ROBUST ESTIMATION OF RESERVE RISK

BY

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ABSTRACT

We tackle problems that appear in the practical application of the Mack method for the estimation of reserving risk and the bootstrapping of ultimate reserve distributions. More specifically, we design a filter for outliers and large jumps, and present a robust version of Mack's variance estimator. A combination of these guarantees a reasonable Mack and bootstrap error even for deficient data. Furthermore, a method is derived that allows us to remove the influence of fluctuations in earning patterns from the reserve risk estimate. It is thereby shown that the relation between underwriting and accident year based loss development patterns is given by a convolution. A numerically stable inversion thereof is obtained by means of a Tikhonov regularization. The reliability of the presented methods is verified with several loss triangles.

KEYWORDS

Reserve risk, stochastic reserves, dynamic financial analysis, Mack method, bootstrap, robust statistics, Tikhonov regularization, inverse problems.

1. INTRODUCTION

In the balance sheet of a P&C insurance company reserves are reported at best estimate of the ultimate loss. This is why many methods have been developed to calculate this quantity. In the recent years though, new regulations and discussions on new accounting rules are pushing for reporting also the reserve risk. This is the risk that the ultimate loss will significantly vary from the best estimate. For instance, the Swiss Solvency Test (SST) requires adding a market value margin to the discounted reserves. This margin is computed from the risk based capital needed to back those reserves. In other words, actuaries need to estimate the risk of reserves (i.e. the uncertainty in the estimates of ultimate losses of underwriting reserves). The subject has thus become quite topical in our field. This paper deals with various aspects of estimating risk from P&C reserve triangles.

The Mack method (Mack, 1993) is one of the most prominent methods for estimating reserve risk. The main reasons are its simplicity and the suitability

Astin Bulletin 40(2), 453-489. doi: 10.2143/AST.40.2.2061124

of its underlying stochastic model. Another popular approach is the bootstrap (England and Verrall, 1999). It was recently shown (England and Verrall, 2006) that the bootstrap can be based on the same stochastic model as the Mack method. Unfortunately, as many actuaries discovered, the straightforward application of both methods to realistic data sets is not always possible. Real data are often affected by problems such as wrong bookings causing outliers or jumps. This paper proposes several corrections to both methods that increase the accuracy of the estimation. As mentioned above, the resulting reserve risk is not directly reported in the accounting balance sheet, but it matters for Asset-Liability Management (ALM), the risk-adjusted valuation of companies and internal models to compute the market value margin required by solvency tests such as the SST.

There are two major sources of inaccuracy when directly applying the standard methods: outliers or large artificial jumps in the data and fluctuations in the earning patterns due to the wrong representation of the triangles in underwriting years instead of accident years. To remedy the first one, we propose in this paper the application of filters and/or a robust modification of the Mack's variance estimator. The corresponding modification of the bootstrap algorithm is also described. If the reserve triangles describe loss development per underwriting year, which is often the case in reinsurance, a straightforward application of the Mack or bootstrap method will treat the fluctuations in earning patterns as noise in the loss developments. Hence the reserve risk is overestimated. We present here a method for separating fluctuations in earning patterns from those in claim settlements. In general, we aim at separating true reserve uncertainty from noise artificially introduced by the method or wrongly booked data.

In Section 2, we briefly present the standard techniques as far as they are relevant to our study. Their application to real data with errors is discussed in Section 3, followed by the conclusion in Section 4. In Appendix A, we present the data used to produce the results and to demonstrate the methods.

2. Reserve Estimates and Risk: Standard Solutions

Let L_{jk} be the accumulated incurred claims (losses) of the accident year with index j, $1 \le j \le N$, either paid or reported up to development year k, $1 \le k \le N$. The exact definition of the term accident year will be the subject of an explicit discussion later in the paper. One has claim observations if $k \le N + 1 - j$, so the available data form a triangle. The goal is to estimate the reserve amount R, see Section 2.1, and the mean squared error of the reserve estimator, denoted by $\text{mse}(\hat{R})$, see Sections 2.2 and 2.3.

2.1. The Chain Ladder Algorithm

A popular method for estimating claim reserves is the chain ladder method (Taylor, 2000). Estimates of unobserved (future) losses are obtained recursively,

$\hat{L}_{j,k+1} = \hat{L}_{j,k}\hat{f}_k$, starting from the latest observation $\hat{L}_{j,N+1-j} = L_{j,N+1-j}$. The chain ladder factors $\hat{f}_k$ are given by the weighted average of the individual developments, $f_{jk} := L_{j,k+1}/L_{jk}$, over the accident years j,

$$\hat{f}_k = \sum_{j=1}^{N-k} \frac{L_{jk}}{\sum_{l=1}^{N-k} L_{lk}} f_{jk} = \frac{\sum_{j=1}^{N-k} L_{j,k+1}}{\sum_{j=1}^{N-k} L_{jk}}, \tag{1}$$

for $1 \le k \le N-1$. The ultimate claim amount is $U_j := L_{jN}$. Here the triangle is assumed to be large enough to cover the full business development, so U_j is really ultimate. It can be estimated as

$$\hat{U}_j = L_{j,N+1-j} \cdot \hat{f}_{N+1-j} \cdot \ldots \cdot \hat{f}_{N-1}. \tag{2}$$

The current reserve amount R_j (at the end of development year $k = N+1-j$) is given by

$$\hat{R}_j = L_{j,N+1-j}\,(\hat{f}_{N+1-j} \cdot \ldots \cdot \hat{f}_{N-1} - 1) + C_{j,N+1-j}, \tag{3}$$

where C_{jk} denotes the case reserve, that is the claims of accident year j which have been reported but not been paid up to development year k. The first summand of Eq. (3) is the estimate of incurred but not reported (IBNR) losses.

2.2. Mack Method

The underlying uncertainty of the reserve estimation $\hat{R}_j$ is assessed in terms of its mean square error. An analytic estimate thereof can be obtained by using the assumptions of the Mack model (Mack, 1993):

$$E(L_{j,k+1} \mid L_{j1}, \ldots, L_{jk}) = L_{jk} f_k, \tag{4}$$

$$\mathrm{Var}(L_{j,k+1} \mid L_{j1}, \ldots, L_{jk}) = L_{jk}\sigma_k^2, \tag{5}$$

$$\{L_{i1}, \ldots, L_{iN}\},\ \{L_{j1}, \ldots, L_{jN}\},\ i \neq j,\ \text{are independent.} \tag{6}$$

The values of f_k and σ_k can be estimated by the chain ladder factors (1) and the variance estimator (Mack, 1993)

$$\hat{\sigma}_k^2 = \frac{1}{N-k-1} \sum_{j=1}^{N-k} L_{jk} \left(\frac{L_{j,k+1}}{L_{jk}} - \hat{f}_k \right)^2, \ 1 \le k \le N-2, \tag{7}$$

$$\hat{\sigma}_{N-1}^2 = \min\left[\hat{\sigma}_{N-2}^4 / \hat{\sigma}_{N-3}^2,\ \min\left(\hat{\sigma}_{N-3}^2, \hat{\sigma}_{N-2}^2 \right) \right]. \tag{8}$$

The final formula (Mack, 1993) for the mean squared error estimate reads

$$\widehat{\text{mse}}\left(\hat{R}_j\right) = \hat{U}_j^2 \sum_{k=N+1-j}^{N-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2}\left(\frac{1}{\hat{L}_{jk}} + \frac{1}{\sum_{l=1}^{N-k} L_{lk}}\right). \tag{9}$$

In addition, the mean squared error of the overall reserve estimate, $\hat{R} = \sum_{j=2}^{N} \hat{R}_j$, can be calculated

$$\widehat{\text{mse}}(\hat{R}) = \sum_{j=2}^{N}\left\{\widehat{\text{mse}}\left(\hat{R}_j\right) + \hat{U}_j\left(\sum_{l=j+1}^{N} \hat{U}_l\right)\sum_{k=N+1-j}^{N-1} \frac{2\hat{\sigma}_k^2 / \hat{f}_k^2}{\sum_{n=1}^{N-k} L_{nk}}\right\}. \tag{10}$$

2.3. Bootstrapping of Reserve Risk

Bootstrapping, introduced in (Efron, 1979), is a general approach to statistical inference. Its first application to stochastic reserves can be found in (England and Verrall, 1999). This approach is based on the over-dispersed Poisson model. The extension to arbitrary generalized linear models (McCullagh and Nelder, 1989), including the Mack model, was demonstrated recently (England and Verrall, 2006). Bootstrapping as a general method is reviewed in Section 2.3.1. The application to reserve risk, in particular to the Mack model, is discussed in Section 2.3.2.

2.3.1. *Bootstrapping in General*

The task of bootstrapping is as follows: given an independent and identically distributed (*iid*) sample of size n, $\mathbf{x} = (x_1, \ldots, x_n)$, from an unknown distribution F, and a statistic $\hat{\theta}(\mathbf{x})$, such as an estimator, find out the induced distribution $P(\hat{\theta})$ of $\hat{\theta}$.

The main idea behind the bootstrap method is to approximate the distribution F by the empirical distribution $\hat{F} = \sum_{i=1}^{n} \delta(x - x_i)/n$. An *iid* sample of size n is drawn from $\hat{F}$, called bootstrapped sample $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$. In practice, this is done by drawing random samples with replacement from $(x_1, \ldots, x_n)$. The sampling is repeated B times giving $\{\mathbf{x}_1^*, \ldots, \mathbf{x}_B^*\}$, and the statistic is evaluated for each sample. Finally, the desired distribution $P(\hat{\theta})$ is approximated by the bootstrapped distribution $\hat{P} = \sum_{i=1}^{B} \delta(\hat{\theta} - \hat{\theta}(\mathbf{x}_1^*))/B$.

2.3.2. *Application to Reserve Risk*

The goal is to predict the distribution of the chain ladder reserve estimator $\hat{R}$, i.e. $\hat{\theta} = \hat{R}$. To start with, one has to choose a sample $\mathbf{x}$. The suggestion of (England and Verrall, 2006) is to take scaled Pearson residuals, which, assuming the Mack model, are given by

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$$r_{jk} = \frac{\sqrt{L_{jk}}\left(f_{jk} - \hat{f}_k\right)}{\hat{\sigma}_k}, \ j \le N-k, \ k \le N-1. \tag{11}$$

This triangle of residuals is bootstrapped (resampled with replacement) to form a triangle of bootstrapped residuals $\mathbf{r}^*$. The sampling is repeated B times giving $\{\mathbf{r}_1^*, ..., \mathbf{r}_B^*\}$ and for each of the bootstrapped triangles $\mathbf{r}_i^*$ one evaluates the chain ladder reserve estimation $\hat{R}(\mathbf{r}_i^*)$. To this end, the residual definition (11) is inverted to form a triangle of bootstrapped development factors f_{jk}^*,

$$f_{jk}^* = r_{jk}^* \frac{\hat{\sigma}_k}{\sqrt{L_{jk}}} + \hat{f}_k. \tag{12}$$

The corresponding bootstrapped chain ladder factors read

$$\hat{f}_k^* = \sum_{j=1}^{N-k} \frac{L_{jk}}{\sum_{l=1}^{N-k} L_{lk}} f_{jk}^*. \tag{13}$$

Then the bootstrapped future losses are obtained recursively by drawing samples from the process distribution (England and Verrall, 2006). In this paper, a lognormal distribution is assumed for the cumulative losses L_{jk}, such that the bootstrapped future losses are given by

$$L_{j,k+1}^* \sim \text{Lognormal}(\hat{f}_k^* L_{jk}, \hat{\sigma}_k^2 L_{jk}), \quad k = N-j+1, \tag{14}$$

$$L_{j,k+1}^* \sim \text{Lognormal}(\hat{f}_k^* L_{jk}^*, \hat{\sigma}_k^2 L_{jk}^*), \quad k \ge N-j+2. \tag{15}$$

Finally, the desired distribution of the chain ladder reserve estimator is approximated by the empirical distribution $\hat{P} = \sum_{i=1}^{B} \delta(\hat{R} - \hat{R}(\mathbf{r}_i^*))/B$ of the bootstrapped reserve estimate $\hat{R}(\mathbf{r}_i^*)$. In addition, the mean squared error of the chain ladder estimator can be assessed by means of

$$\widehat{\text{mse}}(\hat{R}) = \frac{1}{B-1}\sum_{i=1}^{B}\left(\hat{R}\left(\mathbf{r}_i^*\right) - \bar{R}^*\right)^2, \text{ with } \bar{R}^* = \frac{1}{B}\sum_{i=1}^{B}\hat{R}\left(\mathbf{r}_i^*\right). \tag{16}$$

3. Applying the Mack and Bootstrapping Methods to real data with errors

3.1. Data Errors, Data Problems

Real data often have errors of different kinds. Applying the prescribed methods to such data leads to false estimates of reserve uncertainties. In this section,

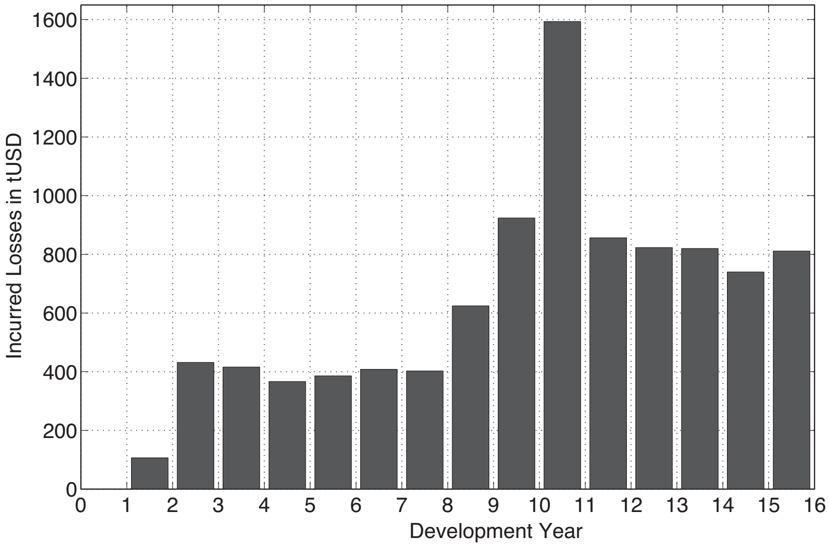

FIGURE 1: Upward outlier in the loss development of line of business A, underwriting year 2, Table 10 Appendix A.

we will propose different means of avoiding or eliminating such errors. The following types of errors are typical for data of (re)insurance companies.

(A) Incomplete data: Parts of the loss triangle are missing, usually in the upper left corner, reflecting past calendar years with poor data coverage.

(B) Booking errors: The data contain one-time errors, e.g. isolated bookings which differ drastically from the level given by both the previous and the subsequent development year, see Fig. 1. Typically these are errors that were corrected the following year. Even if these bookings made sense for pure accounting, they do not reflect the true reserving risks and policies.

(C) Small numbers and large jumps: For long-tail business, the incurred losses L_{jk} of early development years can be either zero or very small for some accident years j, see Fig. 2. Note these values appear in the denominator of the variance estimators (7) of both the original Mack method and the bootstrapping of the Mack model. While most authors agree that zero denominators leading to infinite terms must be omitted from the analysis, some very low values (such as a few dollars) may lead to huge variance estimates. Resulting reserve risk estimates are extremely sensitive against small variations of these low values, which is in sharp contrast to their low importance.

(D) Underwriting-year based triangles: Some (re)insurers only provide triangles per underwriting year, not per accident year (to be exactly defined in Section 3.5). A straightforward application of methods that were introduced for accident-year

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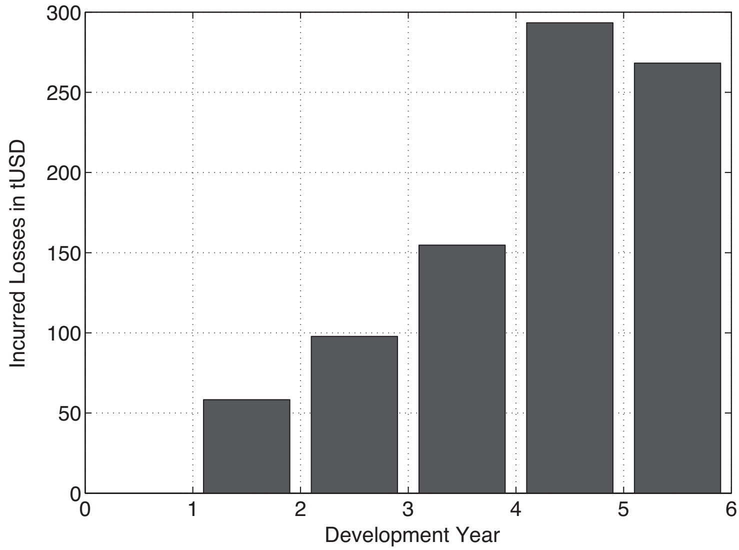

FIGURE 2: Large jump in the loss development of line of business A, underwriting year 9, Table 10 Appendix A. The loss of the second period is larger then the first one by a factor of 10^3.

triangles (such as Mack or bootstrapping) leads to incorrect results, typically overestimating the reserving risk. The effect is eminent in particular for short-tail lines of business.

The problems (A) and (B) can be treated on the same footing by data filtering which was first discussed by (Mack, 1999). We will extend these ideas in Section 3.2. In the presence of the data problem (C), a robust modification of the Mack method and the bootstrapping as proposed in Section 3.3 is recommended. The methods are illustrated in Section 3.4 using the data shown in Appendix A. Problem (D) is addressed in Section 3.5.

Aside from problems (A)-(D), practitioners are confronted with further data problems that are not analyzed here. Examples are fluctuations in the nature of the underlying business over different accident years and trends and cycles in claim development related to calendar years rather than development years. These effects violate the assumption of independent claim developments and may therefore lead to further estimation errors.

3.2. Data Filtering

3.2.1. *Detection of Defective Data*

Data gaps of type (A) and data errors of type (B) as defined in Section 3.1 need to be detected and then treated by a data filter. Error detection is not trivial. One has to distinguish errors from plausible jumps of observed quantities such as incurred losses to a new level confirmed by the subsequent bookings. Here we discuss the automatic detection of defective data points. Of course, an actuary should finally decide whether the data are indeed defective or not.

One type of error in the triangles is that of outliers, i.e. isolated bookings which drastically exceed the level given by both the previous and the subsequent development years, see Fig. 1. Usually, this effect is due to booking errors. We suggest detecting an outlier by comparing its size with the estimate of the ultimate loss $\hat{U}_j$. Thus, L_{jk} is detected as an upward outlier if it fulfills both of the following two inequalities:

$$L_{jk} - L_{j,k-1} \geq a\hat{U}_j \quad \text{and} \quad L_{j,k+1} - L_{jk} \leq -a\hat{U}_j. \tag{17}$$

A downward outlier is similarly detected as follows:

$$L_{jk} - L_{j,k-1} \leq -a\hat{U}_j \quad \text{and} \quad L_{j,k+1} - L_{jk} \geq a\hat{U}_j. \tag{18}$$

The parameter a determines the threshold of the outlier detection, and we suggest choosing a between 10% and 30%.

Another possible anomaly in the data is that of huge jumps in the loss development, see Fig. 2, where the factor between values matters more than the difference. We suggest detecting jumps by a comparison of the development and the chain ladder factors, which gives the following criterion for the detection of a large jump at the development factor f_{jk}:

$$f_{jk} \geq b\hat{f}_k. \tag{19}$$

The threshold can be controlled by the parameter b, and we suggest choosing b between 10 and 100. Of course, the arbitrariness of this choice also demonstrates the limits of data filtering. In particular in the jump case, it can be very difficult to decide whether the data are indeed false. We therefore suggest in Section 3.3 a modification of the Mack method and the bootstrap that makes these methods robust against data errors.

3.2.2. *Filter Functions*

Suppose one has detected a defective data point L_{mn} which one wants to exclude from the measurement of the reserve risk. This data point could be an outlier or simply $L_{mn} = 0$. In this case there are two development factors, namely $f_{mn} = L_{m,n+1} / L_{mn}$ and $f_{m,n-1} = L_{mn} / L_{m,n-1}$, which are defective as well. We therefore introduce two kind of filter functions,

$$v_{jk} = 1 - \delta_{jm}\delta_{kn}, \tag{20}$$

$$w_{jk} = 1 - \delta_{jm}(\delta_{kn} + \delta_{k,n-1}), \tag{21}$$

where δ_{ij} denotes the Kronecker delta. Here v_{jk} is used to suppress L_{mn} itself and w_{jk} is used to exclude f_{mn} and $f_{m,n-1}$.

Suppose on the other hand one has detected a defective development factor f_{mn}, e.g. a large jump in the loss development. In this case we set the filter functions to

$$v_{jk} = 1, \tag{22}$$

$$w_{jk} = 1 - \delta_{jm}\,\delta_{kn}. \tag{23}$$

3.2.3. *Filtering and the Mack method*

The chain ladder factors (1) are weighted averages of the development coefficients, while the variance estimators (7) are weighted averages of the deviation of f_{jk} from the mean development. One thus has to suppress in both cases f_{mn} and $f_{m,n-1}$ from the averaging which yields

$$\hat{f}_k' = \sum_{j=1}^{N-k} \frac{L_{jk}}{\sum_{l=1}^{N-k} w_{lk} L_{lk}}\, w_{jk} f_{jk}, \tag{24}$$

$$\hat{\sigma}_k^{2\prime} = \frac{1}{\sum_{j=1}^{N-k} w_{jk} - 1} \sum_{j=1}^{N-k} w_{jk}\, L_{jk}\,(f_{jk} - \hat{f}_k)^2. \tag{25}$$

Similar equations can be found in (Mack, 1999) with a different prefactor, $1/(N-k-1)$ instead of $1/\left(\sum_{j=1}^{N-k} w_{jk} - 1\right)$, in (25). We choose the latter in order to obtain an unbiased estimate. Finally, the mean squared error is obtained by (9) and (10), substituting $\hat{f}_k'$ and $\hat{\sigma}_k^{2\prime}$ for $\hat{f}_k$ and $\hat{\sigma}_k^2$.

3.2.4. *Filtering and the Bootstrap*

The residuals r_{jk}, see Eq. (11), are computed from the development factors f_{jk}. Thus, both r_{mn} and $r_{m,n-1}$ have to be eliminated from the empirical distribution before the sampling. In other words, random samples are drawn only from those residuals r_{jk} where the filter function w_{jk} is equal to unity. We have to suppress the defective data L_{mn} in the formulas for the bootstrapped development and chain ladder factors, Eq. (12) and (13). We therefore replace (13) by

$$\hat{f}_k^{*\prime} = \sum_{j=1}^{N-k} \frac{v_{jk} L_{jk}}{\sum_{l=1}^{N-k} v_{lk} L_{lk}}\, f_{jk}^*, \tag{26}$$

and use these filtered chain ladder factors for the forecasting step in Eqs. (14) and (15). Moreover, $\hat{f}_k'$ has to be substituted for $\hat{f}_k$ in Eqs. (11), (12) and $\hat{\sigma}_k^2$ is replaced by $\hat{\sigma}_k^{2\prime}$ in Eqs. (11), (12), (14) and (15).

3.3. Robust Estimation of Reserve Risk

Data jumps of type (C) as defined in Section 3.1 may lead to absurdly high variance estimates. One way of correcting this is to filter jumps from very small to high losses. Here another method is proposed: an essentially unbiased robust estimator.

3.3.1. *Robust Mack Method*

Mack's variance estimator (7), which can be rewritten as

$$\hat{\sigma}_k^2 = \frac{1}{N-k-1} \sum_{j=1}^{N-k} \frac{1}{L_{jk}} \left(L_{j,k+1} - \hat{f}_k L_{jk} \right)^2, \tag{27}$$

displays a singularity at $L_{jk} = 0$. Since the incurred losses L_{jk} appear in the denominator, the Mack estimator is very sensitive to small L_{jk} values and errors of these. To make the estimator more robust, we suggest replacing the denominator by an expectation value,

$$\left(\hat{\sigma}_k^2 \right)^{\mathrm{r}} = \frac{1}{N-k-1} \sum_{j=1}^{N-k} \frac{1}{E\left(L_{jk} \right)} \left(L_{j,k+1} - \hat{f}_k L_{jk} \right)^2, \tag{28}$$

which will be justified below by a study on the estimation bias. In practice, the theoretical loss expectation value $E(L_{jk})$ is unavailable, implying that one has to insert an appropriate estimate $\hat{E}(L_{jk})$ for $E(L_{jk})$. According to (England and Verrall, 2002), $E(L_{jk})$ can be assessed by backward recursion starting with the observed incurred losses to date in the latest diagonal,

$$\hat{E}(L_{j,N-j+1}) = L_{j,N-j+1}, \tag{29}$$

$$\hat{E}(L_{j,k-1}) = \hat{E}(L_{jk}) \; \hat{f}_{k-1}^{-1}, \quad k \le N-j+1. \tag{30}$$

It can be shown[1] that this estimator has the same form as $E(L_{jk} | \mathbf{L}_f)$, i.e.

$$E(L_{jk} | \mathbf{L}_f) = L_{j,N+1-j} \; f_{N-j}^{-1} \cdot \ldots \cdot f_k^{-1}, \quad k \le N+1-j, \tag{31}$$

which is the best prediction of L_{jk} given the future triangle $\mathbf{L}_f = \{L_{jk} | j + k \ge N + 1\}$.

The losses on the latest diagonal, as well as the chain ladder factors, are typically less corrupted by data errors than the individual losses L_{jk} of early development years. Hence, the expectation value $E(L_{jk})$ in the denominator of (28) is more resilient than the individual losses L_{jk}. The estimator $(\hat{\sigma}_k^2)^{\mathrm{r}}$ is

[1] The proof, which relies on the Mack assumptions, is available on request.

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therefore robust, in the sense of "outlier resistance" (Huber and Ronchetti, 2009), and can replace $\hat{\sigma}_k^2$ in all derived calculations such as Eq. (9).

However, the robustness comes with the price of a bias. We will spend the rest of this section computing this bias and evaluating its order of magnitude. Let us start with the expectation value

$$E\left(\left(\hat{\sigma}_k^2\right)^{\mathrm{r}}\right) = \frac{1}{N-k-1}\sum_{j=1}^{N-k}\frac{E\left(L_{j,k+1}^2 - 2\hat{f}_k L_{jk} L_{j,k+1} + \hat{f}_k^2 L_{jk}^2\right)}{E(L_{jk})}. \tag{32}$$

The first summand of the expectation value in the numerator reads

$$\begin{aligned} E(L_{j,k+1}^2) &= E(E(L_{j,k+1}^2 \mid L_{j1}, \ldots, L_{jk})) \\ &= E(\mathrm{Var}(L_{j,k+1} \mid L_{j1}, \ldots, L_{jk}) + E(L_{j,k+1} \mid L_{j1}, \ldots, L_{jk})^2) \\ &= \sigma_k^2 E(L_{jk}) + f_k^2 E(L_{jk}^2), \end{aligned} \tag{33}$$

where the Mack model (4), (5) is used in the third line. Using the notation $\mathbf{L}_k = \{L_{ij} \mid j \le k, j \le N+1-i\}$, $1 \le k \le N$, the second summand reads

$$\begin{aligned} E(\hat{f}_k L_{jk} L_{j,k+1}) &= E\left(L_{jk} E\left[L_{j,k+1}\frac{\sum_{l=1}^{N-k} L_{l,k+1}}{\sum_{l=1}^{N-k} L_{lk}}\middle| \mathbf{L}_k\right]\right) \\ &= E\left(\frac{L_{jk}}{\sum_{l=1}^{N-k} L_{lk}} E\left[L_{j,k+1}\sum_{l=1, l\neq j}^{N-k} L_{l,k+1} + L_{j,k+1}^2 \middle| \mathbf{L}_k\right]\right) \\ &= E\left(\frac{L_{jk}}{\sum_{l=1}^{N-k} L_{lk}} f_k L_{jk}\sum_{l=1, l\neq j}^{N-k} f_k L_{lk}\right) + E\left(\frac{L_{jk}}{\sum_{l=1}^{N-k} L_{lk}}\left[\sigma_k^2 L_{jk} + f_k^2 L_{jk}^2\right]\right) \\ &= f_k^2 E(L_{jk}^2) + \sigma_k^2 E\left(\frac{L_{jk}^2}{\sum_{l=1}^{N-k} L_{lk}}\right). \end{aligned} \tag{34}$$

Here the third line makes use of the Mack assumptions (4) to (6). Finally, the last summand of (32) reads

$$\begin{aligned} E\left(\hat{f}_k^2 L_{jk}^2\right) &= E\left(L_{jk}^2\left[\mathrm{Var}(\hat{f}_k \mid \mathbf{L}_k) + E^2(\hat{f}_k \mid \mathbf{L}_k)\right]\right) \\ &= E\left(L_{jk}^2\left[\frac{\sum_{j=1}^{N-k}\mathrm{Var}(L_{j,k+1} \mid \mathbf{L}_k)}{\left(\sum_{j=1}^{N-k} L_{jk}\right)^2} + f_k^2\right]\right) \\ &= \sigma_k^2 E\left(\frac{L_{jk}^2}{\sum_{l=1}^{N-k} L_{lk}}\right) + f_k^2 E(L_{jk}^2). \end{aligned} \tag{35}$$

By inserting the results (33)-(35) into Eq. (32), one obtains

$$
\begin{aligned}
E\left(\left(\hat{\sigma}_k^2\right)^{\mathrm{r}}\right) &= \frac{1}{N-k-1}\sum_{j=1}^{N-k}\frac{1}{E(L_{jk})}\left[\sigma_k^2 E(L_{jk}) - \sigma_k^2\, E\left(\frac{L_{jk}^2}{\sum_{l=1}^{N-k} L_{lk}}\right)\right] \\
&= \sigma_k^2 + \frac{\sigma_k^2}{N-k-1}\left[1 - \sum_{j=1}^{N-k}\frac{1}{E(L_{jk})}\, E\left(\frac{L_{jk}^2}{\sum_{l=1}^{N-k} L_{lk}}\right)\right].
\end{aligned} \tag{36}
$$

The estimator $(\hat{\sigma}_k^2)^{\mathrm{r}}$ has therefore a non-vanishing bias

$$
\begin{aligned}
\mathcal{B}_{(\hat{\sigma}_k^2)^{\mathrm{r}}} &= E\left(\left(\hat{\sigma}_k^2\right)^{\mathrm{r}}\right) - \sigma_k^2 \\
&= \frac{\sigma_k^2}{N-k-1}\left[1 - \sum_{j=1}^{N-k}\frac{1}{E(L_{jk})}\, E\left(\frac{L_{jk}^2}{\sum_{l=1}^{N-k} L_{lk}}\right)\right].
\end{aligned} \tag{37}
$$

In order to simplify this expression let us make the approximation

$$
E\left(\frac{L_{jk}^2}{\sum_{l=1}^{N-k} L_{lk}}\right) \simeq \frac{E(L_{jk}^2)}{E\left(\sum_{l=1}^{N-k} L_{lk}\right)}, \tag{38}
$$

which can be justified in the case where the losses L_{jk} are independent and identically normal distributed with mean μ_k and variance $\sigma_k{}^2$. This assumption implies that the relative error of the above assumption reads as[2]

$$
\text{Error} = \frac{2}{N-k}\,\frac{\sigma_k^2}{\mu_k^2 + \sigma_k^2}. \tag{39}
$$

For instance, the relative error is less than 4% for $N-k \geq 3$ and $\sigma_k/\mu_k = 0.25$ or less then 6% for $N-k \geq 5$ and $\sigma_k/\mu_k = 0.4$. By using (38), one obtains

$$
\begin{aligned}
\mathcal{B}_{(\hat{\sigma}_k^2)^{\mathrm{r}}} &\approx \frac{\sigma_k^2}{N-k-1}\left[1 - \sum_{j=1}^{N-k}\frac{\mathrm{Var}(L_{jk}) + E^2(L_{jk})}{E(L_{jk})E\left(\sum_{l=1}^{N-k} L_{lk}\right)}\right] \\
&= \frac{-\sigma_k^2}{N-k-1}\sum_{j=1}^{N-k}\frac{\mathrm{Var}(L_{jk})}{E(L_{jk})\sum_{l=1}^{N-k} E(L_{lk})}.
\end{aligned} \tag{40}
$$

[2] The derivation is available on request.

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The order of magnitude of this expression can be assessed by assuming the fluctuations in the losses L_{jk} to be bounded by a certain fraction μ of the expected losses $E(L_{jk})$,

$$[\mathrm{Var}(L_{jk})]^{1/2} \leq \mu E(L_{jk}). \tag{41}$$

By combining this inequality with Eq. (40), we find

$$\frac{|\mathcal{B}|}{\sigma_k^2} \leq \frac{\mu^2}{N-k-1}. \tag{42}$$

It follows that the bias $\mathcal{B}_{(\hat{\sigma}_k^2)^{\mathrm{r}}}$ corresponding to the first development years k, $k \ll N$, is negligible for large and moderately distorted triangles (with $N \geq 10$ and $\mu \leq 0.5$). For later development years k, $k \gg 1$, this bias may be significant; its contribution to the final Mack error is however small, implying that the bias is acceptable.

Even though one has to assume in this argumentation regular and moderately distorted triangles, our tests have shown that we can recommend using the robust estimator also for triangles with serious data problems. Then we argue that shifting from the original to the robust estimate essentially means a justified correction rather than a bias.

The above findings are substantiated by a Monte Carlo simulation summarized in Appendix B. There the bias and the root mean squared error of the robust estimator (28) are calculated stochastically, by generating random triangles in accordance with the Mack assumptions (4) to (6). This simulation reveals that the bias of (28) stays less than 5%, which is much smaller than the corresponding root mean squared error. The latter turns out to be identical to the one of Mack's variance estimator (27), so that these two estimators are equal in that respect. Moreover, we test their stability by introducing artificial errors in the Monte Carlo simulations, which shows that the gain in robustness is huge.

3.3.2. *Robust Bootstrapping*

Bootstrapping in the context of Mack's model has a similar sensitivity to small values of incurred losses to that of its analytic counterpart. This is no surprise since Mack's variance estimator is part of the algorithm, see Section 2.3.2. Like in the previous section, we suggest stabilizing the procedure by using the robust estimator (28) instead of the original one, that is σ_k^2 has to be replaced by $(\sigma_k^2)^{\mathrm{r}}$ in Eqs. (11), (12), (14) and (15). Furthermore, the residuals (11), which can be rewritten as

$$r_{jk} = \frac{L_{j,k+1} - \hat{f}_k L_{jk}}{\hat{\sigma}_k \sqrt{L_{jk}}}, \tag{43}$$

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may display singularities. In order to exclude the losses from the denominator, we suggest transforming the residuals prior to bootstrapping,

$$r'_{jk} = r_{jk}\sqrt{\frac{L_{jk}}{\hat{E}(L_{jk})}}. \tag{44}$$

Here the estimate of the expectation value $\hat{E}(L_{jk})$ is obtained by (29) and (30). After resampling, the bootstrapped residual $r'_{j'k'}$ with randomly picked indices j' and k' is transformed inversely,

$$r^*_{jk} = r'_{j'k'}\sqrt{\frac{\hat{E}(L_{jk})}{L_{jk}}}. \tag{45}$$

It is shown in the examples of Section 3.4 that this procedure leads to a similar reserve risk estimate to that of the robust Mack method described in the previous section.

3.4. Examples

3.4.1. *Line of Business with no Apparent Data Errors*

This section will demonstrate the robust methods which were proposed in Section 3.2 and 3.3. Our first example, shown in Appendix A Table 9, is the smooth triangle A which exhibits neither outliers nor large jumps. It therefore provides a probe which allows us to examine the impact of the robust estimator $(\hat{\sigma}_k^2)^{\mathrm{r}}$ on the Mack or the bootstrap error. Table 1 shows the results of the Mack and the bootstrap analysis. We used 10000 iterations for the bootstrap algorithm which permits convergence to the tenth decimal place. The first row of Table 1 shows the results of the standard Mack and bootstrap algorithm as described in Section 2. The results for both methods are almost identical since the Mack model is used for the bootstrapping. The second row shows the results of the robust methods described in Section 3.3.1 and 3.3.2. The relative difference in the mean squared error of the standard and the robust

TABLE 1

COMPARISON OF THE STANDARD AND THE ROBUST METHODS (SEE SECTION 3.3.1 AND 3.3.2) IN CASE OF THE SMOOTH TRIANGLE A.

LoBA	Reserve (USD)	Mack error (USD)	Mack error (%)	Bootstrap error (USD)	Bootstrap error (%)
Original Method	9900	793	8.0	780	7.9
Robust Method	9900	741	7.5	740	7.5

TABLE 2

Application of the outlier filter and the robust estimator (28) to the erroneous loss triangle B.

LoBB	Reserve (mUSD)	Mack error (mUSD)	Mack error (%)	Bootstrap error (mUSD)	Bootstrap error (%)
Original Method	25	9.0	36	9.8	37
Outlier Filter	27	7.9	29	8.1	29
Robust Estimator	27	4.9	18	4.8	18

techniques is (793 – 741) / 793 ≈ 6.6% in case of the Mack method and (780 – 740) / 740 ≈ 5.4% in case of the bootstrapping. Of course, it is not clear whether this difference stems from the bias or the variance of the estimators $\hat{\sigma}_k^2$ and $(\hat{\sigma}_k^2)^{\mathrm{r}}$. Nevertheless, we conclude that the order of magnitude of the impact of the bias of $(\hat{\sigma}_k^2)^{\mathrm{r}}$ on the reserve risk estimate is no larger than 5%.

3.4.2. *Line of Business with Data Errors*

The loss triangle B, see Appendix A Table 10, is the prime example of a defective data set. It contains vanishing entries, outliers as well as large jumps, and it thus leads to a huge reserve risk estimate. Let us apply the robust methods of Section 3.2 and 3.3 step by step in order to obtain a reasonable result.

First, one has to handle the four vanishing entries of the first development year. The information regarding these losses is missing and we accordingly treat them as defective data points which are filtered using the methods described in Section 3.2.3 and 3.2.4. Alternatively, it is feasible to fill these data points with backward projections using the chain ladder factors. However, this would artificially smooth the triangle and thus underestimate the reserve risk. The resulting Mack and bootstrap errors are shown in Table 2. We show integer percentage figures as the bootstrap algorithm does not converge to values with more precise digits, even for a large number (e.g. 25'000) of iterations. We assume that this poor convergence is due to the irregularity of the data. The outliers are detected using the criteria (17), (18) and a detection threshold of $a = 20\%$. This identifies three upward outliers at $(i = 13, k = 6)$, $(i = 2, k = 11)$, $(i = 8, k = 11)$ which we also exclude from the Mack and the bootstrap analysis using the methods of Section 3.2.3 and 3.2.4. Here the relative reserve risk estimate drops to 29%, see the second row of Table 2. The estimate is still huge since the data exhibit large jumps. The jump at $(i = 9, k = 1,2)$ is particularly dominant with a loss increase by a factor of 10^3. We accordingly apply the robust methods of Section 3.3.1 and 3.3.2 which both lead to a relative reserve risk estimate of 18%, see the third row of Table 2. Alternatively, one can obtain robust results by combining the standard Mack and bootstrap method

TABLE 3

APPLICATION OF THE JUMP FILTER TO THE ERRONEOUS LOSS TRIANGLE B.
THE FIRST LINE WAS OBTAINED BY THE STANDARD METHOD INCLUDING AN OUTLIER FILTER.
THE JUMP FILTER WAS APPLIED IN THE SECOND AND THIRD ROW WITH DECREASING THRESHOLD b.

Jump filter threshold b	Reserve (mUSD)	Mack error (mUSD)	Mack error (%)	Bootstrap error (mUSD)	Bootstrap error (%)
∞	27	7.9	29	8.1	29
10	27	5.9	22	6.2	23
2	26	4.7	18	4.8	18

with a jump filter (see Section 3.2.1). Table 3 presents the corresponding results for different detection thresholds b. A comparison with Table 2 shows that the effect of the robust estimator is comparable to that of a strong jump filter (with threshold $b = 2$) in the example of LoBB.

3.5. Fluctuations in Earning Patterns and the Estimation of Reserve Risk

Individual defective data points are one possible source of inaccuracy in the measurement of reserve risk. Another reason for miscalculation is a systematic shift in the nature of the loss development patterns over the years.

As an example let us take data problem (D) as defined in Section 3.1. There some variations in the timing of the risk exposure for different underwriting years lead to additional volatility in the data. The measured reserve risk does not only reflect the stochastic nature of the claim settlements but also has a component which is due to the volatility of the earning pattern over the years. For instance, PartnerRe publish their loss triangles both per underwriting and per accident year. The difference in the Mack error is 501 mUSD – 468 mUSD = 33 mUSD, as to be discussed in Section 3.5.6, and we argue that this difference is mainly due to the influence of fluctuations in earning patterns.

The term accident year needs a clearer definition at this point. We mean the calendar year during which a loss was primarily triggered, mainly regarding contracts of the "risk attaching" type, irrespective of the fact that some financial consequences and the reporting may have occurred in later years. The accident year of a certain loss event may coincide with the underwriting year or may be one or more years later, reflecting the earning pattern of the contract. At the same time the earning pattern describes the pace with which reserves for a certain new contract will be built up.

In some cases the accident-year based data are not available. We have therefore developed a method that allows for removal of the impact of the earning patterns from underwriting-year based reserve risk estimates. The procedure is independent from the method used to predict the reserve risk. We will use it with the Mack method, but it is possible to combine it with any other estimation method such as bootstrapping.

An overview of the procedure is given in Section 3.5.1 followed by the derivation of the method in the next three subsections. The results are illustrated using two examples in Section 3.5.6.

3.5.1. *Description of the Method*

We assume that fluctuations in the claims settlement are independent of variations of the earning pattern. Hence the variances (and their estimates) of the ultimate reserve estimates are additive,

$$\hat{\sigma}_{\mathrm{UY}}^2 = \hat{\sigma}_{\mathrm{AY}}^2 + \hat{\sigma}_{\mathrm{EP}}^2. \tag{46}$$

Here the following abbreviations are used:

- $\hat{\sigma}_{\mathrm{UY}}$: Mack error of the original underwriting year based triangle which is subject to both effects: uncertainty in the size of claims and volatility in the earning pattern.
- $\hat{\sigma}_{\mathrm{AY}}$: Estimate of the true reserving risk that stems only from variations in the claims development.

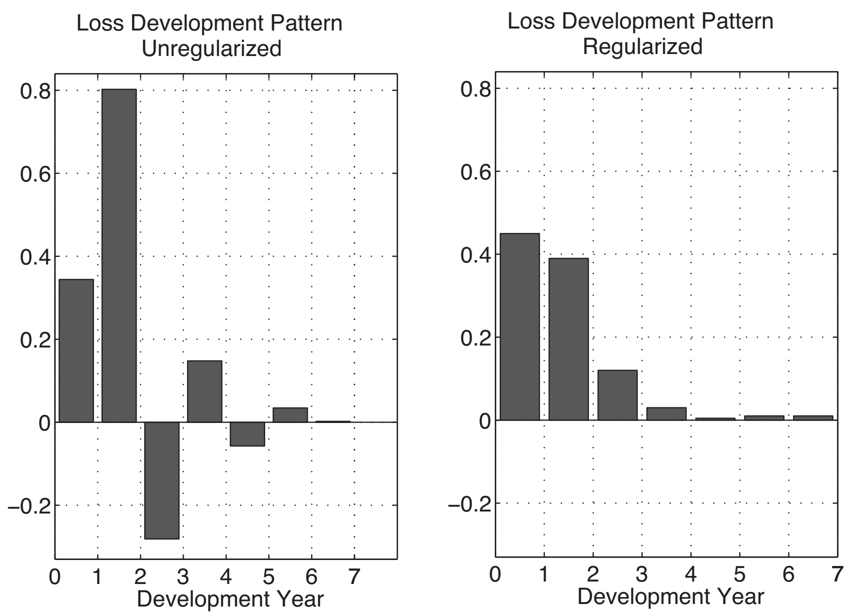

FIGURE 3: Application of the Tikhonov regularization.
The figure on the left shows the straightforward solution of (50).
The regularized solution, obtained by Eq. (52), is shown in the second figure.

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- $\hat{\sigma}_{\mathrm{EP}}$: Mack error of an auxiliary triangle $\mathcal{L}_{ik}$ that has fluctuations only due to the volatility of the earning pattern. The ultimate claims are kept constant.

The true reserve risk σ_{AY} can thus be estimated by

$$\hat{\sigma}_{\mathrm{AY}} = \sqrt{\hat{\sigma}_{\mathrm{UY}}^2 - \hat{\sigma}_{\mathrm{EP}}^2}. \tag{47}$$

The remainder of this subsection will explain the construction of the auxiliary triangle $\mathcal{L}_{ik}$.

First let us calculate the earned premium patterns. We assume that most of the premium of an underwriting year is earned after a period of l years, where l is typically small. The values of the incremental earning patterns are then defined as

$$p_{ik} = \frac{P_{ik} - P_{i,k-1}}{P_{il}}, \; i \leq N, \, k \leq l, \tag{48}$$

where P_{ik} is the accumulated earned premium of underwriting year i, earned up to development year k. We have a triangle of observed earned premiums with $k \leq N + 1 - i$. For $k \geq N + 1 - i$, we choose the projections obtained by the chain ladder method.

Next let us evaluate the average incremental accident year pattern $\mathbf{d}$ which we define as

$$d_k := \frac{1}{N} \sum_{j=1}^{N} \frac{L_{jk} - L_{j,k-1}}{\hat{U}_j}, \tag{49}$$

setting $L_{jk} = 0$ and $P_{ik} = 0$ for $k < 1$. This pattern is not directly available in our case. It is however related to the observable average incremental underwriting year pattern $\tilde{\mathbf{d}}$ via a convolution,

$$\tilde{d}_k = \sum_{j=1}^{l} p_j \, d_{k-j+1}. \tag{50}$$

Here p_j denotes an average earning pattern $p_j = \frac{1}{N}\sum_{i=1}^{N} p_{ij}$, and the average underwriting year pattern $\tilde{\mathbf{d}}$ is defined as

$$\tilde{d}_k := \frac{1}{N} \sum_{i=1}^{N} \frac{\tilde{L}_{ik} - \tilde{L}_{i,k-1}}{\hat{U}_i}. \tag{51}$$

The abbreviation $\tilde{L}_{ik}$ denotes the accumulated total claims of underwriting year i, $1 \leq i \leq N$, either paid or reported up to development year k, $1 \leq k \leq N$.

A straightforward inversion of (50) is numerically very sensitive to noise in the patterns $\tilde{\mathbf{d}}$ and $\mathbf{p}$. Usually, this leads to unreasonable accident patterns $\mathbf{d}$,

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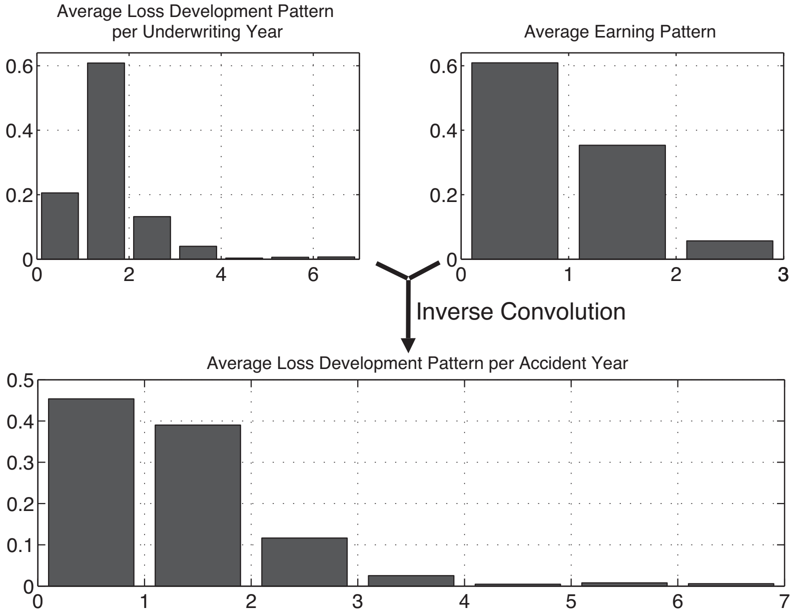

FIGURE 4: First step of the construction of the auxiliary triangle $\mathcal{L}_{ik}$ for line of business C. The inverse convolution leads to the average loss development per accident year. The x-axis shows the development year in each of the plots.

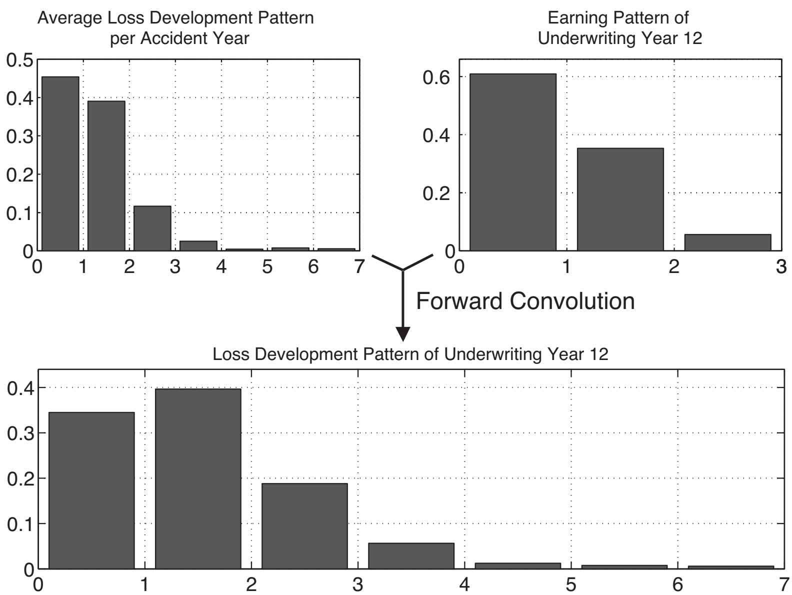

FIGURE 5: Second step of the construction of the auxiliary triangle $\mathcal{L}_{ik}$ for line of business C. A forward convolution leads to the loss development pattern for a specific underwriting year, $i = 12$ in the displayed example. This is done for all underwriting years. The x-axis shows the development year in each plot.

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see Fig. 3. However, a robust solution can be found with the Tikhonov regularization (Tikhonov, 1963), see Section 3.5.4. A stable pattern **d** is here obtained, see Fig. 4, by the solution of the linear equation

$$(\lambda^2 \Delta^T \Delta + \mathbf{A}^T \mathbf{A})\,\mathbf{d} = \mathbf{A}^T \tilde{\mathbf{d}}, \tag{52}$$

where Δ denotes the operator

$$\Delta = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & & \vdots \\ 0 & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}. \tag{53}$$

The matrix **A** is determined by the average earning pattern

$$\mathbf{A} = \begin{pmatrix} p_1 & 0 & \cdots & & & 0 \\ p_2 & p_1 & \ddots & & & \vdots \\ p_3 & p_2 & p_1 & & & \\ 0 & p_3 & p_2 & p_1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_3 & p_2 & p_1 \end{pmatrix}. \tag{54}$$

This formulation is for the case $l = 3$; other choices of l can be handled analogously. The parameter λ determines the degree to which the solution is regularized. As explained in Section 3.5.5 and Appendix B, a reasonable choice for λ is given by the mean of the singular value spectrum of **A**, i.e.

$$\lambda = \frac{1}{N}\sum_{i=1}^{N} s_i\,. \tag{55}$$

The singular values of **A**, denoted by $\{s_1, \ldots, s_N\}$, are the square roots of the eigenvalues of $\mathbf{A}^\dagger\mathbf{A}$.

Finally, the incremental auxiliary triangle $\mathcal{L}_{ik}$ can be constructed via N convolutions, see Fig. 5, of the earning patterns and the average development pattern per accident year **d**,

$$\mathcal{L}_{ik} = \hat{U}_i \sum_{j=1}^{l} p_{ij}\, d_{k-j+1}. \tag{56}$$

3.5.2. *From Underwriting to Accident Years*

Equations (50) and (56) permit the construction of the auxiliary triangle $\mathcal{L}_{ik}$. The derivation of these relations is shown in the following. To start with, we

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define $\mathcal{L}_{ijk}$ as the incremental total claims of underwriting year i, $1 \le i \le N$, and accident year $i+j-1$, $1 \le j \le l$, either paid or reported in the year $i+k-1$, $1 \le k \le N$. Here the indices j and k both count the years from the underwriting year i onwards. Then we define the incremental pattern d_{ijk} as

$$d_{ijk} := \frac{\mathcal{L}_{ijk}}{\sum_{k=1}^{N} \mathcal{L}_{ijk}}. \tag{57}$$

This describes the development of claims that stem from a fixed underwriting and accident year.

In order to derive (56), let us consider the artificial scenario in which there is no variability in the claim development. "Claim variability" is understood here as the variations in the incremental accident year pattern

$$d_{jk} = \frac{L_{jk} - L_{j,k-1}}{\hat{U}_j}. \tag{58}$$

Thus, a triangle has no variations in the claim development if the above pattern d_{jk} is independent of j, such that there is only one pattern d_k. In order to relate d_{ijk} and d_k, one has to take into account that the index k in the definition of d_k (49) is defined relative to the index j. In contrast, the index k in (57) refers to the underwriting year i. One therefore has the identity

$$d_{ijk} = d_{k-j+1}. \tag{59}$$

Furthermore, note that the ultimate loss amount of underwriting year i and accident year j is approximately the fraction of the ultimate loss of underwriting year i which was earned in the year j,

$$\sum_{k=1}^{N} \mathcal{L}_{ijk} \approx p_{ij} \hat{U}_i. \tag{60}$$

By inserting the results (59) and (60) into Eq. (57), one finds

$$\mathcal{L}_{ijk} = \hat{U}_i \, p_{ij} \, d_{k-j+1}. \tag{61}$$

Finally, the auxiliary triangle $\mathcal{L}_{ik}$ results from a reduction of the accident year index j,

$$\mathcal{L}_{ik} = \sum_{j=1}^{l} \mathcal{L}_{ijk} = \hat{U}_i \sum_{j=1}^{l} p_{ij} \, d_{k-j+1}. \tag{62}$$

Now Eq. (50) is derived with similar arguments. Instead of the previously treated artificial scenario, let us consider a realistic case in which there is variability

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in the claim development. Eq. (59) therefore does not hold as the development patterns of this realistic triangle have a dependency on i and j. However, the patterns will mostly depend on the portfolio structure which is determined by the underwriting year i. Hence we make the approximation

$$d_{ijk} \approx d_{i,k-j+1}. \tag{63}$$

Furthermore, let us approximate the ultimate loss amount of underwriting year i and accident year j by the fraction of the ultimate loss of underwriting year i which was on average earned in the year j,

$$\sum_{k=1}^{N} \mathcal{L}_{ijk} \approx p_j \hat{U}_i. \tag{64}$$

Upon inserting the results (63) and (64) into Eq. (57), one obtains

$$\mathcal{L}_{ijk} = \hat{U}_i \, p_j \, d_{i,k-j+1}. \tag{65}$$

The reduction of the index j yields

$$\tilde{L}_{ik} - \tilde{L}_{i,k-1} = \sum_{j=1}^{I} \mathcal{L}_{ijk} = \hat{U}_i \sum_{j=1}^{I} p_j \, d_{i,k-j+1}. \tag{66}$$

This equation can be rewritten as

$$\tilde{d}_{ik} = \sum_{j=1}^{I} p_j \, d_{i,k-j+1}, \tag{67}$$

where $\tilde{d}_{ik} := (\tilde{L}_{ik} - \tilde{L}_{i,k-1}) \,/\, \hat{U}_i$ denotes the underwriting year development pattern. Taking the average over the underwriting years, one finds

$$\tilde{d}_k = \sum_{j=1}^{I} p_j \left(\frac{1}{N} \sum_{i=1}^{N} d_{i,k-j+1} \right), \tag{68}$$

where the definition (51) is used. The average $\sum_{i=1}^{N} d_{i,k-j+1} / N$ can in turn be used as an estimate for the average incremental accident year pattern (49),

$$\frac{1}{N} \sum_{i=1}^{N} d_{i,k-j+1} \approx d_{k-j+1}. \tag{69}$$

The average underwriting and accident year patterns are thus related via a convolution,

$$\tilde{d}_k = \sum_{j=1}^{I} p_j \, d_{k-j+1}. \tag{70}$$

3.5.3. *Inverse Convolution*

The convolution (70) will be needed in inverted form. For that purpose let us rewrite it in matrix notation,

$$\tilde{\mathbf{d}} = \mathbf{A} \cdot \mathbf{d}, \tag{71}$$

where **A** is the Toeplitz matrix defined in Eq. (54). On a first glance one might suggest solving (71) by a matrix inversion of **A**,

$$\hat{\mathbf{d}} = \mathbf{A}^{-1} \cdot \tilde{\mathbf{d}}, \tag{72}$$

where $\hat{\mathbf{d}}$ denotes the estimate of the solution. However, this usually leads to oscillating or even divergent development patterns $\hat{\mathbf{d}}$, see Fig. 3. The cause of this effect is the presence of noise in the observed pattern $\tilde{\mathbf{d}}$. To account for this noise one has to replace the relation between the different development patterns (71) by

$$\tilde{\mathbf{d}} = \mathbf{A} \cdot \mathbf{d} + \mathbf{n}, \tag{73}$$

where **n** is an unknown noise term, that is a random vector with zero mean and finite variance. A straightforward matrix inversion,

$$\hat{\mathbf{d}} = \mathbf{A}^{-1} \cdot \tilde{\mathbf{d}} = \mathbf{d} + \mathbf{A}^{-1} \cdot \mathbf{n}, \tag{74}$$

leads therefore to an error term, $\mathbf{A}^{-1} \cdot \mathbf{n}$, which can cause oscillations.

3.5.4. *Tikhonov Regularization*

A method which allows one to find a stable solution of (73) is the Tikhonov regularization (Tikhonov, 1943; Tikhonov, 1963; Foster, 1961). The goal is to find a smooth development pattern **d** that is approximately in line with the observation $\tilde{\mathbf{d}}$. The smoothing of **d** does not imply any smoothing of the original triangle or a lowering of the reserve risk estimate, as to be shown in Fig. 11. The data misfit function implied by **d** is defined in terms of the two-norm

$$\text{misfit}\,(\mathbf{d}) = \|\tilde{\mathbf{d}} - \mathbf{A} \cdot \mathbf{d}\|^2. \tag{75}$$

The smoothness of the solution can be quantified by the two-norm of its "first derivative" (Hansen, 1998), which reads

$$\|\Delta \mathbf{d}\|^2 = \sum_{k=1}^{N-1} (d_{k+1} - d_k)^2,$$

with Δ the "derivative operator" defined in (53). A compromise between data misfit and smoothness can be found by the minimization of a weighted sum of (75) and (76), i.e.

$$\widehat{\mathbf{d}}_\lambda = \operatorname{argmin}\{\|\tilde{\mathbf{d}} - \mathbf{A}\cdot\mathbf{d}\|^2 + \lambda^2\|\Delta\,\mathbf{d}\|^2\}, \tag{76}$$

where λ is the regularization parameter. The function in (76) is a quadratic form. Hence its unique minimum is the null of its derivative,

$$\frac{\partial}{\partial d_k}\{(\tilde{\mathbf{d}} - \mathbf{A}\mathbf{d})^T\cdot(\tilde{\mathbf{d}} - \mathbf{A}\mathbf{d}) + \lambda^2\mathbf{d}^T\Delta^T\Delta\mathbf{d}\} = 0, \quad k = 1, \ldots, N.$$

The regularized solution of (73) is therefore obtained by the solution of the set of linear equations

$$(\lambda^2\Delta^T\Delta + \mathbf{A}^T\mathbf{A})\,\widehat{\mathbf{d}}_\lambda = \mathbf{A}^T\tilde{\mathbf{d}}. \tag{77}$$

3.5.5. *Choosing the Regularization Parameter*

To fix the parameter λ, it is helpful to analyze the reason for the noise sensitivity of matrix inversions. This in turn can best be explored by the singular value decomposition: any matrix $\mathbf{A} \in \mathbb{C}^{n\times m}$ can be decomposed as

$$\mathbf{A} = \sum_{i=1}^{r} s_i\,\mathbf{u}_i\mathbf{v}_i^\dagger, \tag{78}$$

where r denotes the rank of $\mathbf{A}$, s_i is the square root of the i'th eigenvalue of $\mathbf{A}\mathbf{A}^\dagger$ (i'th singular value), $\mathbf{u}_i$ is the left singular vector (given by the i'th eigenvector of $\mathbf{A}\mathbf{A}^\dagger$), and $\mathbf{v}_i$ denotes the right singular vector (given by the i'th eigenvector of $\mathbf{A}^\dagger\mathbf{A}$).

Armed with the above decomposition, one can rewrite the straightforward solution of the noisy inverse problem (73) as

$$\widehat{\mathbf{d}} = \mathbf{A}^{-1}\cdot\hat{\mathbf{d}} = \mathbf{d} + \sum_{i=1}^{r}\frac{\mathbf{u}_i^\dagger\mathbf{n}}{s_i}\mathbf{v}_i. \tag{79}$$

This demonstrates that small singular values in the spectrum of $\mathbf{A}$ are responsible for the noise sensitivity of $\mathbf{A}$'s inverse. This affects in particular inverse convolutions, since the corresponding Toeplitz matrices (54) tend to have tiny singular values, see Fig. 6. However, the noise sensitivity can be avoided by a filter f_i which damps down the components with small singular values:

$$\widehat{\mathbf{d}_\mathbf{f}} = \sum_{i=1}^{r} f_i(\mathbf{v}_i^\dagger\mathbf{d})\mathbf{v}_i + \sum_{i=1}^{r} f_i\frac{(\mathbf{u}_i^\dagger\mathbf{n})}{s_i}\mathbf{v}_i. \tag{80}$$

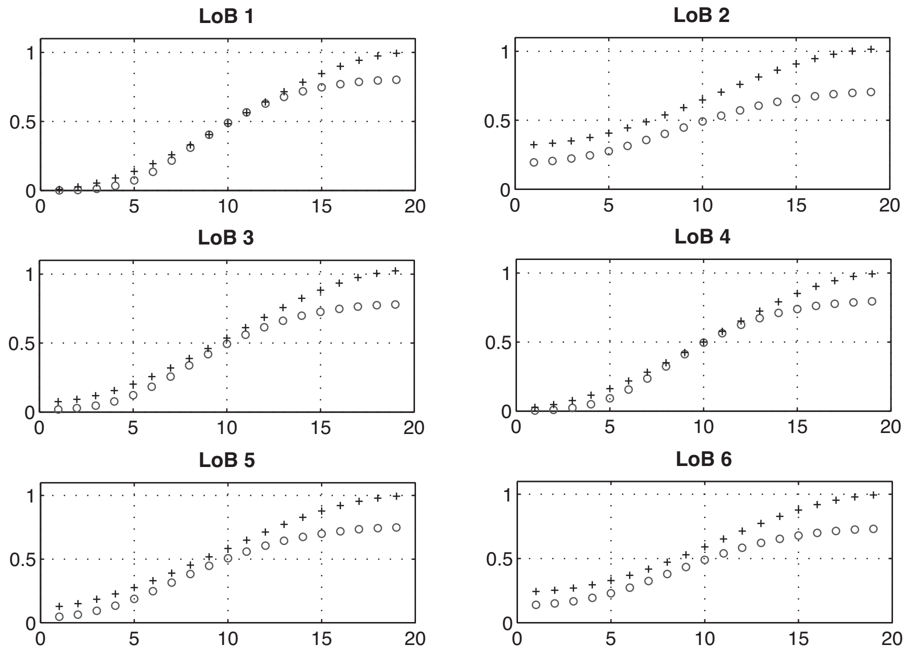

FIGURE 6: Singular value spectrum (plus signs) and filter function (circles) for different LoBs. The y-axis denotes the value of the singular value s_i or the filter f_i, and the x-axis shows the index i.

A common choice for the filter function reads

$$f_i(\lambda) = \frac{s_i^2}{s_i^2 + \lambda^2}, \tag{81}$$

which leads to the same solution as the Tikhonov regularization (Foster, 1961; Hansen, 1998).

In conclusion, the regularization parameter λ controls the shape of a filter, whose purpose is to suppress components with small singular values. Therefore, we suggest basing the choice of λ on the singular value spectrum of **A**. Figure 6 shows the spectrum of **A** (plus signs) for different LoBs. Apparently, these spectra have a very similar shape which is most likely due to the Toeplitz structure of **A**. It should therefore be possible to determine the appropriate position of the filter (relative to the spectrum) once and for all and to use the same relative position for all LoBs. For specific LoBs, we find that the mean of the singular values is an appropriate choice for λ,

$$\lambda = \frac{1}{N}\sum_{i=1}^{N} s_i. \tag{82}$$

Appendix B and the fact that the filter approximately reproduces the spectrum, see Fig. 6, supports the choice of a numerical criterion based on Eq. (82). Furthermore, the spectrum of **A** is quite uniform for the investigated triangles, so we suggest using Eq. (82) for regularizing the triangles of all LoBs.

3.5.6. *Examples*

(A) Short-tail line of business

This section will demonstrate the earned premium correction which was described in Section 3.5.1. The first example is the short-tail line of business C, shown in Appendix A. It exhibits an outlier at ($i = 1$, $k = 2$) and we therefore start with the application of the robust methods, see Table 4. Here we have filtered the outlier mentioned above, and we used 10000 iterations for the bootstrap algorithm which permits convergence to the tenth decimal point. Now the main task is the construction of the auxiliary triangle $\mathcal{L}_{ik}$, see Eq. (56). The first step is to evaluate the average loss development pattern per accident year, which is obtained by a single inverse convolution, see Fig. 4. However, a straightforward inverse convolution, that is the exact solution of (50), leads to an unreasonable pattern, shown in Fig. 3 on the left-hand side. The right-hand side of Fig. 3, and the bottom of Fig. 4, show the result of the Tikhonov regularization, i.e. the solution of Equation (52). This yields a smooth result for the loss development pattern.

TABLE 4

Robust methods and earning pattern (EP) correction applied to line of business C.

LoBC	Reserve (tUSD)	Mack error (tUSD)	Mack error (%)	Bootstrap error (tUSD)	Bootstrap error (%)
Original Method	7.4	1.85	25.1	1.85	25.1
Outlier Filter	7.4	1.32	17.9	1.32	18.0
Robust Estimator	7.4	1.23	16.7	1.25	16.9
Auxiliary Triangle	–	0.73	9.9	0.72	9.8
After EP Correction	7.4	0.99	13.4	1.02	13.8

Figure 5 shows the second step of the construction of $\mathcal{L}_{ik}$. The average loss development pattern per accident year is convoluted with the i'th earning pattern, shown in Appendix A Table 13. This convolution is repeated for all underwriting years i. The Mack or bootstrap error of the resulting auxiliary triangle $\mathcal{L}_{ik}$ yields $\hat{\sigma}_{\mathrm{EP}}$. Together with the reserve risk estimate of the original triangle, $\hat{\sigma}_{\mathrm{UY}}$, and Equation (47), we can evaluate the earning pattern correction, see Table 4.

(B) PartnerRe's overall portfolio

PartnerRe has published its loss development triangles (PartnerRe, 2006) on underwriting and on accident year bases. It therefore offers the possibility of testing the earning pattern correction. However, the corresponding earned premium patterns are not published. Hence we first have to reconstruct them from the given incurred loss triangles. We therefore take a modification of Eq. (67), namely

$$\tilde{d}_{ik} = \sum_{j=1}^{l} p_{ij}\, d_{i,k-j+1}, \tag{83}$$

which can be obtained by replacing (64) with (60) in the derivation of (67). The linear system of equations (83) can be used to reconstruct the earning pattern p_{ij}. It is however over-determined in p_{ij}. To find a unique solution, we truncate the equation system after the first three development years, reflecting the fact that the premium is fully earned after three (or sometimes even two) years. One obtains

$$\begin{pmatrix} \tilde{d}_{i1} \\ \tilde{d}_{i2} \\ \tilde{d}_{i3} \end{pmatrix} = \begin{pmatrix} d_{i1} & 0 & 0 \\ d_{i2} & d_{i1} & 0 \\ d_{i3} & d_{i2} & d_{i1} \end{pmatrix} \begin{pmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{pmatrix} =: \mathbf{d}_i \cdot \mathbf{p}_i . \tag{84}$$

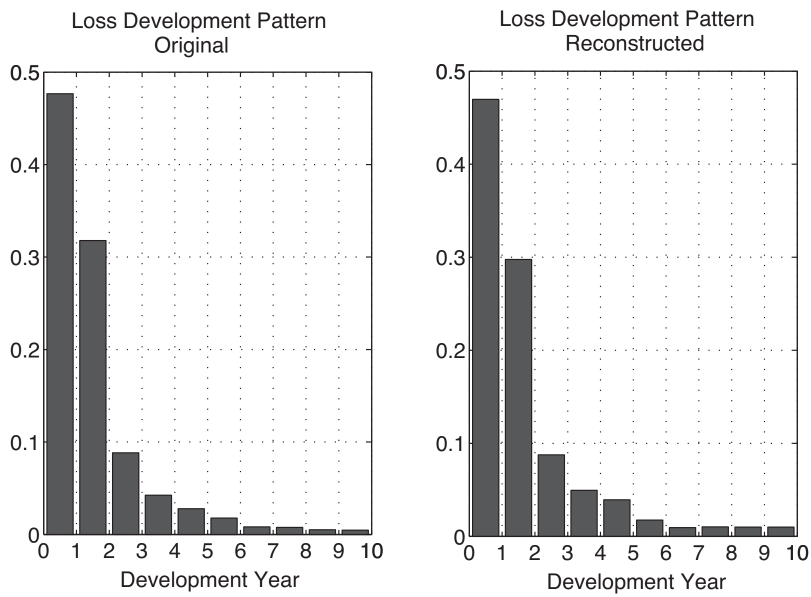

FIGURE 7: Comparison of the true average loss development pattern per accident year (left-hand side), and the reconstructed one (right hand side), for PartnerRe's overall portfolio.

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TABLE 5

TEST OF THE EARNED PREMIUM CORRECTION. MACK ERROR OF PARTNERRE'S OVERALL PORTFOLIO IS EVALUATED ON UNDERWRITING AND ON ACCIDENT YEAR BASED TRIANGLES. THE RESULT OF THE EARNED PREMIUM CORRECTION IS SHOWN IN BETWEEN.

	Mack error (mUSD)	Reserve (mUSD)
Original triangle per underwriting years	501	$26.2 \cdot 10^3$
Earning pattern corrected result	479	″
Original triangle per accident years	469	″

The matrices $\mathbf{d}_i$ do not have any small singular values ($s_i \geq 0.3$). Hence, the straightforward inversion of (84) is numerically stable, and it is not necessary to perform a regularization. The reconstructed earning pattern is shown in Table 6. The values p_{i3} are small and scattered around zero, suggesting that assuming only two instead of three earning years would be a valid alternative.

Given Equation (50) and the earned premium pattern, one can reconstruct the average loss development pattern per accident year $\mathbf{d}$. The result, shown on the right hand side of Fig. 7, is close to the true average loss development pattern per accident year. The latter, which was obtained from the accident year based triangle, is shown on the left-hand side of Fig. 7. We have used in total eleven inverse convolutions to obtain the reconstructed pattern. Figure 7 therefore demonstrates that the relation between underwriting and accident year patterns is well described by a convolution. Finally, the earning pattern corrected Mack error is obtained by means of Equation (47) and (56). The results are shown in Table 5. The earning pattern corrected Mack error is a good and slightly conservative estimate of the true value: it differs from the accident year based Mack error by around 10 mUSD. We explain this difference mainly by the fact that we had to reconstruct PartnerRe's earning patterns.

TABLE 6

RECONSTRUCTED EARNING PATTERN OF PARTNERRE'S OVERALL PORTFOLIO.

	Period				Period		
Underwriting Year	1	2	3	Underwriting Year	1	2	3
1996	0.90	0.08	0.02	2001	0.86	0.20	–0.06
1997	0.92	0.12	–0.05	2002	0.90	0.12	–0.01
1998	0.89	0.13	0.02	2003	0.81	0.22	–0.03
1999	0.89	0.15	0.04	2004	0.86	0.15	0.01
2000	0.78	0.20	0.02	2005	0.88	0.14	–0.01

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4. Conclusion

The aim of this paper is the development of robust and accurate solutions for the assessment of reserve risk. This is accomplished for the Mack method and the bootstrapping method based on Mack's assumptions. More specifically, we have developed a filter for outliers and large jumps as well as a robust version of the variance estimator which is used in the Mack and the bootstrap methods. These procedures guarantee reasonable Mack and bootstrap estimates even for partially deficient data. The robust variance estimator leads to a substantial gain in stability at the price of a small bias. Its root mean squared error is similar to the one of Mack's variance estimator.

TABLE 7

Application of the outlier filter, the robust estimator and the earning pattern correction to various lines of businesses. The earning pattern correction is not significant for line of business A since it is long-tailed, in contrast to the short-tail line of business C.

	Original Method	Outlier Filter	Robust Estimator	Earning Pattern Correction
LoBA Mack error (%)	8.0	8.0	7.5	6.9
LoBA Bootstrap error (%)	7.9	7.9	7.5	6.9
LoBB Mack error (%)	36	29	18	18
LoBB Bootstrap error (%)	37	29	18	18
LoBC Mack error (%)	25.1	17.9	16.7	13.4
LoBC Bootstrap error (%)	25.1	18.0	16.9	13.8

As a further result, we have designed a method that corrects the error introduced by applying the Mack or bootstrapping method to underwriting year based triangles. The influence of fluctuations in earning patterns is thereby removed from the reserve risk estimate. As a by-product, one finds that the relation between loss development patterns based on underwriting year and accident year is approximately given by a convolution. A numerically stable inversion thereof is obtained through the Tikhonov regularization. We have demonstrated the different methods with the aid of the triangles shown in Appendix A. The results are summarized in Table 7.

Acknowledgments

We would like to thank the referee for many helpful comments on the manuscript.

https://doi.org/10.2143/AST.40.2.2061124 Published online by Cambridge University Press

APPENDIX

Appendix A – Triangles and patterns

This Appendix shows the data which were used in the examples of Section 3.4 and 3.5.6. The data were obtained from three different lines of business A-C. The accumulated incurred losses L_{ik} of underwriting year i which are either paid or reported up to development year k are shown in Table 9, 10 and 12. The total current case reserve, $\sum_{i=1}^{N} C_{i,N+1-i}$, which has been reported but not been paid, is summarized in Table 8. This quantity is required for the estimation of the total reserve R, see Eq. (3).

TABLE 8

CURRENT CASE RESERVE: TOTAL CLAIM AMOUNT, $\sum_{i=1}^{N} C_{i,N+1-i}$, WHICH HAS BEEN REPORTED BUT NOT BEEN PAID.

Case Losses	LoBA	LoBB	LoBC
$\sum_{i=1}^{N} C_{i,N+1-i}$ (tUSD)	3.2	$14.3 \cdot 10^3$	2.3

TABLE 9

LINE OF BUSINESS A.
INCURRED LOSSES L_{ik} PER UNDERWRITING YEAR i AND DEVELOPMENT YEAR k, IN USD.

i	L_{i1}	L_{i2}	L_{i3}	L_{i4}	L_{i5}	L_{i6}	L_{i7}	L_{i8}	L_{i9}	L_{i10}	L_{i11}	L_{i12}	L_{i13}	L_{i14}	L_{i15}	L_{i16}	L_{i17}	L_{i18}	L_{i19}
1	31	190	229	253	285	426	470	499	528	539	548	552	559	567	568	569	572	569	571
2	29	132	193	226	337	350	362	367	373	383	384	382	387	386	387	386	387	388	
3	41	168	248	411	440	462	477	484	493	491	493	499	497	498	498	498	498		
4	55	196	421	464	496	516	530	544	542	547	556	556	557	557	554	555			
5	58	285	385	433	462	482	500	499	505	515	514	517	518	518	519				
6	94	385	512	563	602	635	639	646	660	660	661	661	661	662					
7	72	327	464	529	575	599	602	619	624	631	633	637	634						
8	89	350	509	599	648	689	718	719	732	735	742	750							
9	78	384	586	640	718	775	788	799	809	827	827								
10	86	385	490	603	705	743	785	804	806	810									
11	73	316	458	591	653	703	737	753	776										
12	87	231	329	439	469	514	544	582											
13	122	441	639	789	886	978	987												
14	131	446	683	821	934	948													
15	217	830	1,264	1,565	1,681														
16	188	1,052	1,782	1,980															
17	236	1,317	1,966																
18	292	1,455																	
19	250																		

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The incremental earning patterns p_{ij}, see Eq. (48), are shown for the lines of business A and C in Table 11 and 13. Most of the premium is earned after three years. The earning pattern is required for the earning pattern correction introduced in Section 3.5.

TABLE 10

LINE OF BUSINESS B.
INCURRED LOSSES L_{ik} PER UNDERWRITING YEAR i AND DEVELOPMENT YEAR k, IN UNITS OF 10'000 USD.

i	L_{i1}	L_{i2}	L_{i3}	L_{i4}	L_{i5}	L_{i6}	L_{i7}	L_{i8}	L_{i9}	L_{i10}	L_{i11}	L_{i12}	L_{i13}	L_{i14}	L_{i15}	L_{i16}	L_{i17}	L_{i18}	L_{i19}
1	0.000	6	6	10	10	44	44	45	51	54	54	56	55	54	54	54	62	61	64
2	0.000	11	43	42	37	39	41	40	62	92	159	86	82	82	74	81	84	86	
3	1.520	35	56	80	71	66	74	78	73	73	74	73	72	77	77	78	79		
4	4.330	10	74	92	94	86	87	82	81	80	78	77	77	78	77	76			
5	0.000	97	116	113	117	128	127	131	125	123	119	117	116	114	114				
6	5.507	10	21	26	54	49	49	55	58	54	55	54	53	53					
7	0.264	6	10	13	17	15	17	17	17	18	18	20	19						
8	0.000	12	13	25	31	28	27	26	25	23	39	23							
9	0.004	6	10	15	29	27	31	30	27	29	24								
10	1.874	17	25	46	43	64	61	53	55	56									
11	14.34	24	29	30	67	73	100	103	168										
12	41.30	41	68	123	99	129	183	194											
13	69.02	90	133	102	159	319	124												
14	57.58	163	145	147	170	127													
15	62.83	158	422	679	695														
16	93.76	144	219	232															
17	79.72	199	166																
18	88.07	96																	
19	34.56																		

TABLE 11

LINE OF BUSINESS A. EARNING PATTERN p_{ij} PER UNDERWRITING YEAR i AND DEVELOPMENT YEAR j.

	Period				Period		
Underwriting Year	1	2	3	Underwriting Year	1	2	3
1	0.31	0.52	0.17	11	0.49	0.42	0.09
2	0.34	0.53	0.13	12	0.51	0.39	0.10
3	0.32	0.50	0.18	13	0.40	0.46	0.14
4	0.20	0.31	0.48	14	0.43	0.43	0.15
5	0.26	0.62	0.12	15	0.41	0.47	0.13
6	0.37	0.50	0.13	16	0.34	0.54	0.12
7	0.38	0.49	0.13	17	0.45	0.47	0.08
8	0.38	0.51	0.11	18	0.49	0.39	0.12
9	0.34	0.57	0.09	19	0.41	0.47	0.12
10	0.42	0.50	0.08				

TABLE 12

LINE OF BUSINESS C.
INCURRED LOSSES L_{ik} PER UNDERWRITING YEAR i AND DEVELOPMENT YEAR k, IN UNITS OF 10 USD.

i	L_{i1}	L_{i2}	L_{i3}	L_{i4}	L_{i5}	L_{i6}	L_{i7}	L_{i8}	L_{i9}	L_{i10}	L_{i11}	L_{i12}	L_{i13}	L_{i14}	L_{i15}	L_{i16}	L_{i17}	L_{i18}	L_{i19}
1	15	211	169	174	175	178	191	192	192	192	192	192	192	192	192	192	192	192	192
2	14	83	103	106	109	110	110	110	110	111	111	111	111	111	110	110	110	110	
3	36	140	168	178	183	186	187	188	186	187	186	186	186	186	186	186	186		
4	60	170	194	194	195	196	200	201	202	202	202	201	201	201	201	201			
5	51	144	168	176	177	176	176	178	179	178	179	178	179	178	178				
6	26	202	234	247	246	248	249	249	249	245	244	244	243	243					
7	19	95	124	127	126	127	127	127	131	132	132	132	131						
8	17	88	98	97	99	99	100	100	101	100	100	101							
9	16	88	97	97	98	99	100	100	100	100	100								
10	17	92	102	105	105	107	107	106	108	108									
11	16	69	78	79	80	81	82	81	81										
12	25	82	89	93	95	96	96	95											
13	47	126	138	141	144	144	140												
14	49	135	183	192	184	181													
15	40	101	144	192	180														
16	52	231	301	299															
17	89	228	262																
18	78	347																	
19	107																		

TABLE 13

LINE OF BUSINESS C. EARNING PATTERN p_{ij} PER UNDERWRITING YEAR i AND DEVELOPMENT YEAR j.

	Period				Period		
Underwriting Year	1	2	3	Underwriting Year	1	2	3
1	0.29	0.58	0.13	11	0.72	0.25	0.03
2	0.32	0.60	0.09	12	0.76	0.22	0.03
3	0.31	0.59	0.10	13	0.62	0.31	0.07
4	0.35	0.57	0.08	14	0.59	0.33	0.08
5	0.36	0.56	0.08	15	0.62	0.32	0.05
6	0.43	0.47	0.10	16	0.57	0.40	0.03
7	0.52	0.39	0.09	17	0.64	0.33	0.03
8	0.56	0.39	0.05	18	0.66	0.29	0.05
9	0.56	0.40	0.04	19	0.59	0.35	0.05
10	0.67	0.29	0.04				

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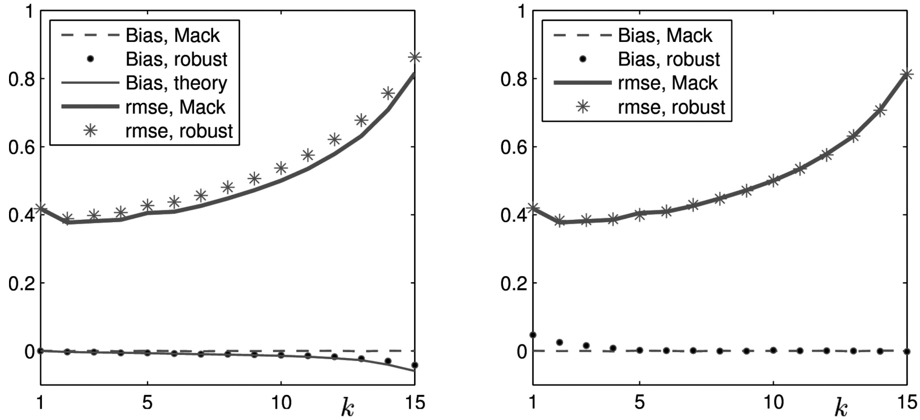

FIGURE 8: The bias and the root mean squared error (rmse) of Mack's variance estimator (27) and the robust one (28) as a function of the development year k. The loss expectation value in (28) is calculated either exactly (left) or by backward projection (right). The bias and the rmse are given in units of σ_k^2. The bias of the robust estimator is much smaller than the rmse. The analytical prediction of the bias is given by the thin line.

Appendix B – Monte Carlo simulation for the bias estimation of the robust Mack estimator

In Section 3.3.1, we have proposed a modification of Mack's variance estimator, see Eq. (28), which is supposed to be more robust against data errors than the original version. The bias of one version of this estimator was evaluated with the analytical expression (40), which was argued to be sufficiently small. Here, a Monte Carlo simulation is presented which allows us to analyze the bias and, in addition, the root mean squared error.

The Monte Carlo method. The stochastic simulation is based on the generation of random "Mack triangles" L^*_{jk}. To produce these random triangles, we identify the first column of L^*_{jk} with the one of a real triangle L_{jk}, that is

$$L^*_{j1} = L_{j1}, \quad j = 1, \ldots, N. \tag{85}$$

The subsequent columns $L^*_{j,k+1}$, $k \geq 1$, are generated recursively by drawing random numbers from a lognormal distribution with mean $f_k L^*_{jk}$ and variance $\sigma_k^2 L^*_{jk}$, that is

$$L^*_{j,k+1} \sim \text{Lognormal}(f_k L^*_{jk}, \sigma_k^2 L^*_{jk}), \tag{86}$$

where the parameters f_k and σ_k^2 are obtained by using the chain ladder factors $\hat{f}_k$ and the variance factors $\hat{\sigma}_k^2$ of the real triangle L_{jk}, i.e. $f_k \equiv \hat{f}_k$ and $\sigma_k^2 \equiv \hat{\sigma}_k^2$. Due to this construction, the random triangles satisfy the Mack assumptions, (4) to (6), thus justifying the naming "Mack triangles".

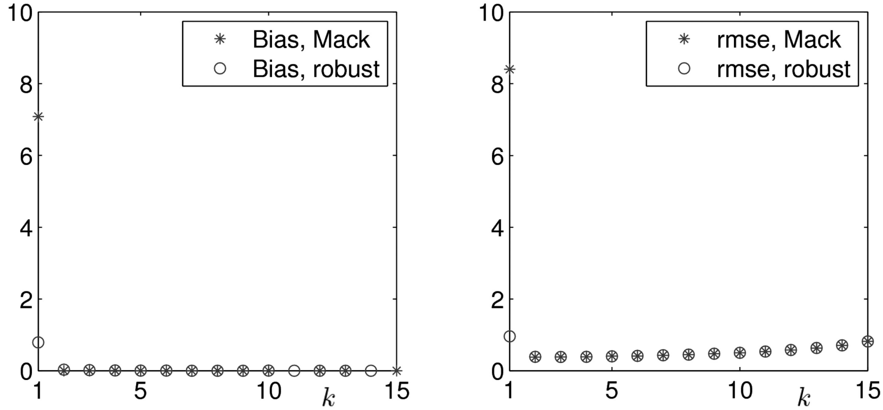

FIGURE 9: Estimation of the bias (left) and the root mean squared error (right) for a scenario with an artificial data error: the entry $L^*_{1,5}$ is multiplied with a factor 0.1 in the Monte Carlo simulation. Circles correspond to the robust estimator (28), while the asterisks are obtained with the original Mack estimator (27). The bias and the rmse are given in units of σ_k^2. The figure clearly shows that the robust Mack estimator is more resilient than the original one.

The reliability of the different variance estimators is then obtained by calculating the bias

$$\mathcal{B} = E(\hat{\sigma}_k^2) - \sigma_k^2, \tag{87}$$

and the root mean squared error

$$\text{rmse} = \sqrt{E\left(\left[\hat{\sigma}_k^2 - \sigma_k^2\right]^2\right)}. \tag{88}$$

Estimation of the bias and the mean squared error. Figure 8 shows the result of the Monte Carlo simulation using line of business A for the real triangle L_{jk}. Here the left-hand side gives a comparison of the reliability of Mack's variance estimator (27) and the robust estimator (28), with $E(L_{jk})$ calculated exactly. The dashed line denotes the relative bias $\mathcal{B}/\sigma_k^2$ of Mack's variance estimator and the dots show the same quantity using the robust version. As expected, Mack's estimator is unbiased, while the robust estimator exhibits a small bias (less than 5%), which agrees up to a small error with the analytical prediction Eq. (40) (thin line). This bias is much smaller than the relative root mean squared error rmse$/\sigma_k^2$ represented by the thick line (Mack's estimator) and the asterisks (robust estimator). Similar results are obtained if the expectation value $E(L_{jk})$ in (28) is replaced by the forward projection

$$\hat{E}(L_{j1}) = L_{j1}, \tag{89}$$

$$\hat{E}(L_{j,k+1}) = \hat{f}_k L_{jk}, \quad k \geq 1. \tag{90}$$

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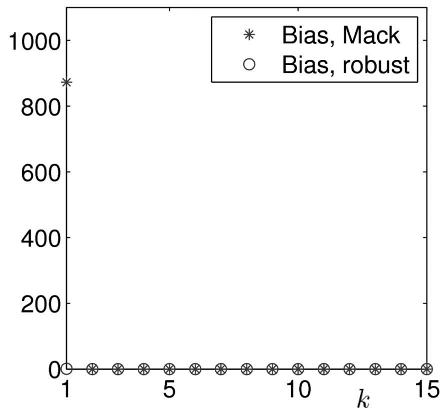

FIGURE 10: Similar to Fig. 9 but with a more severe data error: the entry $L^*_{1,5}$ is multiplied with a factor of 10^{-3} in the Monte Carlo simulation.

However, we do not recommend forward projection, because of the large stochastic error in the first losses L_{j1}, especially for long-tail business.

When using the backward projection (29) and (30) for the estimation of $E(L_{jk})$, one obtains the right-hand side of Fig. 8. This result is similar to the one shown on the left, apart from the sign and the orientation of $\mathcal{B}$. The bias is here positive (= conservative) for small k's, while it tends to zero for $k \to N$. This difference can be explained by noting that the bias is zero whenever $E(L_{jk})$ agrees with the losses L_{jk}. In the case of the forward projection, $E(L_{jk})$ is close to L_{jk} for small k's, while for the backward projection one has $E(L_{jk}) \simeq L_{jk}$ for $k \to N$.

Similar results are obtained, when the real triangle L_{jk} is replaced by those of other lines of business.

Test of robustness. To test the stability of the estimator (28), with $E(L_{jk})$ calculated by backward projection, we now introduce errors in the random "Mack triangles". To this end, a particular entry, $L^*_{1,5}$ in our example, is multiplied by a factor $c = 0.1$ (Fig. 9) or $c = 10^{-3}$ (Fig. 10), which leads to errors similar to those encountered in line of business C. Clearly, the figures demonstrate a large gain in stability.

Conclusions. The simulations reveal that the bias of the robust estimator (29) stays less than 5%, which is much smaller than the corresponding root mean squared error. The latter is similar to the one of the original Mack estimator, demonstrating the reliability of the proposed estimator. Moreover, it turns out that the estimation of $E(L_{jk})$ by backward projection leads to a slightly positive bias, which is preferable from the point of view of risk management. We conclude that the robust estimator has a performance comparable to the Mack estimator; the gain in stability, however, is enormous.

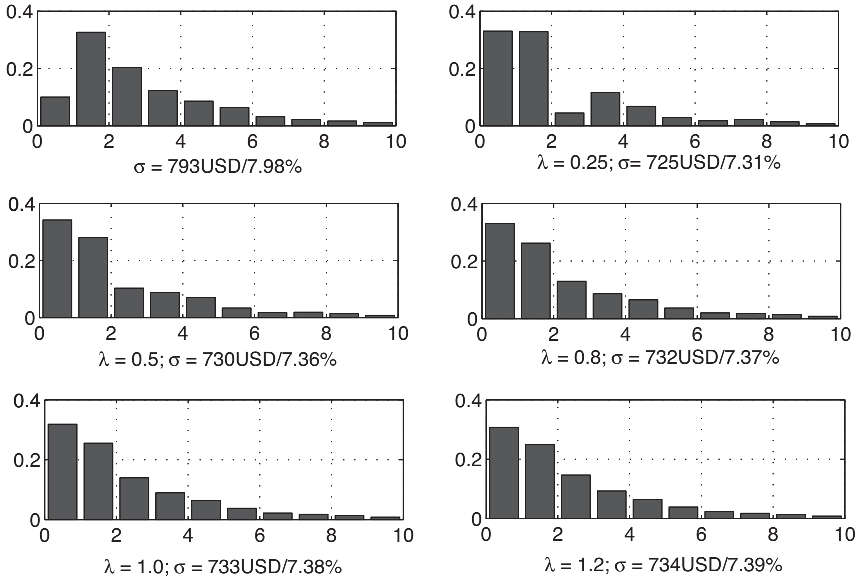

FIGURE 11: Effect of λ on the inverted accident year based loss development pattern for line of business A. The top left plot shows the average underwriting year based pattern. The other five patterns are the corresponding accident year based patterns resulting from an inverse convolution, using different λ's. The λ values are given in units of the mean of the singular values of **A**. The top left σ denotes the Mack error of the triangle per underwriting years. The other σ's are the earning pattern corrected Mack errors of the reconstructed accident year based triangles for the corresponding choices of λ.

Appendix C – Calibration of the Regularization Factor

In order to determine an appropriate choice for the Tikhonov regularization parameter λ, we analyze the patterns of line of business A, see Fig. 11. The general applicability of this choice has been explored for further LoBs. To interpret the quality of the inversion one can use the following rough rule: the underwriting year based pattern $\tilde{\mathbf{d}}$ and the inverted pattern per accident year $\mathbf{d}$ can roughly be compared by a shift of one development year, $d_k \approx \tilde{d}_{k+1}$.

We conclude from Fig. 11 that $\lambda \geq 0.8\bar{s}$ is necessary to obtain a smooth pattern, where $\bar{s}$ is the mean of the singular values. However, a very large regularization parameter, $\lambda \gg \bar{s}$, is not reasonable since the corresponding filter becomes nontransparent even for large singular values. For instance, the choice $\lambda = 10\bar{s}$ leads to the filter function $f \approx (2\bar{s})^2 / ((2\bar{s})^2 + (10\bar{s})^2) \approx 4\%$ for the largest singular value. Hence the inverse map is almost the zero map. It is therefore necessary to choose λ in the vicinity of $\bar{s}$. Since the Mack error is nearly constant in this region (see Fig. 11), we suggest choosing precisely $\lambda = \bar{s}$. Studies of different lines of business have confirmed that the inverted loss development pattern is in general smooth for $\lambda \geq 0.8\bar{s}$ and the resulting reserve risk estimate is almost independent of λ as long as $\lambda \geq 0.5\bar{s}$. The choice of Eq. (82) is thus suitable for a large class of lines of business.

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https://doi.org/10.2143/AST.40.2.2061124 Published online by Cambridge University Press