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Abstract

We study the McKay correspondence for representations of the cyclic group of order p in characteristic p. The main tool is the motivic integration generalized to quotient stacks associated to representations. Our version of the change of variables formula leads to an explicit computation of the stringy invariant of the quotient variety. A consequence is that a crepant resolution of the quotient variety (if any) has topological Euler characteristic p as in the tame case. Also, we link a crepant resolution with a count of Artin–Schreier extensions of the power series field with respect to weights determined by ramification jumps and the representation.

1. Introduction

The McKay correspondence generally means, for a finite subgroup G of $SL_d(\mathbb{C})$, an equality between an invariant of the representation $G \curvearrowright \mathbb{C}^d$ and an invariant of a crepant resolution of the quotient variety $\mathbb{C}^d/G$ (see [Rei02]). The aim of this paper is to take a step toward the *wild McKay correspondence*, that is, the McKay correspondence for a finite subgroup $G \subset SL_d(k)$ such that the characteristic of a field k divides the order of G. We will study the simplest possible case where G is the cyclic group of prime order. Gonzalez-Springberg and Verdier [GV85] and Schröer [Sch09] also worked on the McKay correspondence in the wild case, but on different aspects.

Our McKay correspondence will be formulated in a similar way to that of Batyrev [Bat99], which is an equality of an orbifold invariant of the G-variety $\mathbb{C}^d$ and a stringy invariant of the quotient variety. Denef and Loeser [DL02] gave an alternative proof, using the motivic integration and giving a more direct link between the invariants. We follow this approach with a stacky language by the author [Yas04, Yas06].

Let k be a perfect field of characteristic $p > 0$ and G the cyclic group of order p. In this paper, we will study the McKay correspondence for a finite-dimensional G-representation V. If for $1 \leqslant i \leqslant p$, V_i denotes the indecomposable G-representation of dimension i, then V is decomposed as $V = \bigoplus_{\lambda=1}^{l} V_{d_\lambda}$, $1 \leqslant d_\lambda \leqslant p$. We define a numerical invariant D_V of V by

$$D_V := \sum_{\lambda=1}^{l} \frac{(d_\lambda - 1)d_\lambda}{2}.$$

When $D_V \geqslant p$, a *stringy motivic invariant* of the quotient variety $X := V/G$, denoted $M_{\mathrm{st}}(X)$, will be defined in the same way as in [DL02] to be some motivic integral over the arc space of X.

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If X admits a resolution of singularities with simple normal crossing relative canonical divisor, then the invariant coincides with the one defined with resolution data as in [Bat98, Bat99]. The following is our main result. For a positive integer j with $p \nmid j$ or for $j = 0$, we put

$$\mathrm{sht}_V(j) := \sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} \left\lfloor \frac{ij}{p} \right\rfloor.$$

Theorem 1.1 (Proposition 6.9 and Corollary 6.19). *If $D_V \geqslant p$, then*

$$M_{\mathrm{st}}(X) = \mathbb{L}^d + \frac{\mathbb{L}^{l-1}(\mathbb{L}-1)(\sum_{s=1}^{p-1} \mathbb{L}^{s-\mathrm{sht}_V(s)})}{1 - \mathbb{L}^{p-1-D_V}}.$$

When X has a crepant resolution $Y \longrightarrow X$, the theorem shows that $D_V = p$ and

$$[Y] = \mathbb{L}^d + \mathbb{L}^l \sum_{s=1}^{p-1} \mathbb{L}^{s-\mathrm{sht}_V(s)}.$$

In particular, Y has topological Euler characteristic p, which is, in characteristic zero, conjectured by Reid [Rei02] and proved by Batyrev [Bat99]. We will also define the *stringy motivic invariant* of the 'projectivization' $[(X\backslash\{0\})/\mathbb{G}_m]$ of X and prove that it satisfies the Poincaré duality, which was originally proved by Batyrev [Bat98] for $\mathbb{Q}$-Gorenstein projective varieties with log terminal singularities in characteristic zero.

The proof of Theorem 1.1 is based on the motivic integration suitably generalized to the quotient stack $\mathcal{X} := [V/G]$ and the change of variables formula for the morphism $\mathcal{X} \longrightarrow X$. Following [Yas04, Yas06], we will define *twisted arcs* of $\mathcal{X}$ and develop the motivic integration over the space of them. A twist of a twisted arc comes from an Artin–Schreier extension of $k((t))$, that is, a Galois extension of the power series field $k((t))$ with Galois group $G \cong \mathbb{Z}/(p)$. Not only do there exist infinitely many distinct twists, but also they are parameterized by an infinite-dimensional space. This contrasts strikingly the situation in the tame case, where we have only finitely many twists.

Let $J_\infty X$ be the arc space of X and $\mathcal{J}_\infty \mathcal{X}$ the space of twisted arcs of $\mathcal{X}$. Then the map $\phi : \mathcal{X} \longrightarrow X$ induces a map $\phi_\infty : \mathcal{J}_\infty \mathcal{X} \longrightarrow J_\infty X$, which is bijective outside measure-zero subsets. The *change of variables formula* for ϕ_∞ will be formulated as

$$\int_A \mathbb{L}^F \, d\mu_X = \int_{\phi_\infty^{-1}(A)} \mathbb{L}^{F \circ \phi_\infty - \mathrm{ord}\, \mathrm{Jac}_\phi - \mathfrak{s}_\mathcal{X}} \, d\mu_\mathcal{X}$$

(for details, see Theorem 5.20). Here for $\gamma \in \mathcal{J}_\infty \mathcal{X}$, if j is the ramification jump of the associated Artin–Schreier extension of $k((t))$, then $\mathfrak{s}_\mathcal{X}(\gamma) := \mathrm{sht}_V(j)$. An interesting consequence of Theorem 1.1 is the following: suppose that k is a finite field, and that $Y \longrightarrow X$ is a crepant resolution. For each finite extension $\mathbb{F}_q/k$ with q a power of p, let $N_{q,j}$ be the number of Artin–Schreier extensions of $\mathbb{F}_q((t))$ with ramification jump j. Let $E_0 \subset Y$ be the preimage of the origin $0 \in X$. Then we have the following equality (Corollary 6.28, cf. [Ros07]):

$$\sharp E_0(\mathbb{F}_q) = 1 + \frac{p-1}{p} \sum_{j>0,\, p \nmid j} \frac{N_{q,j}}{q^{\mathrm{sht}_V(j)}}.$$

This result would provide new insight into the link between the singularity theory and the Galois theory of local fields.

The paper is organized as follows. In § 2 we construct the moduli space of G-covers of the formal disk and study its structure. In § 3 we proceed with the study of twisted arcs and jets,

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and their moduli spaces. Section 4 is devoted to introducing the motivic integration over the space of twisted arcs. Section 5 contains the proof of the change of variables formula, which is the technical heart of the paper. In § 6 we define stringy invariants and conclude various versions of the McKay correspondence from the change of variables formula. Finally, in § 7, we make some remarks on future problems.

1.1 Convention

Throughout the paper, we work over a perfect field k of characteristic $p > 0$. A *k-field* means a field containing k. We set $G = \mathbb{Z}/(p)$ and denote by σ the residue class of 1 in G.

2. G-covers of the formal disk

The main objective of this section is to construct the moduli spaces of G-covers of the formal disk $D := \operatorname{Spec} k[[t]]$. This will be used in the next section in the construction of the moduli space of twisted arcs.

2.1 G-covers of the formal punctured disk

Let $D^* := \operatorname{Spec} k((t))$ be the formal punctured disk. We will first examine the set of étale G-covers of D^*, denoted by $G\text{-Cov}(D^*)$. It is classified by the étale cohomology group $H^1(D^*, G)$ (see [Mil80, p. 127]). Then from the Artin–Schreier sequence of étale sheaves,

$$0 \longrightarrow G \longrightarrow \mathcal{O}_{D^*} \xrightarrow{\wp: f \longmapsto f^p - f} \mathcal{O}_{D^*} \longrightarrow 0,$$

we have

$$H^1(D^*, G) = \operatorname{coker}(H^0(\mathcal{O}_{D^*}) \xrightarrow{\wp} H^0(\mathcal{O}_{D^*})).$$

Consequently, we have the one-to-one correspondence

$$G\text{-Cov}(D^*) \longleftrightarrow \frac{k((t))}{\wp(k((t)))}.$$

More explicitly, this correspondence is described as follows: for a k-algebra A and $f \in A$, we define a ring extension

$$A[\wp^{-1} f] := \frac{A[u]}{(u^p - u + f)}$$

endowed with the G-action by $\sigma(u) = u + 1$. Then the G-cover corresponding to the class of $f \in k((t))$ is

$$E_f^* := \operatorname{Spec} k((t))[\wp^{-1} f].$$

We next describe the set $k((t))/\wp(k((t)))$. Given $f \in k((t))$, we denote by f_i the coefficient of t^i in f so that $f = \sum_{i \in \mathbb{Z}} f_i t^i$ with $f_i = 0$ for $i \ll 0$.

Lemma 2.1. *We have $\wp(k[[t]]) = \wp(k) \cdot 1 \oplus k[[t]] \cdot t$. In particular, if k is algebraically closed, then $k[[t]] = \wp(k[[t]])$.*

Proof. For $f \in k[[t]]$, we have

$$\wp(f) = \sum_{p \nmid i} -f_i t^i + \sum_{p \mid i} (f_{i/p}^p - f_i) t^i.$$

Hence $\wp(k[[t]]) \subset \wp(k) \cdot 1 \oplus k[[t]] \cdot t$. For the converse, let $g \in \wp(k) \cdot 1 \oplus k[[t]] \cdot t$. Then we can inductively choose the coefficients f_i of f such that $\wp(f) = g$ as follows. First, take f_0 in $\wp^{-1} g_0$.

If we have chosen $f_0, f_1, \dots, f_{i-1}$ such that $\wp(f) \equiv g \mod t^i$, then we set either $f_i := -g_i$ or $f_i := f_{i/p}^p - g_i$ depending on whether p divides i. This shows the first assertion. The second assertion follows from the fact that if k is algebraically closed, then $\wp(k) = k$. □

Notation 2.2. We put $\mathbb{N}' := \{j \in \mathbb{Z} \mid j > 0, p \nmid j\}$ and $\mathbb{N}'_0 := \mathbb{N}' \cup \{0\}$.

Lemma 2.3. *For $f \in k((t))$, there exists $g = \sum_{i \in \mathbb{N}'_0} g_{-i} t^{-i} \in k[t^{-1}] \subset k((t))$ such that $f - g \in \wp(k((t)))$. Moreover, such g_i, $i < 0$, are uniquely determined and the class of g_0 in $k/\wp(k)$ is also uniquely determined.*

Proof. From the previous lemma, we may eliminate the terms of positive degrees in f and assume that $f_i = 0$ for $i > 0$. Let pi $(i > 0)$ be the largest multiple of p such that $f_{-pi} \neq 0$ if any. Then replacing f with $f - \wp(f_{-pi}^{1/p} t^{-i})$, we get that $f_{-pi} = 0$ without changing f_i for $i < -pi$. (Since k is perfect, $f_{-pi}^{1/p}$ exists in k.) Iterating this procedure, we eventually get a polynomial g of the desired form.

For the uniqueness, let $g' \in k[t^{-1}]$ have the same property. From the conditions on g and g', we have either $h := g - g' \in k$ or $-\operatorname{ord} h \in \mathbb{N}'$. However, we have $h \in \wp(k((t)))$ and every element of $\wp(k((t)))$ of negative order has order $-pn$ with n a positive integer. Thus we conclude $h \in \wp(k)$. This shows the uniqueness of the lemma. □

Definition 2.4. Let A be a k-algebra. A *representative polynomial over A* is a Laurent polynomial of the form

$$f = \sum_{i \in \mathbb{N}'} f_{-i} t^{-i} \in A[t^{-1}], \quad f_{-i} \in A.$$

We note that there is no constant term. We denote by RP_A the set of representative polynomials over A.

Lemma 2.3 shows the following proposition.

Proposition 2.5. *We have a one-to-one correspondence,*

$$G\text{-Cov}(D^*) \longleftrightarrow \mathrm{RP}_k \times \frac{k}{\wp(k)}.$$

In particular, if k is algebraically closed, then

$$G\text{-Cov}(D^*) \longleftrightarrow \mathrm{RP}_k.$$

Definition 2.6. Let $\bar{k}$ be an algebraic closure of k. We say that $E_1^*, E_2^* \in G\text{-Cov}(D^*)$ are *geometrically equivalent* and write $E_1^* \sim_{\mathrm{geo}} E_2^*$ if their complete base changes $E_1^* \hat{\times}_k \bar{k}$ and $E_2^* \hat{\times}_k \bar{k}$ are isomorphic G-covers of $D^* \hat{\times}_k \bar{k} = \operatorname{Spec} \bar{k}((t))$.

Obviously

$$G\text{-Cov}(D^*)/{\sim_{\mathrm{geo}}} \longleftrightarrow \mathrm{RP}_{\bar{k}}.$$

If k is a finite field, then $k/\wp(k)$ has p elements. Hence the quotient map

$$G\text{-Cov}(D^*) \longrightarrow G\text{-Cov}(D^*)/{\sim_{\mathrm{geo}}}$$

is a p-to-one surjection.

Definition 2.7. We say that $E^* \in G\text{-Cov}(D^*)$ is *representative* if E^* is isomorphic to E_f^* for $f \in \mathrm{RP}_k$. We denote the set of representative G-covers of D^* by $G\text{-Cov}^{\mathrm{rep}}(D^*)$.

By construction, we have the following proposition.

PROPOSITION 2.8. *The composition*

$$G\text{-Cov}^{\mathrm{rep}}(D^*) \hookrightarrow G\text{-Cov}(D^*) \twoheadrightarrow G\text{-Cov}(D^*)/{\sim_{\mathrm{geo}}}$$

is bijective. Moreover, the right map is p-to-one if k is a finite field.

2.2 The stratification by the ramification jump

The spaces G-Cov(D^*), G-Cov$(D^*)/{\sim_{\mathrm{geo}}}$ and RP_k are all infinite-dimensional. We will construct stratifications of them with finite-dimensional strata, which will help to control these spaces.

We say that $E^* \in G$-Cov(D^*) is *trivial* if E^* is the disjoint union of p copies of D^*, equivalently if E^* corresponds to 0 by the correspondence in Proposition 2.5. For a non-trivial $E^* \in G$-Cov(D^*), let E be the normalization of $D := \operatorname{Spec} k[[t]]$ in $\mathcal{O}_{E^*}$ and $\mathfrak{m}_E$ the maximal ideal of $\mathcal{O}_E$. Then G acts on $\mathcal{O}_E/\mathfrak{m}_E^i$ for all $i \in \mathbb{N}$.

DEFINITION 2.9. The *ramification jump* of E^* (and of E), denoted by $\mathrm{rj}(E^*) = \mathrm{rj}(E)$, is defined as follows. If E is unramified over D, then we put $\mathrm{rj}(E) = 0$. Otherwise $\mathrm{rj}(E)$ is the positive integer j such that the G-action on $\mathcal{O}_E/\mathfrak{m}_E^i$ is trivial if $i \leqslant j+1$, and non-trivial if $i \geqslant j+2$.[1] We thus have a function

$$\mathrm{rj} : G\text{-Cov}(D^*) \longrightarrow \mathbb{Z}_{\geqslant 0}.$$

PROPOSITION 2.10. *Let $f \in k((t))$. Suppose that $j := -\operatorname{ord} f \in \mathbb{N}_0'$. (This condition holds in particular if f is a representative polynomial.) By convention, if $f = 0$, then we put $j = 0$. Then $\mathrm{rj}(E_f^*) = j$. In particular, the function* rj *takes values in $\mathbb{N}_0'$.*

Proof. Let $L := \mathcal{O}_{E_f^*} = k((t))[\wp^{-1} f]$ and $g := \wp^{-1} f \in L$. If $j = 0$, then $\mathcal{O}_{E_f^*}$ is isomorphic to the product of p copies of $k((t))$ or to $k'((t))$ for an Artin–Schreier extension k'/k. Hence the assertion holds. Next we suppose that $j > 0$ and write $j = pq - r$, where q and r are integers with $1 \leqslant r \leqslant p-1$. If v_L denotes the normalized valuation on L, then

$$v_L(g) = -j = -pq + r.$$

Let $l \in \{1, 2, \dots, p-1\}$ be such that $lr = pc + 1$ for some non-negative integer c. Since

$$v_L(t^{lq-c} g^l) = p(lq-c) - lj = (lpq - pc) - lpq + pc + 1 = 1,$$

$s := t^{lq-c} g^l$ is a uniformizer of L. We now have

$$\sigma(s) = t^{lq-c}(g+1)^l = t^{lq-c} g^l + l t^{lq-c} g^{l-1} + (\text{higher-degree terms}).$$

Therefore

$$\sigma(s) - s = l t^{lq-c} g^{l-1} + (\text{higher-degree terms})$$

and

$$\begin{aligned} v_L(\sigma(s) - s) &= p(lq-c) + (l-1)(-pq+r) \\ &= p(lq-c) - lpq + pc + 1 + pq - r \\ &= pq - r + 1 \\ &= j + 1. \end{aligned}$$

This proves the proposition. □

[1] Since $\mathcal{O}_{E^*}/k((t))$ is a cyclic extension of prime degree, the ramification jump is unique and equal in both lower and upper numberings. See, for instance, [Tho08, § 2].

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For $j \in \mathbb{N}'_0$, we set

$$G\text{-Cov}(D^*, j) := \{E^* \in G\text{-Cov}(D^*) \mid \text{rj}(E^*) = j\},$$
$$G\text{-Cov}^{\text{rep}}(D^*, j) := \{E^* \in G\text{-Cov}^{\text{rep}}(D^*) \mid \text{rj}(E^*) = j\},$$
$$\text{RP}_{k,j} := \{f \in \text{RP}_k \mid \text{ord}\, f = -j\}.$$

Proposition 2.11. *For $j \in \mathbb{N}'$, we have*

$$G\text{-Cov}^{\text{rep}}(D^*, j) \longleftrightarrow G\text{-Cov}(D^*, j)/\sim_{\text{geo}} \longleftrightarrow \text{RP}_{k,j} \longleftrightarrow k^* \times k^{j-1-\lfloor j/p \rfloor}.$$

Here $\lfloor \cdot \rfloor$ denotes the floor function, which assigns to a real number a the largest integer not exceeding a.

Proof. The left and middle correspondences are clear. We note that

$$\sharp\{i \in \mathbb{N}' \mid i \leqslant j\} = j - \lfloor j/p \rfloor.$$

The right correspondence sends $\sum_{i \in \mathbb{N}', i \leqslant j} g_{-i} t^{-i}$, $g_{-j} \neq 0$, to $(g_{-i})_{i \in \mathbb{N}', i < j}$. □

Thus, for instance, the infinite-dimensional space $G\text{-Cov}^{\text{rep}}(D^*)$ admits a stratification

$$G\text{-Cov}^{\text{rep}}(D^*) = \bigsqcup_{j \in \mathbb{N}'_0} G\text{-Cov}^{\text{rep}}(D^*, j),$$

whose strata are all finite-dimensional.

For later use, we also define $G\text{-Cov}(D^*, \leqslant j) := \bigcup_{j' \leqslant j} G\text{-Cov}(D^*, j')$, and similarly for $G\text{-Cov}^{\text{rep}}(D^*, \leqslant j)$ and $\text{RP}_{k, \leqslant j}$. Then

$$G\text{-Cov}^{\text{rep}}(D^*, \leqslant j) \longleftrightarrow G\text{-Cov}(D^*, \leqslant j)/\sim_{\text{geo}} \longleftrightarrow \text{RP}_{k,\leqslant j} \longleftrightarrow k^{j-\lfloor j/p \rfloor}.$$

2.3 Moduli spaces of G-covers of D^*

Harbater [Har80, § 2] constructed the coarse moduli space of G-covers of the formal disk $D = \text{Spec}\, k[[t]]$ when k is algebraically closed.[2] He also illustrates with an example why the moduli space cannot have a universal family [Har80, Remark 2.2]. Since we would like to still have a ‘universal family’ and work over a non-algebraically closed field, we will take a different approach.

In view of his example, to have a universal family, it seems that we need an additional structure on G-covers. We will take representative polynomials as such a structure, or rather consider the moduli space of representative polynomials.

For each j, the functor

$$\{\text{affine } k\text{-scheme}\} \longrightarrow \{\text{set}\}, \quad \text{Spec}\, A \longmapsto \text{RP}_{A,\leqslant j}$$

is obviously represented by a scheme isomorphic to $\mathbb{A}_k^{j-\lfloor j/p \rfloor}$, which we denote by $\mathbf{RP}_{k,\leqslant j}$. We can write its coordinate ring explicitly as

$$B_{\leqslant j} := k[x_i \mid i \in \mathbb{N}', i \leqslant j].$$

[2] To be precise, he constructed the coarse moduli space of *pointed* principal G-covers. In our case where G is abelian, it is equal to the coarse moduli space of *unpointed* principal G-covers.

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Then the identity morphism of $\mathbf{RP}_{k,\leqslant j}$ corresponds to the *universal representative polynomial*,

$$f_j^{\mathrm{univ}} := \sum_{j' \in \mathbb{N}', j' \leqslant j} x_{j'} t^{-j'} \in \mathrm{RP}_{B_{\leqslant j}, \leqslant j}.$$

For $j_1 \leqslant j_2$, we have a canonical closed embedding $\mathbf{RP}_{k,\leqslant j_1} \hookrightarrow \mathbf{RP}_{k,\leqslant j_2}$.

Then the functor

$$\{\text{affine } k\text{-scheme}\} \longrightarrow \{\text{set}\}, \quad \operatorname{Spec} A \longmapsto \mathrm{RP}_A$$

is represented by the union $\mathbf{RP}_k := \bigcup_j \mathbf{RP}_{k,\leqslant j}$ which should be understood as the limit in the category of ind-schemes [KV04, § 1.1]. In particular, for a k-field K, we have $\mathrm{RP}_K \longleftrightarrow \mathbf{RP}_k(K)$. Readers who are not familiar with ind-schemes can just ignore them as they will not be used below.

DEFINITION 2.12. A *representative family of G-covers of D^* of ramification jump $\leqslant j$ over an affine scheme* $S = \operatorname{Spec} A$ is an étale G-torsor over $S \hat{\times} D^* = \operatorname{Spec} A((t))$ which is isomorphic to $\operatorname{Spec} A((t))[\wp^{-1} f]$ with $f \in \mathrm{RP}_{A,\leqslant j}$. We denote the set of isomorphism classes of those families by $G\text{-}\mathrm{Cov}^{\mathrm{rep}}(D^*, \leqslant j)(S)$.

The functor

$$\begin{aligned} \{\text{affine } k\text{-scheme}\} &\longrightarrow \{\text{set}\} \\ S &\longmapsto G\text{-}\mathrm{Cov}^{\mathrm{rep}}(D^*, \leqslant j)(S) \end{aligned}$$

is represented by a scheme canonically isomorphic to $\mathbf{RP}_{k,\leqslant j}$, which is denoted by $\mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*, \leqslant j)$. We have the universal family of representative G-covers of ramification jump $\leqslant j$:

$$\begin{array}{c} E^{*,\mathrm{univ}}_{\leqslant j} := \operatorname{Spec} B_{\leqslant j}((t))[\wp^{-1} f_j^{\mathrm{univ}}] \\ \downarrow \\ \operatorname{Spec} B_{\leqslant j}((t)) \\ \downarrow \\ \operatorname{Spec} B_{\leqslant j} = \mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*, \leqslant j). \end{array}$$

We define

$$\mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*) := \bigcup_{j \in \mathbb{N}'_0} \mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*, \leqslant j),$$

which is again regarded as an ind-scheme. For a perfect k-field K, we have a one-to-one correspondence

$$\mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*)(K) = G\text{-}\mathrm{Cov}^{\mathrm{rep}}(D^* \hat{\times}_k K) \longleftrightarrow G\text{-}\mathrm{Cov}(D^* \hat{\times}_k K)/\sim_{\mathrm{geo}}.$$

The universal family $E^{*,\mathrm{univ}}_{\infty}$ over $\mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*)$ is defined as the union of $E^{*,\mathrm{univ}}_{\leqslant j}$.

Putting $\mathbf{RP}_{k,j} := \mathbf{RP}_{k,\leqslant j} \backslash \mathbf{RP}_{k,\leqslant j-1}$, we have a stratification $\mathbf{RP}_k = \bigsqcup_j \mathbf{RP}_{k,j}$. Similarly, we have $\mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*) = \bigsqcup_j \mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*, j)$. Then for $j > 0$,

$$\mathbf{RP}_{k,j} \cong \mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D^*, j) \cong \mathbb{G}_m \times \mathbb{A}_k^{j-1-\lfloor j/p \rfloor}. \tag{2.1}$$

For a k-algebra A, the A-points of $\mathbf{RP}_{k,j}$ correspond to

$$\mathrm{RP}_{A,j} := \{f \in \mathrm{RP}_{A,\leqslant j} \mid f_j \in A^*\}.$$

2.4 The stratified moduli space of G-covers of the formal disk

What we will really need is the moduli space of (ramified) G-covers of the formal *non-punctured* disk $D = \operatorname{Spec} k[[t]]$. A *$G$-cover* of D means the normalization E of $\operatorname{Spec} k[[t]]$ in a G-cover $E^* \longrightarrow D^*$. If it exists, such a moduli space should bijectively correspond to the moduli space of G-covers of D^* at the level of points. The author does not know so far if such a moduli space exists. Instead we will construct strata of the hypothetical moduli space, which are sufficient for application to the motivic integration.

We define $\text{G-}\mathbf{Cov}^{\text{rep}}(D, j)$ to be $\text{G-}\mathbf{Cov}^{\text{rep}}(D^*, j)$ endowed with a different universal family constructed as follows. The coordinate ring of this moduli space is $B_j := B_{\leqslant j}[x_j^{-1}]$. Then the universal family of $\text{G-}\mathbf{Cov}^{\text{rep}}(D^*, j)$ is written as

$$E_j^{*,\text{univ}} := \operatorname{Spec} B_j((t))[\wp^{-1} f_j^{\text{univ}}] \longrightarrow \operatorname{Spec} B_j((t)) \longrightarrow \text{G-}\mathbf{Cov}^{\text{rep}}(D^*, j).$$

Let $g := \wp^{-1} f_j^{\text{univ}} \in B_j((t))[\wp^{-1} f_j^{\text{univ}}]$. With the notation in the proof of Proposition 2.10, we put $s := t^{lq-c} g^l$. Then s is a uniformizer on each fiber of the projection $E_j^{*,\text{univ}} \longrightarrow \text{G-}\mathbf{Cov}^{\text{rep}}(D^*, j)$. We define C_j to be the $B_j[[t]]$-subalgebra of $B_j((t))[g]$ generated by s. Then $\operatorname{Spec} C_j \longrightarrow \operatorname{Spec} B_j[[t]]$ is a family of G-covers of D over $\text{G-}\mathbf{Cov}^{\text{rep}}(D^*, j)$.

Definition 2.13. We define the *moduli space of representative G-covers of D of ramification jump j*, denoted by $\text{G-}\mathbf{Cov}^{\text{rep}}(D, j)$, to be $\operatorname{Spec} B_j$ with the *universal family*

$$E_j^{\text{univ}} := \operatorname{Spec} C_j \longrightarrow \operatorname{Spec} B_j[[t]] \longrightarrow \text{G-}\mathbf{Cov}^{\text{rep}}(D, j).$$

2.5 Details of the G-actions on $\mathcal{O}_{E^*}$ and $\mathcal{O}_E$

Let $0 \neq f \in \mathrm{RP}_k$ be a representative polynomial of order $-j$. Let E and E^* be the corresponding G-covers of D and D^* respectively, and let $g = \wp^{-1} f \in \mathcal{O}_{E^*}$. Then $\mathcal{O}_{E^*}$ has a basis $1, g, \dots, g^{p-1}$ over $k((t))$.

Notation 2.14. In what follows, for a k-algebra or module M endowed with a G-action, we denote by δ the k-linear operator $\sigma - \mathrm{id}_M$ on M. For $a \in \mathbb{N}$, we denote by $M^{\delta^a = 0}$ the kernel of $\delta^a : M \longrightarrow M$.

Sometimes it is more useful to use δ rather than σ in order to study G-actions.

Lemma 2.15. *For any integer i with $1 \leqslant i \leqslant p-1$ and for any $0 \neq h \in k((t))$, we have $\delta^i(g^i h) \neq 0$ and $\delta^{i+1}(g^i h) = 0$. Therefore, for each integer a with $0 \leqslant a \leqslant p$, we have*

$$\mathcal{O}_{E^*}^{\delta^a = 0} = \bigoplus_{i=0}^{a-1} k((t)) \cdot g^i.$$

Proof. We will prove this by induction on i. For $i = 1$, since $\sigma(g) = g + 1$, we have $\delta(gh) = h(\sigma(g) - g) = h$ and $\delta^2(gh) = \delta(h) = 0$. For $i > 1$, we have

$$\sigma(g^i h) = h(g+1)^i = h(g^i + i g^{i-1} + \cdots + ig + 1)$$

and

$$\delta(g^i h) = h(i g^{i-1} + \cdots + ig + 1). \tag{2.2}$$

Applying δ^{i-1} and δ^i to this, we obtain the lemma. □

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COROLLARY 2.16. *We have*

$$\mathcal{O}_E = \prod_{\substack{0 \leqslant i < p \\ -ij+np \geqslant 0}} k \cdot g^i t^n.$$

Moreover, for each integer a with $0 \leqslant a \leqslant p$, we have

$$\mathcal{O}_E^{\delta^a = 0} = \prod_{\substack{0 \leqslant i < a \\ -ij+np \geqslant 0}} k \cdot g^i t^n.$$

Proof. Let v be the normalized valuation on $\mathcal{O}_{E^*}$. Then $v(g^i t^n) = -ij + np$. For every non-negative integer r, there exists a unique pair (i, n) of integers such that $0 \leqslant i < p$ and $r = -ij + np$. This proves the first assertion. Then the second follows from the previous lemma. □

COROLLARY 2.17. *For $h \in \mathcal{O}_E$ with $p \nmid v_E(h)$, we have $v_E(\delta(h)) = v_E(h) + \mathrm{rj}(E)$. Here v_E denotes the normalized valuation of $\mathcal{O}_E$.*

Proof. We can write h as a $k((t))$-linear combination of g^i, $0 \leqslant i < p$. Then the corollary follows from (2.2). □

3. Twisted arcs and jets

To a G-representation V, we will associate the quotient stack $\mathcal{X} = [V/G]$ and the quotient variety $X = V/G$. The McKay correspondence follows from the change of variables formula of motivic integrals for the morphism $\mathcal{X} \longrightarrow X$. To obtain the formula, we need an almost bijection between the arc spaces of X and $\mathcal{X}$. However, general arcs of X lift to $\mathcal{X}$ not as ordinary arcs but as *twisted arcs*. In this section, we will construct the spaces of twisted arcs and jets, and examine their structures. Our use of stacks is not really necessary. However, it puts everything on an equal footing in the framework of the birational geometry of stacks.

3.1 Ordinary arcs and jets of a scheme

Let X be a variety, that is, a separated scheme of finite type over k. An *n-jet* of X is a morphism $\operatorname{Spec} k[[t]]/(t^{n+1}) \longrightarrow X$. There exists a fine moduli scheme $J_n X$ of n-jets of X, called the *n-jet scheme* of X. Thus, for a k-algebra A,

$$(J_n X)(A) = \operatorname{Hom}(\operatorname{Spec} A[[t]]/(t^{n+1}), X).$$

There is a natural morphism $J_n X \longrightarrow X$, defined by reduction modulo t. Also for $n' \geqslant n$, we have a *truncation map* $J_{n'} X \longrightarrow J_n X$. The projective limit, $J_\infty X := \varprojlim J_n X$, is called the *arc space* of X. For every k-field K,

$$(J_\infty X)(K) = \operatorname{Hom}(\operatorname{Spec} K[[t]], X).$$

We denote the *truncation map* $J_\infty X \longrightarrow J_n X$ *by* π_n.

3.2 A G-representation

From now on, we denote by V a d-dimensional G-representation and suppose that V is decomposed into indecomposables as

$$V = \bigoplus_{\lambda=1}^{l} V_{d_\lambda} \left(1 \leqslant d_\lambda \leqslant p, \sum_{\lambda=1}^{l} d_\lambda = d \right),$$

where V_a denotes the unique indecomposable G-representation of dimension a. We suppose that V is non-trivial, that is, $(d_1, \dots, d_l) \neq (1, \dots, 1)$.

We denote the coordinate ring of the affine space V by

$$k[\mathbf{x}] = k[x_{\lambda,i} \mid 1 \leqslant \lambda \leqslant l,\, 1 \leqslant i \leqslant d_\lambda],$$

and fix the G-action on it by

$$\sigma(x_{\lambda,i}) = \begin{cases} x_{\lambda,i} + x_{\lambda,i+1} & (i \neq d_\lambda) \\ x_{\lambda,d} & (i = d_\lambda). \end{cases}$$

This is equivalent to saying that

$$\delta(x_{\lambda,i}) = \begin{cases} x_{\lambda,i+1} & (i \neq d_\lambda) \\ 0 & (i = d_\lambda). \end{cases}$$

(Recall that the operator δ is defined as $\sigma - \mathrm{id}$.)

Most arguments below can be reduced to the case where V is indecomposable. In that case, $l = 1$ and $d_1 = d$. Then we simply write $x_i = x_{1,i}$.

3.3 *G*-arcs and jets

For $0 \neq f \in \mathrm{RP}_k$, we define $E_{f,n}$ to be $\operatorname{Spec} \mathcal{O}_{E_f}/\mathfrak{m}_{E_f}^{np+1}$, which is a closed subscheme of E_f. Since $k[[t]]/(t^{n+1}) \subset (\mathcal{O}_E/\mathfrak{m}_{E_f}^{pn+1})^G$ (the equality does not generally hold), we have a natural morphism

$$E_{f,n} \longrightarrow D_n := \operatorname{Spec} k[[t]]/(t^{n+1}).$$

If $f = 0$, then E_f has p connected components and each component is identified with D via the projection $E_f \longrightarrow D := \operatorname{Spec} k[[t]]$. In this case, we just define $E_{f,n}$ to be the disjoint union of p copies of D_n.

Definition 3.1. We define a *G-arc* (respectively, *G-n-jet*) of V as a G-equivariant morphism $E_f \longrightarrow V$ (respectively, $E_{f,n} \longrightarrow V$) for some $f \in \mathrm{RP}_k$. More generally, for $f \in \mathrm{RP}_{A,j}$, let $E_f \longrightarrow \operatorname{Spec} A[[t]]$ be the corresponding G-cover and let $E_{f,n} \subset E_f$ be as above. Then we define a *G-arc of V of ramification jump j over A* as a G-equivariant morphism $E_f \longrightarrow V$. Two G-arcs over A are regarded as the same if the associated representative polynomials are the same and the morphisms are the same. Similarly for G-n-jets. (If $f \neq f'$, then two G-n-jets $E_{f,n} \longrightarrow V$ and $E_{f',n} \longrightarrow V$ must always be distinguished, even when there is an isomorphism $E_{f,n} \cong E_{f',n}$ compatible with morphisms to V and D_n.)

Lemma 3.2. *For any k-algebra B endowed with a G-action, we have a bijection:*

$$\{G\text{-equivariant } k\text{-algebra map } k[\mathbf{x}] \longrightarrow B\} \longrightarrow \prod_{\lambda=1}^{l} B^{\delta^{d_\lambda}=0}$$
$$\alpha \longmapsto (\alpha(x_{1,1}), \dots, \alpha(x_{l,1})).$$

Proof. Let $\alpha : k[\mathbf{x}] \longrightarrow B$ be a G-equivariant k-algebra map. Then for every λ and i, we have $\alpha(\delta(x_{\lambda,i})) = \delta(\alpha(x_{\lambda,i}))$. In particular, $\alpha(x_{\lambda,i}) = \delta^{i-1}(\alpha(x_{\lambda,1}))$ and $\delta^{d_\lambda}(x_{\lambda,1}) = 0$. This shows that α is determined by $\alpha(x_{\lambda,1})$, $1 \leqslant \lambda \leqslant l$, and the map of the lemma is well defined.

Conversely, if $(f_1, \dots, f_l) \in \prod_{\lambda=1}^{l} B^{\delta^{d_\lambda}=0}$ is given, then we define a k-algebra map $\alpha : k[\mathbf{x}] \longrightarrow B$ by $\alpha(x_{\lambda,i}) = \delta^{i-1}(f_\lambda)$. We can easily see that α is the unique G-equivariant k-algebra map with $\alpha(x_{\lambda,1}) = f_\lambda$. Hence this construction gives the inverse map. □

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PROPOSITION 3.3. *For each $0 \leqslant n < \infty$ and for each $j \in \mathbb{N}'_0$, there exists a fine moduli scheme $J^G_{n,j}V$ of G-n-jets of V of ramification jump j.*

Proof. We prove this only when V is indecomposable. We first consider the case $j > 0$. From the previous lemma, for a fixed f, G-n-jets $E_f \longrightarrow V$ correspond to elements of $(\mathcal{O}_{E_f}/\mathfrak{m}_{E_f}^{np+1})^{\delta^d=0}$. With the notion as in § 2.5, we have

$$\mathcal{O}_{E_f}/\mathfrak{m}_{E_f}^{np+1} = \bigoplus_{\substack{0\leqslant i<p \\ 0\leqslant -ij+np\leqslant np}} k\cdot [g^i t^n].$$

Then $(\mathcal{O}_{E_f}/\mathfrak{m}_{E_f}^{np+1})^{\delta^d=0}$ is the linear subspace generated by the elements $g^i t^n$ from the basis such that either $i < d$ or $-ij + np + dj > np$. If we denote by $\nu_{n,j}$ the dimension of the subspace, then G-n-jets are parameterized by $k^{\nu_{n,j}}$. This argument can apply to families, in particular, to the universal family over G-$\mathbf{Cov}^{\mathrm{rep}}(D, j)$. With the notation from § 2.4, let $\mathfrak{m}_j \subset C_j$ be the ideal generated by s. Then G-n-jets over G-$\mathbf{Cov}^{\mathrm{rep}}(D, j)$,

$$\operatorname{Spec} C_j/\mathfrak{m}_j^{np+1} \longrightarrow V,$$

correspond to elements of the module $(C_j/\mathfrak{m}_j^{np+1})^{\delta^d=0}$, which is isomorphic to $B_j^{\nu_{n,j}}$ as a B_j-module. This shows that the desired moduli space $J^G_{n,j}V$ is isomorphic to $\mathbb{A}_k^{\nu_{n,j}} \times$ G-$\mathbf{Cov}^{\mathrm{rep}}(D, j)$.

The case where $j = 0$ is easier. Then $f = 0$ and $E_{0,n}$ is the union of p-copies of D_n. We fix one connected component of $E_{0,n}$, identify it with D_n and write $D_n \hookrightarrow E_{0,n}$. Then a G-n-jet $E_{0,n} \longrightarrow V$ is uniquely determined by its restriction to D_n. Conversely, an ordinary n-jet $D_n \longrightarrow V$ uniquely extends to a G-n-jet $E_{0,n} \longrightarrow V$. Therefore we can identify $J^G_{n,0}V$ with J_nV. □

PROPOSITION 3.4. *The following hold:*

(i) *for every n and j, $J^G_{n,j}V \cong \mathbb{A}_k^m \times$ G-$\mathbf{Cov}^{\mathrm{rep}}(D, j)$ for some m;*

(ii) *for $n = 0$, $J^G_{0,j}V \cong \mathbb{A}_k^l \times$ G-$\mathbf{Cov}^{\mathrm{rep}}(D, j)$ $(j \in \mathbb{N}')$ and $J^G_{0,0}V = \mathbb{A}_k^d$;*

(iii) *for $n' \geqslant n$, truncation maps $J^G_{n',j}V \longrightarrow J^G_{n,j}V$ are induced by a (not necessarily surjective) linear map $\mathbb{A}_k^{m'} \longrightarrow \mathbb{A}_k^m$.*

Proof. The assertions follow from the proof of the previous proposition. □

Now the space of G-arcs of ramification jump j, denoted $J^G_{\infty,j}V$, is constructed as the projective limit of $J^G_{n,j}V$, $n \in \mathbb{Z}_{\geqslant 0}$. Hence it is isomorphic to $(\prod_{i=1}^{\infty} \mathbb{A}_k^1) \times$ G-$\mathbf{Cov}^{\mathrm{rep}}(D, j)$. Let $\pi_n : J^G_{\infty,j}V \longrightarrow J^G_{n,j}V$ denote truncation maps.

COROLLARY 3.5. *For $0 \leqslant n < \infty$,*

$$\pi_n(J^G_{\infty,j}V) \cong \begin{cases} \mathbb{A}_k^{nd+l} \times \text{G-}\mathbf{Cov}^{\mathrm{rep}}(D, j) & (j \in \mathbb{N}') \\ \mathbb{A}_k^{(n+1)d} & (j = 0). \end{cases}$$

Moreover, the truncation map $\pi_{n+1}(J^G_{\infty,j}V) \longrightarrow \pi_n(J^G_{\infty,j}V)$ is a trivial fibration with fiber $\mathbb{A}_k^d$.

Proof. The case $j = 0$ is obvious from $J^G_{n,0}V = J_nV$. For $j > 0$, with the notation as in the proof of Proposition 3.3, G-n-jets in $\pi_n(J^G_{\infty,j}V)$ correspond to elements of the linear subspace of $\mathcal{O}_{E_f}/\mathfrak{m}_{E_f}^{np+1}$ generated by $g^i t^n$ with $i < d$. This shows the first assertion. The second assertion follows from the first. □

Definition 3.6. For $0 \leqslant n \leqslant \infty$, we put $J_n^G V := \bigsqcup_{j \geqslant 0} J_{n,j}^G V$. (Here, for each j, $J_{n,j}^G V$ is a connected component of $J_n^G V$.)

The following is obvious from the definition.

Corollary 3.7. *The truncation maps $\pi_{n+1}(J_\infty^G V) \longrightarrow \pi_n(J_\infty^G V)$ are trivial fibrations with fiber $\mathbb{A}_k^d$.*

3.4 Twisted arcs and jets

Let $\mathcal{X}$ be the quotient stack $[V/G]$. For an algebraically closed k-field K and for a representative polynomial $f \in \mathrm{RP}_{K,j}$, we set

$$\mathcal{D}_f := [E_f/G] \quad \text{and} \quad \mathcal{D}_{f,n} := [E_{f,n}/G].$$

Definition 3.8. We define a *twisted arc* (respectively, *twisted n-jet*) of $\mathcal{X}$ over K as a morphism

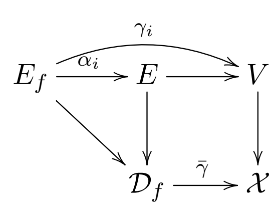
$$\mathcal{D}_f \longrightarrow \mathcal{X} \text{ (respectively, } \mathcal{D}_{f,n} \longrightarrow \mathcal{X})$$

which is induced from a G-arc $E_f \longrightarrow V$ (respectively, G-n-jet $E_{f,n} \longrightarrow V$). We say that two twisted arcs $\gamma : \mathcal{D}_f \longrightarrow \mathcal{X}$ and $\gamma' : \mathcal{D}_{f'} \longrightarrow \mathcal{X}$ (over K) are *isomorphic* if $f = f'$ and if two morphisms $\gamma, \gamma' : \mathcal{D}_f \rightrightarrows \mathcal{X}$ are 2-isomorphic. (Recall that stacks form a 2-category and hence morphisms between two stacks form a usual category.)

Clearly twisted arcs (jets) are closely related to G-arcs (jets). To each G-arc $\gamma : E_f \longrightarrow V$, we can associate a twisted arc $\bar{\gamma} : \mathcal{D}_f \longrightarrow \mathcal{X}$. Conversely, given a twisted arc $\mathcal{D}_f \longrightarrow \mathcal{X}$, then there exists a G-arc $E_f \longrightarrow V$ whose associated twisted arc is the given one.

Proposition 3.9. *The set of twisted arcs of $\mathcal{X}$ over an algebraically closed k-field K is in one-to-one correspondence with $(J_\infty^G V)(K)/G$ in such a way that the class of $\gamma \in (J_\infty^G V)(K)$ corresponds to $\bar{\gamma}$. Here G acts on $J_\infty^G V$ by $\sigma(\gamma) := \gamma \circ \sigma = \sigma \circ \gamma$. Similarly, the set of twisted n-jets of $\mathcal{X}$ over K is in one-to-one correspondence with $(J_n^G V)(K)/G$.*

Proof. Let $\gamma_i : E_f \longrightarrow V$ $(i = 1, 2)$ be G-arcs such that $\bar{\gamma} := \bar{\gamma}_1 = \bar{\gamma}_2$. We have to show that γ_1 and γ_2 are in the same G-orbit. Let $E := \mathcal{D}_f \times_{\bar{\gamma}, \mathcal{X}} V$. Then for each i, there exists an isomorphism $\alpha_i : E_f \longrightarrow E$ which fits into the following 2-commutative diagram.



Then $\gamma_2 = \gamma_1 \circ \alpha_1^{-1} \circ \alpha_2$. For G-n-jets, the corresponding assertion holds. All that remains is to show that $\alpha_1^{-1} \circ \alpha_2 = \tau$ for some $\tau \in G$. This will be done in the following lemma in a more general setting. □

Lemma 3.10. *Let U be a G-scheme and $[U/G]$ the quotient stack with the natural morphism $\alpha : U \longrightarrow [U/G]$. Suppose that $\beta : U \longrightarrow U$ is an isomorphism such that $\beta \circ \alpha$ and α are isomorphic. Then $\beta = \tau$ for some $\tau \in G$.*

Proof. Let $m : G \times U \longrightarrow U$ be the morphism defining the G-action and $\pi : G \times U \longrightarrow U$ the projection. From the definition of quotient stacks, there exists a G-equivariant isomorphism

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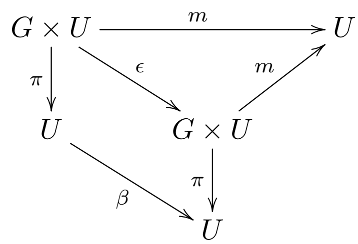
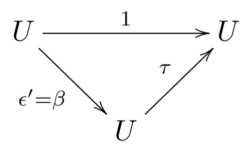
$\epsilon : G \times U \longrightarrow G \times U$ making the following diagram commutative.



Suppose that ϵ maps $\{1\} \times U$ onto $\{\tau\} \times U$. If ϵ' denotes the restriction of ϵ to $\{1\} \times U$ and if we identify $\{1\} \times U$ and $\{\tau\} \times U$ with U, then we have the following commutative diagram.



This shows that $\beta = \epsilon' = \tau^{-1}$. □

Definition 3.11. We define the *space of twisted arcs* and *twisted n-jets* of $\mathcal{X}$ as the quotient schemes

$$\mathcal{J}_\infty \mathcal{X} := (J_\infty^G V)/G \quad \text{and} \quad \mathcal{J}_n \mathcal{X} := (J_n^G V)/G.$$

Then for $0 \leqslant n \leqslant \infty$, we write $\mathcal{J}_n \mathcal{X} = \bigsqcup_{j \in \mathbb{N}_0'} \mathcal{J}_{n,j}\mathcal{X}$, where the subscript j indicates ramification jumps. We define the function

$$\mathrm{rj} : \mathcal{J}_n \mathcal{X} \longrightarrow \mathbb{N}_0'$$

by $\mathrm{rj}(\gamma) := j$ for $\gamma \in \mathcal{J}_{n,j}\mathcal{X}$.

Remark 3.12. The genuine moduli spaces of twisted arcs or jets must be constructed as stacks as in [Yas06].

Definition 3.13. For schemes X and Y of finite type, a morphism $f : Y \longrightarrow X$ is called a *universal homeomorphism* if one of the following equivalent conditions holds:

(i) f is finite, surjective and universally injective;

(ii) for every morphism $X' \longrightarrow X$ of schemes, the induced morphism $Y \times_X X' \longrightarrow X'$ is a homeomorphism.

We say that two schemes X and Y of finite type are *universally homeomorphic* if there exists a universal homeomorphism between them in either direction. For instance, see [NS11, § 3.8] for more details.

If T is a G-variety and $S \subset T$ is a G-stable closed subvariety, then the map $S/G \longrightarrow T/G$ is not a closed embedding but only a universal homeomorphism onto its image. This is why this notion is necessary below.

We note that the G-action on $J_{n,j}^G V = \mathbb{A}_k^m \times \text{G-}\mathbf{Cov}^{\mathrm{rep}}(D, j)$ is trivial on $\text{G-}\mathbf{Cov}^{\mathrm{rep}}(D, j)$ and linear on $\mathbb{A}_k^m$. Indeed the linearity follows from the proof of Lemma 2.15. Hence we have the following fact which is essential to define the motivic measure on $\mathcal{J}_\infty \mathcal{X}$ below.

Corollary 3.14. *Every geometric fiber of the truncation $\pi_{n+1}(\mathcal{J}_\infty \mathcal{X}) \longrightarrow \pi_n(\mathcal{J}_\infty \mathcal{X})$ is universally homeomorphic to the quotient of $\mathbb{A}_K^d$ by some linear G-action with K an algebraically* *closed k-field. Moreover,*

$$\pi_0(\mathcal{J}_{\infty,j}\mathcal{X}) = \mathcal{J}_{0,j}\mathcal{X} = \begin{cases} \mathbb{A}^l \times \text{G-}\mathbf{Cov}^{\text{rep}}(D,j) & (j \in \mathbb{N}') \\ V/G & (j = 0). \end{cases}$$

3.5 Push-forward maps for twisted arcs and jets

Let $X := V/G$ be the quotient variety. Let $\phi : \mathcal{X} \longrightarrow X$ and $\psi : V \longrightarrow X$ be the natural morphisms. For a twisted arc $\mathcal{D} \longrightarrow \mathcal{X}$, taking the coarse moduli spaces, we get an arc $D \longrightarrow X$. This defines a *push-forward map*

$$\phi_\infty : \mathcal{J}_\infty\mathcal{X} \longrightarrow J_\infty X.$$

We can see that this is actually a scheme morphism as follows. Let the solid arrows of

$$\begin{array}{ccc} E & \longrightarrow & V \\ \downarrow & & \downarrow \psi \\ D \hat{\times} J_\infty^G V & \overset{\alpha}{\dashrightarrow} & X \\ \downarrow & & \\ J_\infty^G V & & \end{array}$$

be the universal family of G-arcs. Then there exists the dashed arrow α which makes the whole diagram commutative. This morphism α is a family of arcs of X over $J_\infty^G V$. From the universality of $J_\infty X$, this induces a morphism $J_\infty^G V \longrightarrow J_\infty X$. Then we can easily see that this factors through $\mathcal{J}_\infty\mathcal{X} = (J_\infty^G V)/G$, and obtain the desired morphism $\mathcal{J}_\infty\mathcal{X} \longrightarrow J_\infty X$.

Notation 3.15. From now on, for γ in $J_\infty X$ or $\mathcal{J}_\infty\mathcal{X}$, we denote by γ_n its truncation at level n: $\gamma_n = \pi_n(\gamma)$.

Let $\gamma : \mathcal{D} \longrightarrow \mathcal{X}$ be a twisted arc and $\gamma_n : \mathcal{D}_n \hookrightarrow \mathcal{D} \longrightarrow \mathcal{X}$ its truncation at level n. Then we have an arc $\phi_\infty\gamma : D \longrightarrow X$ and its truncation at level n, $(\phi_\infty\gamma)_n : D_n \hookrightarrow D \longrightarrow X$. This n-jet of X depends only on γ_n, hence we have a *push-forward map*

$$\phi_n : \pi_n(\mathcal{J}_\infty\mathcal{X}) \longrightarrow J_n X, \quad \gamma_n \longmapsto (\phi_\infty\gamma)_n.$$

We can easily see that this is a scheme morphism and compatible with truncation maps.

Remark 3.16. Unlike the tame case, we do not have a map $\mathcal{J}_n\mathcal{X} \longrightarrow J_n X$. This is because D_n is not the coarse moduli space of $\mathcal{D}_n$.

Let $V^G \subset V$ be the fixed point locus and $\mathcal{Y} := [V^G/G] \subset \mathcal{X}$. Since we have supposed that V is non-trivial, ϕ is proper and birational. Then $\mathcal{Y}$ is the exceptional locus of ϕ. We define $\mathcal{J}_\infty\mathcal{Y}$ to be the subset of $\mathcal{J}_\infty\mathcal{X}$ consisting of those twisted arcs that factor through $\mathcal{Y}$. Let $Y \subset X$ be the image of $\mathcal{Y}$. Then the arc space $J_\infty Y$ of Y is regarded as a subscheme of $J_\infty X$.

Proposition 3.17. *The map*

$$\phi_\infty : \mathcal{J}_\infty\mathcal{X} \backslash \mathcal{J}_\infty\mathcal{Y} \longrightarrow J_\infty X \backslash J_\infty Y$$

is bijective.

Proof. We will show that $\gamma \in \mathcal{J}_\infty\mathcal{X} \backslash \mathcal{J}_\infty\mathcal{Y}$ can be reconstructed from $\bar{\gamma} := \phi_*\gamma$. Let $E^* \longrightarrow D^*$ be a G-cover obtained as the base change of $V \longrightarrow X$ by $\bar{\gamma}|_{D^*}$. If E is the normalization of D in E^*, then the morphism $E^* \longrightarrow V$ uniquely extends to $E \longrightarrow V$, thanks to the valuative criterion of properness. This is a G-arc and induces a twisted arc $\mathcal{D} := [E/G] \longrightarrow \mathcal{X}$. Now it is straightforward to check that this twisted arc is isomorphic to γ. □

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4. Motivic integration

In this section we introduce the motivic measure on the space of twisted arcs and define integrals relative to this. This is mostly a matter of repeating material from the literature [Bat98, DL99, DL02, Seb04] with a slight modification.

4.1 The Grothendieck ring of varieties and a variant

Let Var_k denote the set of isomorphism classes of k-varieties. The *Grothendieck ring of varieties over* k, denoted $K_0(\mathrm{Var}_k)$, is the abelian group generated by $[Y] \in \mathrm{Var}_k$ subject to the following relation: If Z is a closed subvariety of Y, then $[Y] = [Y \setminus Z] + [Z]$. It has a ring structure where the product is simply defined by $[Y][Z] := [Y \times Z]$. We denote by $\mathbb{L}$ the class $[\mathbb{A}^1_k]$ of the affine line.

For our purposes, we also need the following relation.

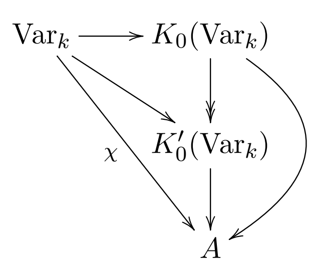
Condition 4.1. Let $f : Y \longrightarrow Z$ be a morphism of varieties. If every geometric fiber of f is universally homeomorphic to the quotient of $\mathbb{A}^n_K$ with K an algebraically closed k-field by some linear G-action, then $[Y] = \mathbb{L}^n[Z]$.

Definition 4.2. We define $K'_0(\mathrm{Var}_k)$ to be the quotient of $K_0(\mathrm{Var}_k)$ by imposing Condition 4.1.

Let A be an abelian group and let $\chi : \mathrm{Var}_k \longrightarrow A$ be a map satisfying the following property: For every variety Z and every closed subvariety $Y \subset Z$, $\chi(Z) = \chi(Z \setminus Y) + \chi(Y)$. (Such a map is called a *generalized Euler characteristic*.) Then there exists a unique group homomorphism

$$K_0(\mathrm{Var}_k) \longrightarrow A$$

through which χ factors. Additionally, suppose that for every morphism $f : Y \longrightarrow Z$ as in Condition 4.1, $\chi(Y) = \chi(\mathbb{A}^n_k)\chi(Z)$. Then there exists a group homomorphism $K'_0(\mathrm{Var}_k) \longrightarrow A$ which fits into the commutative diagram.



The maps $K_0^{(\prime)}(\mathrm{Var}_k) \longrightarrow A$ are ring maps if A is a ring and if $\chi(Y)\chi(Z) = \chi(Y \times Z)$ for any Y and Z.

4.2 Various realizations

4.2.1 *Counting rational points.* In this paragraph, we suppose that k is a finite field. Then for a finite extension $\mathbb{F}_q/k$, associating to a variety X the number of $\mathbb{F}_q$-points $\sharp X(\mathbb{F}_q)$, we obtain a map

$$\sharp_q : \mathrm{Var}_k \longrightarrow \mathbb{Z}, \quad X \longmapsto \sharp X(\mathbb{F}_q).$$

This is a generalized Euler characteristic and defines

$$\sharp_q : K_0(\mathrm{Var}_k) \longrightarrow \mathbb{Z}.$$

Let $\bar{k}$ be a fixed algebraic closure of k. For a variety Y over k, we denote by $Y_{\bar{k}}$ the variety over $\bar{k}$ obtained from Y by extension of scalars. Then, fixing a prime $l \neq p$, we write (compactly supported) l-adic étale cohomology groups as $H^i(Y_{\bar{k}}) = H^i(Y_{\bar{k}}, \mathbb{Q}_l)$ and $H^i_c(Y_{\bar{k}}) = H^i_c(Y_{\bar{k}}, \mathbb{Q}_l)$.

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Lemma 4.3. *For a G-representation V of dimension d, we have isomorphisms of* $\mathrm{Gal}(\bar{k}/k)$*-representations:*

$$H_c^i((V/G)_{\bar{k}}) \cong H_c^i(V_{\bar{k}}) \cong \begin{cases} \mathbb{Q}_l(-d) & (i = 2d) \\ 0 & (\textit{otherwise}). \end{cases}$$

Proof. In this proof, we omit the subscript $\bar{k}$. Let $W := V^G \subset V$ be the fixed point locus and $U := V \backslash W$. From the exact G-equivariant sequence

$$\cdots \longrightarrow H_c^i(U) \longrightarrow H_c^i(V) \longrightarrow H_c^i(W) \longrightarrow H_c^{i+1}(U) \longrightarrow \cdots \tag{4.1}$$

we have equivariant isomorphisms

$$H_c^i(U) \cong \begin{cases} H_c^i(V) & (i = 2d) \\ H_c^{i-1}(W) & (i = 2\dim W + 1) \\ 0 & (\text{otherwise}). \end{cases}$$

We claim that $H^i(U/G) = H^i(U)^G$. Indeed since $U \longrightarrow U/G$ is an étale Galois covering, we have the Hochschild–Serre spectral sequence [Mil80, p. 105, Theorem 2.20],

$$H^i(G, H^j(U, \mathbb{Z}/l^n)) \Rightarrow H^{i+j}(U/G, \mathbb{Z}/l^n).$$

Then since $\sharp G = p \neq l$, the group cohomology groups $H^i(G, H^j(U_{\bar{k}}, \mathbb{Z}/l^n))$ vanish for $i \neq 0$ and the spectral sequence degenerates. Hence for each j,

$$H^j(U, \mathbb{Z}/l^n)^G = H^j(U/G, \mathbb{Z}/l^n).$$

Then passing to the limits and tensoring with $\mathbb{Q}_l$, we can show the claim.

Now, since the G-action on $H_c^{2\dim W}(W)$ is trivial, from the Poincaré duality, we have $H_c^i(U/G) = H_c^i(U)$ for every i. Let $\bar{W} \subset V/G$ be the image of W. Then the map $W \longrightarrow \bar{W}$ is a universal homeomorphism and hence $H_c^i(W) = H_c^i(\bar{W})$ (see, for instance, [NS11, § 4.2]). From the five lemma, the long exact sequence

$$\cdots \longrightarrow H_c^i(U/G) \longrightarrow H_c^i(V/G) \longrightarrow H_c^i(\bar{W}) \longrightarrow H_c^{i+1}(U/G) \longrightarrow \cdots$$

is isomorphic to (4.1). In particular, $H_c^i(V/G) \cong H_c^i(V)$ for every i. The lemma follows again from the Poincaré duality. □

Lemma 4.4. *Let $f : Y \longrightarrow Z$ be a morphism as in Condition 4.1. Then we have an isomorphism of* $\mathrm{Gal}(\bar{k}/k)$*-representations,*

$$H_c^i(Y_{\bar{k}}) \cong H_c^i(Z_{\bar{k}}) \otimes \mathbb{Q}(-n).$$

Proof. From the previous lemma and the invariance of étale cohomology under universal homeomorphisms, we have

$$R^i f_! \mathbb{Q}_l = \begin{cases} \mathbb{Q}_l(-n) & (i = 2n) \\ 0 & (\text{otherwise}), \end{cases}$$

which proves the lemma. □

Proposition 4.5. *The map $\sharp_q$ factors through* $K_0'(\mathrm{Var}_k)$.

Proof. This follows from the previous lemma and the Lefschetz trace formula. □

4.2.2 *Poincaré polynomials.* The *Poincaré polynomial* of a smooth proper variety X is

$$P(X;T) = \sum_i (-1)^i b_i(X) T^i \in \mathbb{Z}[T].$$

Here $b_i(X) = \dim H^i(X_{\bar{k}})$. Following Nicaise [Nic11, Appendix], we generalize this to any variety. Indeed there exists a map

$$P : K_0(\mathrm{Var}_k) \longrightarrow \mathbb{Z}[T],$$

and for a variety X we simply write $P([X])$ as $P(X)$. Making the variable T explicit, we also write it as $P(X;T)$. Two important properties of the generalized Poincaré polynomial are as follows. Firstly, for a variety X, the degree of $P(X)$ equals twice the dimension of X. Secondly, $P(X;1)$ equals the *topological Euler characteristic*

$$e_{\mathrm{top}}(X) := \sum_i \dim H_c^i(X_{\bar{k}}).$$

PROPOSITION 4.6. *The map P factors through $K_0'(\mathrm{Var}_k)$.*

Proof. Let f be a morphism $Z \longrightarrow Y$ as in Condition 4.1. We need to show that $P(Z) = P(Y)T^{2n}$. Let $A \subset k$ be a finitely generated $\mathbb{F}_p$-subalgebra such that f is obtained from an A-morphism $f_A : Z_A \longrightarrow Y_A$ by extension of scalars. Let $a : \operatorname{Spec} \mathbb{F}_q \longrightarrow \operatorname{Spec} A$ be a general closed point. Let Y_a be the fiber of $Y \longrightarrow \operatorname{Spec} A$ over a, and similarly for Z_a. Then from [Nic11], $P(Y) = P(Y_a)$ and $P(Z) = P(Z_a)$. Moreover, $P(Y_a)$ is computed from the weight filtrations on $H_c^i(Y_a \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q)$. Similarly for $P(Z_a)$. From Lemma 4.4,

$$P(Z) = P(Z_a) = P(Y_a)T^{2n} = P(Y)T^{2n}.$$

This proves the proposition. □

4.3 Localization and completion

We need to further extend our modified Grothendieck ring $K_0'(\mathrm{Var}_k)$. We first consider its localization by $\mathbb{L}$, $\mathcal{M}' := K_0'(\mathrm{Var}_k)[\mathbb{L}^{-1}]$. Then we define its *dimensional completion* $\hat{\mathcal{M}}'$ as follows. Let $F^m\mathcal{M}'$ be the subgroup of $\mathcal{M}'$ generated by $[X]\mathbb{L}^i$ with $\dim X + i < -m$. Then $\{F^m\mathcal{M}'\}_{m\in\mathbb{Z}}$ is a descending filtration of $\mathcal{M}'$. We define

$$\hat{\mathcal{M}}' := \varprojlim \mathcal{M}'/F^m\mathcal{M}'.$$

This inherits the ring structure and the filtration from $\mathcal{M}'$. For later use, we define a *norm* $\|\cdot\|$ on $\hat{\mathcal{M}}'$ by

$$\begin{aligned} \|\cdot\| : \hat{\mathcal{M}}' &\longrightarrow \mathbb{R}_{\geqslant 0} \\ a &\longmapsto \|a\| := 2^{-n}, \end{aligned}$$

where $n := \sup\{m \mid a \in F^m\hat{\mathcal{M}}'\}$.

Let $\bar{\mathcal{M}}'$ be the image of $\mathcal{M}'$ in $\hat{\mathcal{M}}'$ and consider the following subring of $\hat{\mathcal{M}}'$:

$$\hat{\mathcal{M}}'_0 := \bar{\mathcal{M}}'\left[\frac{1}{1-\mathbb{L}^{-n}} \,\middle|\, n \in \mathbb{Z}_{>0}\right].$$

Every element $a \in \hat{\mathcal{M}}'_0$ has the expression

$$a = b + b\mathbb{L}^{-n} + b\mathbb{L}^{-2n} + \cdots = \frac{b}{1-\mathbb{L}^{-n}} \tag{4.2}$$

for some $b \in \bar{\mathcal{M}}'$ and $n > 0$. In this paper, every explicitly computed element of $\hat{\mathcal{M}}'$ actually lies in $\hat{\mathcal{M}}'_0$.

The map $P : K'_0(\mathrm{Var}_k) \longrightarrow \mathbb{Z}[T]$ extends to

$$P : \hat{\mathcal{M}}' \longrightarrow \mathbb{Z}((T^{-1}))$$

and we have

$$P(\hat{\mathcal{M}}'_0) \subset \mathbb{Z}\left[T, T^{-1}, \frac{1}{1-T^{-n}} \,\middle|\, n \in \mathbb{Z}_{>0}\right] \subset \mathbb{Z}(T).$$

For $a \in \hat{\mathcal{M}}'_0$, if the rational function $P(a;T)$ has no pole at $T = 1$, then we define the *topological Euler characteristic* of a by

$$e_{\mathrm{top}}(a) := P(a;1) \in \mathbb{Q}.$$

For a variety X, we have $e_{\mathrm{top}}([X]) = e_{\mathrm{top}}(X)$.

When k is a finite field contained in $\mathbb{F}_q$, we can easily see that the map $\sharp_q$ uniquely extends to the ring map $\mathcal{M}' \longrightarrow \mathbb{Q}$. This map further descends to $\bar{\mathcal{M}}' \longrightarrow \mathbb{Q}$,[3] which we still denote by $\sharp_q$. Then, for $a \in \hat{\mathcal{M}}'_0$ expressed as in (4.2), we put

$$\sharp_q(a) := \sharp_q(b) + \sharp_q(b)q^{-n} + \sharp_q(b)q^{-2n} + \cdots = \frac{\sharp_q(b)}{1-q^{-n}} \in \mathbb{Q}.$$

Note that the first expression as a series converges in $\mathbb{R}$ and coincides with the second expression. This is also independent of the expression of a, since, for another expression

$$a = \frac{b'}{1-\mathbb{L}^{-n'}}$$

with $n \mid n'$, there exists a polynomial $f(\mathbb{L}^{-1})$ in $\mathbb{L}^{-1}$ with integer coefficients such that $b' = b \cdot f(\mathbb{L}^{-1})$ and $1 - \mathbb{L}^{-n'} = (1-\mathbb{L}^{-n})f(\mathbb{L}^{-1})$. For a variety X, we have $\sharp_q([X]) = \sharp X(\mathbb{F}_q)$.

4.4 Motivic integration over a variety

We briefly review the motivic integration over singular varieties. The original reference for the theory in characteristic zero is [DL99]. For the positive characteristic case, see [Seb04].

Let X be a reduced k-variety of pure dimension d.

Definition 4.7. A subset $C \subset J_\infty X$ is called a *cylinder* if for some $0 \leqslant n < \infty$, $\pi_n(C) \subset J_n X$ is a constructible subset and $C = \pi_n^{-1}(\pi_n(C))$. A subset $C \subset J_\infty X$ is called *stable* if, for some $0 \leqslant n < \infty$, $\pi_n(C) \subset J_n X$ is a constructible subset and, for all $n' \geqslant n$, the map $\pi_{n'+1}(C) \longrightarrow \pi_{n'}(C)$ is a piecewise trivial fibration with fiber $\mathbb{A}^d_k$. For a stable subset $C \subset J_\infty X$, we define its *measure* by

$$\mu_X(C) := [\pi_n(C)]\mathbb{L}^{-nd} \in \hat{\mathcal{M}}' \; (n \gg 0).$$

Remark 4.8. In some literature, the value of measure and hence all computations following it differ by a factor $\mathbb{L}^d$.

[3] Indeed, by the realization map of $\hat{\mathcal{M}}'$ to the completed Grothendieck ring $\hat{K}_0(MR(\mathrm{Gal}(\bar{k}/k), \mathbb{Q}_l))$ of mixed Galois representations constructed in the same way as in [Yas06, pp. 728–730], the subring $\bar{\mathcal{M}}'$ maps to the non-completed Grothendieck ring $K_0(MR(\mathrm{Gal}(\bar{k}/k), \mathbb{Q}_l))$, which is a subring of the completed one. Then we can compose the map $\bar{\mathcal{M}}' \longrightarrow K_0(MR(\mathrm{Gal}(\bar{k}/k), \mathbb{Q}_l))$ with the map $K_0(MR(\mathrm{Gal}(\bar{k}/k), \mathbb{Q}_l)) \longrightarrow \mathbb{Q}$ defined by the alternating trace of the Frobenius action, and obtain the desired map.

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DEFINITION 4.9. For an ideal sheaf $I \subset \mathcal{O}_X$, we define a function

$$\begin{aligned} \operatorname{ord} I : J_\infty X &\longrightarrow \mathbb{Z}_{\geqslant 0} \cup \{\infty\} \\ \gamma &\longmapsto \operatorname{length} k[[t]]/\gamma^{-1} I. \end{aligned}$$

The *Jacobian ideal sheaf* $\operatorname{Jac}_X \subset \mathcal{O}_X$ of X is defined to be the dth Fitting ideal of the sheaf of differentials, $\Omega_{X/k}$. This defines a closed subscheme supported on the singular locus. For a cylinder $C \subset J_\infty X$ and for each $e \in \mathbb{N}$, the subset $C \cap (\operatorname{ord} \operatorname{Jac}_X)^{-1}(e)$ is stable, and we define

$$\mu_X(C) := \sum_{e=0}^{\infty} \mu_X(C \cap (\operatorname{ord} \operatorname{Jac}_X)^{-1}(e)).$$

This indeed converges in $\hat{\mathcal{M}}'$.

DEFINITION 4.10. A subset $C \subset J_\infty X$ is *measurable* if for any $\epsilon \in \mathbb{R}_{>0}$, there exists a sequence of cylinders, $C_0(\epsilon)$, $C_1(\epsilon), \ldots,$ such that

$$C \Delta C_0(\epsilon) \subset \bigcup_{i \geqslant 1} C_i(\epsilon)$$

and $\|\mu_X(C_i(\epsilon))\| < \epsilon$ for all $i \geqslant 1$. Here Δ denotes the symmetric difference. If we can take $C_0(\epsilon) \subset C$, we call C *strongly measurable.*

We define the measure of a measurable subset $C \subset J_\infty X$ by

$$\mu_X(C) := \lim_{\epsilon \longrightarrow 0} \mu_X(C_0(\epsilon))$$

with $C_0(\epsilon)$ as above. This converges and the limit is independent of the choice of $C_0(\epsilon)$.

DEFINITION 4.11. Let $A \subset J_\infty X$ be a subset and $F : A \longrightarrow \mathbb{Z} \cup \{\infty\}$ a function on it. We say that F is *measurable* if every fiber $F^{-1}(n)$ is measurable. We say that F is *exponentially integrable*[4] if:

(i) F is measurable;
(ii) $F^{-1}(\infty)$ has measure zero; and
(iii) for every $\epsilon > 0$, there exist at most finitely many $n \in \mathbb{Z}$ such that $\|\mu_X(F^{-1}(n))\| > \epsilon$.

For an exponentially integrable function $F : J_\infty X \supset A \longrightarrow \mathbb{Z} \cup \{\infty\}$, the *integral* of $\mathbb{L}^F$ is defined as

$$\int_A \mathbb{L}^F \, d\mu_X := \sum_{n \in \mathbb{Z}} \mu_X(F^{-1}(n)) \mathbb{L}^n \in \hat{\mathcal{M}}'.$$

4.5 Motivic integration over the quotient stack $\mathcal{X}$

Let V be a d-dimensional non-trivial G-representation and $\mathcal{X} := [V/G]$. The following arguments contain a lot of repetition from the previous subsection and from the literature. However, we have to pay attention to slight differences coming from the fact that the space of twisted arcs, $\mathcal{J}_\infty \mathcal{X}$, is a projective limit of inductive limits of varieties, while $J_\infty X$ is only a projective limit.

DEFINITION 4.12. A subset $C \subset \mathcal{J}_n \mathcal{X}$ is called *constructible* if it is a constructible subset of $\mathcal{J}_{n, \leqslant j} \mathcal{X}$ for some $j \in \mathbb{N}$. A subset $C \subset \mathcal{J}_\infty \mathcal{X}$ is called a *cylinder* if for some n, $\pi_n(C)$ is constructible and $C = \pi_n^{-1}(\pi_n(C))$.

[4] In the literature, $-F$ is called exponentially integrable when the same condition holds.

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For a cylinder C, we define its measure by

$$\mu_{\mathcal{X}}(C) := [\pi_n(C)]\mathbb{L}^{-nd} \in \hat{\mathcal{M}}' \, (n \gg 0).$$

This is well defined from Corollary 3.14.

Definition 4.13. A subset $C \subset \mathcal{J}_\infty \mathcal{X}$ is *measurable* if for any $\epsilon \in \mathbb{R}_{>0}$, there exists a sequence of cylinders, $C_0(\epsilon)$, $C_1(\epsilon), \ldots$, such that

$$C \Delta C_0(\epsilon) \subset \bigcup_{i \geqslant 1} C_i(\epsilon)$$

and $\|\mu_{\mathcal{X}}(C_i(\epsilon))\| < \epsilon$ for all $i \geqslant 1$. If we can take $C_0(\epsilon) \subset C$, then we say that C is *strongly measurable.*

We define the measure of a measurable subset $C \subset \mathcal{J}_\infty \mathcal{X}$ by

$$\mu_{\mathcal{X}}(C) := \lim_{\epsilon \longrightarrow 0} \mu_{\mathcal{X}}(C_0(\epsilon)).$$

We can show that the limit is independent of the choice of $C_0(\epsilon)$ in the same way as the proof of [Bat98, Theorem 6.18] using the following lemma.

Lemma 4.14. *Let C and C_i, $i \in \mathbb{N}$, be cylinders in $\mathcal{J}_\infty \mathcal{X}$. If $C \subset \bigcup_{i \in \mathbb{N}} C_i$, then for some $m \in \mathbb{N}$, we have $C \subset \bigcup_{i=0}^m C_i$.*

Proof. The proof follows that of [Seb04, Lemma 4.3.7]. Suppose that $C \subset \mathcal{J}_{\infty, \leqslant j} \mathcal{X}$. Then, replacing C_i with $C_i \cap \mathcal{J}_{\infty, \leqslant j} \mathcal{X}$, we may suppose also that $C_i \subset \mathcal{J}_{\infty, \leqslant j} \mathcal{X}$. Then since $\mathcal{J}_{\infty, \leqslant j} \mathcal{X}$ is affine and hence quasi-compact, the lemma follows from the quasi-compactness of the constructible topology [GD71, § 7, Proposition 7.2.13]. □

We now define *measurable* and *exponentially integrable functions* defined on subsets of $\mathcal{J}_\infty \mathcal{X}$, and the *integral* of an exponentially integrable function in the exactly same way as in Definition 4.11. Following [Yas04, Yas06], we define the order function associated to an ideal sheaf on $\mathcal{X}$ as follows.

Definition 4.15. For a coherent ideal sheaf $I \subset \mathcal{O}_{\mathcal{X}}$ and for a twisted arc $\gamma : \mathcal{D} \longrightarrow \mathcal{X}$, we define a function $\operatorname{ord} I : \mathcal{J}_\infty \mathcal{X} \longrightarrow (1/p)\mathbb{Z} \cup \{\infty\}$ as follows: Let $E \xrightarrow{\alpha} \mathcal{D} \longrightarrow D$ be the associated G-cover of D. Then

$$\operatorname{ord} I(\gamma) := \frac{1}{p} \cdot \operatorname{length}\left(\frac{\mathcal{O}_E}{(\gamma \circ \alpha)^{-1} I} \right).$$

If $\mathcal{Y}$ is the closed substack of $\mathcal{X}$ defined by the ideal sheaf I, then we write $\operatorname{ord} I$ also as $\operatorname{ord} \mathcal{Y}$.

4.6 Some technical results

Here we collect technical results on the measurability and integrability which will be needed below.

Lemma 4.16. *Let $\mathcal{Y} \subset \mathcal{X}$ be a closed substack. Then for every $n \in (1/p)\mathbb{Z}_{\geqslant 0}$ and for every $j \in \mathbb{N}'_0$, $(\operatorname{ord} \mathcal{Y})^{-1}(n) \cap \mathcal{J}_{\infty, j} \mathcal{X}$ is a cylinder.*

Proof. Let $n' := \lceil n \rceil$, where $\lceil \cdot \rceil$ is the ceiling function. Let

$$\begin{array}{ccc} E & \xrightarrow{\gamma} & V \\ {\scriptstyle \xi} \downarrow & & \\ J^G_{n',j} V & & \end{array}$$

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be the universal G-n'-jet of ramification jump j. Then we consider the coherent sheaf $\mathcal{F} := \xi_*(\gamma^*\mathcal{O}_{\mathcal{Y}})$ over $J^G_{n',j}V$. From the semicontinuity,

$$W := \{x \in J^G_{n',j}V \mid \operatorname{length} \mathcal{F} \otimes \kappa(x) \geqslant pn\}$$

is a closed subset. Let $\bar{W}$ be the image of W in $\mathcal{J}_{n',j}\mathcal{X}$. Then

$$(\operatorname{ord} \mathcal{Y})^{-1}(\geqslant n) \cap \mathcal{J}_{\infty,j}\mathcal{X} = \pi_n^{-1}(\bar{W}),$$

which is a cylinder. Hence $(\operatorname{ord} \mathcal{Y})^{-1}(n) \cap \mathcal{J}_{\infty,j}\mathcal{X}$ is also a cylinder. □

LEMMA 4.17. *Let $\mathcal{Y} \subset \mathcal{X}$ be a closed substack of positive codimension. Then the subset $\mathcal{J}_\infty\mathcal{Y} := (\operatorname{ord} \mathcal{Y})^{-1}(\infty)$ of $\mathcal{J}_\infty\mathcal{X}$ is measurable and has measure zero.*

Proof. For any $\epsilon > 0$, we choose n, j and n_i $(i > j)$ so that $n \gg j \gg 0$ and $n_{i_1} \gg n_{i_2}$ $(i_1 > i_2)$. Then

$$\begin{aligned}&(\mathcal{J}_\infty\mathcal{Y})\Delta((\operatorname{ord} \mathcal{Y})^{-1}(\geqslant n) \cap \mathcal{J}_{\infty,\leqslant j}\mathcal{X})\\ &\quad\subset ((\operatorname{ord} \mathcal{Y})^{-1}(\geqslant n) \cap \mathcal{J}_{\infty,\leqslant j}\mathcal{X}) \cup \bigcup_{i>j}((\operatorname{ord} \mathcal{Y})^{-1}(\geqslant n_i) \cap \mathcal{J}_{\infty,i}\mathcal{X}).\end{aligned}$$

This shows the lemma. □

DEFINITION 4.18. We define a subsemiring $\mathcal{N} \subset \hat{\mathcal{M}}'$ by

$$\mathcal{N} := \left\{ \sum_{i\in\mathbb{N}} [X_i]\mathbb{L}^{n_i} \in \hat{\mathcal{M}}' \;\middle|\; X_i \in \mathrm{Var}_k,\ \lim_{i\longrightarrow\infty} \dim X_i + n_i = -\infty \right\}.$$

(Notice that there is no minus sign in the above series.)

We need this semiring for a technical reason. In fact, motivic measures and motivic integrals take values in $\mathcal{N}$ (or its variant added with $\mathbb{L}^{1/r}$, defined below).

LEMMA 4.19. *For $a, b \in \mathcal{N}$, we have that $\|a + b\| = \max\{\|a\|, \|b\|\}$.*

Proof. Let us write $a = \sum_{i\in\mathbb{N}} [X_i]\mathbb{L}^{n_i}$ and put $n := \max\{\dim X_i + n_i \mid i \in \mathbb{N}\}$. Then $\|a\| = 2^n$. This fact proves the lemma. (In the semiring $\mathcal{N}$, we can avoid difficulties coming from cancellation of terms.) □

LEMMA 4.20. *For $i, j \in \mathbb{N}$, let $a_{ij} \in \mathcal{N}$. Then the following are equivalent:*

(i) *for every i, $\lim_{j\longrightarrow\infty} \|a_{ij}\| = 0$ and $\lim_{i\longrightarrow 0} \|\sum_{j\in\mathbb{N}} a_{ij}\| = 0$;*
(ii) *for every j, $\lim_{i\longrightarrow\infty} \|a_{ij}\| = 0$ and $\lim_{j\longrightarrow 0} \|\sum_{i\in\mathbb{N}} a_{ij}\| = 0$;*
(iii) *for every $\epsilon > 0$, there exist at most finitely many pairs $(i, j) \in \mathbb{N}^2$ such that $\|a_{ij}\| > \epsilon$.*

Moreover, if one of the above conditions holds, then

$$\sum_{i,j\in\mathbb{N}^2} a_{ij} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij}.$$

Proof. Following the definition, we can translate the first condition as follows: for every $\epsilon > 0$ and every i, there exist at most finitely many j with $\|a_{ij}\| > \epsilon$, and for every ϵ', there exist at most finitely many i with

$$\left\| \sum_{j=0}^{\infty} a_{ij} \right\| = \max\{\|a_{ij}\| \mid j \in \mathbb{N}\} > \epsilon'.$$

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(The equality above follows from the previous lemma.) Then this is equivalent to the third condition. We can similarly prove the equivalence of the second and third. □

Proposition 4.21. *Let A_i, $i \in \mathbb{N}$, be mutually disjoint subsets of $\mathcal{J}_\infty \mathcal{X}$ and let $A := \bigsqcup_{i=0}^{\infty} A_i$. Let $F : A \longrightarrow \mathbb{Z} \cup \{\infty\}$ be a function such that for every i, $F|_{A_i}$ is measurable. Then F is exponentially integrable if and only if for every i, $F|_{A_i}$ is exponentially integrable and*

$$\lim_{i \longrightarrow 0} \left\| \int_{A_i} \mathbb{L}^F \, d\mu_{\mathcal{X}} \right\| = 0.$$

Moreover, if this is the case, then

$$\int_A \mathbb{L}^F \, d\mu_{\mathcal{X}} = \sum_{i=0}^{\infty} \int_{A_i} \mathbb{L}^F \, d\mu_{\mathcal{X}}.$$

Proof. To prove the 'if' part, we can easily see that

$$\lim_{i \longrightarrow \infty} \| \mu_{\mathcal{X}}(F^{-1}(n) \cap A_i) \| = 0.$$

It follows that $F^{-1}(n) = \bigsqcup_i F^{-1}(n) \cap A_i$ is measurable. From the previous lemma, we conclude that F is exponentially integrable.

We now turn to the 'only if' part. Obviously $F|_{A_i}$ is exponentially integrable. Then the assertion that

$$\lim_{i \longrightarrow 0} \left\| \int_{A_i} \mathbb{L}^F \, d\mu_{\mathcal{X}} \right\| = 0$$

follows from the previous lemma.

The last assertion also follows from the previous lemma. □

Lemma 4.22. *Let $F : \mathcal{J}_\infty \mathcal{X} \supset A \longrightarrow \mathbb{Z} \cup \{\infty\}$. Then F is exponentially integrable if and only if there exist measurable subsets A_i, $i \in \mathbb{N}$ such that:*

(i) *$A = \bigsqcup_{i \in \mathbb{N}} A_i$;*
(ii) *A_0 has measure zero;*
(iii) *F has a finite constant value on each A_i, $i > 0$, and*

$$\lim_{i \longrightarrow \infty} \| \mu_{\mathcal{X}}(A_i) \mathbb{L}^{F(A_i)} \| = 0.$$

Proof. The 'only if' part is obvious. Suppose that there exist such measurable subsets A_i. Then for each $n \in \mathbb{Z} \cup \{\infty\}$,

$$F^{-1}(n) = \bigsqcup_{F(A_i) = n} A_i.$$

Then by assumption, F is exponentially integrable on $F^{-1}(n)$ and

$$\lim_{n \longrightarrow \infty} \left\| \int_{F^{-1}(n)} \mathbb{L}^F \, d\mu_{\mathcal{X}} \right\| = 0.$$

From Lemma 4.20, F is exponentially integrable. □

LEMMA 4.23. *Let $C \subset \mathcal{J}_\infty \mathcal{X}$ be a strongly measurable subset and $F : C \longrightarrow \mathbb{Z} \cup \{\infty\}$ an exponentially integrable function. Let $C_0(\epsilon)$, $\epsilon \in \mathbb{R}_{>0}$ as in Definition 4.13. Then*

$$\int_C \mathbb{L}^F \, d\mu_{\mathcal{X}} = \lim_{\epsilon \longrightarrow 0} \int_{C_0(\epsilon)} \mathbb{L}^F \, d\mu_{\mathcal{X}}.$$

Proof. We see that

$$\lim_{\epsilon \longrightarrow 0} \left\| \int_{C \setminus C_0(\epsilon)} \mathbb{L}^F \, d\mu_{\mathcal{X}} \right\| = 0.$$

Now the lemma follows from the obvious equality

$$\int_C \mathbb{L}^F \, d\mu_{\mathcal{X}} = \int_{C \setminus C_0(\epsilon)} \mathbb{L}^F \, d\mu_{\mathcal{X}} + \int_{C_0(\epsilon)} \mathbb{L}^F \, d\mu_{\mathcal{X}}.$$ □

4.7 Adding fractional powers of $\mathbb{L}$

In applications, we often consider functions on arc spaces with fractional values. For this reason, we need to add fractional powers of $\mathbb{L}$ to Grothendieck rings. For a positive integer r, we put

$$\mathcal{M}'_{1/r} := \mathcal{M}'[\mathbb{L}^{1/r}] = \mathcal{M}'[x]/(x^r - \mathbb{L}).$$

Then its dimensional completion $\hat{\mathcal{M}}'_{1/r}$ is defined similarly. Now for an exponentially integrable function

$$F : J_\infty X \ (\text{or } \mathcal{J}_\infty \mathcal{X}) \supset A \longrightarrow \frac{1}{r}\mathbb{Z} \cup \{\infty\},$$

its integral $\int_A \mathbb{L}^F \, d\mu_X$ (or $\int_A \mathbb{L}^F \, d\mu_{\mathcal{X}}$) is defined as an element of $\hat{\mathcal{M}}'_{1/r}$.

If r divides r', then we can identify $\hat{\mathcal{M}}'_{1/r}$ with a subring of $\hat{\mathcal{M}}'_{1/r'}$. Then the values of measures and integrals are independent of which ring we consider. Therefore, in what follows, we will not make the value ring explicit.

4.8 Motivic integration on G-$\mathbf{Cov}^{\mathrm{rep}}(D)$

For a constructible subset C of G-$\mathbf{Cov}^{\mathrm{rep}}(D, \leqslant j)$, we define its *measure* simply as

$$\nu(C) := [C] \in \hat{\mathcal{M}}'.$$

Let $F :$ G-$\mathbf{Cov}^{\mathrm{rep}}(D) \longrightarrow \mathbb{Z}$ be a function which is constant on each stratum G-$\mathbf{Cov}^{\mathrm{rep}}(D, j)$. Then we write $F($G-$\mathbf{Cov}^{\mathrm{rep}}(D, j))$ as $F(j)$. Suppose that

$$\lim_{j \longrightarrow \infty} F(j) + j - \lfloor j/p \rfloor = -\infty.$$

Then we define the *integral* of $\mathbb{L}^F$ by

$$\begin{aligned} \int_{\text{G-}\mathbf{Cov}^{\mathrm{rep}}(D)} \mathbb{L}^F \, d\nu &= \sum_{j \in \mathbb{N}'_0} \nu(\text{G-}\mathbf{Cov}^{\mathrm{rep}}(D, j)) \mathbb{L}^{F(j)} \\ &= (\mathbb{L} - 1)\mathbb{L}^{-1} \sum_{j \in \mathbb{N}'_0} \mathbb{L}^{j - \lfloor j/p \rfloor + F(j)}. \end{aligned}$$

PROPOSITION 4.24. *Let $(\mathcal{J}_\infty \mathcal{X})_0$ be the preimage of the origin by the projection $\mathcal{J}_\infty \mathcal{X} \longrightarrow \mathcal{X}$. Let $\pi : \mathcal{J}_\infty \mathcal{X} \longrightarrow$ G-$\mathbf{Cov}^{\mathrm{rep}}(D)$ be the projection. Then we have*

$$\int_{\text{G-}\mathbf{Cov}^{\mathrm{rep}}(D)} \mathbb{L}^F \, d\nu = \int_{(\mathcal{J}_\infty \mathcal{X})_0} \mathbb{L}^{F \circ \pi} \, d\mu_{\mathcal{X}}.$$

Proof. From Corollary 3.14, the preimage of $\mathcal{J}_{0,j} \mathcal{X} \longrightarrow \mathcal{X}$ of the origin is isomorphic to G-$\mathbf{Cov}^{\mathrm{rep}}(D, j)$. The proposition follows from this. □

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5. The change of variables formula

We keep the notation from the previous section: V is a non-trivial G-representation, $\mathcal{X} = [V/G]$ and $X = V/G$. In this section, we will prove the change of variables formula for the map $\phi_\infty : \mathcal{J}_\infty\mathcal{X} \longrightarrow J_\infty X$, which enables us to express integrals on $\mathcal{J}_\infty\mathcal{X}$ as ones on $J_\infty X$, and vice versa.

5.1 Preliminary results

Definition 5.1. Let $S = k[\mathbf{x}]$ be the coordinate ring of V and $R := S^G$ that of X. The *Jacobian ideal* $\mathrm{Jac}_\psi \subset S$ of the quotient map $\psi : V \longrightarrow X$ is defined as the zeroth Fitting ideal of the module of differentials, $\Omega_{V/X}$. This defines a coherent ideal sheaf on $\mathcal{X}$, which we call the *Jacobian ideal* of $\phi : \mathcal{X} \longrightarrow X$ and denote it by $\mathrm{Jac}_\phi \subset \mathcal{O}_\mathcal{X}$.

Lemma 5.2. *Let $f = \sum_{i\geqslant r} a_i t^i \in k[[t]]$, $a_r \neq 0$, be a power series of order r and let $f^{-1} = \sum_{i\geqslant -r} b_i t^i \in k((t))$ be its inverse. Then the negative part, $\sum_{i=-r}^{-1} b_i t^i$, of f^{-1} depends only on the class of f in $k[[t]]/(t^{2r})$.*

Proof. The classes of $t^{-r} f$ and $t^r f^{-1}$ in $k[[t]]/(t^r)$ are mutual inverses. The negative part of f^{-1} depends only on the class of $t^r f^{-1}$ in $k[[t]]/(t^r)$. Then it depends only on the class of $t^{-r} f$ in $k[[t]]/(t^r)$ and depends only on the class of f in $k[[t]]/(t^{2r})$. □

Proposition 5.3. *There exists a constant $c \geqslant 2$ depending only on V such that if $\gamma, \gamma' \in \mathcal{J}_\infty\mathcal{X}$ and if $\phi_n\gamma_n = \phi_n\gamma'_n$ for some n with*

$$n \geqslant c \cdot \mathrm{ord}\,\mathrm{Jac}_\phi(\gamma),$$

then γ and γ' have the same associated G-covers of D.

Proof. From Lemmas 6.17 and 6.29 below, Jac_ψ is generated by elements of R, say $f_1, \dots, f_l \in R$. Set $\bar{\gamma} := \phi_\infty\gamma$ and $\tilde{f}_i := \bar{\gamma}^* f_i \in k[[t]]$. Then $e := \mathrm{ord}\,\mathrm{Jac}_\phi(\gamma)$ is equal to the minimum of $\mathrm{ord}\,\tilde{f}_i$, say $\mathrm{ord}\,\tilde{f}_1$. Let S_{f_1} and R_{f_1} be localizations of S and R by f_1. Then $\mathrm{Spec}\, S_{f_1} \longrightarrow \mathrm{Spec}\, R_{f_1}$ is an étale G-cover and we have

$$S_{f_1} = R_{f_1}[\wp^{-1}(g/f_1^m)]$$

for some $g \in R$ and $m \geqslant 0$. Then the G-cover $E^* \longrightarrow D^*$ associated to γ is given by

$$\mathcal{O}_{E^*} = k((t))[\wp^{-1}(\tilde{g}/\tilde{f}_1{}^m)],$$

with $\tilde{g} := \bar{\gamma}^* g$. Hence E^* is determined by the negative part of the Laurent power series $\tilde{g}/\tilde{f}_1{}^m$. From the previous lemma, the negative part of $(\tilde{f}_1)^{-m}$ is determined by $\tilde{f}_1^m$ modulo t^{2me}. The terms of $\tilde{g}$ of degree at least me do not contribute to the negative part of $\tilde{g}/\tilde{f}_1{}^m$. This shows that E^* depends only on $\phi_n\gamma_n$ if $n \geqslant 2me$. The proposition follows. □

Proposition 5.4. *Let c be a constant as in the previous proposition. Then there exists a constant $c' > 0$ depending only on X such that for $\gamma, \gamma' \in \mathcal{J}_\infty\mathcal{X}$, if we put $e := \mathrm{ord}\,\mathrm{Jac}_\phi(\gamma)$ and if $\phi_\infty\gamma_n = \phi_\infty\gamma'_n$ for some n with*

$$n \geqslant \max\{ce, c' \cdot \mathrm{ord}\,\mathrm{Jac}_X(\phi_\infty\gamma)\},$$

then $\gamma_{n-e} = \gamma'_{n-e}$.

Proof. Let $\tilde{\gamma}, \tilde{\gamma'} \in J_\infty^G V$ be liftings of γ, γ', respectively. From the previous proposition, γ and γ' have the same G-cover E of D. Fixing an isomorphism $E \cong \operatorname{Spec} k[[s]]$, we can think of $\tilde{\gamma}, \tilde{\gamma'}$ as elements of the ordinary arc space $J_\infty V$ of V. Let c' be the constant c_X in [Seb04, Lemme 7.1.1]. Then there exists $\beta \in J_\infty V$ such that $\phi_\infty \beta = \phi_\infty \tilde{\gamma'}$ and $\beta_{pn-pe} = \tilde{\gamma}_{pn-pe}$. (Here $\tilde{\gamma}$ is now considered as an element of $J_\infty V$ and so $\tilde{\gamma}_{pn}$ is the image of $\tilde{\gamma}$ by $J_\infty V \longrightarrow J_{pn} V$, which is the same as the image of $\tilde{\gamma}$ by $J_\infty^G V \longrightarrow J_n^G V$.) The equality $\phi_\infty \beta = \phi_\infty \tilde{\gamma'}$ shows that β is actually a G-arc and in the same G-orbit as $\tilde{\gamma'}$. Then the equality $\beta_{pn-pe} = \tilde{\gamma}_{pn-pe}$ implies $\gamma_{n-e} = \gamma'_{n-e}$. □

We can rephrase the proposition as the following corollary.

Corollary 5.5. *Let $\gamma \in \mathcal{J}_\infty \mathcal{X}$, $e := \operatorname{ord} \operatorname{Jac}_\phi(\gamma)$ and $e' := \operatorname{ord} \operatorname{Jac}_X(\phi_\infty \gamma)$. Let c and c' be positive constants as above. Then if $n \geqslant \max\{ce, c'e'\}$, then $\phi_n^{-1}(\phi_n \gamma_n)$ is included in the fiber of $\pi_n(\mathcal{J}_\infty \mathcal{X}) \longrightarrow \pi_{n-e}(\mathcal{J}_\infty \mathcal{X})$ over γ_{n-e}.*

Corollary 5.6. *Let $C \subset \mathcal{J}_\infty \mathcal{X}$ be a cylinder with $C \cap (\operatorname{ord} \operatorname{Jac}_\phi)^{-1}(\infty) = \emptyset$. Then $\phi_\infty(C) \subset J_\infty X$ is a stable subset.*

Proof. From Lemma 4.14, without loss of generality, we may suppose that the functions $\operatorname{ord} \operatorname{Jac}_\phi$ and $\operatorname{ord} \phi^{-1} \operatorname{Jac}_X$ take constant values, say e and e', on C. Let $n \in \mathbb{N}$ be such that C is a cylinder at level n and $n \geqslant \max\{ce, c'e'\}$. Then from the previous corollary, $\phi_\infty(C)$ is a cylinder at level $n + e$. Moreover, since the function $\operatorname{ord} \operatorname{Jac}_X$ is constant on it, $\phi_\infty(C)$ is stable. □

Corollary 5.7. *If $C \subset \mathcal{J}_\infty \mathcal{X}$ is a (strongly) measurable subset, then so is $\phi_\infty(C)$.*

Proof. If $\phi_\infty(C) \backslash J_\infty Y$ is strongly measurable, so is $\phi_\infty(C)$. Hence we may suppose that C is disjoint from $\mathcal{J}_\infty \mathcal{Y}$. Then, let $C_i(\epsilon) \subset \mathcal{J}_\infty \mathcal{X}$ be cylinders as in Definition 4.13. Replacing $C_i(\epsilon)$, $i > 0$, with their intersections with $(\operatorname{ord} \mathcal{Y})^{-1}(n) \cap \mathcal{J}_{\infty,j} \mathcal{X}$, $n \in \mathbb{N}$, $j \in \mathbb{N}'_0$, we may suppose that $C_i(\epsilon)$ are all disjoint from $\mathcal{J}_\infty \mathcal{Y}$. Then from the previous corollary, $\phi_\infty(C_i(\epsilon))$ are cylinders as well. Moreover, we can easily see that

$$\|\mu_{\mathcal{X}}(C_i(\epsilon))\| \geqslant \|\mu_X(\phi_\infty(C_i(\epsilon)))\|.$$

This shows that $\phi_\infty(C)$ is strongly measurable. □

5.2 The key dimension count

The essential part in the proof of the change of variables formula is counting the dimension of $\phi_n^{-1}(\phi_n \gamma_n)$ for $n \gg 0$. To do this, we will follow Looijenga's argument [Loo02].

5.2.1 *Identifying $\phi_n^{-1}(\phi_n \gamma_n)$ with a certain* Hom *module.* For simplicity, we first suppose that V is indecomposable and that the G-action on the coordinate ring $k[\mathbf{x}] = k[x_1, \ldots, x_d]$ is given by $\sigma(x_i) = x_i + x_{i+1}$ $(i < d)$ and $\sigma(x_d) = x_d$. Let $\gamma, \gamma' \in \mathcal{J}_\infty \mathcal{X}$ be such that $\phi_n \gamma_n = \phi_n \gamma'_n$ for $n \geqslant \max\{ce, c'e'\}$ with the notation as above. Then we can choose their liftings $\beta, \beta' \in J_\infty^G V$ such that $\beta_{n-e} = \beta'_{n-e}$. Let E be the G-cover of D associated with β and β'. Then β^* and $(\beta')^*$ induce the same S-module structure on $\mathfrak{m}_E^{(n-e)p+1} / \mathfrak{m}_E^{2(n-e)p+2}$. Since $np + 1 \leqslant 2(n-e)p + 2$, it induces an S-module structure on $M_{n,e} := \mathfrak{m}_E^{(n-e)p+1} / \mathfrak{m}_E^{np+1}$. Then the induced map

$$\beta^* - (\beta')^* : S \longrightarrow M_{n,e}$$

is a k-derivation and corresponds to an S-linear map

$$\Omega_{S/k} \longrightarrow M_{n,e}$$

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and an $\mathcal{O}_E$-linear map

$$\Delta_{\beta,\beta'} : \beta^*\Omega_{S/k} \longrightarrow M_{n,e},$$

which are G-equivariant. Moreover, from the construction,

$$\Delta_{\beta,\beta'}(dx_1) \in M^\natural_{n,e} := \mathrm{Im}((\mathfrak{m}_E^{(n-e)p+1})^{\delta^d=0} \longrightarrow M_{n,e}).$$

Let $F_{n,e,\beta}$ be the fiber of the map $\pi_n(J^G_\infty V) \longrightarrow \pi_{n-e}(J^G_\infty V)$ over β_{n-e}. Let

$$\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e}) := \{\alpha \in \mathrm{Hom}_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e}) \mid \alpha \text{ is } G\text{-equiv. and } \alpha(dx_1) \in M^\natural_{n,e}\}.$$

Then we have an injection

$$\begin{aligned} F_{n,e,\beta} &\longrightarrow \mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e}) \\ \beta'_n &\longmapsto \Delta_{\beta,\beta'}. \end{aligned}$$

Since a G-equivariant map α is determined by $\alpha(dx_1)$, $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e})$ is identified with $M^\natural_{n,e}$. Comparing the dimensions, we conclude that $F_{n,e,\beta}$ is identified with $M^\natural_{n,e}$ and with $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e})$.

Consider the exact sequence

$$0 \longrightarrow \mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e}) \longrightarrow \mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_E}((\psi \circ \beta)^*\Omega_{R/k}, M_{n,e}),$$

where $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e})$ is the preimage of $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e})$ in $\mathrm{Hom}_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e})$. Then β and β' have the same image in J_nX if and only if $\Delta_{\beta,\beta'}$ maps to $0 \in \mathrm{Hom}_{\mathcal{O}_E}((\psi\circ\beta)^*\Omega_{R/k}, M_{n,e})$. This shows the following proposition.

Proposition 5.8. *The fiber of the map $F_{n,e,\beta} \longrightarrow J_nX$ over $\phi_n\gamma_n$ is identified with $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e})$. Hence $\phi_n^{-1}(\phi_n\gamma_n)$ is universally homeomorphic to the quotient of $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e})$ by some G-linear action. In particular,*

$$[\phi_*^{-1}(\phi_*\gamma_n)] = [\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e})] \in \hat{\mathcal{M}}'.$$

When V is decomposable, we define $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e})$ to be the submodule of $\mathrm{Hom}_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_{n,e})$ consisting of those G-equivariant maps α with $\alpha(dx_{\lambda,1}) \in M^\natural_{n,e}$, $1 \leqslant \lambda \leqslant l$. Then we similarly define $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e})$. Now Proposition 5.8 holds also in the decomposable case by the same reasoning.

5.2.2 *Counting the dimension of* $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}M_{n,e})$*: the indecomposable case.* We now suppose that V is indecomposable. The decomposable case will be discussed in the next subsection. To count the dimension of $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e})$, we have to know the precise structure of the module $\beta^*\Omega_{S/R}$. It is the quotient of a free module $\beta^*\Omega_{S/k} = \bigoplus_i \mathcal{O}_E \cdot dx_i$ by the submodule $\mathrm{Im}((\psi \circ \beta)^*\Omega_{R/k} \longrightarrow \beta^*\Omega_{S/k})$. Then we first note that the submodule is generated by G-invariant elements. Let $\omega = \sum_{i=1}^d \omega_i dx_i \in \beta^*\Omega_{S/k}$ be G-invariant. Then

$$\begin{aligned} \sigma(\omega) &= \sum_{i=1}^{d-1} \sigma(\omega_i)(dx_i + dx_{i+1}) + \sigma(\omega_d)dx_d \\ &= \sigma(\omega_1)dx_1 + \sum_{i=2}^{d}(\sigma(\omega_{i-1}) + \sigma(\omega_i))dx_i \\ &= \omega. \end{aligned}$$

Hence $\sigma(\omega_1) = \omega_1$ and $\sigma(\omega_{i-1}) + \sigma(\omega_i) = \omega_i$, $i \geqslant 2$.

Notation 5.9. For an abelian group M endowed with a G-action, we define an operator δ_- on M by

$$\delta_- := \sigma^{-1} - \mathrm{id}_M = -\sigma^{-1}\delta.$$

Then $\omega_{i-1} = \delta_-(\omega_i)$ and ω_1 is G-invariant. Furthermore, this is equivalent to

$$\omega_i = \delta_-^{d-i}(\omega_d) \quad \text{and} \quad \delta_-^d(\omega_d) = 0.$$

Notation 5.10. For $f \in \mathcal{O}_E^{\delta^d=0}$, we define a G-invariant element $\omega_f \in \beta^*\Omega_{S/k}$ by

$$\omega_f := \sum_{i=1}^{d} \delta_-^{d-i}(f) \cdot dx_i.$$

We note that if $h \in k[[t]]$, then we have

$$\omega_{hf} = h \cdot \omega_f.$$

As before, we write $\mathcal{O}_{E^*} = k((t))[\wp^{-1}f]$ and put $g := \wp^{-1}f$. Then $1, g, \ldots, g^{p-1}$ form a basis of $\mathcal{O}_{E^*}$ as a $k((t))$-vector space. Hence every $f \in \mathcal{O}_{E^*}$ is uniquely written as $f = \sum_{\lambda=0}^{p-1} f^{(\lambda)}$, $f^{(\lambda)} \in k((t)) \cdot g^\lambda$. Suppose that $\delta_-^d(f) = 0$, or equivalently that $f^{(\lambda)} = 0$ for $\lambda \geqslant d$. Then from Lemma 2.15, we have

$$\omega_f = \sum_{i=1}^{d}\biggl(\sum_{\lambda \geqslant d-i} \delta_-^{d-i}(f^{(\lambda)})\biggr) dx_i.$$

LEMMA 5.11. *There exists an $\mathcal{O}_E$-basis $\omega_{f_1}, \ldots, \omega_{f_d}$ of $\mathrm{Im}((\psi \circ \beta)^*\Omega_{R/k} \longrightarrow \beta^*\Omega_{S/k})$ such that $f_i^{(d-i)} \neq 0$ and $f_i^{(\lambda)} = 0$, $\lambda > d-i$. Namely for every $1 \leqslant i \leqslant d$, the terms of $dx_{i'}$, $i' < i$, in ω_{f_i} vanish, hence we have*

$$\begin{pmatrix} \omega_{f_1} \\ \omega_{f_2} \\ \vdots \\ \omega_{f_{d-1}} \\ \omega_{f_d} \end{pmatrix} = \begin{pmatrix} \delta_-^{d-1}(f_1) & \delta_-^{d-2}(f_1) & \ldots & \delta_-(f_1) & f_1 \\ & \delta_-^{d-2}(f_2) & \ldots & \delta_-(f_2) & f_2 \\ & & \ddots & \vdots & \vdots \\ & & & \delta_-(f_{d-1}) & f_{d-1} \\ & & & & f_d \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_{d-1} \\ dx_d \end{pmatrix}.$$

Proof. The proof is more or less standard linear algebra. Since $\mathcal{O}_E$ is a PID and $\beta^*\Omega_{S/R}$ is a torsion module, there exists a basis of $\mathrm{Im}((\psi \circ \beta)^*\Omega_{R/k} \longrightarrow \beta^*\Omega_{S/k})$ consisting of d elements, say $\omega_{f_1}, \ldots, \omega_{f_d}$. For some i, $f_i^{(d-1)} \neq 0$, say $f_1^{(d-1)} \neq 0$. Moreover, we may and shall suppose that $f_1^{(d-1)}$ has the least order among non-zero $f_i^{(d-1)}$. Then replacing f_i, $i \geqslant 2$ with $f_i - g_i f_1$ for suitable $g_i \in k[[t]]$, we may suppose that $f_i^{(d-1)} = 0$ for $i \geqslant 2$. Repeating this procedure for ω_{f_i}, $i \geqslant 2$, we may suppose that $f_2^{(d-2)} \neq 0$ and $f_i^{(d-2)} = 0$, $i \geqslant 3$. Repeating this, we eventually get a basis of the expected form. □

LEMMA 5.12. *Let $f_1, \ldots, f_d$ be as above. Then*

$$\operatorname{ord} \mathrm{Jac}_\phi(\gamma) = \frac{1}{p} \sum_{i=1}^{d} v_E(\delta_-^{d-i}(f_i)).$$

Proof. Since Fitting ideals are compatible with pullbacks, the ideal, $\gamma^{-1}\mathrm{Jac}_\phi = \beta^{-1}\mathrm{Jac}_\psi \subset \mathcal{O}_E$, is equal to the zeroth Fitting ideal of $\beta^*\Omega_{S/R}$. This module is isomorphic to the cokernel of a map $\mathcal{O}_E^d \longrightarrow \mathcal{O}_E^d$ defined by the matrix

$$\begin{pmatrix} \delta_-^{d-1}(f_1) & \delta_-^{d-2}(f_1) & \dots & \delta_-(f_1) & f_1 \\ & \delta_-^{d-2}(f_2) & \dots & \delta_-(f_2) & f_2 \\ & & \ddots & \vdots & \vdots \\ & & & \delta_-(f_{d-1}) & f_{d-1} \\ & & & & f_d \end{pmatrix}.$$

Now, by the definition of Fitting ideals, $\beta^{-1}\mathrm{Jac}_\psi$ is the determinant of the matrix and equal to $\prod_{i=1}^d \delta^{d-i}(f_i)$. Hence

$$\mathrm{length}\, \mathcal{O}_E/\gamma^{-1}\mathrm{Jac}_\phi = \sum_{i=1}^d v_E(\delta_-^{d-i}(f_i)),$$

which proves the lemma. □

Definition 5.13. Suppose that V is an indecomposable G-representation of dimension d. Then we define the *shift number of $j \in \mathbb{N}_0'$ with respect to V* to be

$$\mathrm{sht}_V(j) := \sum_{i=1}^{d-1} \left\lfloor \frac{ij}{p} \right\rfloor.$$

Proposition 5.14. *Let the assumption be as in Corollary 5.5. Additionally we suppose that V is indecomposable (although this assumption will be removed in the next subsection). Then $\phi_n^{-1}(\phi_n\gamma_n)$ is universally homeomorphic to the quotient of $\mathbb{A}_k^{e+\mathrm{sht}_V(j)}$ by some linear G-action.*

Proof. Let $M_n := \mathcal{O}_{E^*}/\mathfrak{m}_E^{np+1}$ and let $M_n^\natural \subset M_n$ be the image of $\mathcal{O}_{E^*}^{\delta^d=0} = \bigoplus_{i=0}^{d-1} k((t))g^i$. Then $M_{n,e}^\natural \subset M_n^\natural$ and we can identify $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_{n,e})$ with the similarly defined $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_n)$. Indeed since $\beta^*\Omega_{S/R}$ has length pe, every $\mathcal{O}_E$-linear map $\beta^*\Omega_{S/R} \longrightarrow M_n$ has its image in $M_{n,e}$. Therefore we may count the dimension of $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_n)$ instead. This trick will make the following arguments easier.

Let $\omega_{f_1}, \dots, \omega_{f_d}$ be as in Lemma 5.11. Then

$$\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_n) = \{\alpha \in \mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta\Omega_{S/k}, M_n) \mid \alpha(\omega_{f_i}) = 0\,(i = 1, \dots, d)\}.$$

Identifying $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_n)$ with $M_n^\natural$, we can identify $\mathrm{Hom}^\natural_{\mathcal{O}_E}(\beta^*\Omega_{S/R}, M_n)$ with the set of $h \in M_n^\natural$ satisfying

$$\begin{pmatrix} \delta_-^{d-1}(f_1) & \delta_-^{d-2}(f_1) & \dots & \delta_-(f_1) & f_1 \\ & \delta_-^{d-2}(f_2) & \dots & \delta_-(f_2) & f_2 \\ & & \ddots & \vdots & \vdots \\ & & & \delta_-(f_{d-1}) & f_{d-1} \\ & & & & f_d \end{pmatrix} \begin{pmatrix} h \\ \delta(h) \\ \vdots \\ \delta^{d-2}(h) \\ \delta^{d-1}(h) \end{pmatrix} = 0 \mod \mathfrak{m}_E^{np+1}. \tag{5.1}$$

Let us write $h = \sum_{\lambda=0}^{d-1} h^{[\lambda]} g^\lambda$, $h^{[\lambda]} \in k((t))$, such that if we write $h^{[\lambda]} = \sum_i h_i^{[\lambda]} t^i$, then $h_i^{[\lambda]} = 0$ for i with

$$pi - j\lambda > np \left(\Leftrightarrow i > n + \left\lfloor \frac{j\lambda}{p} \right\rfloor \right).$$

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Then

$$\delta^{d-i}(h) = \sum_{\lambda=d-i}^{d-1} \delta^{d-i}(g^\lambda) \cdot h^{[\lambda]}.$$

We note that $v_E(\delta^{d-i}(g^{d-i})) = 0$. From the bottom row of (5.1),

$$f_d \cdot \delta^{d-1}(g^{d-1}) \cdot h^{[d-1]} = 0 \mod \mathfrak{m}_E^{np+1}.$$

Hence the coefficients $h_i^{[d-1]}$ of $h_i^{[d-1]}$ are zero for i such that

$$v_E(f_d) + ip \leqslant np \left(\Leftrightarrow i \leqslant n - \frac{v_E(f_d)}{p} \right).$$

The other $v_E(f_d)/p + \lfloor (d-1)j/p \rfloor$ coefficients,

$$h_i^{[d-1]}, \quad n - \frac{v_E(f_d)}{p} < i \leqslant n + \left\lfloor \frac{(d-1)j}{p} \right\rfloor,$$

can take arbitrary values. Suppose now that $h^{[d-1]}, \dots, h^{[d-l]}$ are fixed. Then we consider the $(l+1)$th row from the bottom of equation (5.1),

$$\delta_-^l(f_{d-l}) \cdot \delta^{d-l-1}(g^{d-l-1}) \cdot h^{[d-l-1]} + (\text{fixed terms}) = 0 \mod \mathfrak{m}_E^{np+1}.$$

The left-hand side is the image of $\omega_{f_{d-l}}$ by the G-equivariant map $\alpha \in \mathrm{Hom}_{\mathcal{O}_E}(\beta^*\Omega_{S/k}, M_n)$ with $dx_1 \longmapsto h$. In particular, it is G-invariant for any h. Hence the equality holds for at least one choice of $h^{[d-l-1]} \in k((t))$. Indeed we can choose $h^{[d-l-1]}$ as

$$-\frac{(\text{fixed terms})}{\delta_-^l(f_{d-l}) \cdot \delta^{d-l-1}(g^{d-l-1})} \in k((t))$$

with coefficients of degree greater than $n + \lfloor j(d-l-1)/p \rfloor$ eliminated. (This is the point where we use the trick of replacing $M_{n,e}$ with M_n.) Once a solution exists, then the equation uniquely determines the coefficients $h_i^{[d-l-1]}$ of $h^{[d-l-1]}$ for i such that

$$v_E(\delta_-^l(f_{d-l})) + ip \leqslant np \left(\Leftrightarrow i \leqslant n - \frac{v_E(\delta_-^l(f_{d-l}))}{p} \right).$$

The other $v_E(\delta_-^l(f_{d-l}))/p + \lfloor (d-l-1)j/p \rfloor$ coefficients,

$$h_i^{[d-l-1]}, \quad n - \frac{v_E(\delta_-^l(f_{d-l}))}{p} < i \leqslant n + \left\lfloor \frac{(d-l-1)j}{p} \right\rfloor$$

can take arbitrary values. Hence the solution space of (5.1) has dimension

$$\sum_{l=0}^{d-1} \left(\frac{v_E(\delta_-^l(f_{d-l}))}{p} + \left\lfloor \frac{(d-l-1)j}{p} \right\rfloor \right) = e + \mathrm{sht}_V(j).$$

We have completed the proof. □

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5.2.3 *Counting the dimension of* $\mathrm{Hom}^{\natural}_{\mathcal{O}_E}(\beta^*\Omega_{S/R}M_{n,e})$*: the decomposable case.* We generalize the shift number to the decomposable case in the following definition.

Definition 5.15. Suppose that $V = \bigoplus_{\lambda=1}^{l} V_{d_\lambda}$. Then we define the *shift number* of $j \in \mathbb{N}_0'$ *with respect to* V to be

$$\mathrm{sht}_V(j) := \sum_{\lambda=1}^{l} \mathrm{sht}_{V_{d_\lambda}}(j) = \sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} \left\lfloor \frac{ij}{p} \right\rfloor .$$

With this definition, Proposition 5.14 holds also in the decomposable case. Now we sketch how the above arguments can be generalized to this case. Let

$$k[\mathbf{x}] = k[x_{\lambda,i} \mid 1 \leqslant \lambda \leqslant l,\, 1 \leqslant i \leqslant d_\lambda]$$

be the coordinate ring of V as in § 3.2. Then $\mathrm{Im}((\psi \circ \beta)^*\Omega_{R/k} \longrightarrow \beta^*\Omega_{S/k})$ is again generated by G-invariant elements. A G-invariant element of $\beta^*\Omega_{S/k}$ is of the form

$$\omega_{\mathbf{f}} := \sum_{\lambda} \sum_{i} \delta_-^{d_\lambda - i}(f_\lambda)\, dx_i$$

for some

$$\mathbf{f} = (f_1, \dots, f_l) \in \prod_{\lambda=1}^{l} \mathcal{O}_E^{\delta_-^{d_\lambda}=0}.$$

Then we can generalize Lemma 5.11 as follows.

Lemma 5.16. *There exists a basis of* $\mathrm{Im}((\psi \circ \beta)^*\Omega_{R/k} \longrightarrow \beta^*\Omega_{S/k})$,

$$\omega_{\mathbf{f}_{\lambda,i}}\ (1 \leqslant \lambda \leqslant l,\, 1 \leqslant i \leqslant d_\lambda),$$

such that the terms of $dx_{\lambda',i'}$ *in* $\omega_{\mathbf{f}_{\lambda,i}}$ *vanish either if* $\lambda' < \lambda$ *or if* $\lambda' = \lambda$ *and* $i' < i$.

Proof. The proof is almost the same as that of Lemma 5.11. We use induction on l. We first take any basis $\omega_{\mathbf{f}_{\lambda,i}}$. Next we eliminate the $dx_{1,1}$ terms of $\omega_{\mathbf{f}_{\lambda,i}}$ with $(\lambda, i) \neq (1,1)$, suitably replacing the $\mathbf{f}_{\lambda,i}$. Then we eliminate the $dx_{1,2}$ terms of $\omega_{\mathbf{f}_{\lambda,i}}$ with $(\lambda, i) \neq (1,1), (1,2)$. Repeating this procedure d_1 times, we get a basis such that $\omega_{\mathbf{f}_{1,i}}$, $1 \leqslant i \leqslant d_1$, satisfy the condition of the assertion. Applying the assumption of induction to $\omega_{\mathbf{f}_{\lambda,i}}$, $\lambda \geqslant 2$, we prove the lemma. □

Once the above lemma is proved, the rest of arguments in the indecomposable case work also in the decomposable case.

Proposition 5.17. *The assertion of Proposition 5.14 holds without the assumption that* V *is indecomposable.*

5.3 The change of variables formula

Definition 5.18. We define a function $\mathfrak{s}_{\mathcal{X}}$ on $\mathcal{J}_\infty\mathcal{X}$ as the composition $\mathrm{sht}_V \circ \mathrm{rj}$:

$$\mathfrak{s}_{\mathcal{X}} : \mathcal{J}_\infty\mathcal{X} \xrightarrow{\mathrm{rj}} \mathbb{N}_0' \xrightarrow{\mathrm{sht}_V} \mathbb{Z}.$$

Lemma 5.19. *Let* $A \subset J_\infty X$ *be a cylinder. Then the function* $-\mathrm{ord}\,\mathrm{Jac}_\phi - \mathfrak{s}_{\mathcal{X}}$ *on* $\phi_\infty^{-1}(A)$ *is exponentially measurable and*

$$\int_{\phi_\infty^{-1}(A)} \mathbb{L}^{-\mathrm{ord}\,\mathrm{Jac}_\phi - \mathfrak{s}_{\mathcal{X}}}\, d\mu_{\mathcal{X}} = \mu_X(A).$$

Proof. Let $\mathcal{Y} \subset \mathcal{X}$ be the exceptional locus of $\phi : \mathcal{X} \longrightarrow X$. Then there exists a stratification $\phi_\infty^{-1}(A) \backslash \mathcal{J}_\infty\mathcal{Y} = \bigsqcup_{i \in \mathbb{N}} B_i$ such that:

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(i) for every i, B_i is a cylinder; and
(ii) for every i, the functions $\operatorname{ord}\mathrm{Jac}_\phi$, $\operatorname{ord}\phi^{-1}\mathrm{Jac}_X$ and rj are constant on B_i.

From Lemma 5.6, for every $i > 0$, $\phi_\infty(B_i)$ is a cylinder. Let $n := -\operatorname{ord}\mathrm{Jac}_\phi(B_i) - \mathfrak{s}_\mathcal{X}(B_i)$. Then from Propositions 5.14 and 5.17, $\mu_\mathcal{X}(B_i)\mathbb{L}^n = \mu_X(\phi_\infty(B_i))$. Hence

$$\mu_X(A) = \sum_i \mu_X(\phi_\infty(B_i)) = \sum_i \int_{B_i} \mathbb{L}^{-\operatorname{ord}\mathrm{Jac}_\phi - \mathfrak{s}_\mathcal{X}}\, d\mu_\mathcal{X}.$$

Now the lemma follows from Proposition 4.21. □

THEOREM 5.20 (The change of variables formula, cf. [DL02, Yas04, Yas06]). *Let $A \subset J_\infty X$ be a strongly measurable subset and let $B = \bigsqcup_i B_i \subset \mathcal{J}_\infty\mathcal{X}$ be a countable disjoint union of strongly measurable subsets B_i such that $B = \phi_\infty^{-1}(A)$. (For instance, if A is a cylinder, then $B = \phi_\infty^{-1}(A)$ satisfies this condition.) Let $F : J_\infty X \supset A \longrightarrow (1/r)\mathbb{Z} \cup \{\infty\}$ be a measurable function. Then F is exponentially integrable if and only if the function $F \circ \phi_\infty - \operatorname{ord}\mathrm{Jac}_\phi - \mathfrak{s}_\mathcal{X}$ on B is exponentially integrable. Moreover, if this is the case, then we have*

$$\int_A \mathbb{L}^F\, d\mu_X = \int_B \mathbb{L}^{F\circ\phi_\infty - \operatorname{ord}\mathrm{Jac}_\phi - \mathfrak{s}_\mathcal{X}}\, d\mu_\mathcal{X}.$$

Proof. Using Proposition 4.21, we will reduce the situation to an easier one step by step.

(i) We may suppose that B is disjoint from $\mathcal{J}_\infty\mathcal{Y}$. From Lemma 4.17, $\mathcal{J}_\infty\mathcal{Y}$ is measurable and of measure zero. Also $J_\infty Y$ is measurable and of measure zero. Moreover, $B_i\backslash\mathcal{J}_\infty\mathcal{Y}$ and $A\backslash J_\infty Y$ are strongly measurable. Therefore we may replace B with $B\backslash\mathcal{J}_\infty\mathcal{Y}$, and A with $A\backslash J_\infty Y$.

(ii) We may suppose that B is strongly measurable. From Corollary 5.7, for each i, $\phi_\infty(B_i)$ is strongly measurable. From Proposition 4.21, it is enough to prove the theorem for $B = B_i$ and $A = \phi_\infty(B_i)$.

(iii) We may suppose that B is a cylinder and A is stable. Let $B_0(\epsilon) \subset B$ be cylinders as in Definition 4.13. Then for a sequence $\epsilon_i \in \mathbb{R}_{>0}$, $i \in \mathbb{N}$, with $\lim_{i\longrightarrow\infty}\epsilon_i = 0$, $B\backslash\bigcup_{i\in\mathbb{N}} B_0(\epsilon_i)$ is measurable and has measure zero. Its image in A is also measurable and has measure zero. Hence it is enough to prove the theorem for $B = B_0(\epsilon_i)$ and $A = \phi_\infty(B_0(\epsilon))$. As we have supposed that B is disjoint from $\mathcal{J}_\infty\mathcal{Y}$, $B_0(\epsilon_i)$ is also disjoint from $\mathcal{J}_\infty\mathcal{Y}$. Hence $\phi_\infty(B_0(\epsilon))$ is stable.

(iv) We may suppose that the functions $\operatorname{ord}\mathrm{Jac}_\phi$, $\operatorname{ord}\phi^{-1}\mathrm{Jac}_X$ and rj are constant on B. We take the stratification $B = \bigsqcup_i B_i$ of B with respect to the values of these functions. Then the stratification has only finite strata. It suffices to show the theorem for $B = B_i$ and $A = \phi_\infty(B_i)$.

Now we fix $n \in (1/r)\mathbb{Z} \cup \{\infty\}$. Then $C := F^{-1}(n)$ is measurable. Let $C_i(\epsilon)$ be as in Definition 4.13. For every i and ϵ, we can take $C_i(\epsilon) \subset A$. Let $e := \operatorname{ord}\mathrm{Jac}_\phi(B)$ and $s := \mathfrak{s}_\mathcal{X}(B)$. Then

$$\mu_\mathcal{X}(\phi_\infty^{-1}(C_i(\epsilon))) = \mu_X(C_i(\epsilon))\mathbb{L}^{-e-s}.$$

This shows that

$$\lim_{\epsilon\longrightarrow 0}\mu_\mathcal{X}(\phi_\infty^{-1}(C_0(\epsilon))) = \mu_\mathcal{X}(\phi_\infty^{-1}(C)).$$

Hence

$$\begin{aligned}\int_C \mathbb{L}^F\, d\mu_X &= \mu_X(C)\mathbb{L}^n \\ &= \lim_{\epsilon\longrightarrow 0}\mu_X(C_0(\epsilon))\mathbb{L}^n\end{aligned}$$

$$= \lim_{\epsilon \longrightarrow 0} \mu_{\mathcal{X}}(\phi_\infty^{-1}(C_0(\epsilon)))\mathbb{L}^{n-e-s}$$
$$= \mu_{\mathcal{X}}(\phi_\infty^{-1}(C))\mathbb{L}^{n-e-s}$$
$$= \int_{\phi_\infty^{-1}(C)} \mathbb{L}^{F \circ \phi_\infty - \operatorname{ord} \operatorname{Jac}_\phi - \mathfrak{s}_{\mathcal{X}}} \, d\mu_{\mathcal{X}}.$$

Now the theorem follows again from Proposition 4.21. □

6. Stringy invariants and the McKay correspondence

In this section, we will define stringy invariants as certain motivic integrals. Then we will obtain several versions of the McKay correspondence, which are consequences of the change of variables formula above. We still suppose that V is a non-trivial G-representation, $X = V/G$ and $\mathcal{X} = [V/G]$.

6.1 Stringy invariants of stringily Kawamata log terminal pairs

Let ω_X be the canonical sheaf of X, which is defined as the double dual of $\bigwedge^d \Omega_{X/k}$. Since R is a UFD (see, for instance, [CW11, Theorem 3.8.1]), in particular, X is 1-Gorenstein. Namely ω_X is invertible. The *ω-Jacobian ideal* $\operatorname{Jac}_X^\omega \subset \mathcal{O}_X$ of X is defined by the equality

$$\operatorname{Im}\left(\bigwedge^d \Omega_{X/k} \longrightarrow \omega_X\right) = \operatorname{Jac}_X^\omega \cdot \omega_X.$$

Remark 6.1. It is known that if X is a local complete intersection, then $\operatorname{Jac}_X = \operatorname{Jac}_X^\omega$. In our setting, this is the case only when V is isomorphic to $V_2 \oplus V_1^{\oplus d-2}$, $V_2^{\oplus 2} \oplus V_1^{\oplus d-4}$ or $V_3 \oplus V_1^{\oplus d-3}$. Indeed, X is a hypersurface in these cases (see, for instance, [CW11]). Otherwise X is not even Cohen–Macaulay from [ES80].

In [Bat98, Bat99], Batyrev introduced *stringy invariants* for Kawamata log terminal pairs in characteristic zero. To define these, he used the resolution of singularities. However, we are working in positive characteristic and it is not yet known that a resolution of singularities always exists. Therefore, following Denef and Loeser [DL02], we will define the stringy invariant as an integral on $J_\infty X$.

Definition 6.2. Let $Z = \sum_{i=1}^m a_i Z_i$ be a formal $\mathbb{Q}$-linear combination of closed subschemes $Z_i \subsetneq X$. Then we define a function

$$\operatorname{ord} Z := \sum_{i=1}^m a_i \cdot \operatorname{ord} Z_i : J_\infty X \longrightarrow \mathbb{Q} \cup \{\infty\}.$$

Here we suppose that $\operatorname{ord} Z$ takes the constant value ∞ on the measure-zero subset $\bigcup_{i=1}^m (\operatorname{ord} Z_i)^{-1}(\infty)$. We say that the pair (X, Z) is *stringily Kawamata log terminal* if the function $\operatorname{ord} Z + \operatorname{ord} \operatorname{Jac}_X^\omega$ on $J_\infty X$ is exponentially integrable. If this is the case, we define its *stringy motivic invariant* by

$$M_{\mathrm{st}}(X, Z) := \int_{J_\infty X} \mathbb{L}^{\operatorname{ord} Z + \operatorname{ord} \operatorname{Jac}_X^\omega} \, d\mu_X.$$

Then we define the *stringy Poincaré function* by

$$P_{\mathrm{st}}(X, Z) = P(M_{\mathrm{st}}(X, Z)) \in \bigcup_{r=1}^\infty \mathbb{Z}((T^{-1/r})).$$

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Let $f : Y \longrightarrow X$ be a proper birational morphism with Y smooth. Then we define the *canonical divisor* of f, denoted K_f, to be the divisor such that

$$\mathrm{Im}(f^*\omega_X \longrightarrow \omega_Y \otimes K(Y)) = \mathcal{O}_Y(-K_f) \cdot \omega_Y.$$

Here $K(Y)$ denotes the function field of Y. This divisor has support in the exceptional locus of f. We define the *pullback* f^*Z of Z by f as the formal linear combination $\sum_{i=1}^m a_i \cdot f^{-1}Z_i$ of the scheme-theoretic preimages $f^{-1}Z_i \subset Y$.

PROPOSITION 6.3. *With the notation as above, we have*

$$M_{\mathrm{st}}(X, Z) = M_{\mathrm{st}}(Y, f^*Z - K_f).$$

Proof. Since

$$f^{-}\mathrm{Jac}_X^{\omega} \cdot \mathcal{O}_Y(-K_f) = \mathrm{Jac}_f,$$

we have

$$f^{-1}\mathrm{ord}\,\mathrm{Jac}_X^{\omega} - \mathrm{ord}\,\mathrm{Jac}_f = -\mathrm{ord}\,K_f.$$

The proposition follows from this and the change of variables formula for varieties in positive characteristic [Seb04]. □

COROLLARY 6.4. *Suppose that $K_f - f^*Z$ is a simple normal crossing divisor written as $\sum_{i=1}^m a_i E_i$, $a_i \in \mathbb{Q}$, with E_i prime divisors. Then (X, Z) is stringily Kawamata log terminal if and only if for every i, $a_i > -1$. Moreover, if this is the case, then*

$$M_{\mathrm{st}}(X, Z) = M_{\mathrm{st}}(Y, f^*Z - K_f) = \sum_{I \subset \{1,\dots,m\}} [E_I^{\circ}] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{1+a_i} - 1}, \quad \text{where } E_I^{\circ} := \bigcap_{i \in I} E_i \Big\backslash \bigcup_{i \notin I} E_i.$$

Proof. The corollary follows from the previous proposition and the explicit computation of motivic integrals (see, for instance, [Cra04]). □

DEFINITION 6.5. Let (X, Z) be a pair as above. Let $f : Y \longrightarrow X$ be a proper birational morphism such that Y is normal and f^*Z is a Cartier divisor. Let E be a prime divisor on Y. Then the canonical divisor K_f of f is similarly defined on the smooth locus of Y (and extends to Y as a Weil divisor). The *discrepancy* of (X, Z) at E, denoted $a(E; X, Z)$, is defined to be the coefficient of E in the divisor $K_f - f^*Z$. We say that (X, Z) is *Kawamata log terminal* if for every Y and E as above, $a(E; X, Z) > -1$.

PROPOSITION 6.6. *If (X, Z) is stringily Kawamata log terminal, then it is Kawamata log terminal. Additionally, if there exists a resolution $f : Y \longrightarrow X$ with $K_f - f^*Z$ a simple normal crossing $\mathbb{Q}$-Cartier divisor, then the converse is also true.*

Proof. For the first assertion, let $f : Y \longrightarrow X$ and E be as in Definition 6.5. Let $U \subset Y$ be an open dense subset such that

$$(K_f - f^*Z)|_U = a(E; X, Z) \cdot E|_U.$$

Then from the change of variables formula in [Seb04], $\mathrm{ord}\, f^*Z - \mathrm{ord}\, K_f$ is exponentially integrable on $J_\infty U$. Hence $a(E; X, Z) > -1$. This shows that (X, Z) is Kawamata log terminal.

The second assertion follows from Corollary 6.4. □

Next we will define stringy invariants of stacky pairs.

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Definition 6.7. Let $\mathcal{Z} = \sum_{i=1}^{m} a_i \mathcal{Z}_i$ be a formal $\mathbb{Q}$-linear combination of closed substacks $Z_i \subsetneq \mathcal{X}$. Then we define a function

$$\operatorname{ord} Z := \sum_{i=1}^{m} a_i \cdot \operatorname{ord} Z_i : \mathcal{J}_\infty \mathcal{X} \longrightarrow \mathbb{Q} \cup \{\infty\}.$$

We say that the pair $(\mathcal{X}, \mathcal{Z})$ is *stringily Kawamata log terminal* if the function $\operatorname{ord} \mathcal{Z} - \mathfrak{s}_{\mathcal{X}}$ on $\mathcal{J}_\infty \mathcal{X}$ is exponentially integrable. If this is the case, we define its *stringy motivic invariant by*

$$M_{\mathrm{st}}(\mathcal{X}, \mathcal{Z}) := \int_{\mathcal{J}_\infty \mathcal{X}} \mathbb{L}^{\operatorname{ord} \mathcal{Z} - \mathfrak{s}_{\mathcal{X}}} \, d\mu_{\mathcal{X}}.$$

6.2 An explicit formula for $M_{\mathrm{st}}(\mathcal{X})$

Definition 6.8. For a G-representation $V = \bigoplus_{\lambda=1}^{l} V_{d_\lambda}$, we put

$$D_V := \sum_{\lambda=1}^{l} \frac{(d_\lambda - 1) d_\lambda}{2} = \sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} i \in \mathbb{N}.$$

Proposition 6.9. *The pair $(\mathcal{X}, 0)$ is stringily Kawamata log terminal if and only if $D_V \geqslant p$. Moreover, if this is the case, then*

$$M_{\mathrm{st}}(\mathcal{X}) := M_{\mathrm{st}}(\mathcal{X}, 0) = \mathbb{L}^d + \frac{\mathbb{L}^{l-1}(\mathbb{L}-1)(\sum_{s=1}^{p-1} \mathbb{L}^{s - \mathrm{sht}_V(s)})}{1 - \mathbb{L}^{p-1-D_V}}.$$

Proof. For an integer s with $1 \leqslant s \leqslant p-1$ and a non-negative integer n, we have

$$\begin{aligned} \mathrm{sht}_V(np+s) &= \sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} \left(in + \left\lfloor \frac{is}{p} \right\rfloor \right) \\ &= \left(\sum_{\lambda=1}^{l} \frac{(d_\lambda - 1) d_\lambda}{2} \right) n + \sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} \left\lfloor \frac{is}{p} \right\rfloor \\ &= D_V \cdot n + \mathrm{sht}_V(s). \end{aligned}$$

Hence we have

$$\begin{aligned} M_{\mathrm{st}}(\mathcal{X}) &= \int_{\mathcal{J}_\infty \mathcal{X}} \mathbb{L}^{-\mathfrak{s}_{\mathcal{X}}} \, d\mu_{\mathcal{X}} \\ &= \sum_{j \in \mathbb{N}_0'} \mu_{\mathcal{X}}(\mathcal{J}_{\infty, j} \mathcal{X}) \mathbb{L}^{-\mathrm{sht}_V(j)} \\ &= \mathbb{L}^d + \sum_{j \in \mathbb{N}'} [\mathbb{A}_k^l \times \text{G-}\mathbf{Cov}^{\mathrm{rep}}(D, j)] \mathbb{L}^{-\mathrm{sht}_V(j)} \\ &= \mathbb{L}^d + \sum_{j > 0} (\mathbb{L} - 1) \mathbb{L}^{l + j - 1 - \lfloor j/p \rfloor - \mathrm{sht}_V(j)} \\ &= \mathbb{L}^d + (\mathbb{L} - 1) \mathbb{L}^{l-1} \sum_{s=1}^{p-1} \sum_{n=0}^{\infty} \mathbb{L}^{(p-1-D_V)n + s - \mathrm{sht}_V(s)}. \end{aligned}$$

This converges if and only if $D_V \geqslant p$. Now the formula of the proposition easily follows. □

Corollary 6.10. *If $D_V = p$, then*

$$M_{\mathrm{st}}(\mathcal{X}) = \mathbb{L}^d + \mathbb{L}^l \sum_{s=1}^{p-1} \mathbb{L}^{s - \mathrm{sht}_V(s)} \in \mathbb{Z}[\mathbb{L}].$$

Proof. The equality is a direct consequence of the previous proposition. Moreover, in this case,

$$\mathrm{sht}_V(s) \leqslant \frac{s}{p}\sum_{\lambda=1}^{l}\sum_{i=1}^{d_\lambda-1}\lambda = \frac{s}{p}D_V = s.$$

Hence $M_{\mathrm{st}}(\mathcal{X}) \in \mathbb{Z}[\mathbb{L}]$. □

DEFINITION 6.11. When $D_V \geqslant p$, we define the *stringy Euler number* of $\mathcal{X}$ by

$$e_{\mathrm{st}}(\mathcal{X}) := e_{\mathrm{top}}(M_{\mathrm{st}}(\mathcal{X})).$$

(See § 4.3.)

COROLLARY 6.12. *If $D_V \geqslant p$, then*

$$e_{\mathrm{st}}(\mathcal{X}) = 1 + \frac{p-1}{D_V - p + 1}.$$

In particular, if $D_V = p$, then $e_{\mathrm{st}}(\mathcal{X}) = p$.

Proof. Obvious. □

In the following, we compute $M_{\mathrm{st}}(\mathcal{X})$ in several cases.

Example 6.13. Suppose that $V = V_p^{\oplus l}$. Then $\mathrm{sht}_V(s) = l(s-1)(p-1)/2$ (see, for instance, [GKP89, p. 94]). Hence

$$M_{\mathrm{st}}(\mathcal{X}) = \mathbb{L}^d + (\mathbb{L}-1)\mathbb{L}^{l-1}\sum_{s=1}^{p-1}\frac{\mathbb{L}^{s-l(s-1)(p-1)/2}}{1-\mathbb{L}^{p-1-lp(p-1)/2}}.$$

In the following cases, we have $D_V = p$.

Example 6.14. If $p = 3$ and $V = V_3$, then $M_{\mathrm{st}}(\mathcal{X}) = \mathbb{L}^3 + 2\mathbb{L}^2$.

Example 6.15. If $p = 2$ and $V = V_2^{\oplus 2}$, then $M_{\mathrm{st}}(\mathcal{X}) = \mathbb{L}^4 + \mathbb{L}^3$.

Example 6.16. If $V = V_2^{\oplus p}$, then

$$M_{\mathrm{st}}(\mathcal{X}) = \mathbb{L}^{2p} + \mathbb{L}^p(\mathbb{L}^{p-1} + \mathbb{L}^{p-2} + \cdots + \mathbb{L}) = \mathbb{L}^{2p} + \mathbb{L}^{2p+1} + \cdots + \mathbb{L}^{p+1}.$$

6.3 The McKay correspondence

We say that V has *reflections* if the fixed point locus V^G has codimension one. A non-trivial G-representation V has reflection if and only if $V \cong V_2 \oplus V_1^{\oplus d-2}$.

6.3.1 *The no reflection case.*

LEMMA 6.17. *Suppose that V has no reflection. Then $\phi^{-1}\mathrm{Jac}_X^{\omega} = \mathrm{Jac}_\phi$.*

Proof. Since the quotient map $\psi : V \longrightarrow X$ is étale in codimension one, $\psi^*\omega_X = \omega_V$, and the lemma follows. □

COROLLARY 6.18. *Let $Z = \sum_i a_i Z_i$ be a formal $\mathbb{Q}$-linear combination of closed subschemes $Z_i \subsetneq X$. Then (X, Z) is stringily Kawamata log terminal if and only if so is $(\mathcal{X}, \phi^* Z)$. Moreover, if this is the case, then*

$$M_{\mathrm{st}}(X, Z) = M_{\mathrm{st}}(\mathcal{X}, \phi^* Z).$$

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Proof. From the change of variables formula and the previous lemma, we have

$$M_{\mathrm{st}}(X,Z)=\int_{J_\infty X}\mathbb{L}^{\mathrm{ord}\,Z+\mathrm{ord}\,\mathrm{Jac}_X^{\omega}}\,d\mu_X=\int_{\mathcal{J}_\infty\mathcal{X}}\mathbb{L}^{\mathrm{ord}\,\phi^*Z-\mathfrak{s}_{\mathcal{X}}}\,d\mu_{\mathcal{X}}=M_{\mathrm{st}}(\mathcal{X},\phi^*Z).\qquad\square$$

Corollary 6.19. *The pair $(X,0)$ is stringily Kawamata log terminal if and only if $D_V\geqslant p$. Moreover, if this is the case, then*

$$M_{\mathrm{st}}(X)=M_{\mathrm{st}}(\mathcal{X}).$$

Proof. This follows from the previous corollary and Proposition 6.9. □

Remark 6.20. In particular, if $D_V\geqslant p$, then X has only canonical singularities. Namely all discrepancies are non-negative. On the other hand, from [Yas12], X does not satisfy a closely related property, strong F-regularity.

Corollary 6.21 (The p-cyclic McKay correspondence). *Suppose that V has no reflection and that there exists a crepant resolution $f:Y\longrightarrow X$. Then the following hold:*

(i) $M_{\mathrm{st}}(\mathcal{X})=[Y]$;
(ii) $D_V=p$;
(iii) $e_{\mathrm{top}}(Y)=p$.

Proof. For the first assertion, we have

$$M_{\mathrm{st}}(\mathcal{X})=M_{\mathrm{st}}(X)=[Y].$$

Then, in particular, $\mathcal{X}$ is stringily Kawamata log terminal. Hence $d\geqslant p$. Also from the first assertion, we have

$$e_{\mathrm{st}}(\mathcal{X})=e_{\mathrm{top}}(Y),$$

which is an integer. Now the second and third assertions follow from Corollary 6.12. □

Remark 6.22. The third assertion of Corollary 6.21 does not hold if we replace V with a *non-linear* G-action on $\mathrm{Spec}\,k[[x_1,\dots,x_d]]$. For instance, let $0\in X$ be either an E_8^2-singularity in characteristic two, an E_6^1-singularity in characteristic three, or an E_8^1-singularity in characteristic five (for the notation, see [Art77]). Then X is the quotient of $\mathrm{Spec}\,k[[x,y]]$ by a G-action such that the associated covering $\mathrm{Spec}\,k[[x,y]]\longrightarrow X$ is étale outside 0. The minimal resolution of X is a crepant resolution. (Indeed from [Lip69, Theorem 4.1] and [Băd01, Corollary 4.19], the blow-up of X at the singular point has only rational double points. Hence discrepancies at the exceptional curves are zero.) However, the topological Euler characteristic of the minimal resolution is not p.

Example 6.23. Suppose that $V=V_3$. If we suppose that G acts on the coordinate ring $k[x,y,z]$ of V, by $x\longmapsto x$, $y\longmapsto -x+y$, $z\longmapsto x-y+z$, then the invariant subring is

$$k[x,y,z]^G=k[x,N_y,N_z,d],$$

where $d=y^2+xz-xy$, $N_y=\prod_{g\in G}g(y)$, $N_z=\prod_{g\in G}g(z)$ (see [CW11, Theorem 4.10.1]). In particular, X is a hypersurface. In an abuse of notation, we let x,N_y,N_z,d correspond to variables X,Y,Z,W, respectively. Then according to computations with Macaulay2 [GS] for small primes p, the defining equation of X seems to be

$$2X^pZ+W^p-Y^2+\sum_{i=2}^{(p+1)/2}(-1)^iC_{i-1}X^{2(p-i)}W^i.$$

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Here C_i denotes the ith Catalan number modulo p. The author does not know if this equation is known.

Now suppose that $p = 3$. Then the equation becomes

$$-X^3Z + W^3 - Y^2 + X^2W^2 = 0.$$

If k is algebraically closed, then by a suitable coordinate transformation, X is defined by

$$Z^2 + X^3 + Y^4 + X^2Y^2 + Y^3W = 0.$$

This equation defines the compound E_6^1-singularity as studied by Hirokado *et al.* [HIS13]. The blow-up X_1 of X along the singular locus is singular along a line. Then the blow-up Y of X_1 along the singular locus is now smooth and a crepant resolution of X. Moreover, the exceptional locus of $Y \longrightarrow X$ is a simple normal crossing divisor with two irreducible components, say $E_1 \cup E_2$. Then the E_i are universally homeomorphic to $\mathbb{A}_k^1 \times \mathbb{P}_k^1$ and $E_1 \cap E_2 \cong \mathbb{A}_k^1$. Therefore $[Y] = \mathbb{L}^3 + 2\mathbb{L}^2$, which agrees with our previous computation in Example 6.14.

Example 6.24. Suppose that $p = 2$ and $V = V_2 \oplus V_2$. Using the description of the invariant subring in [CW11, Theorem 1.12.1], we can see that X is a hypersurface defined by

$$W^2X + V^2Y + VWZ + Z^2.$$

By direct computation, the blow-up Y of X along the singular locus is a crepant resolution, and its exceptional locus is universally homeomorphic to $\mathbb{A}_k^2 \times \mathbb{P}_k^1$. Therefore

$$M_{\mathrm{st}}(X) = [Y] = \mathbb{L}^4 + \mathbb{L}^3,$$

which coincides with Example 6.15.

In fact, as the following corollary shows, the last two examples are essentially the only examples in dimension up to 4 where X has a crepant resolution.

COROLLARY 6.25. *Suppose that V is of dimension up to 4 and has no reflection. Then the following are equivalent:*

(i) X has a crepant resolution;

(ii) $D_V = p$;

(iii) one of the following holds:

  (a) $p = 2$ and $V \cong V_2 \oplus V_2$;

  (b) $p = 3$ and $V \cong V_3$;

  (c) $p = 3$ and $V \cong V_3 \oplus V_1$.

Proof. Since V has no reflection, V is isomorphic to either V_3, $V_2 \oplus V_2$, $V_3 \oplus V_1$ or V_4. Then an easy computation shows that (ii) and (iii) are equivalent. From Corollary 6.21, (i) implies (ii). The converse follows from the examples above. □

COROLLARY 6.26. *Suppose that V has no reflection. Let $f : Y \longrightarrow X$ be a crepant resolution and $E_0 := f^{-1}(0)$. Then, in the notation of § 4.8, we have*

$$\int_{\mathrm{G}\text{-}\mathbf{Cov}^{\mathrm{rep}}(D)} \mathbb{L}^{-\mathrm{sht}_V}\, d\nu = [E_0].$$

Moreover, if k is a finite field, then for each finite extension $\mathbb{F}_q/k$, we have

$$\sharp E_0(\mathbb{F}_q) = \sum_{j \in \mathbb{N}'_0} \frac{\sharp \text{G-}\mathbf{Cov}^{\text{rep}}(D,j)(\mathbb{F}_q)}{q^{\text{sht}_V(j)}}.$$

Proof. We have

$$\begin{aligned}\int_{\text{G-}\mathbf{Cov}^{\text{rep}}(D)} \mathbb{L}^{-\text{sht}_V}\, d\nu &= \int_{(\mathcal{J}_\infty \mathcal{X})_0} \mathbb{L}^{-\mathfrak{s}_\mathcal{X}}\, d\mu_\mathcal{X} \\ &= \int_{\pi_0^{-1}(0)} \mathbb{L}^{\text{ord Jac}_X^\omega}\, d\mu_X \\ &= \int_{\pi_0^{-1}(E_0)} 1\, d\mu_Y \\ &= [E_0].\end{aligned}$$

This shows the first assertion. Then the second assertion is obtained by applying the counting-points realization $\sharp_q$. □

Example 6.27. Let $k = \mathbb{F}_2$ and $V = V_2^{\oplus 2}$. Let $Y \longrightarrow X$ be a crepant resolution as in Example 6.24. Then $E_0 = \mathbb{P}^1_k$ and for each finite extension $\mathbb{F}_q/\mathbb{F}_2$, $\sharp E_0(\mathbb{F}_q) = q + 1$. On the other hand,

$$\begin{aligned}\sum_{j \in \mathbb{N}'_0} \frac{\sharp \text{G-}\mathbf{Cov}^{\text{rep}}(D,j)(\mathbb{F}_q)}{q^{\text{sht}_V(j)}} &= 1 + \sum_{n=0}^{\infty} \frac{\text{G-}\mathbf{Cov}^{\text{rep}}(D,2n+1)(\mathbb{F}_q)}{q^{\text{sht}_V(2n+1)}} \\ &= 1 + \sum_{n=0}^{\infty} (q-1)\cdot q^{-n} \\ &= 1 + q.\end{aligned}$$

The second assertion of Corollary 6.26 will be slightly differently formulated in terms of Artin–Schreier extensions of $k((t))$. We have

$$\sharp \text{G-}\mathbf{Cov}^{\text{rep}}(D,j)(\mathbb{F}_q) = \frac{1}{p} \sharp G\text{-Cov}(D \times_k \mathbb{F}_q, j).$$

Let $N_{q,j}$ be the number of Galois extensions of $\mathbb{F}_q((t))$ in a fixed algebraic closure $\overline{k((t))}$ of $k((t))$ with ramification jump j. Then for $j \in \mathbb{N}'$, we have

$$\sharp G\text{-Cov}(D \times_k \mathbb{F}_q, j) = (p-1) N_{q,j}.$$

The factor $p-1$ comes from $p-1$ choices of isomorphism of G to the Galois group. Hence we have the following corollary.

Corollary 6.28. *With the same notation as above, we have*

$$\sharp E_0(\mathbb{F}_q) = 1 + \frac{p-1}{p} \sum_{j \in \mathbb{N}'} \frac{N_{q,j}}{q^{\text{sht}_V(j)}}.$$

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6.3.2 *The reflection case.* Next we consider the case where V has reflections. We write the coordinate ring of V as $S = k[x, y, z_1, \dots, z_{d-2}]$ with the G-action given by

$$\sigma(x) = x + y,$$
$$\sigma(y) = y,$$
$$\sigma(z_i) = z_i.$$

Then

$$R := S^G = k[x^p - xy^{p-1}, y, z_1, \dots, z_{d-2}].$$

(See, for instance, [CW11, Theorem 1.11.2].) Thus X is again an affine d-space; in particular, it is smooth.

LEMMA 6.29. *We have* $\mathrm{Jac}_\psi = (y^{p-1}) \subset S$.

Proof. From the explicit generators above of R, the module $\Omega_{S/R}$ is isomorphic to the cokernel of the map $S^d \longrightarrow S^d$ represented by the Jacobian matrix

$$\begin{pmatrix} -y^{p-1} & & & & \\ (1-p)xy^{p-2} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Hence by definition, Jac_ψ is generated by its determinant $-y^{p-1}$, which proves the lemma. $\square$

The fixed point locus V^G is defined by the ideal $(y) \subset S$. We define $\mathcal{Y}$ to be the quotient stack $[V^G/G]$, which is a closed reduced substack of $\mathcal{X}$, and define Y to be the image of $\mathcal{Y}$ on X, which is defined by $(y) \subset R$.

COROLLARY 6.30. *Suppose that V has reflections. Let $Z = \sum a_i Z_i$ be a formal $\mathbb{Q}$-linear combination of closed subschemes $Z_i \subsetneq X$. Then (X, Z) is stringily Kawamata log terminal if and only if so is $(\mathcal{X}, \phi^* Z + (1-p)\mathcal{Y})$. Moreover, if this is the case, then*

$$M_{\mathrm{st}}(X, Z) = M_{\mathrm{st}}(\mathcal{X}, \phi^* Z + (1-p)\mathcal{Y}).$$

In particular, for $a < 1$,

$$M_{\mathrm{st}}(X, aY) = M_{\mathrm{st}}(\mathcal{X}, (a + 1 - p)\mathcal{Y}).$$

Proof. In this case, since X is smooth, $\mathrm{Jac}^\omega_X = R$. Hence

$$\begin{aligned} M_{\mathrm{st}}(X, Z) &= \int_{J_\infty X} \mathbb{L}^{\mathrm{ord}\, Z}\, d\mu_X \\ &= \int_{\mathcal{J}_\infty \mathcal{X}} \mathbb{L}^{\mathrm{ord}\, \phi^* Z - \mathrm{ord}\, \mathrm{Jac}_\phi - \mathfrak{s}_\mathcal{X}}\, d\mu_\mathcal{X} \\ &= \int_{\mathcal{J}_\infty \mathcal{X}} \mathbb{L}^{\mathrm{ord}\, \phi^* Z - \mathrm{ord}\, (p-1)\mathcal{Y} - \mathfrak{s}_\mathcal{X}}\, d\mu_\mathcal{X} \\ &= M_{\mathrm{st}}(\mathcal{X}, \phi^* Z + (1-p)\mathcal{Y}). \end{aligned}$$

$\square$

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Example 6.31. We suppose that $V = V_2$. From Corollary 6.4, for $a < 1$, we have

$$M_{\mathrm{st}}(X, aY) = \frac{\mathbb{L}^2 - \mathbb{L}}{1 - \mathbb{L}^{a-1}}.$$

On the other hand, for $a < 2 - p$, we can compute $M_{\mathrm{st}}(\mathcal{X}, a\mathcal{Y})$ from the definition as follows. First, we have

$$\begin{aligned} M_{\mathrm{st}}(\mathcal{X}, a\mathcal{Y}) &= \int_{\mathcal{J}_{\infty,0}\mathcal{X}} \mathbb{L}^{\operatorname{ord} a\mathcal{Y}} \, d\mu_{\mathcal{X}} + \sum_{j>0} \int_{\mathcal{J}_{\infty,j}} \mathbb{L}^{\operatorname{ord} a\mathcal{Y} - \mathfrak{s}_{\mathcal{X}}} \, d\mu_{\mathcal{X}} \\ &= \frac{\mathbb{L}^2 - \mathbb{L}}{1 - \mathbb{L}^{a-1}} + \sum_{j \in \mathbb{N}'} \int_{\mathcal{J}_{\infty,j}} \mathbb{L}^{\operatorname{ord} a\mathcal{Y} - \mathfrak{s}_{\mathcal{X}}} \, d\mu_{\mathcal{X}}. \end{aligned}$$

Now we compute the second term. Let us fix $E \in \text{G-}\mathbf{Cov}^{\mathrm{rep}}(D, j)(\bar{k})$, and for G-arcs $\gamma : E \longrightarrow V$ we denote associated twisted arcs by $\bar{\gamma}$. Then, with the notation as in § 2.5, those G-arcs $\gamma : E \longrightarrow V$ with $\operatorname{ord} \mathcal{Y}(\bar{\gamma}) = n$ correspond, by $\gamma \longmapsto \gamma^*(x)$, to

$$\bar{k}[[t]] \times \bar{k}^* \cdot gt^n \times \prod_{m>n} \bar{k} \cdot gt^m,$$

provided that $np - j > 0$. Otherwise there is no such G-arc. Hence for $np > j$,

$$\begin{aligned} \mu_{\mathcal{X}}((\operatorname{ord} \mathcal{Y})^{-1}(n) \cap \mathcal{J}_{\infty,j}\mathcal{X}) &= (\mathbb{L} - 1)\mathbb{L}^{n+1+\lfloor j/p \rfloor} \times (\mathbb{L} - 1)\mathbb{L}^{j-1-\lfloor j/p \rfloor} \\ &= \mathbb{L}^{-n+j}(\mathbb{L} - 1)^2. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{j \in \mathbb{N}'} \int_{\mathcal{J}_{\infty,j}\mathcal{X}} \mathbb{L}^{\operatorname{ord} a\mathcal{Y} - \mathfrak{s}_{\mathcal{X}}} \, d\mu &= \sum_{n=1}^{\infty} \sum_{j \in \mathbb{N}'} \mu_{\mathcal{X}}((\operatorname{ord} \mathcal{Y})^{-1}(n) \cap \mathcal{J}_{\infty,j}\mathcal{X}) \mathbb{L}^{an - \lfloor j/p \rfloor} \\ &= \sum_{n=1}^{\infty} \sum_{\substack{j \in \mathbb{N}' \\ j < np}} \mathbb{L}^{(a-1)n + j - \lfloor j/p \rfloor} (\mathbb{L} - 1)^2. \end{aligned}$$

Then, putting $j = rp + i$ $(0 \leqslant r < n, 0 < i < p)$, we continue:

$$\begin{aligned} &= (\mathbb{L} - 1)^2 \sum_{n=1}^{\infty} \left(\sum_{i=1}^{p-1} \sum_{r=0}^{n-1} \mathbb{L}^{(a-1)n + rp + i - r} \right) \\ &= (\mathbb{L} - 1)^2 \sum_{n=1}^{\infty} \sum_{i=1}^{p-1} \mathbb{L}^{(a-1)n+i} \frac{1 - \mathbb{L}^{n(p-1)}}{1 - \mathbb{L}^{p-1}} \\ &= (\mathbb{L} - 1)\mathbb{L} \sum_{n=1}^{\infty} \mathbb{L}^{(a-1)n} (\mathbb{L}^{n(p-1)} - 1) \\ &= (\mathbb{L} - 1)\mathbb{L} \left(\frac{\mathbb{L}^{a+p-2}}{1 - \mathbb{L}^{a+p-2}} - \frac{\mathbb{L}^{a-1}}{1 - \mathbb{L}^{a-1}} \right). \end{aligned}$$

As expected, we now have

$$\begin{aligned} M_{\mathrm{st}}(\mathcal{X}, a\mathcal{Y}) &= \frac{\mathbb{L}^2 - \mathbb{L}}{1 - \mathbb{L}^{a-1}} + (\mathbb{L} - 1)\mathbb{L} \left(\frac{\mathbb{L}^{a+p-2}}{1 - \mathbb{L}^{a+p-2}} - \frac{\mathbb{L}^{a-1}}{1 - \mathbb{L}^{a-1}} \right) \\ &= \frac{\mathbb{L}^2 - \mathbb{L}}{1 - \mathbb{L}^{a+p-2}}. \end{aligned}$$

Remark 6.32. For the pair $(\mathcal{X}, a\mathcal{Y})$ being stringily Kawamata log terminal, the coefficient a must be negative. In particular, $(\mathcal{X}, 0)$ is not stringily Kawamata log terminal, and its stringy invariant $M_{\mathrm{st}}(\mathcal{X})$ is not defined in this paper. However, stringy invariants should be generalized beyond log terminal singularities to some extent, as Veys [Vey04] confirmed for surface singularities in characteristic zero. Then, it appears meaningful to claim, for instance, that

$$M_{\mathrm{st}}(\mathcal{X}, a\mathcal{Y}) = \frac{\mathbb{L}^2 - \mathbb{L}}{1 - \mathbb{L}^{a+p-2}} \quad \text{and} \quad e_{\mathrm{st}}(\mathcal{X}, a\mathcal{Y}) = \frac{2}{2-a-p},$$

unless $a + p - 2 = 0$. In this way, we would be able to relate weighted counts of Artin–Schreier extensions of $k((t))$ to stringy invariants of singularities even when $D_V < p$.

6.4 Pseudo-projectivization and the Poincaré duality

Batyrev proved that the stringy invariant of a log terminal projective variety in characteristic zero satisfies the *Poincaré duality*. We will obtain a similar result for the 'projectivization' of our quotient variety X.

Let $\mathbb{G}_m = \operatorname{Spec} k[t^{\pm}]$ be the multiplicative group scheme over k. We have natural $\mathbb{G}_m$-actions on V and X, which are compatible with the quotient map $V \longrightarrow X$. Moreover, the action on V commutes with the G-action, inducing a $\mathbb{G}_m \times G$-action on V. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the quotient stacks $[(V\backslash\{0\})/\mathbb{G}_m \times G]$ and $[(X\backslash\{0\})/\mathbb{G}_m]$, respectively. The former is a smooth Deligne–Mumford stack and isomorphic to $[\mathbb{P}(V)/G]$. The latter is not a variety but a (singular) Artin stack with finite stabilizers. Indeed, from the following lemma, the $\mathbb{G}_m$-action on $X\backslash\{0\}$ have non-reduced stabilizers $\operatorname{Spec} k[t]/(t^p - 1) \subset \mathbb{G}_m$ at singular points.

Lemma 6.33. *Let $W \subset V$ be the fixed point locus of the G-action and $\bar{W} \subset X$ its image. Then the morphism $W \longrightarrow \bar{W}$ is isomorphic to the Frobenius morphism of W.*

Proof. We prove only the indecomposable case. Let $k[x_1, \dots, x_d]$ be the coordinate ring of V with the G-action as in § 3.2. Then W is defined by $x_2 = \cdots = x_d = 0$. Hence the coordinate ring $k[\bar{W}]$ of $\bar{W}$ is identified with the image of $k[x_1, \dots, x_d]^G$ on $k[x_1] = k[x_1, \dots, x_d]/(x_2, \dots, x_d)$. We can easily see that $x_1 \notin k[\bar{W}]$. On the other hand, x_1^p, the image of the norm of x_1, is in $k[\bar{W}]$. Since $k[W]$ is purely inseparable over $k[\bar{W}]$, we have $k[\bar{W}] = k[x_1^p]$, which shows the lemma. □

The stacks $\mathcal{W}_1$ and $\mathcal{W}_2$ have the same coarse moduli space $\bar{X} = (V\backslash\{0\})/\mathbb{G}_m \times G$. Moreover, $\mathcal{W}_1$ and $\mathcal{W}_2$ have open dense subsets isomorphic to $(V\backslash V^G)/\mathbb{G}_m$ and are birational. Since the morphisms $\mathcal{X}\backslash\{0\} \longrightarrow \mathcal{W}_1$ and $X\backslash\{0\} \longrightarrow \mathcal{W}_2$ are $\mathbb{G}_m$-torsors, it seems natural to define stringy invariants of $\mathcal{W}_1$ and $\mathcal{W}_2$ as follows.

Definition 6.34. We define the *stringy motivic invariant* of $\mathcal{W}_1$ and $\mathcal{W}_2$ by

$$M_{\mathrm{st}}(\mathcal{W}_1) = M_{\mathrm{st}}(\mathcal{W}_2) := \frac{\mathbb{L}^d - \mathbb{L}^l}{\mathbb{L} - 1} + (M_{\mathrm{st}}(X) - (\mathbb{L}^d - \mathbb{L}^l))\frac{\mathbb{L}^l - 1}{\mathbb{L}^l(\mathbb{L} - 1)}.$$

Let $\mathcal{W}$ denote either $\mathcal{W}_1$ or $\mathcal{W}_2$.

Proposition 6.35. *Suppose that $D_V \geqslant p$. Then*

$$M_{\mathrm{st}}(\mathcal{W}) = \frac{\mathbb{L}^d - 1}{\mathbb{L} - 1} + \frac{(\mathbb{L}^l - 1)(\sum_{s=1}^{p-1} \mathbb{L}^{s - \mathrm{sht}_V(s)})}{\mathbb{L}(1 - \mathbb{L}^{p-1-D_V})}.$$

Proof. We have

$$M_{\mathrm{st}}(\mathcal{W}) = \frac{\mathbb{L}^d - \mathbb{L}^l}{\mathbb{L}-1} + \left(\mathbb{L}^d + \frac{\mathbb{L}^{l-1}(\mathbb{L}-1)(\sum_{s=1}^{p-1}\mathbb{L}^{s-\mathrm{sht}_V(s)})}{1-\mathbb{L}^{p-1-D_V}} - (\mathbb{L}^d - \mathbb{L}^l)\right)\frac{\mathbb{L}^l - 1}{\mathbb{L}^l(\mathbb{L}-1)}$$
$$= \frac{\mathbb{L}^d - 1}{\mathbb{L}-1} + \frac{(\mathbb{L}^l - 1)(\sum_{s=1}^{p-1}\mathbb{L}^{s-\mathrm{sht}_V(s)})}{\mathbb{L}(1-\mathbb{L}^{p-1-D_V})}. \qquad \square$$

Proposition 6.36 (Poincaré duality). *Let us write $M_{\mathrm{st}}(\mathcal{W})$ as $M_{\mathrm{st}}(\mathcal{W};\mathbb{L})$ to clarify that it is a rational function in $\mathbb{L}$. Then we have*

$$M_{\mathrm{st}}(\mathcal{W};\mathbb{L}^{-1})\mathbb{L}^{d-1} = M_{\mathrm{st}}(\mathcal{W};\mathbb{L}).$$

Proof. The first term of the expression in Proposition 6.35 equals $[\mathbb{P}_k^{d-1}]$, which obviously satisfies the Poincaré duality. For the second term, substituting $\mathbb{L}^{-1}$ for $\mathbb{L}$ and multiplying by $\mathbb{L}^{d-1}$, we obtain

$$\frac{(\mathbb{L}^{-l}-1)(\sum_{s=1}^{p-1}\mathbb{L}^{\mathrm{sht}_V(s)-s})}{\mathbb{L}^{-1}(1-\mathbb{L}^{-p+1+D_V})}\mathbb{L}^{d-1} = \frac{(\mathbb{L}^l - 1)(\sum_{s=1}^{p-1}\mathbb{L}^{\mathrm{sht}_V(s)-s})\mathbb{L}^{p+d-l-D_V}}{\mathbb{L}(1-\mathbb{L}^{p-1-D_V})}.$$

Then the Poincaré duality follows from the following equations: for $1 \leqslant s \leqslant p-1$,

$$\mathrm{sht}_V(p-s) - (p-s) + p + d - l - D_V = s + \left(\sum_{\lambda=1}^{l}\sum_{i=1}^{d_\lambda - 1} i + \left\lfloor -\frac{is}{p}\right\rfloor\right) + d - l - D_V$$
$$= s + \left(\sum_{\lambda=1}^{l}\sum_{i=1}^{d_\lambda - 1} - \left\lfloor \frac{is}{p}\right\rfloor - 1\right) + d - l$$
$$= s - \mathrm{sht}_V(s). \qquad \square$$

7. Remarks on future problems

7.1 Generalizations

This study should be a toy model for the *wild McKay correspondence.* The following are possible directions of generalization.

(i) General groups and non-linear actions. If we similarly define twisted arcs, then the almost bijection between twisted arcs of $\mathcal{X}$ and arcs of X should be valid in general. Looking at Harbater's work [Har80], we should be able to construct the spaces of twisted arcs or jets at least for p-groups, whether their detailed structure can be understood or not. As explained in Remark 6.22, the non-linear case will be quite different from the linear case even in dimension two. Some non-linear action appears as the projectivization of a linear one. Then we may apply some results in the linear case to such cases.

(ii) General local fields. Sebag [Seb04] generalized the motivic integration to formal schemes over a discrete valuation ring. Replacing $k((t))$ with a general local field along this line, we might be able to get, for instance, a result on weighted counts of Galois extensions of the local field.

(iii) General proper birational morphisms of general Deligne–Mumford stacks. We proved the change of variables formula only for the morphism $[V/G] \longrightarrow V/G$. However, ultimately, it should be generalized to an arbitrary proper birational morphism of Deligne–Mumford stacks (with a mild finiteness condition). It was obtained in [Yas06] when the morphism is tame and stacks are smooth.

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7.2 Other related problems

As we saw in Corollary 6.21, if there exists a crepant resolution of X, then $D_V = p$. From Corollary 6.25, the converse holds in dimension up to 4. What about higher dimensions? The only known examples of crepant resolutions of X are Examples 6.14 and 6.15. For instance, if $V = V_2^{\oplus p}$, then from Example 6.16, $M_{\mathrm{st}}(X) = \mathbb{L}^{2p} + \mathbb{L}^p \cdot [\mathbb{P}^{p-1}]$. This seems to suggest that there exists a crepant resolution $Y \longrightarrow X$ such that the exceptional locus is universally homeomorphic to $\mathbb{A}_k^p \times \mathbb{P}_k^{p-1}$.

In characteristic zero, the McKay correspondence is proved at the derived category level [BK04, BKR01, CI04, Kaw05]. However, from [Yi94], the skew group algebra $k[\mathbf{x}] * G$ always has *infinite* global dimension in the wild case. Then, if there exists something like the *derived wild McKay correspondence*, then what would replace the derived category of $k[\mathbf{x}] * G$-modules?

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References

Art77 M. Artin, *Coverings of the rational double points in characteristic p*, in *Complex analysis and algebraic geometry* (Iwanami Shoten, Tokyo, 1977), 11–22.

Băd01 L. Bădescu, *Algebraic surfaces*, Universitext (Springer, New York, 2001).

Bat98 V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, in *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)* (World Scientific, River Edge, NJ, 1998), 1–32.

Bat99 V. V. Batyrev, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math. Soc. (JEMS) **1** (1999), 5–33.

BK04 R. V. Bezrukavnikov and D. B. Kaledin, *McKay equivalence for symplectic resolutions of quotient singularities*, Tr. Mat. Inst. Steklova **246** (2004), 20–42 (in Algebr. Geom. Metody, Svyazi i Prilozh).

BKR01 T. Bridgeland, A. King and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), 535–554; (electronic).

CW11 H. E. A. E. Campbell and D. L. Wehlau, *Modular invariant theory*, in *Encyclopaedia of mathematical sciences, Vol. 139*, Invariant Theory and Algebraic Transformation Groups, vol. 8 (Springer, Berlin, 2011).

Cra04 A. Craw, *An introduction to motivic integration*, in *Strings and geometry*, Clay Mathematics Proceedings, vol. 3 (American Mathematical Society, Providence, RI, 2004), 203–225.

CI04 A. Craw and A. Ishii, *Flops of G-Hilb and equivalences of derived categories by variation of GIT quotient*, Duke Math. J. **124** (2004), 259–307.

DL99 J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), 201–232.

DL02 J. Denef and F. Loeser, *Motivic integration, quotient singularities and the McKay correspondence*, Compositio Math. **131** (2002), 267–290.

ES80 G. Ellingsrud and T. Skjelbred, *Profondeur d'anneaux d'invariants en caractéristique p*, Compositio Math. **41** (1980), 233–244.

GV85 G. Gonzalez-Springberg and J.-L. Verdier, *Sur la règle de McKay en caractéristique positive*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), 585–587.

GKP89 R. L. Graham, D. E. Knuth and O. Patashnik, *A foundation for computer science*, in *Concrete mathematics* (Addison-Wesley, Reading, MA, 1989).

GS D. R. Grayson and M. E. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.

GD71 A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique. I*, in *Grundlehren der mathematischen Wissenschaften, Vol. 166* (Springer, Berlin, 1971).

Har80 D. Harbater, *Moduli of p-covers of curves*, Comm. Algebra **8** (1980), 1095–1122.

HIS13 M. Hirokado, H. Ito and N. Saito, *Three dimensional canonical singularities in codimension two in positive characteristic*, J. Algebra **373** (2013), 207–222.

KV04 M. Kapranov and E. Vasserot, *Vertex algebras and the formal loop space*, Publ. Math. Inst. Hautes Études Sci. **100** (2004), 209–269.

Kaw05 Y. Kawamata, *Log crepant birational maps and derived categories*, J. Math. Sci. Univ. Tokyo **12** (2005), 211–231.

Lip69 J. Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Publ. Math. Inst. Hautes Études Sci. **36** (1969), 195–279.

Loo02 E. Looijenga, *Motivic measures*, Astérisque **276** (2002), 267–297; Séminaire Bourbaki, Vol. 1999/2000.

Mil80 J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33 (Princeton University Press, Princeton, NJ, 1980).

Nic11 J. Nicaise, *A trace formula for varieties over a discretely valued field*, J. Reine Angew. Math. **650** (2011), 193–238.

NS11 J. Nicaise and J. Sebag, *The Grothendieck ring of varieties*, in *Motivic integration and its interactions with model theory and non-Archimedean geometry, Vol. I*, London Mathematical Society Lecture Note Series, vol. 383 (Cambridge University Press, Cambridge, 2011), 145–188.

Rei02 M. Reid, *La correspondance de McKay*, Astérisque **276** (2002), 53–72; Séminaire Bourbaki, Vol. 1999/2000.

Ros07 M. A. Rose, *Frobenius action on l-adic Chen–Ruan cohomology*, Commun. Number Theory Phys. **1** (2007), 513–537.

Sch09 S. Schröer, *The Hilbert scheme of points for supersingular abelian surfaces*, Ark. Mat. **47** (2009), 143–181.

Seb04 J. Sebag, *Intégration motivique sur les schémas formels*, Bull. Soc. Math. France **132** (2004), 1–54.

Tho08 L. Thomas, *A valuation criterion for normal basis generators in equal positive characteristic*, J. Algebra **320** (2008), 3811–3820.

Vey04 W. Veys, *Stringy invariants of normal surfaces*, J. Algebraic Geom. **13** (2004), 115–141.

Yas04 T. Yasuda, *Twisted jets, motivic measures and orbifold cohomology*, Compositio Math. **140** (2004), 396–422.

Yas06 T. Yasuda, *Motivic integration over Deligne–Mumford stacks*, Adv. Math. **207** (2006), 707–761.

Yas12 T. Yasuda, *Pure subrings of regular local rings, endomorphism rings and Frobenius morphisms*, J. Algebra **370** (2012), 15–31.

Yi94 Z. Yi, *Homological dimension of skew group rings and crossed products*, J. Algebra **164** (1994), 101–123.

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