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Lower bound for the Perron–Frobenius degrees of Perron numbers

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Abstract. Using an idea of Doug Lind, we give a lower bound for the Perron–Frobenius degree of a Perron number that is not totally real, in terms of the layout of its Galois conjugates in the complex plane. As an application, we prove that there are cubic Perron numbers whose Perron–Frobenius degrees are arbitrary large, a result known to Lind, McMullen and Thurston. A similar result is proved for bi-Perron numbers.

Key words: symbolic dynamics, Perron numbers, Perron–Frobenius degree, non-negative matrices, entropy
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1. *Introduction*

Let A be a non-negative, integral, *aperiodic* matrix, meaning that some power of A has strictly positive entries. One can associate to A a subshift of finite type with topological entropy equal to $\log(\lambda)$, where λ is the spectral radius of A. By the Perron–Frobenius theorem, λ is a *Perron* number [**3**]; a real algebraic integer $p \geq 1$ is called Perron if it is strictly greater than the absolute value of its other Galois conjugates. Lind proved a converse, namely, that any Perron number is the spectral radius of a non-negative, integral, aperiodic matrix [**7**]. As a result, Perron numbers naturally appear in the study of entropies of different classes of maps such as post-critically finite self-maps of the interval [**10**], pseudo-Anosov surface homeomorphisms [**2**], geodesic flows, and Anosov and Axiom A diffeomorphisms [**7**].

Given a Perron number p, its *Perron–Frobenius degree*, $d_{PF}(p)$, is defined as the smallest size of a non-negative, integral, aperiodic matrix with spectral radius equal to p. In other words, the logarithms of Perron numbers are exactly the topological entropies of mixing subshifts of finite type, and the Perron–Frobenius degree of a Perron number is the smallest 'size' of a mixing subshift of finite type realizing that number. Our main result gives a lower bound for the Perron–Frobenius degree of a Perron number, which is not totally real. See the related work of Boyle and Lind, which gives an upper bound in the context of non-negative polynomial matrices [**1**].

THEOREM 1.1. *Let $p > 0$ be a Perron number. Assume that some Galois conjugate p' of p is not real, and $\eta := \tan^{-1}((p - \mathrm{Re}(p'))/|\mathrm{Im}(p')|) \leq 1$. Then*

$$d_{PF}(p) \geq \frac{2\pi}{3\eta}.$$

To visualize the angle η geometrically, see the left-hand side of Figure 3 for $t = p'/p$. It was known to Lind, McMullen [**8**] and Thurston [**10**, note on page 6]) that there are examples of Perron numbers of constant algebraic degree (in fact cubics), whose Perron–Frobenius degrees are arbitrary large. Their proofs are not published, to the best of the author's knowledge. As a first application, we give a proof of their result.

COROLLARY 1.2. (Lind, McMullen, Thurston) *For any $N > 0$, there are cubic Perron numbers whose Perron–Frobenius degrees are larger than N.*

The second application is a similar result for a class of algebraic integers called *bi-Perron* numbers. A unit algebraic integer $\alpha > 1$ is called bi-Perron if all other Galois conjugates of α lie in the annulus $\{z \in \mathbb{C} \mid 1/\alpha < |z| < \alpha\}$, except possibly for α^{-1}.

Bi-Perron numbers appear in the study of stretch factors of pseudo-Anosov homeomorphisms, in particular *the surface entropy conjecture* (also known as Fried's conjecture). Fried proved that the stretch factor of any pseudo-Anosov homeomorphism on a closed, orientable surface S is a bi-Perron number [**2**], and Penner showed the Perron–Frobenius degree of the stretch factor is at most $6|\chi(S)|$ (see [**9**, page 5]). The strong form of the surface entropy conjecture states that the set of stretch factors of pseudo-Anosov homeomorphisms over all closed, orientable surfaces is exactly the set of bi-Perron numbers (see [**2**, Problem 2] or [**8**]). In §3 we prove the following corollary.

COROLLARY 1.3. *For any $N > 0$, there are bi-Perron numbers of algebraic degree at most 6 whose Perron–Frobenius degrees are larger than N.*

We do not know if the examples in the above corollary arise as stretch factors of pseudo-Anosov maps (see Question 4.2).

1.1. *Outline.*

In §2 we recall the proof of Lind's theorem and prove Theorem 1.1. In §3 we prove two applications of the main theorem, namely Corollaries 1.2 and 1.3. In §4 we suggest further questions regarding Perron numbers arising as stretch factors of pseudo-Anosov homeomorphisms.

2. *Perron–Frobenius degree*

Given an algebraic integer λ of degree d over $\mathbb{Q}$ and minimal polynomial $f(x) = x^d - c_1x^{d-1} - \cdots - c_d$, define its companion matrix as

$$B = \begin{bmatrix} 0 & 0 & \cdots & c_d \\ 1 & 0 & \cdots & c_{d-1} \\ 0 & 1 & \cdots & c_{d-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_1 \end{bmatrix}.$$

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Note that the characteristic polynomial of B is equal to $f(x)$ up to sign. The Jordan form of B shows that $\mathbb{R}^d$ splits into a direct sum of one-dimensional and two-dimensional B-invariant subspaces corresponding to real roots and pairs of conjugate complex roots of $f(x)$. If λ' is a root of $f(x)$, denote the B-invariant subspace corresponding to λ' by $E_{\lambda'}$, and let $\pi_{\lambda'}: \mathbb{R}^d \to E_{\lambda'}$ be the projection to $E_{\lambda'}$ along the complementary direct sum. As λ is real, E_λ is one-dimensional. Fixing a point $w \in E_\lambda$, we identify rw with r for $r \in \mathbb{R}$. Let E be the *positive half-space corresponding to* λ, that is, the set of points such that their projection under π_λ is a positive multiple of w. By an integral point in E we mean an integral point with respect to the standard basis of $\mathbb{R}^d$.

THEOREM 2.1. (Lind [**7**]) *Let λ be a Perron number, with the companion matrix $B: \mathbb{R}^d \to \mathbb{R}^d$. Let E be the positive half-space corresponding to λ. There are integral points $z_1, \ldots, z_n$ in E such that for each $1 \le i \le n$, $Bz_i = \sum_{j=1}^n a_{ij} z_j$ with $a_{ij} \in \mathbb{N} \cup \{0\}$, and any irreducible component of the matrix $A = [a_{ij}]$ is an aperiodic matrix whose spectral radius is equal to λ.*

The next theorem, also due to Lind, gives a converse to the previous theorem.

THEOREM 2.2. (Lind [**7**]) *Let λ be a Perron number, with the companion matrix $B: \mathbb{R}^d \to \mathbb{R}^d$. Let E be the positive half-space corresponding to λ. If A is an $n \times n$ aperiodic, non-negative, integral matrix with spectral radius equal to λ, then there are integral points $z_1, \ldots, z_n \in E$ such that for each $1 \le i \le n$ we have $Bz_i = \sum_{j=1}^n a_{ij} z_j$.*

We recall Lind's proof of Theorem 2.2.

Proof. Consider $A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. By Perron–Frobenius theory, there is a *positive* eigenvector $v \in \mathbb{R}^n$ corresponding to λ. By working over the field $\mathbb{Q}(\lambda)$, we can assume that $v \in \mathbb{Q}(\lambda)^n$. Let

$$v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n,$$

and assume that for $1 \le i \le n$,

$$v_i = z_{i1} + z_{i2}\lambda + \cdots + z_{id}\lambda^{d-1} > 0,$$

where the numbers z_{ij} are integers. Define, for $1 \le i \le n$,

$$z_i = (z_{i1}, z_{i2}, \ldots, z_{id})^T \in \mathbb{Z}^d.$$

Since v is an eigenvector for A,

$$\lambda v_i = (Av)_i = \sum_j a_{ij} v_j. \qquad (*)$$

Let $\Psi : \mathbb{Q}(\lambda) \longrightarrow \mathbb{Q}^d$ be the map

$$\Psi(a_0 + a_1\lambda + \cdots + a_{d-1}\lambda^{d-1}) = (a_0, \ldots, a_{d-1})^T.$$

In particular, $z_i = \Psi(v_i)$ for $1 \le i \le n$. Taking Ψ from both sides of equation $(*)$ gives

$$\Psi(\lambda v_i) = \sum_j a_{ij} \Psi(v_j).$$

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Note that multiplication by λ on $\mathbb{Q}(\lambda)$ has matrix B with respect to the basis $\{1, \lambda, \ldots, \lambda^{d-1}\}$. Hence, we obtain

$$Bz_i = \sum_j a_{ij} z_j.$$

Finally, we need to verify that the points z_i belong to the positive half-space E. Note that

$$w^* = (1, \lambda, \ldots, \lambda^{d-1}) \in \mathbb{R}^d,$$

is a *left* eigenvector for the linear map B corresponding to the eigenvalue λ. Let E_λ be the one-dimensional invariant subspace of $\mathbb{R}^d$ corresponding to λ and C be its invariant complement. Therefore,

$$C = \{x \in \mathbb{R}^d \mid w^* x = 0\}.$$

Let π_λ be the projection map from $\mathbb{R}^d$ onto E_λ along the complementary direct sum. Define a map $m_{w^*} : \mathbb{R}^d \longrightarrow \mathbb{R}$ that is multiplication by w^* from the left. Then m_{w^*} should be a multiple of the map π_λ. On the other hand,

$$m_{w^*}(z_i) = w^* z_i = z_{i1} + z_{i2}\lambda + \cdots + z_{id}\lambda^{d-1} = v_i > 0.$$

Hence, replacing each z_i by $-z_i$ if necessary (in case m_{w^*} is a negative multiple of π_λ), we have $z_i \in E$ for each i and the proof is complete. □

Remark 2.3. $\mathbb{R}^d$ can be identified with $\mathbb{Q}(\lambda) \otimes_{\mathbb{Q}} \mathbb{R}$. Multiplication by λ is a linear map on $\mathbb{Q}(\lambda) \otimes_{\mathbb{Q}} \mathbb{R}$ which has the matrix B with respect to the basis $\{1, \lambda, \ldots, \lambda^{d-1}\}$. Therefore, an integral point in the standard basis of $\mathbb{R}^d$ can be considered as a point in $\mathbb{Z}[\lambda]$.

The following lemma and propositions will be used in the proof of Theorem 1.1.

LEMMA 2.4. *Let λ be a Perron number, with the companion matrix $B : \mathbb{R}^d \to \mathbb{R}^d$. Let δ be a Galois conjugate of λ, and π_δ be the projection onto E_δ along the complementary direct sum. Then for any non-zero integral point $z \in \mathbb{R}^d$, $\pi_\delta(z) \neq 0$.*

Proof. Assume to the contrary that $\pi_\delta(z) = 0$. Therefore z lies in the invariant complementary direct sum of E_δ in $\mathbb{R}^d$. Set $z = (x_1, \ldots, x_d)^T \in \mathbb{Z}^d$. Working in the complexification $\mathbb{R}^d \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathbb{R}^d$, we obtain that $w^* z = 0$, where $w^* = [1, \delta, \ldots, \delta^{d-1}]$ is a *left* eigenvector for B corresponding to the eigenvalue δ. Therefore,

$$x_1 + x_2\delta + \cdots + x_d\delta^{d-1} = 0.$$

However, this means that δ satisfies an integral polynomial equation with degree less than d. Therefore all x_j should be zero. This contradicts the fact that $z \in E$. □

The idea of using the next proposition has been generously suggested by Douglas Lind in a Mathoverflow post (see https://mathoverflow.net/questions/228826/lower-bound-for-perron-frobenius-degree-of-a-perron-number). In this post the author had asked for a way of finding a lower bound for the Perron–Frobenius degree of a Perron number. This was the answer that Lind gave:

> If a Perron number λ has negative trace, then any Perron–Frobenius matrix must have size strictly greater than the algebraic degree of λ, for example the largest root of $x^3 + 3x^2 - 15x - 46$. If B denotes the $d \times d$ companion matrix of the minimal polynomial of λ (which of course can have negative entries), then $\mathbb{R}^d$ splits into the dominant 1-dimensional eigenspace D and the direct sum E of all the other generalized eigenspace.
>
> Although I've not worked this out in detail, roughly speaking the smallest size of a Perron–Frobenius matrix for λ should be at least as large as the smallest number of sides of a polyhedral cone lying on one side of E (positive D-coordinate) and invariant (mapped into itself) under B. This is purely a geometrical condition, and there are likely further arithmetic constraints as well. For example, if λ has all its other algebraic conjugates of roughly the same absolute value, then B acts projectively as nearly a rotation, and this forces any invariant polyhedral cone to have many sides, so the geometric lower bound will be quite large.

The following proposition is only one way of using the above idea, and it would be nice to weaken the geometric assumptions about the roots or to explore the arithmetic constraints that Lind mentions.

PROPOSITION 2.5. *Let $\hat{B} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear map. Assume that the eigenvalues of $\hat{B}$ are λ, δ and θ such that:*

(1) *$\lambda > 1$ is a positive real number, δ, θ are a pair of conjugate complex numbers with non-zero imaginary parts and positive real parts, and $|\delta| < \lambda$;*

(2) *if we set $t = \delta/\lambda$, then $\eta := |(1 - \mathrm{Re}(t))/\mathrm{Im}(t)| \leq 1$.*

Define E as the positive half-space corresponding to the eigenvalue λ. Let M be the minimum number of sides for an arbitrary non-degenerate polygonal cone $\hat{\mathcal{C}} \subset E$ that is invariant under the map $\hat{B}$, that is, $\hat{B}(\hat{\mathcal{C}}) \subset \hat{\mathcal{C}}$. Then $M \geq 2\pi/3\eta$.

Proof. Let $\hat{\mathcal{C}}$ be an invariant polygonal cone for the map $\hat{B}$ with M sides. Let E_λ be the one-dimensional invariant subspace in $\mathbb{R}^3$ corresponding to λ. Pick an eigenvector $w \in E$ corresponding to the eigenvalue λ, and let H be the set of points whose projection under π_λ is the constant vector w. Define

$$\mathcal{P} := \hat{\mathcal{C}} \cap H.$$

Hence, $\mathcal{P}$ is a polygon with M sides and $\hat{\mathcal{C}}$ is the cone over $\mathcal{P}$ (Figure 1).

Let G be the two-dimensional invariant subspace corresponding to δ. One can think about G as the complex plane with the action of $\hat{B}$ on G being multiplication by the complex number δ.

Now if $w + w'$ is a vector in H where $w' \in G$, then

$$\hat{B}(w + w') = \lambda w + \delta w' = \lambda\left(w + \frac{\delta}{\lambda} w'\right).$$

Here by $\delta w'$ we mean multiplication by δ inside the complex plane G. Note that $w + (\delta/\lambda)w' \in H$, hence the action of $\hat{B}$ on H is multiplication by the complex number

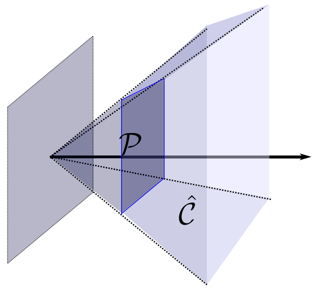

FIGURE 1. The cone $\hat{\mathcal{C}}$ over the polygon $\mathcal{P}$.

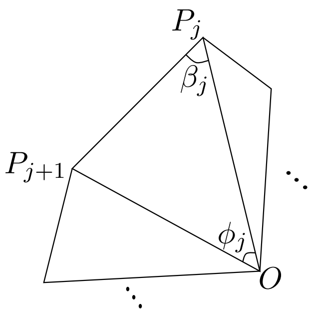

FIGURE 2. The triangle OP_jP_{j+1}.

$t = \delta/\lambda$. As a corollary, the polygon $\mathcal{P}$ is invariant under multiplication by $t = \delta/\lambda$. Note that $0 \in \mathcal{P}$ since $|t| = |\delta|/\lambda < 1$ and successive multiplication by t converges to the origin in H (i.e. the intersection point $E_\lambda \cap H$). Now if we set $\eta = |\tan^{-1}((1 - \mathrm{Re}(t))/\mathrm{Im}(t))|$, by Proposition 2.6 we have $M \geq 2\pi/3\eta$.

Note that this proposition is purely geometric and λ does not need to be an algebraic integer. □

PROPOSITION 2.6. *Let $\mathcal{P}$ be a convex non-degenerate polygon in the complex plane, having M sides and containing the origin. Let t be a complex number with non-zero imaginary part and positive real part. Assume that $\mathcal{P}$ is invariant under multiplication by t, that is, $t\mathcal{P} \subset \mathcal{P}$. If $\eta := |\tan^{-1}((1 - \mathrm{Re}(t))/\mathrm{Im}(t))| \leq 1$, then $M \geq 2\pi/3\eta$.*

Proof. Without loss of generality assume that $|t| \leq 1$ and $\mathrm{Im}(t) > 0$. See Figure 3 to visualize the angle η geometrically. Let $P_1, \ldots, P_M$ be the vertices of $\mathcal{P}$ in counterclockwise order and let O denote the origin. Define $\beta_j := \angle(OP_jP_{j+1})$ and $\phi_j := \angle(P_jOP_{j+1})$ (see Figure 2). The proof is divided into a few steps.

Step 1:

$$\beta_j \geq \frac{\pi}{2} - \eta.$$

This is because if the above condition is not satisfied, then tP_j lies outside the polygon $\mathcal{P}$ (see Figure 3, right), contradicting the assumption that the polygon $\mathcal{P}$ is invariant under multiplication by t.

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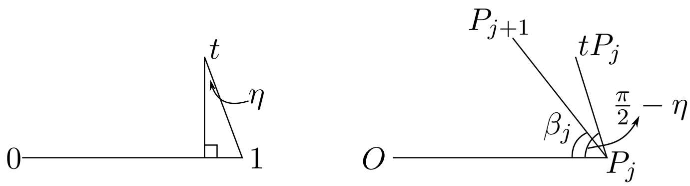

FIGURE 3. Left: the angle η. Right: step 1.

Define $l_j = |OP_j|$. We will work with the values l_{j+1}/l_j. Note that $P := \prod_{j=1}^{M} l_{j+1}/l_j = 1$. The index set $\mathcal{M} := \{1, \ldots, M\}$ can be partitioned into two sets according to whether $\phi_j < \eta$ or not:

$$\mathcal{A} = \{1 \leq j \leq M \mid \phi_j \geq \eta\} \quad \text{and} \quad \mathcal{B} = \{1 \leq j \leq M \mid \phi_j < \eta\}.$$

Define

$$P_{\mathcal{A}} := \prod_{j \in \mathcal{A}} \frac{l_{j+1}}{l_j}, \quad P_{\mathcal{B}} := \prod_{j \in \mathcal{B}} \frac{l_{j+1}}{l_j}.$$

We clearly have $P = P_{\mathcal{A}} \cdot P_{\mathcal{B}} = 1$. We give lower bounds for the values of $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$.

Step 2:

$$\phi_j - \eta < \frac{\pi}{2}.$$

To see this, consider the triangle OP_jP_{j+1} and note that sum of any two angles has to be less than π:

$$\beta_j + \phi_j < \pi \implies \frac{\pi}{2} - \eta + \phi_j < \pi \implies \phi_j - \eta < \frac{\pi}{2}.$$

Here we used step 1 for the first implication.

Step 3: For any $j \in \mathcal{A}$,

$$\frac{l_{j+1}}{l_j} \geq \frac{\cos(\eta)}{\cos(\phi_j - \eta)}.$$

Consider the triangle OP_jP_{j+1}. Let A be the point on the segment OP_{j+1} such that $\angle OP_jA = \pi/2 - \eta$. Such a point exists by step 1, since

$$\angle OP_jA = \frac{\pi}{2} - \eta \leq \beta_j = \angle OP_jP_{j+1}.$$

Let H be the projection of O onto P_jA (see Figure 4). It follows from the assumption $j \in \mathcal{A}$ that the point H lies inside the triangle OP_jP_{j+1}. This is because

$$\angle P_jOH = \eta \leq \phi_j = \angle P_jOA.$$

Then

$$\frac{l_{j+1}}{l_j} = \frac{OP_{j+1}}{OP_j} \geq \frac{OA}{OP_j} = \frac{OA}{OH} \cdot \frac{OH}{OP_j} = \frac{1}{\cos(\phi_j - \eta)} \cdot \cos(\eta).$$

Step 4: For any $j \in \mathcal{B}$, we have:

$$\frac{l_{j+1}}{l_j} \geq \cos(\eta).$$

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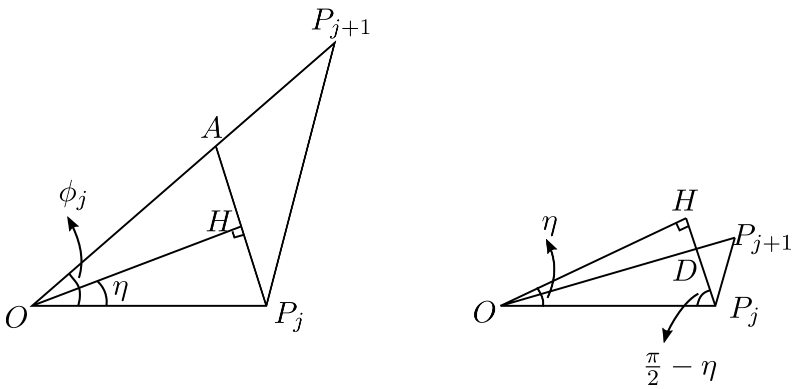

FIGURE 4. Left: step 3. Right: step 4.

Choose the point H such that $\angle P_j O H = \eta$ and $\angle O P_j H = \pi/2 - \eta$. Therefore. $\angle O H P_j = \pi/2$. Let D be the intersection of the lines $O P_{j+1}$ with $P_j H$. Then D lies on the segments $O P_{j+1}$ and $P_j H$ (see Figure 4). To see this, note that

$$\angle O P_j P_{j+1} = \beta_j \geq \frac{\pi}{2} - \eta = \angle O P_j H,$$
$$\angle P_j O H = \eta \geq \phi_j = \angle P_j O P_{j+1}.$$

Here the first inequality is by Step 1, and the second inequality follows from the assumption $j \in \beta$. Now

$$\frac{l_{j+1}}{l_j} = \frac{O P_{j+1}}{O P_j} \geq \frac{O D}{O P_j} \geq \frac{O H}{O P_j} = \cos(\eta).$$

We are now ready to give a lower bound for M. Note that steps 3 and 4 imply that

$$1 = P = P_{\mathcal{A}} \cdot P_{\mathcal{B}} \geq \bigg(\prod_{j \in \mathcal{A}} \frac{\cos(\eta)}{\cos(\phi_j - \eta)} \bigg) \cdot \bigg(\prod_{j \in \mathcal{B}} \cos(\eta) \bigg)$$
$$\implies \cos(\eta)^{M/|\mathcal{A}|} \leq \bigg(\prod_{j \in \mathcal{A}} \cos(\phi_j - \eta) \bigg)^{1/|\mathcal{A}|},$$

where $|\mathcal{A}|$ is the cardinality of $\mathcal{A}$. We now observe that, keeping the sum of ϕ_j for $j \in \mathcal{A}$ fixed, the product of $\cos(\phi_j - \eta)$ is maximized when all ϕ_j are equal. This is simply a consequence of the following inequality, where we take a and b to be the quantities $\phi_j - \eta$:

$$\cos(a) \cdot \cos(b) \leq \bigg(\cos\bigg(\frac{a+b}{2}\bigg)\bigg)^2.$$

Crucially $0 \leq \phi_j - \eta < \pi/2$, for each $j \in \mathcal{A}$ (by step 2 and the definition of the set $\mathcal{A}$), which implies that all $\cos(\cdot)$ involved are non-negative. To see that the inequality holds, note that

$$\begin{aligned} 2\cos(a) \cdot \cos(b) &= \cos(a+b) + \cos(a-b) \\ &= 2\bigg(\cos\bigg(\frac{a+b}{2}\bigg)\bigg)^2 - 1 + \cos(a-b) \leq 2\bigg(\cos\bigg(\frac{a+b}{2}\bigg)\bigg)^2. \end{aligned}$$

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Let $\overline{\phi}$ be the average of the angles $\phi_j - \eta$ for $j \in \mathcal{A}$. Then we have $0 \le \overline{\phi} \le \pi/2$, since each of the angles $\phi_j - \eta$ satisfies the same bounds. By definition of the set $\mathcal{B}$,

$$\text{for all } j \in \mathcal{B}, \quad \phi_j < \eta \implies \sum_{j\in\mathcal{B}} \phi_j < \eta \cdot |\mathcal{B}| = \eta(M - |\mathcal{A}|).$$

Hence

$$\begin{aligned}\bar{\phi} &= \frac{\sum_{j\in\mathcal{A}}(\phi_j - \eta)}{|\mathcal{A}|} = \frac{\sum_{j\in\mathcal{M}} \phi_j - \sum_{j\in\mathcal{B}} \phi_j - |\mathcal{A}|\eta}{|\mathcal{A}|} \\ &\ge \frac{2\pi - \eta(M - |\mathcal{A}|) - |\mathcal{A}|\eta}{|\mathcal{A}|} = \frac{2\pi - M\eta}{|\mathcal{A}|}.\end{aligned}$$

Now if $2\pi - M\eta$ is negative, then there is nothing to prove. Otherwise $0 \le (2\pi - M\eta)/|\mathcal{A}| \le \overline{\phi} \le \pi/2$, and therefore

$$\cos(\eta)^{M/|\mathcal{A}|} \le \left(\prod_{j\in\mathcal{A}} \cos(\phi_j - \eta)\right)^{1/|\mathcal{A}|} \le \cos(\overline{\phi}) \le \cos\left(\frac{2\pi - M\eta}{|\mathcal{A}|}\right).$$

The next step is to give a lower bound for $\cos(\eta)^{M/|\mathcal{A}|}$.

Step 5: For $\alpha \ge 1$ and $0 \le x \le \frac{1}{2}$ the following inequality holds:

$$(1 - x)^{\alpha} \ge 1 - 2\alpha x.$$

This inequality can be proved by noting that the values of both sides agree at $x = 0$ and then checking the signs of derivatives for $0 \le x \le \frac{1}{2}$ and $1 \le \alpha$ (for the variable x).

The assumption $\eta \le 1$ implies that $0 \le \eta^2/2 \le \frac{1}{2}$ and hence $0 \le 1 - \eta^2/2$. The inequality $\cos(y) \ge 1 - y^2/2$ holds for every real number $0 \le y \le 1$, and clearly $M/|\mathcal{A}| \ge 1$. Hence we have the lower bound

$$\cos(\eta)^{M/|\mathcal{A}|} \ge \left(1 - \frac{\eta^2}{2}\right)^{M/|\mathcal{A}|} \ge 1 - 2\left(\frac{M}{|\mathcal{A}|}\right)\frac{\eta^2}{2} = 1 - \frac{M\eta^2}{|\mathcal{A}|},$$

where in the last inequality we have used step 5. Combining with the previous bound, we obtain that

$$\begin{aligned}&1 - \frac{M\eta^2}{|\mathcal{A}|} \le \cos\left(\frac{2\pi - M\eta}{|\mathcal{A}|}\right) \implies 1 - \cos\left(\frac{2\pi - M\eta}{|\mathcal{A}|}\right) \le \frac{M\eta^2}{|\mathcal{A}|} \\ &\implies 2\sin\left(\frac{2\pi - M\eta}{2|\mathcal{A}|}\right)^2 \le \frac{M\eta^2}{|\mathcal{A}|} \implies \sin\left(\frac{2\pi - M\eta}{2|\mathcal{A}|}\right) \le \sqrt{\frac{M}{2|\mathcal{A}|}}\,\eta.\end{aligned}$$

Recall the assumption $0 \le (2\pi - M\eta)/|\mathcal{A}| \le \pi/2$. Therefore the quantity $(2\pi - M\eta)/2|\mathcal{A}|$ lies in the interval $[0, \pi/4]$. In the latter interval, the inequality $\sin(x) \ge x/\sqrt{2}$ holds. Hence,

$$\begin{aligned}&\frac{1}{\sqrt{2}}\frac{2\pi - M\eta}{2|\mathcal{A}|} \le \sin\left(\frac{2\pi - M\eta}{2|\mathcal{A}|}\right) \le \sqrt{\frac{M}{2|\mathcal{A}|}}\,\eta. \\ &\implies 2\pi - M\eta \le 2\sqrt{M|\mathcal{A}|}\,\eta \implies 2\pi - M\eta \le 2M\eta \\ &\implies M\eta \ge \frac{2\pi}{3}.\end{aligned}$$

□

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Proof of Theorem 1.1. Assume that $d_{PF}(p) = n$. Therefore, there is an $n \times n$ non-negative, integral, aperiodic matrix $A = [a_{ij}]$ with spectral radius equal to p. Let $B : \mathbb{R}^d \to \mathbb{R}^d$ be the companion matrix corresponding to the minimal polynomial of p. Let w be an eigenvector for the map B corresponding to the eigenvalue p, and denote by $E \subset \mathbb{R}^d$ the positive half-space containing w. By Theorem 2.2, there are integral points $z_1, \ldots, z_n \in E$ such that for each $1 \le i \le n$,

$$Bz_i = \sum a_{ij} z_j.$$

Let $\mathcal{C}$ be the cone over the points $z_1, \ldots, z_n$, that is,

$$\mathcal{C} = \{\epsilon_1 z_1 + \cdots + \epsilon_n z_n \mid \forall i,\ \epsilon_i \ge 0\} \subset E.$$

The cone $\mathcal{C}$ is invariant under the action of B, that is, $B(\mathcal{C}) \subset \mathcal{C}$. Let E_p and $E_{p'}$ be the one-dimensional and two-dimensional invariant subspaces of $\mathbb{R}^d$ corresponding to p and p', respectively. Set $W := E_p \oplus E_{p'}$, and let $\pi : \mathbb{R}^d \longrightarrow W$ be the projection onto the invariant subspace W along the complementary direct sum. Since the maps B and π commute, we have

$$Bz_i = \sum_j a_{ij} z_j \implies B(\pi(z_i)) = \pi(B(z_i)) = \sum_j a_{ij} \pi(z_j).$$

Therefore, if we set $\hat{z_i} := \pi(z_i)$, then $\hat{z_i} \in W \cap E$ and they satisfy the same linear equations as z_i did. Hence the cone $\hat{\mathcal{C}} \subset W \cap E$ over the points $\hat{z_i}$ is invariant under the linear action of $\hat{B} := B_{|W}$.

By Lemma 2.4, since the z_i are integral points, none of the points $\pi(z_i)$ can lie entirely inside the one-dimensional subspace $E_p \subset W$; otherwise $\pi_{p'}(z_i) = 0$. Therefore, the cone $\hat{\mathcal{C}}$ is non-degenerate. In summary, the cone over the points $\pi(z_i)$ is a non-degenerate polygonal cone $\hat{\mathcal{C}}$ in W, which is invariant under the action of $\hat{B}$. By assumption the map $\hat{B}$ satisfies the conditions of Proposition 2.5. Therefore, the desired bound holds. □

3. *Applications*

COROLLARY 1.2 (Lind, McMullen, Thurston). *For any $N > 0$, there are cubic Perron numbers whose Perron–Frobenius degrees are larger than N.*

Proof. The idea is to construct a cubic polynomial $f(x)$ with exactly one real root $w_1 > 0$, and such that for one of the other roots, say w_2:

(1) the absolute value of w_2 is smaller than but very close to w_1 and the argument of w_2 is very small;

(2) $f(x)$ is irreducible.

It is easy to construct a reducible polynomial of the form $g(x) = (c - x)[(a - x)^2 + b^2]$ that satisfies (1). Moreover, by perturbing $g(x)$, one expects $g(x)$ to become irreducible and still satisfy (1). The details are as follows. Let $\epsilon > 0$. The proof is broken down into several steps.

Step 1: There are natural numbers $a, b, c \gg 0$ satisfying the inequalities

$$\sqrt{a^2 + b^2} < c \le a + \epsilon\, b, \quad \left(\frac{a}{b}\right)^2 \le c. \tag{1}$$

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First by choosing a_0 much larger than b_0, we may arrange that $\sqrt{a_0^2 + b_0^2} < a_0 + \epsilon\, b_0$. Let c_0 be a positive integer satisfying $(a_0/b_0)^2 \le c_0$. Pick $k \gg 0$ such that

$$k(a_0 + \epsilon\, b_0) - k\sqrt{a_0^2 + b_0^2} \ge c_0 + 4,$$

and denote by c the largest integer between $k\sqrt{a_0^2 + b_0^2}$ and $k(a_0 + \epsilon\, b_0)$. Therefore

$$c \ge k(a_0 + \epsilon b_0) - 1 \ge (c_0 + 4) - 1 = c_0 + 3 > c_0.$$

Set $a = ka_0$ and $b = kb_0$. We now check that the desired inequalities hold for a, b, c. The first inequality is satisfied by the definition of c. Moreover,

$$c \ge c_0 \ge \left(\frac{a_0}{b_0}\right)^2 = \left(\frac{a}{b}\right)^2.$$

Note that by choosing $k \gg 0$, we can assume that all of the numbers a, b and c are large. This proves step 1.

Define the cubic polynomial $f(x)$ as $f(x) = (c - x)[(a - x)^2 + b^2] + 1$.

Step 2: The polynomial $f(x)$ has a real root ω_1 satisfying

$$c < \omega_1 < \min\left\{c + 1, c + \frac{c+1}{a^2 + b^2 - 1}\right\}.$$

By the intermediate value theorem, it is enough to show that

$$f(c) > 0, \quad f(c+1) < 0, \quad f\left(c + \frac{c+1}{a^2+b^2-1}\right) < 0.$$

We have $f(c) = 1 > 0$ and

$$f(c+1) = -[(a - c - 1)^2 + b^2] + 1 \le -b^2 + 1 < 0.$$

Here we have used the assumption $b > 1$.

For the last part, set $p = c + ((c+1)/(a^2 + b^2 - 1))$. Hence

$$f(p) = -\frac{c+1}{a^2+b^2-1}[(a-p)^2 + b^2] + 1 \le -\left(\frac{c+1}{a^2+b^2-1}\right) \cdot b^2 + 1 < 0$$
$$\iff 1 < \left(\frac{c+1}{a^2+b^2-1}\right) \cdot b^2 \iff a^2 + b^2 - 1 < c\, b^2 + b^2 \iff a^2 - 1 < c\, b^2.$$

But the last inequality holds by the assumption $(a/b)^2 \le c$. Therefore the step follows.

Step 3: $f(x)$ has exactly one real root.

To see this, assume to the contrary that all roots of $f(x)$ are real. Denote the other two roots by ω_2 and ω_3. We will prove that $((\omega_2 + \omega_3)/2)^2 < \omega_2\omega_3$, which gives a contradiction. After expanding, we deduce that

$$f(x) = -x^3 + (c + 2a)x^2 - (2ac + a^2 + b^2)x + c(a^2 + b^2) + 1.$$

By Vieta's formula,

$$\omega_1 + \omega_2 + \omega_3 = c + 2a,$$
$$\omega_1\omega_2\omega_3 = c(a^2 + b^2) + 1.$$

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Therefore,

$$0 < 2a - 1 < \omega_2 + \omega_3 = c + 2a - \omega_1 < 2a,$$

where we have used the inequality $c < \omega_1 < c + 1$ from step 2. As a result,

$$\omega_2\omega_3 = \frac{c(a^2+b^2)+1}{\omega_1} > a^2 \iff \omega_1 < \frac{c(a^2+b^2)+1}{a^2}.$$

Here the first equality is the application of Vieta's formula. Using step 2, in order to verify the last inequality, it is enough to show that

$$c + \frac{c+1}{a^2+b^2-1} < \frac{c(a^2+b^2)+1}{a^2} = c + \frac{c\,b^2+1}{a^2}$$
$$\iff \frac{c+1}{a^2+b^2-1} < \frac{c\,b^2+1}{a^2} \iff (c+1)a^2 < (c\,b^2+1)(a^2+b^2-1).$$

But we have

$$(c+1) < c\,b^2 + 1, \quad a^2 < (a^2+b^2-1),$$

which imply the last part. Therefore, we have established that $\omega_2\omega_3 > a^2$. Putting it all together, we obtain

$$\left(\frac{\omega_2+\omega_3}{2}\right)^2 < \left(\frac{2a}{2}\right)^2 = a^2 < \omega_2\omega_3.$$

This completes the proof of the step. Therefore, ω_2 and ω_3 are both non-real and $\omega_3 = \overline{\omega_2}$.

Step 4: ω_1 is a Perron number.

Since $\omega_3 = \overline{\omega_2}$, we have $|\omega_2|^2 = \omega_2\omega_3$. Therefore, by Vieta's formula,

$$|\omega_2|^2 = \omega_2\omega_3 = \frac{c(a^2+b^2)+1}{\omega_1} \leq \frac{c(a^2+b^2)+1}{c}$$
$$= a^2 + b^2 + \frac{1}{c} \leq a^2 + b^2 + 1 \leq c^2 < \omega_1^2.$$

Here we have used $\omega_1 > c$ for the first and last inequalities. The relation $a^2 + b^2 + 1 \leq c^2$ follows from $a^2 + b^2 < c^2$ (step 1) and the fact that both $a^2 + b^2$ and c^2 are integers.

Step 5:

$$|\omega_2|^2 \geq a^2 + b^2 - 1.$$

Again using Vieta's formula,

$$|\omega_2|^2 = \omega_2\omega_3 = \frac{c(a^2+b^2)+1}{\omega_1} \geq a^2 + b^2 - 1$$
$$\iff \omega_1 \leq \frac{c(a^2+b^2)+1}{a^2+b^2-1} = c + \frac{c+1}{a^2+b^2-1}.$$

But the last inequality holds by step 2.

Step 6: Denote the real and imaginary part of ω_2 by $\mathrm{Re}(\omega_2)$ and $\mathrm{Im}(\omega_2)$. Then

$$|\mathrm{Re}(\omega_2)| \leq a, \quad 0 < \omega_1 - \mathrm{Re}(\omega_2) < c - a + 2, \quad |\mathrm{Im}(\omega_2)|^2 \geq b^2 - 1.$$

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Since ω_2 and ω_3 are complex conjugates, we have $\omega_2 + \omega_3 = 2\mathrm{Re}(\omega_2)$. By Vieta's formula,

$$\mathrm{Re}(\omega_2) = \frac{\omega_2 + \omega_3}{2} = \frac{\omega_1 + \omega_2 + \omega_3 - \omega_1}{2} = \frac{c + 2a - \omega_1}{2}.$$

Therefore,

$$|\mathrm{Re}(\omega_2)| \leq \frac{c + 2a - c}{2} = a.$$

Here, for the last inequality, we have used $c < \omega_1 < c + 1$ from step 2. This verifies the first part of the step. For the second part, by step 4, $|\omega_2| < \omega_1$, which implies that $0 < \omega_1 - \mathrm{Re}(\omega_2)$. Moreover,

$$\begin{aligned}\omega_1 - \mathrm{Re}(\omega_2) &= \omega_1 - \left(\frac{c + 2a - \omega_1}{2}\right) \\ &= \frac{3\omega_1 - (c + 2a)}{2} \leq \frac{3(c+1) - (c + 2a)}{2} < c - a + 2.\end{aligned}$$

Here we have used $\omega_1 < c + 1$ from step 2. This completes the second part. For the third part,

$$|\mathrm{Im}(\omega_2)^2| = |\omega_2|^2 - |\mathrm{Re}(\omega_2)|^2 \geq (a^2 + b^2 - 1) - a^2 = b^2 - 1.$$

Here we have used step 5, together with the first part of step 6.

Step 7:

$$\left(\frac{\omega_1 - \mathrm{Re}(\omega_2)}{\mathrm{Im}(\omega_2)}\right)^2 \leq \frac{(\epsilon + 2b^{-1})^2}{1 - b^{-2}}.$$

By the second and third parts of step 6,

$$\left(\frac{\omega_1 - \mathrm{Re}(\omega_2)}{\mathrm{Im}(\omega_2)}\right)^2 \leq \frac{(c - a + 2)^2}{b^2 - 1} \leq \frac{(\epsilon\, b + 2)^2}{b^2 - 1} = \frac{(\epsilon + 2b^{-1})^2}{1 - b^{-2}}.$$

Here we have used the condition $0 < c - a \leq \epsilon\, b$ from step 1.

We can now prove the corollary. Consider the algebraic integer ω_1 defined as above. Since, by step 2, $c < \omega_1 < c + 1$, the number ω_1 is not an integer. The other two roots of $f(x)$ are not real by step 3. Hence, ω_1 is a cubic algebraic integer. By step 4, ω_1 is Perron. By Theorem 1.1,

$$d_{PF}(\omega_1) \geq \frac{2\pi}{3\eta}, \qquad \eta := \tan^{-1}\left(\frac{\omega_1 - \mathrm{Re}(\omega_2)}{|\mathrm{Im}(\omega_2)|}\right),$$

whenever $\eta \leq 1$. By step 7 we have

$$\tan(\eta) \leq \frac{\epsilon + 2b^{-1}}{\sqrt{1 - b^{-2}}}.$$

As mentioned in step 1, given any $\epsilon > 0$, we may find a, b, c with the given properties such that they are arbitrary large. Therefore we may assume that $b > \epsilon^{-1}$, or equivalently $b^{-1} < \epsilon$. Hence

$$\tan(\eta) \leq \frac{\epsilon + 2b^{-1}}{\sqrt{1 - b^{-2}}} \leq \frac{3\epsilon}{\sqrt{1 - b^{-2}}} < 6\epsilon.$$

Here the last inequality follows from $b \geq 2$. To sum up, we have $\tan(\eta) < 6\epsilon$, which is equivalent to $\eta < \tan^{-1}(6\epsilon)$ since the tangent function is strictly increasing on the interval $[0, \pi/2]$. As a result,

$$d_{PF}(\omega_1) \geq \frac{2\pi}{3\eta} > \frac{2\pi}{3\tan^{-1}(6\epsilon)}.$$

By choosing $\epsilon > 0$ arbitrary small, we find arbitrary large lower bounds for $d_{PF}(\omega_1)$. This completes the proof. □

Remark 3.1. If λ is a quadratic Perron number, then $d_{PF}(\lambda) = 2$. To see this, assume that the minimal polynomial of λ is of the form $f(x) = x^2 - ux + v$, where $u, v \in \mathbb{Z}$ and $\Delta = u^2 - 4v > 0$. If we denote the other root by λ', then $u = \lambda + \lambda' > 0$, since λ is Perron. Now if u is even, then $4|\Delta$ and we may take

$$A = \begin{bmatrix} \frac{u}{2} & \frac{\Delta}{4} \\ 1 & \frac{u}{2} \end{bmatrix}.$$

Then all the entries of A are positive integers, and its characteristic polynomial is equal to $f(x)$. If u is odd, then $\Delta \equiv 1 \pmod 4$. Moreover, $\Delta \neq 1$ since otherwise the polynomial $f(x)$ would not have been irreducible. Therefore, we may take

$$A = \begin{bmatrix} \frac{u+1}{2} & \frac{\Delta - 1}{4} \\ 1 & \frac{u-1}{2} \end{bmatrix}.$$

The characteristic polynomial of A is equal to $f(x)$. If $u > 1$, then A has positive entries. If $u = 1$, then we should have $v < 0$ since $\Delta > 1$. In this case A^2 has positive entries. □

An algebraic integer is called a *unit*, if the product of all its Galois conjugates is equal to 1 or -1. Equivalently, the constant term of its minimal polynomial should be equal to 1 or -1.

Definition 3.2. A unit algebraic integer $\alpha > 1$ is called *bi-Perron* if all other Galois conjugates of α lie in the annulus $\{z \in \mathbb{C} \mid 1/\alpha < |z| < \alpha\}$, except possibly for α^{-1}.

OBSERVATION 3.3. *Let $\gamma > 2$ be a Perron number, such that for every other Galois conjugate γ' of γ we have $|\gamma'| \leq \gamma - 2$. Then the unique real solution $\alpha > 1$ to $\alpha + 1/\alpha = \gamma$ is bi-Perron.*

Proof. As the Galois conjugates of α come in reciprocal pairs, the product of all the Galois conjugates is equal to 1. Therefore, α is a unit algebraic integer. Assume that $\alpha' \notin \{\alpha, \alpha^{-1}\}$ is a Galois conjugate of α. We need to prove that $1/\alpha < |\alpha'| < \alpha$. There is a Galois conjugate $\gamma' \neq \gamma$ of γ such that $\alpha' + 1/\alpha' = \gamma'$. There are three cases to consider.

(1) If $|\alpha'| > 1$, then by the triangle inequality,

$$|\alpha'| \leq \left|\alpha' + \frac{1}{\alpha'}\right| + \left|\frac{-1}{\alpha'}\right| = |\gamma'| + \left|\frac{1}{\alpha'}\right| \leq \gamma - 2 + \left|\frac{1}{\alpha'}\right| = \left(\alpha + \frac{1}{\alpha}\right) - 2 + \left|\frac{1}{\alpha'}\right| < \alpha.$$

Here the last inequality follows from $|\alpha'| > 1$ and $\alpha > 1$. This proves the upper bound for $|\alpha'|$. The lower bound follows from $1/\alpha < 1 < |\alpha'|$.

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(2) If $|\alpha'| = 1$, the inequalities hold trivially as $1/\alpha < 1 = |\alpha'| = 1 < \alpha$.
(3) If $|\alpha'| < 1$, then $(\alpha')^{-1}$ is also a Galois conjugate. The result follows from (1), since the inequalities are symmetric. □

Remark 3.4. Without the condition on the absolute value of γ', the conclusion is not true. As an example one can take γ to be the Perron root of the polynomial $(x-5)[(x-4)^2 + 3^2] - 1$.

COROLLARY 1.3. *For any $N > 0$, there are bi-Perron numbers of algebraic degree at most* 6 *whose Perron–Frobenius degrees are larger than N.*

Proof. Pick $\epsilon > 0$. Let ω_1 be the cubic Perron number constructed in Corollary 1.2, with Galois conjugates $w_2 = \overline{\omega_3}$. Recall that in the construction, one could take a, b and c to be arbitrarily large. Therefore, we may assume that $b, c > 2$ and $b > \epsilon^{-1}$. Throughout, when we refer to step x, we mean step x in the proof of Corollary 1.2.

By step 2, $\omega_1 > c > 2$, so the first condition of Observation 3.3 is satisfied. To prove the second condition, we want to show that it is possible to choose a, b and c such that $|\omega_2| \leq \omega_1 - 2$. By the proof of step 4, we have $|\omega_2|^2 \leq a^2 + b^2 + 1$. As we know that $c < \omega_1$, it is enough to prove that $a^2 + b^2 + 1 \leq (c-2)^2$. In the proof of step 1, we defined c as the largest integer between $k\sqrt{a_0^2 + b_0^2}$ and $k(a_0 + \epsilon\, b_0)$. Moreover, we had

$$k(a_0 + \epsilon\, b_0) - k\sqrt{a_0^2 + b_0^2} \geq c_0 + 4 \geq 4.$$

Therefore

$$c \geq k\sqrt{a_0^2 + b_0^2} + 3 = \sqrt{a^2 + b^2} + 3.$$

This implies

$$c - 2 \geq \sqrt{a^2 + b^2} + 1 \implies (c-2)^2 \geq \left(\sqrt{a^2 + b^2} + 1\right)^2 > a^2 + b^2 + 1.$$

This shows that the hypotheses of Observation 3.3 are satisfied. Hence, we may define the bi-Perron number $\alpha > 1$ as the solution to $\alpha + 1/\alpha = \omega_1$. The number α satisfies a monic integral polynomial equation of degree 6, obtained by substituting $x = \alpha + \alpha^{-1}$ in the minimal polynomial of ω_1, $f(x)$, and clearing the denominators. Hence the algebraic degree of α is at most 6. In fact the degree can be taken to be equal to 6 but we do not prove it.

The last step is to give a lower bound for the Perron–Frobenius degree of α. Pick a Galois conjugate $\alpha' \neq \alpha$ of α such that $|\alpha'| \geq 1$. By Theorem 1.1, we have

$$d_{PF}(\alpha) \geq \frac{2\pi}{3\hat{\eta}}, \quad \hat{\eta} := \tan^{-1}\left(\frac{\alpha - \mathrm{Re}(\alpha')}{|\mathrm{Im}(\alpha')|}\right),$$

as long as $\hat{\eta} \leq 1$. We have

$$\alpha + \frac{1}{\alpha} = \omega_1 \implies \alpha = \omega_1 - \frac{1}{\alpha} < \omega_1,$$

since $\alpha > 0$. Moreover,

$$\alpha' + \frac{1}{\alpha'} = \omega_2 \implies \mathrm{Re}(\alpha') = \mathrm{Re}(\omega_2) - \mathrm{Re}\left(\frac{1}{\alpha'}\right) \geq \mathrm{Re}(\omega_2) - 1.$$

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Here we have used the fact that $|1/\alpha'| \leq 1$, which implies that $|\mathrm{Re}(1/\alpha')| \leq 1$. Putting the two inequalities together, we obtain

$$\alpha - \mathrm{Re}(\alpha') \leq \omega_1 - \mathrm{Re}(\omega_2) + 1.$$

On the other hand, α is bi-Perron, so

$$0 < \alpha - |\alpha'| \leq \alpha - \mathrm{Re}(\alpha').$$

Similarly,

$$\alpha' + \frac{1}{\alpha'} = \omega_2 \implies |\mathrm{Im}(\alpha')| = \left|\mathrm{Im}(\omega_2) - \mathrm{Im}\left(\frac{1}{\alpha'}\right)\right| \\ \geq |\mathrm{Im}(\omega_2)| - \left|\mathrm{Im}\left(\frac{1}{\alpha'}\right)\right| \geq |\mathrm{Im}(\omega_2)| - 1.$$

We now can give an upper bound for $\tan(\hat{\eta})$:

$$\tan(\hat{\eta}) = \left(\frac{\alpha - \mathrm{Re}(\alpha')}{|\mathrm{Im}(\alpha')|}\right) \leq \frac{\omega_1 - \mathrm{Re}(\omega_2) + 1}{|\mathrm{Im}(\omega_2)| - 1}.$$

Using the second and third parts of step 6 together with step 1, we obtain

$$\tan(\hat{\eta}) \leq \frac{c - a + 3}{\sqrt{b^2 - 1} - 1} \leq \frac{\epsilon\, b + 3}{\sqrt{b^2 - 1} - 1} = \frac{\epsilon + 3b^{-1}}{\sqrt{1 - b^{-2}} - b^{-1}}.$$

The condition $b > \epsilon^{-1}$ is equivalent to $b^{-1} < \epsilon$. We have

$$\frac{\epsilon + 3b^{-1}}{\sqrt{1 - b^{-2}} - b^{-1}} \leq \frac{4\epsilon}{\sqrt{1 - b^{-2}} - b^{-1}} \leq 16\epsilon,$$

where the last inequality follows from $b \geq 2$. Therefore $\hat{\eta} \leq \tan^{-1}(16\epsilon)$, which implies that

$$d_{PF}(\alpha) \geq \frac{2\pi}{3\hat{\eta}} \geq \frac{2\pi}{3\tan^{-1}(16\epsilon)}.$$

By choosing $\epsilon > 0$ to be arbitrary small, we obtain arbitrarily large lower bounds for the Perron–Frobenius degree. □

4. *Questions*

Let λ be the stretch factor of a pseudo-Anosov map ϕ on a closed orientable surface S. Then, by Fried's theorem, λ is bi-Perron. In fact one can construct a non-negative, integral, aperiodic matrix of size at most $6|\chi(S)|$ and with spectral radius λ using an invariant train track for the map ϕ [**9**]. The author's initial motivation for studying the Perron–Frobenius degree of λ was to control the genus of the underlying surface. Unfortunately our bound is not very effective for bi-Perron numbers coming from pseudo-Anosov maps, since 'generic conjugacy classes of pseudo-Anosov maps tend to have totally real stretch factors' (see [**4**] or [**5**, Appendix C5] for a precise statement). This motivates the following question.

Question 4.1. Give an effective lower bound for the Perron–Frobenius degree of a (possibly totally real) Perron number.

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Question 4.2.

(1) Are there pseudo-Anosov stretch factors with constant algebraic degree, and with arbitrary large Perron–Frobenius degree? In particular, can the bi-Perron numbers constructed in Corollary 1.3 be realized as stretch factors?

(2) Fix a closed, orientable surface S. What is the set of possible Perron–Frobenius degrees of pseudo-Anosov maps on S?

Note that a positive answer to Question 4.2(1) gives new counterexamples to a conjecture/question of Farb recently disproved by Leininger and Reid using methods from Teichmüller theory [**6**].

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