullet,t}^j)^2,$$

where  $R_{i,t}^{*,j-1} = R_{i,t}$  if  $I_{i,t} = 1$  and  $R_{i,t}^{*,j-1} = B_{i,\bullet}^{j-1} F_{\bullet,t}^{j-1}$  if  $I_{i,t} = 0$  (see Stock and Watson (1998), p. 11). Thus, the missing data are filled in with the fitted values from the factor model obtained in the previous iteration. By an analysis identical to the balanced panel case, the factor portfolio returns obtained from minimizing expression (14) are equal to (up to a nonsingular rotation  $L$ ) the APC estimate obtained from  $R_{i,t}^{*,j-1}$ . Applying the EM algorithm amounts to an iterative application of APC until convergence.

Analyzing the convergence properties of the unbalanced panel case is a more difficult task than in the balanced case. We are not able to characterize all possible fixed points of either the EM algorithm applied to APC or ITPCSR in this case. We can show only that the EM estimates and ITPCSR estimates share one fixed point.



In simulations, we find that the ITPCSR and iterated APC estimates give identical factor estimates when the simulated data are normally distributed. Using actual return data, we find that the ITPCSR estimates usually converge to the iterated APC estimates, but there is one out of four 10-year subsamples where they do not converge to the same estimates. We discuss these results in the next section.

## IV. Empirical Analysis

We first present results using actual return data from balanced panels. Next, we show results for an unbalanced panel where the return data are from a simulated factor model. Finally, we show results for unbalanced panels of actual return data.

### A. Balanced Panel of Asset Returns

Theorem 1 implies that, for a balanced panel of assets, iterating the two-pass cross-sectional regression converges to the same estimated factor portfolio returns *regardless of the initial prespecified factors*, and those portfolio returns are the APC estimates. This is true even if the prespecified factors have no true population relation with asset returns. To illustrate this point, we compare the ITPCSR factor portfolio estimates to the APC factor portfolio estimates for two different sets of initial factors. For the first initial factors, we chose factors that should have no economic relevance. The first factor is a monthly sunspot number<sup>6</sup> (i.e., the observation for Jan. 2000 is the average of the daily sunspot numbers in that month). The second and third factors are 1-month and 2-month lags of the first factor. For the second initial set, we use the 3 factors from Fama and French (1993), updated on Ken French's Web site.<sup>7</sup> The first factor is the return, in excess of the 1-month Treasury bill return, on the Center for Research in Security Prices (CRSP) value-weighted portfolio. The second factor, HML, is the return on a high book-to-market equity portfolio in excess of the return on a low book-to-market equity portfolio. The third factor, SMB, is the return on a portfolio of small-market-capitalization firms in excess of the return on a portfolio of large-market-capitalization firms.

We show that the ITPCSR factor portfolio estimates converge to a rotation of the APC factor portfolio estimates regardless of whether we start with the non-sensical sunspot factors or the Fama and French (1993) factors.

The correlation between monthly sunspot numbers and the total monthly return on Standard & Poor's (S&P) index (from Morningstar (2010)) is  $-0.03$  over the period from Jan. 1926 to Dec. 2008. Even though the population factor-loading matrix,  $B$ , is probably 0, the sample factor-loading matrix,  $\hat{B}^0$ , will have rank  $k$  with probability 1. This is all that we need to have the ITPCSR converge.

For the balanced panel results, the sample assets are those firms on the monthly CRSP NYSE/AMEX/NASDAQ stock return files that have complete

<sup>6</sup>The data are available at <http://solarscience.msfc.nasa.gov/SunspotCycle.shtml>

<sup>7</sup>The data are available at [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data\\_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html)

return histories over four different 10-year subperiods: Jan. 1969 to Dec. 1978, Jan. 1979 to Dec. 1988, Jan. 1989 to Dec. 1998, and Jan. 1999 to Dec. 2008. When we require a balanced panel, there are 1,290, 2,218, 2,961, and 3,225 firms that meet the criteria in the four subperiods, respectively. Excess stock returns are computed by subtracting the 1-month U.S. Treasury bill return (from Morningstar (2010)).

Let  $F^0$  be the initial  $3 \times 120$  matrix of factors (e.g., 120 time-series observations on the 3 factors). We estimate the factor-loading matrix,  $B^0$ , using equation (3) and a new matrix of factors,  $F^1$ , using equation (4). The process is iterated with

$$(15) \quad \widehat{B}^j = R \widehat{F}^{j-1'} (\widehat{F}^{j-1} \widehat{F}^{j-1'})^{-1}$$

and

$$(16) \quad \widehat{F}^j = (\widehat{B}^{j'} \widehat{B}^j)^{-1} \widehat{B}^{j'} R.$$

After each iteration, we estimate the time-series regression of the  $k$  new factor portfolio estimates on the previous  $k$  factor portfolio estimates:

$$(17) \quad \widehat{F}^j = a + b \widehat{F}^{j-1} + u.$$

With  $k$  factors ( $k = 3$  in our empirical work), this is a system of  $k$  equations. We calculate the  $R^2$  value for each equation and stop the iteration when the smallest of the  $k$   $R^2$  values is greater than 0.99999999. Note that the  $R^2$  measure is used here as a measure of convergence rather than a statistical measure of fit. If the iteratively estimated factors have converged to a fixed point (subject to the rotational indeterminacy), then all of the  $R^2$  values will converge to 1.

To illustrate Theorem 1, we also regress the  $k$  ITPCSR factor estimates on the  $k$  APC factors and, separately, on the  $k$  initial factors:

$$(18) \quad \widehat{F}^j = a + b \widehat{F}^{\text{APC}} + u,$$

$$(19) \quad \widehat{F}^j = a + b \widehat{F}^0 + u.$$

For each iteration, we measure the  $k$   $R^2$  values from equation (18) and plot them against the iteration number in Figure 1 for the initial sunspot factors and in Figure 2 for the initial Fama–French (1993) factors (for equation (18)). Theorem 1 implies that the ITPCSR and APC factor estimates should be the same up to a linear transformation. Thus, the theorem implies that the  $k$   $R^2$  values from equation (18) should converge to 1.0 as we iterate equations (15) and (16). Figures 1 and 2 show that the  $R^2$  values do converge to 1.0, as the theorem predicts. That is, the ITPCSR estimates converge to a rotation of the APC estimates.

Since the initial sunspot factors in this exercise have no real economic content, the  $R^2$  values for equation (19) should essentially converge to 0. For the initial Fama–French (1993) factors, the  $R^2$  values for equation (19) should decline, but not to 0. These  $R^2$  values do behave in this manner and are available from the authors.

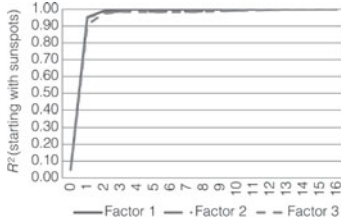
As predicted by Theorem 1, the  $R^2$  values in Figures 1 and 2 converge to 1. Thus, the iterated two-pass cross section regression estimates converge to the APC estimates regardless of whether we begin with the sunspot factors or the Fama–French (1993) factors.

FIGURE 1

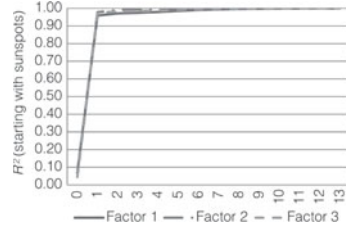
**$R^2$  versus Iteration Number: Balanced Panel Estimation—CRSP Return Data  
(initial factors for the ITPCSR procedure: sunspot factors)**

In Figure 1,  $R^2$  is derived from regressing, for each iteration, ITPCSR factor portfolios on the 3 factors from the one-step APC procedure (plus a constant). The sample consists of 1,290, 2,218, 2,961, and 3,225 firms with complete data available on the CRSP monthly data file for the 10-year periods, Jan. to Dec. 1969–1978 (Graph A), 1979–1988 (Graph B), 1989–1998 (Graph C), and 1999–2008 (Graph D), respectively.

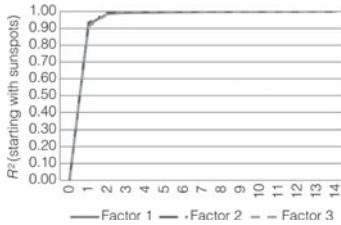
Graph A. 1969–1978



Graph B. 1979–1988



Graph C. 1989–1998



Graph D. 1999–2008

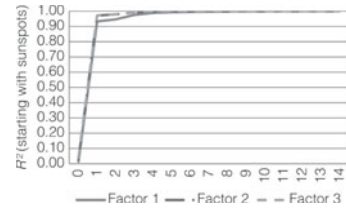
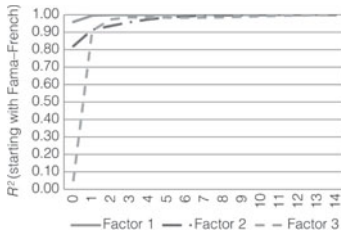


FIGURE 2

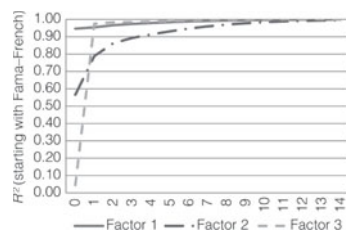
**$R^2$  versus Iteration Number: Balanced Panel Estimation—CRSP Return Data  
(initial factors for the ITPCSR procedure: Fama–French factors)**

In Figure 2,  $R^2$  is derived from regressing, for each iteration, ITPCSR factor portfolios on the 3 factors from the one-step APC procedure (plus a constant). The sample consists of 1,290, 2,218, 2,961, and 3,225 firms with complete data available on the CRSP monthly data file for the 10-year periods, Jan. to Dec. 1969–1978 (Graph A), 1979–1988 (Graph B), 1989–1998 (Graph C), and 1999–2008 (Graph D), respectively.

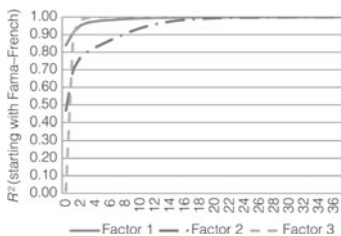
Graph A. 1969–1978



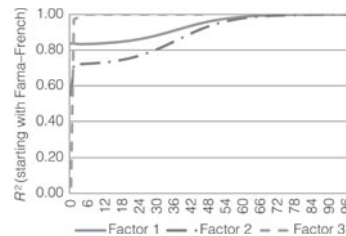
Graph B. 1979–1988



Graph C. 1989–1998



Graph D. 1999–2008



## B. Unbalanced Panel of Simulated Asset Returns

We construct simulated factor returns that have the same pattern of missing data as the actual CRSP return data over one 10-year period. The full sample consists of 11,641 firms, but only 2,642 have complete returns over the 10-year period. In an average month, there are 7,430 firms with return data. We simulate return data for the 11,641 firms from a 3-factor model using a return-generating process similar to that in Connor and Korajczyk ((1993), Sec. III.B). The main difference is that we have doubled the idiosyncratic standard deviation. This four-fold increase in idiosyncratic variance should make it more difficult for the routines to extract the true factors. Factor realizations and idiosyncratic return realization are normally distributed in the simulation.

As in the analysis of the balanced panel above, we start the ITPCSR with two sets of prespecified factors: the sunspot factors and the Fama–French (1993) factors. The initial APC set of factor estimates is obtained by calculating the eigenvectors of  $\Omega^u$  in equation (9) as in Connor and Korajczyk (1987). The APC factor estimates are obtained by the iterative procedure of using the fitted factor model from the APC iteration  $j - 1$  to “fill in” the missing data for iteration  $j$ .

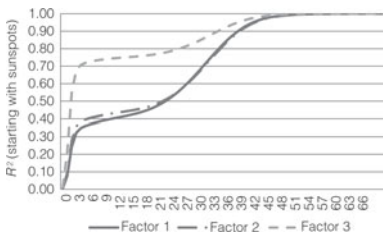
For both the ITPCSR estimates and the APC estimates, the convergence criterion is that the minimal  $R^2$  value from the multivariate regression of estimated factor returns on the factors from the previous iteration (equation (17)) is greater than or equal to 0.9999999999. Figure 3 shows that, as predicted, the ITPCSR estimates converge to the estimates from the iterated APC procedure, up to a linear transformation,  $L$ . This convergence is independent of the initial choice of either the sunspot or Fama–French (1993) factors.

FIGURE 3

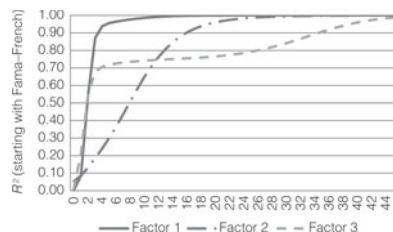
### $R^2$ versus Iteration Number: Unbalanced Panel Estimation–Simulated Return Data

In Figure 3,  $R^2$  is derived from regressing, for each iteration, ITPCSR factor portfolios on the 3 factors from the one-step APC procedure (plus a constant). The initial factors for the ITPCSR procedure are the sunspot (Graph A) or Fama–French (1993) (Graph B) factors. The sample consists of simulated time series from a 3-factor model for 11,641 firms. The pattern of missing data for the simulation is the same as the pattern of missing data in the CRSP sample over a 10-year sample.

Graph A. Sunspots as Initial Factors



Graph B. Fama–French as Initial Factors



## C. Unbalanced Panel of Actual Asset Returns

Actual return data deviate from the normally distributed world assumed in the simulations above. In the case of an unbalanced panel of asset returns, there is some evidence that the ITPCSR estimates, on occasion, are either slow to converge or may converge to a local maximum of the objective function,  $\Lambda$ .

For the unbalanced panel results, the sample assets are those firms on the monthly CRSP NYSE/AMEX/NASDAQ stock return files that have at least 24 months of return histories over the respective 10-year subperiods: Jan. 1969–Dec. 1978, Jan. 1979–Dec. 1988, Jan. 1989–Dec. 1998, and Jan. 1999–Dec. 2008. There are 5,680, 8,884, 11,710, and 10,221 firms that meet the criteria in the 4 subperiods, respectively. Excess stock returns are computed by subtracting the 1-month U.S. Treasury bill return.

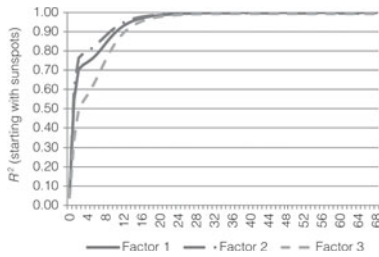
We estimate factor-mimicking portfolios for the full sample of CRSP firms for the ITPCSR and iterated APC approaches. Figures 4 and 5 show how the ITPCSR estimates converge to the iterated APC algorithm estimates. For three of the four 10-year subperiods, the ITPCSR estimates converge to the iterated APC estimates. However, for the 1989–1998 subperiod, the ITPCSR estimates do not converge to the iterated APC estimates.

FIGURE 4

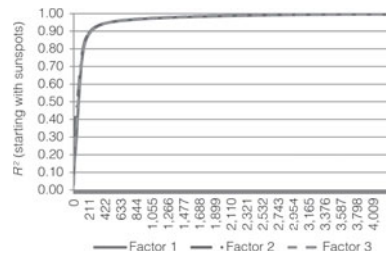
$R^2$  versus Iteration Number: Unbalanced Panel Estimation—CRSP Return Data (initial factors for the ITPCSR procedure: sunspot factors)

In Figure 4,  $R^2$  is derived from regressing, for each iteration, ITPCSR factor portfolios on the 3 factors from the one-step APC procedure (plus a constant). The sample consists of 5,680, 8,884, 11,710, and 10,221 firms available on the CRSP monthly data file for the 10-year periods, Jan. to Dec. 1969–1978 (Graph A), 1979–1988 (Graph B), 1989–1998 (Graph C), and 1999–2008 (Graph D), respectively.

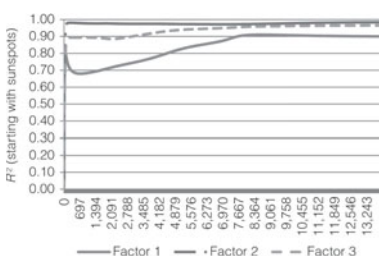
Graph A. 1969–1978



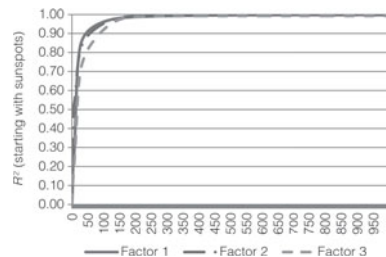
Graph B. 1979–1988



Graph C. 1989–1998



Graph D. 1999–2008



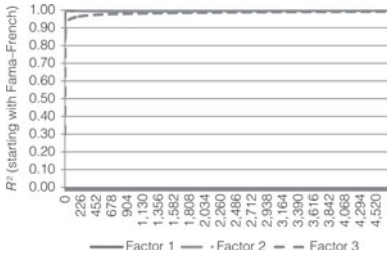
For the nonconverging subperiod, we calculate the least squares objective function,  $\Lambda$ , from equation (10) for the two ITPCSR estimates (starting with the sunspot “factors” and the Fama–French (1993) factors) and the iterative APC estimate for the 1989–1998 subperiod. The iterated APC estimate has the smallest value of  $\Lambda$ , equal to 23,139. The values are 23,230 for the ITPCSR estimate starting with the sunspot factors and 23,179 for the ITPCSR estimate starting with the Fama–French (1993) factors. All three values for  $\Lambda$  are very close (with the largest difference being 0.39%). This seems to indicate that in this unbalanced panel case,

FIGURE 5

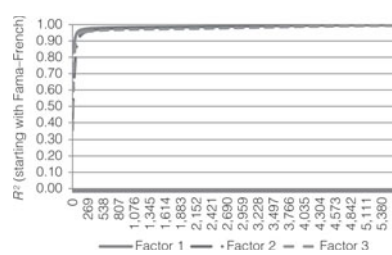
$R^2$  versus Iteration Number: Unbalanced Panel Estimation—CRSP Return Data  
(initial factors for the ITPCSR procedure: Fama–French factors)

In Figure 5,  $R^2$  is derived from regressing, for each iteration, ITPCSR factor portfolios on the 3 factors from the one-step APC procedure (plus a constant). The sample consists of 5,680, 8,884, 11,710, and 10,221 firms available on the CRSP monthly data file for the 10-year periods, Jan. to Dec. 1969–1978 (Graph A), 1979–1988 (Graph B), 1989–1998 (Graph C), and 1999–2008 (Graph D), respectively.

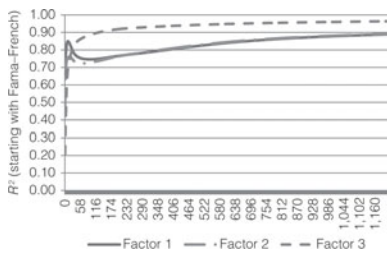
Graph A. 1969–1978



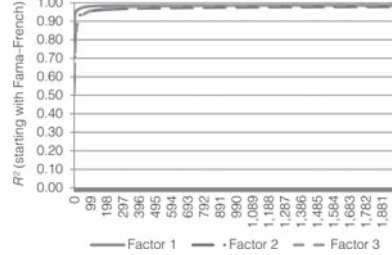
Graph B. 1979–1988



Graph C. 1989–1998



Graph D. 1999–2008



the two estimation algorithms are finding slightly different local maxima for the 1989–1998 period.

D. Comparison of Factor Risk Premia

Due to the rotational indeterminacy in the definition of the factors, individual factor risk premia are not comparable across estimation methods or across subperiods. The vector of factor returns can be tested against the null hypothesis that all three risk premia equal 0; this null hypothesis is invariant to a linear rotation. A  $\chi^2$  test for nonzero means of the vector of risk premia is invariant to factor rotations and simple to aggregate across subperiods. Let  $\hat{\mu}$  denote the 3-vector of sample averages of the estimated factor returns and  $\hat{C}$  the sample covariance matrix of the factor returns over a  $T = 120$  month subperiod. Under the null hypothesis that the true vector of means equals 0, and weak conditions on the time-series process for factor returns:

$$\sqrt{T}\hat{\mu}'\hat{C}^{-1}\hat{\mu} \overset{A}{\sim} \chi^2(3),$$

where  $\overset{A}{\sim}$  denotes the asymptotic distribution for large  $T$ . Note that the test is invariant to rotations of the factors. Since the subperiod returns are independent, the sum of the four  $\chi^2(3)$  subperiod statistics has a  $\chi^2(12)$  distribution.

Table 1 presents the subperiod and aggregate results for four sets of factor return estimates. We use the three sets of factor return estimates illustrated in Figures 4 and 5. We also show the results using “one-step” two-pass cross-sectional regression, that is, the factor returns from the first completed step from the iterated TPCSR estimates, starting with Fama–French (1993) portfolios as initial factors.

TABLE 1  
Tests for Significance of Factor Risk Premia

Factor Estimation Method	Subperiod	$\chi^2(3)$	<i>p</i> -Value	$\chi^2(12)$	<i>p</i> -Value
Asymptotic principal components	1969–1978	1.54	0.67		
	1979–1988	2.73	0.43		
	1989–1998	4.11	0.25		
	1999–2008	1.63	0.65	10.01	0.62
Iterated two-pass cross-sectional regression with Fama–French portfolios as initial factors	1969–1978	1.54	0.67		
	1979–1988	2.75	0.43		
	1989–1998	4.30	0.23		
	1999–2008	1.73	0.63	10.31	0.59
Iterated two-pass cross-sectional regression with sunspot time series as initial factors	1969–1978	1.54	0.67		
	1979–1988	2.76	0.43		
	1989–1998	4.83	0.18		
	1999–2008	1.69	0.64	10.82	0.54
Single-Step two-pass cross-sectional regression with Fama–French portfolios as initial factors	1969–1978	0.72	0.87		
	1979–1988	6.51	0.09		
	1989–1998	13.67	0.00		
	1999–2008	4.30	0.23	25.20	0.01

As predicted by the theory, the three sets of factor return estimates give very similar test results in all three subperiods (the unbalanced panel data prevent the results from being identical across estimation methods). The one-step TPCSR results differ; these estimates have not converged to a linear rotation of the same set of underlying factors.

## V. Conclusion

Two-pass cross-sectional regression (TPCSR) and asymptotic principal components (APC) are two possible methodologies for estimating factor portfolio returns and risk premia in a beta pricing model of asset returns. In this paper, we study an iterative version of TPCSR that overcomes the EIV problem and can be applied to individual firms rather than sets of portfolios (Kruskal (1978), Brown and Weinstein (1983), and Shanken (1983)). This leads to increased precision of the estimator of factor-mimicking portfolio returns (Ang et al. (2010)).

We show an equivalence between the factors estimated by APC and those from ITPCSR when the restrictions of the asset pricing model are imposed. For balanced panels, we show that the ITPCSR factor estimates converge to the APC factor estimates even if the choice of initial prespecified factors makes no economic sense. Here we use sunspot numbers as initial factors as well as the Fama–French (1993) factors, MKT, HML, and SMB. For unbalanced panels that have normally distributed returns, the ITPCSR estimates converge to an iterative version of the APC procedure. For unbalanced panels with actual, nonnormal



data, the ITPCSR estimates converge to an iterative version of APC in three of four 10-year subperiods. In one 10-year subperiod, the ITPCSR estimates are highly, but not perfectly, correlated with the iterative APC estimates. We find evidence that in the unbalanced case, the various estimates may find local maxima of the objective function.

### Appendix. Proof of Theorem 1

*Lemma 1.* Let  $H$  be any matrix of  $k$  eigenvectors of  $R'R$  and  $L$  be any nonsingular  $k \times k$  matrix. Then  $\hat{F} = LH$  is a solution to equations (5) and (6). Conversely, let  $\hat{F}$  denote any solution to equations (5) and (6). Then  $\hat{F} = LH$ , where  $H$  is a matrix of  $k$  eigenvectors of  $R'R$  and  $L$  is a nonsingular  $k \times k$  matrix.

*Proof of Lemma 1.* Assume  $H$  is a set of  $k$  eigenvectors of  $R'R$  and  $L$  is a nonsingular matrix. We must show that  $LH$  solves equations (5) and (6). Substituting equation (6) into equation (5) gives

$$(A-1) \quad \hat{F} = \left[ (LHH'L')^{-1}LHR'RH'L'(LHH'L')^{-1} \right]^{-1} (LHH'L')^{-1}LHR'R.$$

Using the rule that  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$  to eliminate some  $L$ s gives

$$(A-2) \quad \hat{F} = LHH'[HR'RH']^{-1}HR'R.$$

Since  $H$  is a set of eigenvectors of  $R'R$ , we have  $HR'R = AH$ , where  $A$  is a diagonal matrix. Substituting into equation (A-2) gives

$$(A-3) \quad \hat{F} = LHH'[LHH']^{-1}AH = LH,$$

which proves the result.

Now, assume that  $\hat{F}$  is a solution to equations (5) and (6). We will show that  $\hat{F} = LH$ , where  $H$  is a set of eigenvectors and  $L$  is a nonsingular  $k \times k$  matrix. Combining equations (5) and (6) gives

$$(A-4) \quad \hat{F} = \left[ (\hat{F}\hat{F}')^{-1}\hat{F}R'R\hat{F}'(\hat{F}\hat{F}')^{-1} \right]^{-1} (\hat{F}\hat{F}')^{-1}\hat{F}R'R.$$

Note that if  $\hat{F}$  solves equation (A-4) then so does  $F^* = L\hat{F}$  for any nonsingular  $k \times k$  matrix  $L$ . For simplicity, first consider the case  $(\hat{F}\hat{F}') = I$ . Simplifying equation (A-4) gives

$$(A-5) \quad \hat{F} = [\hat{F}R'R\hat{F}']^{-1}\hat{F}R'R = M\hat{F}R'R,$$

where  $M = (\hat{F}R'R\hat{F}')^{-1}$ . Since  $M$  is a real symmetric matrix, we can decompose it as  $M = P^{-1}AP$ , where  $A$  is  $T \times T$  diagonal matrix (Searle (1982), p. 200). Substituting into equation (A-5) gives

$$(A-6) \quad \hat{F} = P^{-1}AP\hat{F}R'R.$$

Multiplying both sides by  $A^{-1}P$  gives

$$(A-7) \quad A^{-1}P\hat{F} = P\hat{F}R'R,$$

which implies that  $H = P\hat{F}$  is a set of  $k$  eigenvectors of  $R'R$ . Hence,  $\hat{F} = LH$ , where  $L = P^{-1}$ .

For the case  $(\widehat{F}\widehat{F}') \neq I$ , construct  $F^* = (\widehat{F}\widehat{F}')^{-1/2}\widehat{F}$  and note that  $(F^*F^{*'}) = I$ . Now, applying the steps above to  $F^*$  proves that  $F^*$  is a linear transformation of a set of  $k$  eigenvectors of  $R'R$  and therefore so is  $\widehat{F} = (\widehat{F}\widehat{F}')^{1/2}F^*$ .  $\square$

*Proof of Theorem 1.* Note that, since  $\Omega = R'R/n$ , the eigenvectors of  $\Omega$  are equal to the eigenvectors of  $R'R$ , since the eigenvectors of a matrix are unaffected by nonzero scalar multiplication. From Lemma 1,  $\widehat{F}$  solves equations (5) and (6) if and only if it consists of a linear transformation of  $k$  eigenvectors of  $R'R$ . The eigenvectors of  $\Omega$  are identical to the eigenvectors of  $R'R$ ; hence, they solve equations (5) and (6). We must show that such an  $\widehat{F}$  obeys equation (7) if and only if it consists of the “first”  $k$  eigenvectors, defined as those associated with the  $k$  largest eigenvalues.

From equation (2), we can write

$$(A-8) \quad R = \widehat{B}\widehat{F} + \widehat{\phi},$$

where  $\widehat{B}\widehat{F}$  and  $\widehat{\phi}$  are orthogonal. Therefore,

$$(A-9) \quad \text{trace}(R'R) = \text{trace}(\widehat{F}'\widehat{B}'\widehat{B}\widehat{F}) + \text{trace}(\widehat{\phi}'\widehat{\phi}).$$

From equation (6) we have

$$(A-10) \quad \begin{aligned} \text{trace}(\widehat{F}'\widehat{B}'\widehat{B}\widehat{F}) &= \text{trace}(\widehat{F}'(\widehat{F}\widehat{F}')^{-1}\widehat{F}R'R\widehat{F}'(\widehat{F}\widehat{F}')^{-1}\widehat{F}) \\ &= \text{trace}(\widehat{F}'(\widehat{F}\widehat{F}')^{-1}\Lambda\widehat{F}), \end{aligned}$$

where  $\Lambda$  is the diagonal matrix of eigenvalues associated with  $\widehat{F}$ . Since  $\text{trace}(XY) = \text{trace}(YX)$  for any conformable matrices  $X$  and  $Y$ ,

$$(A-11) \quad \text{trace}(\widehat{F}'\widehat{B}'\widehat{B}\widehat{F}) = \text{trace}(\widehat{F}\widehat{F}'(\widehat{F}\widehat{F}')^{-1}\Lambda) = \text{trace}(\Lambda).$$

Since  $\text{trace}(\Lambda)$  is the sum of the  $k$  eigenvalues, it is maximized when  $\widehat{F}$  is associated with the  $k$  largest eigenvalues. From equation (A-9), since  $\text{trace}(R'R)$  is fixed, minimizing  $\text{trace}(\widehat{\phi}'\widehat{\phi})$  is equivalent to maximizing  $\text{trace}(\Lambda)$ . Since the eigenvectors of  $R'R$  are equal to the eigenvectors of  $\Omega$ , and their eigenvalues are proportional, the APC factor estimates solve equations (5), (6), and (7). Therefore, the APC estimates are the same as the ITPCSR estimates (up to a linear transformation).  $\square$

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