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Structure of the Fučík spectrum and existence of solutions for equations with asymmetric nonlinearities

A. K. Ben-Naoum
Université catholique de Louvain, CESAME,
4–6 avenue Georges Lemaître, 1348 Louvain-la-Neuve, Belgium
(bennaoum@auto.ucl.ac.be)

C. Fabry and **D. Smets**
Université catholique de Louvain, Département de Mathématique,
2 chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium
(fabry@amm.ucl.ac.be; smets@amm.ucl.ac.be)

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Let $L : \operatorname{dom} L \subset L^2(\Omega) \to L^2(\Omega)$ be a self-adjoint operator, Ω being open and bounded in $\mathbb{R}^N$. We give a description of the Fučík spectrum of L away from the essential spectrum. Let λ be a point in the discrete spectrum of L; provided that some non-degeneracy conditions are satisfied, we prove that the Fučík spectrum consists locally of a finite number of curves crossing at (λ, λ). Each of these curves can be associated to a critical point of the function $H : x \mapsto \langle |x|, x\rangle_{L^2}$ restricted to the unit sphere in $\ker(L - \lambda I)$. The corresponding critical values determine the slopes of these curves. We also give global results describing the Fučík spectrum, and existence results for semilinear equations, by performing degree computations between the Fučík curves.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be open and bounded. We will consider, in the Hilbert space $L^2(\Omega)$ of real-valued functions, semilinear equations involving a self-adjoint operator L. These equations contain an asymmetric nonlinear term, also called a 'jumping nonlinearity',

$$Lu = \alpha u^+ - \beta u^- + f, \tag{1.1}$$

where $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and $f \in L^2(\Omega)$. We will also consider the equation

$$Lu = \alpha u^+ - \beta u^- + g(\cdot, u), \tag{1.2}$$

assuming that $g(x, u)$ has a sublinear growth in u, for $|u| \to \infty$.

The first sections of our paper deal with the homogeneous equation,

$$Lu = \alpha u^+ - \beta u^-. \tag{1.3}$$

The set of points (α, β) for which this equation has non-trivial solutions is called the Fučík or Dancer–Fučík spectrum, Dancer [5] and Fučík [8] having recognized

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its importance in the study of semilinear boundary-value problems. This spectrum will be denoted by $\Sigma(L)$. We will also denote by $\sigma_{\mathrm{d}}(L)$ the discrete spectrum of L (i.e. all the isolated eigenvalues of finite multiplicity), and $\sigma_{\mathrm{ess}}(L)$ the essential spectrum of L, which is the complement of $\sigma_{\mathrm{d}}(L)$ in $\sigma(L)$, the spectrum of L.

We give in § 2 a characterization of the Fučík spectrum. The result is not new, but recalled here in a form adapted to its use in the later sections. Let $I \subset \mathbb{R}$ be an open interval such that $I \cap \sigma(L) = \lambda \in \sigma_{\mathrm{d}}(L)$. Using a Lyapunov–Schmidt decomposition together with contraction mappings arguments, we show that the points of the Fučík spectrum within $I \times I$ can be seen as points where a real-valued function,

$$h_0(\cdot, \alpha, \beta) : \ker(L - \lambda I) \to \mathbb{R},$$

to be defined below, admits 0 as a critical value. This observation has already been made by Gonçalves and Magalhães [12], although under less general hypotheses. As a consequence, the sets

$$F^- = \left\{ (\alpha, \beta) \in I \times I \mid \min_{x \in \ker(L-\lambda I),\ \|x\|=1} h_0(x, \alpha, \beta) = 0 \right\},$$

$$F^+ = \left\{ (\alpha, \beta) \in I \times I \mid \max_{x \in \ker(L-\lambda I),\ \|x\|=1} h_0(x, \alpha, \beta) = 0 \right\}$$

are contained in the Fučík spectrum. In the case where $\dim \ker(L - \lambda I) = 1$, the spectrum (within $I \times I$) is easily seen to be reduced to these sets; this case has been studied by Gallouët and Kavian [11]. Results concerning the general case have been obtained by Gonçalves and Magalhães [12], Magalhães [15], Cac [3] and Schechter [21], through variational methods. Basically, all these works only pay attention to the sets F^+ and F^-.

For what concerns the structure of the Fučík spectrum between F^+ and F^-, examples of Margulies and Margulies [16] have shown that many curves in $\Sigma(L)$ can pass through the point (λ, λ). One of the main purposes of our paper is to provide a general and precise description of this part of the Fučík spectrum. Using a modified problem equivalent to (1.3) when $\alpha \neq \beta$, we start, in § 3, by describing the Fučík spectrum in the neighbourhood of the point (λ, λ). Roughly speaking, we show in theorem 3.1 that, close to (λ, λ), $\Sigma(L)$ is made of curves, each one being associated to a non-degenerate stationary point of the function

$$H : \ker(L - \lambda I) \to \mathbb{R} : x \mapsto \langle |x|, x \rangle$$

restricted to the unit sphere ($\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\Omega)$, the norm being denoted by $\| \cdot \|$). Notice that this local result does not require L to have a compact resolvent. Examples are given with several curves emanating from (λ, λ).

Section 4 is devoted to the global study of the Fučík spectrum. Let $J \subset \mathbb{R}$ be an interval such that $J \cap \sigma_{\mathrm{ess}}(L) = \emptyset$. Using a non-degeneracy condition introduced by Micheletti [18] and Pistoia [19] for elliptic equations, we show that the curves obtained in § 3 for some $\lambda \in J$ can be continued up to the boundary of $J \times J$. Moreover, we show that any element of $\Sigma(L)$ in $J \times J$ belongs to one of these curves. Hence, provided the non-degeneracy conditions are satisfied, the Fučík spectrum can be completely described within $J \times J$. A related result is also presented in § 7, which does not require this non-degeneracy condition.

Section 5 is a remark about situations where the component of the Fučík spectrum containing the point (λ, λ) is reduced to a single curve (at least within the square $I \times I$).

Existence conditions for solutions of the non-homogeneous equation (1.1) and of (1.2) are given in § 8. Results of this type are well known outside the regions between the curves F^- and F^+ (i.e. in type-I regions), where no other part of the Fučík spectrum is present. Some results were obtained by Schechter in [22] concerning the regions between F^- and F^+ (type-II regions), but the conditions he imposes on the forcing term are of a different nature with respect to ours. For instance, his conditions do not apply for (1.1) when f is non-trivial. Instead, our assumptions are similar to those for type-I regions, but apply only in some parts of the type-II regions. These existence conditions are based on results obtained in § 6 concerning topological degree computations. In the neighbourhood of (λ, λ), the computations eventually reduce to the study of the index of each critical point of the function H above, restricted to the unit sphere in $\ker(L-\lambda I)$. As a consequence, if $\dim\ker(L-\lambda I) = 2$, if (α, β) does not belong to the Fučík spectrum and is close to (λ, λ), we are able to show in corollary 8.2, that a solution of (1.1) exists for all $f \in L^2(\Omega)$, if the function

$$S^1 \subset \ker(L-\lambda I) \to \mathbb{R} : x \mapsto \tfrac{1}{2}(\alpha-\beta)H(x) + \lambda - \tfrac{1}{2}(\alpha+\beta)$$

has only simple zeros, the number of zeros being different from 2. This result is obtained by showing that the topological degree with respect to large balls is different from 0. Examples are provided showing that this can occur between some of the Fučík curves obtained by theorem 3.1. This contrasts with the more common situation of a degree 0 between Fučík curves emanating from the same point of the diagonal $\alpha = \beta$. We also study an example where $\dim\ker(L-\lambda I) = 3$ and as many as 14 curves arise from the point (λ, λ). In this last example, the index of the critical points of H restricted to the unit sphere (i.e. the topological index of the gradient field ∇H projected on the unit sphere), as well as the corresponding critical values, are computed.

Being away from the essential spectrum of L allows the use of fairly simple techniques for studying (1.1), (1.2) and (1.3). We use a Lyapunov–Schmidt decomposition, contraction mappings for the component of the equation in $\operatorname{Im}(L-\lambda I)$, and degree arguments for the component in $[\operatorname{Im}(L-\lambda I)]^\perp = \ker(L-\lambda I)$. For the local description of $\Sigma(L)$ in theorems 3.1 and 4.2, we use an appropriate version of the implicit function theorem given in the appendix.

Notice that the results dealing with the restriction of the function H above to the unit sphere in $\ker(L-\lambda I)$ are presented in such a way that $\dim\ker(L-\lambda I) \geqslant 2$ is implicitly assumed. The corresponding statements when $\dim\ker(L-\lambda I) = 1$ are straightforward, and not presented.

2. Reduction to an equivalent problem

Let $J \subset \mathbb{R}$ an open bounded interval such that $J \cap \sigma(L) = \{\lambda_1, \dots, \lambda_p\} \subset \sigma_{\mathrm{d}}(L)$. We will denote by P_i the orthogonal projector onto $\ker(L-\lambda_i I)$ $(1 \leqslant i \leqslant p)$ and by P the projector onto $X := \bigoplus_{i=1}^p \ker(L-\lambda_i I)$. Using a Lyapunov–Schmidt decomposition, we obtain the following lemma.

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LEMMA 2.1. *For any $\alpha, \beta \in J$, $f \in L^2(\Omega)$ and $x \in X$, the problem*

$$Lu = \sum_{i=1}^{p} \lambda_i P_i x + (I - P)[\alpha u^+ - \beta u^- + f], \tag{2.1}$$

$$Pu = x \tag{2.2}$$

has a unique solution $u_x = u_x(f, \alpha, \beta)$. Moreover, the solution $u_x(f, \alpha, \beta)$ is locally Lipschitzian with respect to x, f, α, β.

Proof. Let $\mu = \frac{1}{2}(\alpha + \beta)$. The operator

$$L_\mu = (L - \mu I)_{|X^\perp} : X^\perp \to X^\perp \tag{2.3}$$

is invertible and $\|L_\mu^{-1}\|^{-1} = \operatorname{dist}(\mu, \sigma(L) \setminus \{\lambda_1, \dots, \lambda_p\})$. Equations (2.1) and (2.2) are equivalent to

$$u = x + L_\mu^{-1}[(I - P)(\alpha u^+ - \beta u^- - \mu u + f)]. \tag{2.4}$$

The function

$$u \mapsto \alpha u^+ - \beta u^- - \mu u + f = (\alpha - \mu)u^+ - (\beta - \mu)u^- + f = \tfrac{1}{2}(\alpha - \beta)|u| + f,$$

is Lipschitzian with constant $k = \frac{1}{2}|\beta - \alpha|$. Since $k\|L_\mu^{-1}\| < 1$, the right-hand side of (2.4) is a contraction mapping, so that (2.4) has a unique solution $u_x \in L^2(\Omega)$. The fact that $u_x = u_x(f, \alpha, \beta)$ is Lipschitzian with respect to x, f, α, β follows from standard arguments about fixed points of contraction mappings. □

Let $u_x = u_x(f, \alpha, \beta)$ be the solution of (2.1), (2.2). Define

$$\begin{aligned} c(x, f, \alpha, \beta) &= -P[\alpha u_x^+ - \beta u_x^- + f] + \sum_{i=1}^{p} \lambda_i P_i x \\ &= -\tfrac{1}{2}(\alpha - \beta)P(|u_x|) - P(f) + \sum_{i=1}^{p} (\lambda_i - \mu) P_i x, \end{aligned}$$

so that u_x verifies

$$Lu_x = \alpha u_x^+ - \beta u_x^- + f + c(x, f, \alpha, \beta). \tag{2.5}$$

Equation (1.1) admits a solution if and only if there exists $x \in X$ such that $c(x, f, \alpha, \beta) = 0$. The problem is therefore reduced to a problem in a finite-dimensional space. Notice that $c(rx, rf, \alpha, \beta) = rc(x, f, \alpha, \beta)$ for all $r \geqslant 0$. The existence of the solution u_x can also be obtained through variational techniques (see [3, 11, 12, 15]).

We now turn our attention to the homogeneous equation

$$Lu = \alpha u^+ - \beta u^-. \tag{2.6}$$

The values of (α, β) for which (2.6) has a non-trivial solution form the Fučík spectrum of L. Results about this spectrum can be found in the papers of Gonçalves and Magalhães [12], Magalhães [15], Cac [3] and Schechter [21]. We will study in more detail the structure of the spectrum in the square $J \times J$. We will write $c_0(x, \alpha, \beta)$ for $c(x, 0, \alpha, \beta)$. Hence we have

$$c_0(x, \alpha, \beta) = -P[\alpha u_x^+ - \beta u_x^-] + \sum_{i=1}^{p} \lambda_i P_i x = -\tfrac{1}{2}(\alpha - \beta)P(|u_x|) + \sum_{i=1}^{p} (\lambda_i - \mu)P_i x,$$

where $u_x = u_x(0, \alpha, \beta)$ is given by lemma 2.1.

Such a function c_0 has been introduced by Gallouët and Kavian [10, 11], for the case $p = 1$, with a one-dimensional eigenspace for λ_1. The point (α, β) belongs to the Fučík spectrum if and only if $c_0(x, \alpha, \beta) = 0$ for some $x \neq 0$. The functional

$$h_0 : X \times J \times J \to \mathbb{R} : (x, \alpha, \beta) \mapsto \langle c_0(x, \alpha, \beta), x\rangle$$

will be useful in the sequel. Notice that

$$\begin{aligned} h_0(x, \alpha, \beta) &= \langle c_0(x, \alpha, \beta), x\rangle \\ &= \langle c_0(x, \alpha, \beta), u_x\rangle \\ &= \langle Lu_x, u_x\rangle - \alpha\|u_x^+\|^2 - \beta\|u_x^-\|^2. \end{aligned}$$

This last formula relates h_0 to the energy functional

$$H \to \mathbb{R} : u \mapsto \tfrac{1}{2}\langle Lu, u\rangle - \tfrac{1}{2}\alpha\|u^+\|^2 - \tfrac{1}{2}\beta\|u^-\|^2$$

associated to (2.6), which has been used in [12, 15, 19, 21], in particular when $L = -\Delta$. We present a few properties of the functions c_0 and h_0, starting with an obvious observation.

LEMMA 2.2. *If $\alpha = \beta$, then*

$$c_0(x, \alpha, \beta) = \sum_{i=1}^{p} (\lambda_i - \alpha)P_i x.$$

LEMMA 2.3. *The function h_0 admits partial derivatives with respect to $\alpha, \beta \in J$, is differentiable with respect to $x \in X$ and*

$$\text{(i)} \qquad \frac{\partial}{\partial\alpha} h_0(x, \alpha, \beta) = -\|u_x^+\|^2, \qquad \frac{\partial}{\partial\beta} h_0(x, \alpha, \beta) = -\|u_x^-\|^2,$$

$$\text{(ii)} \qquad \nabla_x h_0(x, \alpha, \beta) = 2c_0(x, \alpha, \beta). \tag{2.7}$$

For the sake of completeness, we provide a proof of lemma 2.3, although the same results can be found in [12, 15], with a different method of proof.

Proof. For (i), we prove, for instance, the first relation. Considering the solutions u_x, v_x corresponding to two different sets (α, β), (α', β) of coefficients, we can write

$$\begin{aligned} Lu_x &= \alpha u_x^+ - \beta u_x^- + c_0(x, \alpha, \beta), \\ Lv_x &= \alpha' v_x^+ - \beta v_x^- + c_0(x, \alpha', \beta). \end{aligned}$$

Multiplying the above equations, respectively, by v_x and u_x and subtracting, we obtain, since L is self-adjoint,

$$\begin{aligned} (\alpha - \alpha')\langle u_x^+, v_x^+\rangle - (\alpha - \beta)\langle u_x^+, v_x^-\rangle + (\alpha' - \beta)\langle u_x^-, v_x^+\rangle \\ + \langle c_0(x, \alpha, \beta) - c_0(x, \alpha', \beta), x\rangle = 0. \end{aligned} \tag{2.8}$$

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But

$$|\langle u_x^+, v_x^- \rangle| \leqslant -\int_{u_x v_x < 0} u_x v_x \leqslant \frac{1}{4} \int_{u_x v_x < 0} [u_x - v_x]^2 \leqslant \tfrac{1}{4} \|u_x - v_x\|^2. \tag{2.9}$$

Since $u_x = u_x(0, \alpha, \beta)$ is Lipschitzian with respect to α, there exists $K > 0$ such that

$$|\langle u_x^+, v_x^- \rangle| \leqslant K|\alpha - \alpha'|^2.$$

A similar result holds for $\langle u_x^-, v_x^+ \rangle$. Dividing (2.8) by $\alpha - \alpha'$ and letting α' tend to α, we obtain

$$\frac{\partial}{\partial \alpha} \langle c_0(x, \alpha, \beta), x \rangle = -\|u_x^+\|^2.$$

For (ii), let u_x and u_y be solutions given by lemma 2.1, respectively, for x and for y in X. We thus have

$$Lu_x = \alpha u_x^+ - \beta u_x^- + c_0(x, \alpha, \beta),$$
$$Lu_y = \alpha u_y^+ - \beta u_y^- + c_0(y, \alpha, \beta).$$

Multiplying the above equations, respectively, by u_y and by u_x, and working as above, it is easy to prove that

$$\langle c_0(x, \alpha, \beta) + c_0(y, \alpha, \beta), x - y \rangle = \langle c_0(x, \alpha, \beta), x \rangle - \langle c_0(y, \alpha, \beta), y \rangle + O(\|x - y\|^2),$$

or, since $c_0(x, \alpha, \beta)$ is Lipschitzian with respect to x,

$$2\langle c_0(x, \alpha, \beta), x - y \rangle = \langle c_0(x, \alpha, \beta), x \rangle - \langle c_0(y, \alpha, \beta), y \rangle + O(\|x - y\|^2). \tag{2.10}$$

This shows that the function $h_0(\cdot, \alpha, \beta) : x \mapsto \langle c_0(x, \alpha, \beta), x \rangle$ is differentiable and that its gradient is given by (2.7). □

Since (α, β) belongs to the Fučík spectrum if and only if $c_0(x, \alpha, \beta) = 0$ for some $x \neq 0$, the following theorem, which provides a characterization of that spectrum within $J \times J$, is an immediate consequence of the previous lemma (see [15]).

Theorem 2.4. *Let $L : \operatorname{dom} L \subset L^2(\Omega) \to L^2(\Omega)$ be self-adjoint and let $J \subset \mathbb{R}$ be such that $J \cap \sigma_{\mathrm{ess}}(L) = \emptyset$. Then the point $(\alpha, \beta) \in J \times J$ belongs to the Fučík spectrum of L if and only if 0 is a critical value of the function*

$$h_0(\cdot, \alpha, \beta) : x \mapsto \langle c_0(x, \alpha, \beta), x \rangle;$$

this critical value being reached at some point $x \neq 0$.

Let us now take for J a smaller interval, say I, such that $I \cap \sigma(L) = \lambda \in \sigma_{\mathrm{d}}(L)$. Theorem 2.4 can be used directly to characterize parts of the Fučík spectrum, which can be considered, in a certain sense, as the outermost parts of that spectrum within the square $I \times I$. Let us introduce the sets

$$F^- = \Big\{ (\alpha, \beta) \in I \times I \mid \min_{x \in \ker(L - \lambda I),\ \|x\| = 1} \langle c_0(x, \alpha, \beta), x \rangle = 0 \Big\},$$

$$F^+ = \Big\{ (\alpha, \beta) \in I \times I \mid \max_{x \in \ker(L - \lambda I),\ \|x\| = 1} \langle c_0(x, \alpha, \beta), x \rangle = 0 \Big\}.$$

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It results from lemma 2.3 and theorem 2.4 that F^-, F^+ are contained in the Fučík spectrum of L. On the other hand, by lemma 2.2, if $\alpha = \beta < \lambda$, we have

$$\langle c_0(x, \alpha, \beta), x\rangle > 0 \quad \text{for all } x \neq 0,$$

whereas, if $\alpha = \beta > \lambda$,

$$\langle c_0(x, \alpha, \beta), x\rangle < 0 \quad \text{for all } x \neq 0.$$

Consequently, the sets F^-, F^+ are non-empty and separate the sets

$$\{(\alpha, \alpha) \in I \times I \mid \alpha < \lambda\} \quad \text{and} \quad \{(\alpha, \alpha) \in I \times I \mid \alpha > \lambda\}.$$

On the other hand, because of lemma 2.3,

$$(\alpha, \beta) \in F^- \quad \Rightarrow \quad (\alpha', \beta') \notin \Sigma(L) \cap (I \times I) \quad \text{if } \alpha' < \alpha,\ \beta' < \beta.$$

A similar result holds for F^+. It is in this sense that F^- and F^+ are the outermost parts of the Fučík spectrum. The same sets have been obtained through a variational approach by Gonçalves and Magalhães [12], Magalhães [15], Cac [3] and Schechter [21], for semilinear elliptic boundary-value problems.

In some problems, it can happen that, for some subspace S of $\ker(L - \lambda I)$, the following hypothesis holds:

$$s \in S \quad \Rightarrow \quad c_0(s, \alpha, \beta) \in S \quad \text{for all } \alpha, \beta \in I. \tag{H}$$

Adapting the arguments of lemma 2.3 and theorem 2.4, it is easy to see that the sets

$$F_S^- = \Big\{(\alpha, \beta) \in I \times I \mid \min_{x \in S,\ \|x\|=1} \langle c_0(x, \alpha, \beta), x\rangle = 0\Big\},$$
$$F_S^+ = \Big\{(\alpha, \beta) \in I \times I \mid \max_{x \in S,\ \|x\|=1} \langle c_0(x, \alpha, \beta), x\rangle = 0\Big\}$$

then also belong to the Fučík spectrum. Because of lemma 2.3, F_S^- must be on the right of (or coincide with) F^-, and F_S^+ on the left of (or coincide with) F^+, in the (α, β)-plane. For the partial differential equation with Laplacian in example 3.2 below, where $\Omega = (0, \pi) \times (0, \pi/\sqrt{6})$, one could take, for instance,

$$S = \{u \in L^2(\Omega) \mid u(x, y) = u(x, \pi/\sqrt{6} - y)\}.$$

It can be seen that the curves F_S^+, F_S^- are distinct from F^+, F^-.

3. The Fučík spectrum close to (λ, λ)

In this section, I is an open bounded interval such that $I \cap \sigma(L) = \lambda \in \sigma_{\mathrm{d}}(L)$. The projection onto $\ker(L - \lambda I)$ is denoted by P. We will construct hereafter some curves belonging to the Fučík spectrum within $I \times I$. This result follows from the application of an implicit function theorem to a system equivalent to

$$Lu = \alpha u^+ - \beta u^-, \tag{3.1}$$
$$\|u\|^2 = 1. \tag{3.2}$$

Since we are interested in values of $(\alpha,\beta)\in\Sigma(L)$ close to (λ,λ), we let

$$\varepsilon=\tfrac{1}{2}(\alpha-\beta),\qquad \tfrac{1}{2}(\alpha+\beta)=\lambda+\varepsilon\eta,$$

and we aim at determining η as a function of ε, for ε 'small'. The system (3.1), (3.2) can be rewritten as

$$Lu=\varepsilon|u|+(\lambda+\varepsilon\eta)u,\qquad \|u\|^2=1.$$

For $\varepsilon\neq 0$, it is equivalent to

$$u=Pu+\varepsilon L_\lambda^{-1}[(I-P)(|u|+\eta u)]+P(|u|+\eta u), \tag{3.3}$$

$$\|u\|^2=1. \tag{3.4}$$

We want to solve this system for u, η as functions of ε, for ε close to 0. A difficulty lies in the fact that the set of points at which the term $P(|u|)$ is differentiable need not be open in $L^2(\Omega)$. For this reason, we will need a version of the implicit function theorem that only requires a (strong) Fréchet differentiability at one point; such a version is presented in the appendix.

For $\varepsilon=0$, the system (3.3), (3.4) reduces to

$$P(|u|+\eta u)=0,\quad u\in\ker(L-\lambda I),$$
$$\|u\|^2=1.$$

Let $(x_0,\eta_0)\in\ker(L-\lambda I)\times\mathbb{R}$ denote a solution of this system. The condition $P(|x_0|)+\eta_0x_0=0$ can be interpreted as expressing the fact that x_0 is a stationary point for the mapping $H:\ker(L-\lambda I)\to\mathbb{R}:x\mapsto\langle|x|,x\rangle$ restricted to the unit sphere, η_0 being then a Lagrange multiplier (the value of η_0 is given by $\eta_0=-\langle|x_0|,x_0\rangle$). On the other hand, provided that $u\neq 0$ almost everywhere in Ω, for all $u\in\ker(L-\lambda I)\setminus\{0\}$, it can be seen that the differentiability condition of the appendix is satisfied for $(\varepsilon,\eta,x)=(0,\eta_0,x_0)$. Indeed, $P(|u|)$ is the only term in (3.3) that may be problematic. Due to the projection P onto the finite-dimensional space $\ker(L-\lambda I)$, this is not the case. It remains to introduce an invertibility condition for the derivative; this non-degeneracy condition can be written

$$y\in\ker(L-\lambda I),\quad \langle x_0,y\rangle=0,\quad P(\operatorname{sgn}(x_0)y)=\langle x_0,|x_0|\rangle y\quad\Rightarrow\quad y=0 \tag{ND1}$$

(notice that $y\in\ker(L-\lambda I)$, $\langle x_0,y\rangle=0$, $P(|x_0|)+\eta_0x_0=0$ imply $\langle|x_0|,y\rangle=0$). The application of the implicit function theorem yields the following result.

THEOREM 3.1. *Let $L:\operatorname{dom}L\subset L^2(\Omega)\to L^2(\Omega)$ be self-adjoint and let $\lambda\in\sigma_{\mathrm{d}}(L)$. Assume that $u\neq 0$ almost everywhere in Ω, for all $u\in\ker(L-\lambda I)\setminus\{0\}$, that the function*

$$H:\ker(L-\lambda I)\to\mathbb{R}:x\mapsto\langle|x|,x\rangle,$$

restricted to the unit sphere, has a stationary point x_0 and that condition (ND1) is satisfied at x_0. Then there are continuous functions $\eta(\cdot),u(\cdot)$, defined in a neighbourhood $\mathcal{E}$ of 0, such that

(i) $u(0)=x_0,\eta(0)=-\langle|x_0|,x_0\rangle$,

(ii) $Lu(\varepsilon)=\varepsilon|u(\varepsilon)|+(\lambda+\varepsilon\eta(\varepsilon))u(\varepsilon)$, $\|u(\varepsilon)\|=1$ *for* $\varepsilon\in\mathcal{E}$.

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Since $\eta(0) = -H(x_0)$, the above theorem means that there is a curve in the Fučík spectrum emanating from the point (λ, λ), with slope

$$(H(x_0) + 1)/(H(x_0) - 1). \tag{3.5}$$

It is obvious that $|H(x_0)| \leqslant 1$; the slopes, as expected, are negative. On the other hand, since the function H is odd, its extrema can be grouped by pairs of extrema of opposite signs. Those pairs correspond to Fučík curves which are symmetric with respect to the line $\alpha = \beta$ (if u is a solution of (2.6), $-u$ is a solution of $Lu = \beta u^+ - \alpha u^-$). The particular case $|H(x_0)| = 1$ occurs when (2.6) admits a solution of constant sign; the lines $\alpha = \lambda$, $\beta = \lambda$ then belong to the Fučík spectrum. However, in this case, the non-degeneracy condition (ND1) will not be satisfied.

Notice that when $\dim\ker(L - \lambda I) = 1$, condition (ND1) is trivially satisfied, but theorem 3.1, reformulated according to the remark in the introduction, brings nothing more than a local description of the sets F^+ and F^- introduced above.

The non-degeneracy condition (ND1) could be replaced by a different one (which is more general for extremum points, but excludes saddle points). This alternative condition requires H to have a 'true' (local) maximum or minimum on the unit sphere S^{n-1} in $\ker(L - \lambda I)$, at the point x_0, meaning, for a maximum, that

$$\left.\begin{array}{l}\text{there exists a neighbourhood } U \subset S^{n-1}, \text{ of } x_0, \text{ such that}\\ \max\{H(x) \mid x \in U\} = H(x_0) \text{ and } H(x) < H(x_0), \text{ for all } x \in \partial U.\end{array}\right\} \tag{ND1$'$}$$

A true minimum is defined similarly. In these cases, the stability of true minima (respectively, true maxima) under perturbation is used instead of an implicit function theorem (see [1]).

If condition (ND1) is satisfied at any $x_0 \in S^{n-1}$, it results from the above theorem that to each stationary point of H restricted to the unit sphere corresponds a Fučík curve emanating from the point (λ, λ). Notice that, with (ND1) satisfied at any $x_0 \in S^{n-1}$, the stationary points of H are necessarily isolated on S^{n-1}, meaning that the Fučík spectrum in the neighbourhood of (λ, λ) consists of a finite number of curves.

The above result is illustrated by examples which deal with situations where $\ker(L - \lambda I)$ is of dimension 2 or 3. In the first case, $\{v^{(1)}, v^{(2)}\}$ will denote an orthonormal basis of $\ker(L - \lambda I)$. It will be convenient to use polar coordinates in $\ker(L - \lambda I)$. We define

$$z_\theta = \cos\theta v^{(1)} + \sin\theta v^{(2)}$$

and introduce the function $h : [0, 2\pi] \to \mathbb{R} : \theta \mapsto H(z_\theta) = \langle |z_\theta|, z_\theta\rangle$. Condition (ND1) is not easy to verify in practice; when $\dim\ker(L - \lambda I) = 2$, it is easier to rely on the alternative non-degeneracy condition (ND1$'$), based on the existence of 'true' maxima or minima. For the third example, condition (ND1) was only checked numerically, by computation of the Hessian along sequences converging to the critical points.

Example 3.2. The following example is inspired by Margulies and Margulies [16]. Consider an equation with Laplacian and an asymmetric nonlinearity, together with

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Dirichlet boundary conditions, i.e.

$$-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=\alpha u^+-\beta u^-, \tag{3.6}$$

$$u_{|\partial\Omega}=0. \tag{3.7}$$

We will take for Ω the rectangle $(0,\pi)\times(0,\pi/\sqrt{6})$. With

$$L=-\Delta, \qquad \operatorname{dom} L=H^2(\Omega)\cap H_0^1(\Omega),$$

the operator L admits an eigenvalue $\lambda=55$ of multiplicity 2, the eigenspace being spanned by the functions $\sin x\sin(3y\sqrt{6})$ and $\sin(7x)\sin(y\sqrt{6})$. The eigenvalues closest to 55 are 49 and 58. It is easy to check that (3.6), (3.7) admit solutions of the form $u(x,y)=\sin xp(y)$, with p verifying

$$p''+(\alpha-1)p^+-(\beta-1)p^-=0, \tag{3.8}$$

$$p(0)=p(\pi\sqrt{6})=0. \tag{3.9}$$

By classical results for the Dirichlet problem for ordinary differential equations with asymmetric nonlinearities (see, for instance, Fučík and Kufner [9]), non-trivial solutions exist if

$$\frac{2}{\sqrt{\alpha-1}}+\frac{1}{\sqrt{\beta-1}}=\frac{1}{\sqrt{6}}, \tag{3.10}$$

or if

$$\frac{1}{\sqrt{\alpha-1}}+\frac{2}{\sqrt{\beta-1}}=\frac{1}{\sqrt{6}}. \tag{3.11}$$

Similarly, the system (3.6), (3.7) also admits solutions of the form

$$u(x,y)=q(x)\sin(y\sqrt{6})$$

if

$$\frac{4}{\sqrt{\alpha-6}}+\frac{3}{\sqrt{\beta-6}}=1, \tag{3.12}$$

or if

$$\frac{3}{\sqrt{\alpha-6}}+\frac{4}{\sqrt{\beta-6}}=1. \tag{3.13}$$

Consequently, the four curves defined by (3.10)–(3.13), which all pass through the point $(55,55)$, belong to the Fučík spectrum of $-\Delta$, with Dirichlet boundary conditions. Using the results of theorem 3.1, we will show that two more curves passing through the point $(55,55)$ also belong to the Fučík spectrum. For this purpose, we have to search the maxima and minima of the function h above. Since

$$\ker(L-\lambda I)=\operatorname{span}\{\sin x\sin(3y\sqrt{6}),\sin(7x)\sin(y\sqrt{6})\},$$

Table 1.

extrema of h at	values of the extrema of h	slopes of the Fučík curves
0	$1/3$	-0.5
$\pi/2$	$1/7$	$-3/4$
1.1381	0.1316	-0.7674
4.2797	-0.1316	-1.3032
$3\pi/2$	$-1/7$	$-4/3$
π	$-1/3$	-2

we have to compute values of

$$\begin{aligned}h(\theta) &= \langle |z_\theta|, z_\theta\rangle \\ &= \frac{4\sqrt{6}}{\pi^2}\int_0^{\pi/\sqrt{6}}\int_0^{\pi} |\cos\theta \sin x \sin(3y\sqrt{6}) + \sin\theta \sin(7x)\sin(y\sqrt{6})| \\ &\qquad\qquad \times [\cos\theta \sin x \sin(3y\sqrt{6}) + \sin\theta \sin(7x)\sin(y\sqrt{6})]\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Numerical computations provide the values given in table 1, for the extrema of h on $[0, 2\pi]$, with the indication of the points where the extrema are obtained. From these values, using formula (3.5), the slopes of the Fučík curves emanating from the point $(55, 55)$ are deduced.

The first and last extrema correspond to solutions of the form $\sin xp(y)$, i.e. to the curves defined by (3.10), (3.11), whereas the second and fifth extrema correspond to solutions of the form $q(x)\ \sin(y\sqrt{6})$, i.e. to the curves given by (3.12), (3.13). The other two extrema lead to supplementary Fučík curves, with respect to those obtained by Margulies and Margulies [16]; as can be seen from the values of the slopes, they are, however, very close to the curves corresponding to solutions of the form $q(x)\sin(y\sqrt{6})$.

Example 3.3. As a second example, we consider a boundary-value problem for an ordinary differential equation of order four,

$$u^{(4)} + (m^2+n^2)u'' = \alpha u^+ - \beta u^-, \tag{3.14}$$

$$u(0) = u(\pi) = 0, \qquad u''(0) = u''(\pi) = 0. \tag{3.15}$$

We assume that m, n are integers with $m \neq n$, so that $\lambda = -m^2n^2$ is an eigenvalue of multiplicity 2 for the operator

$$L : \operatorname{dom} L \subset L^2(0,\pi) \to L^2(0,\pi) : u \mapsto u^{(4)} + (m^2+n^2)u'';$$

we take $\operatorname{dom} L = H^4(0,\pi)$ (the real Sobolev space of order 4). The eigenspace associated to λ is spanned by the functions $\sin mx$, $\sin nx$. According to theorem 3.1, the Fučík curves for (3.14), (3.15), passing through the point $(-m^2n^2, -m^2n^2)$, can be related to the extrema of the function $h : [0, 2\pi] \to \mathbb{R}$,

$$\theta \mapsto \frac{2}{\pi}\int_0^{\pi} |\cos\theta \sin mx + \sin\theta \sin nx|(\cos\theta \sin mx + \sin\theta \sin nx)\,\mathrm{d}x.$$

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Table 2.

extremal values	slopes
$\pm 1/15$	$-7/8$ and $-8/7$
± 0.0437	-0.9163 and -1.0914
± 0.0348	-0.9327 and -1.0721

Choosing, for instance, $m = 15$, $n = 22$, it is observed that h has six extremal values. Table 2 gives the extrema of h. From these values, using (3.5), the slopes of the Fučík curves emanating from the point $(108\,900, 108\,900)$ are obtained.

Notice that, in this problem, some of the extremal values are obtained at two different points, since $h(\theta) = h(2\pi - \theta)$.

Example 3.4. We now consider another equation with an elliptic operator and an asymmetric nonlinearity, together with Dirichlet boundary conditions, i.e.

$$-\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y^2} = \alpha u^+ - \beta u^-, \tag{3.16}$$

$$u_{|\partial\Omega} = 0. \tag{3.17}$$

We will take for Ω the square $(0,\pi) \times (0,\pi)$. With

$$L = -\partial_x^2 - 2\partial_y^2, \qquad \operatorname{dom} L = H^2(\Omega) \cap H_0^1(\Omega),$$

the operator L has a purely discrete spectrum. It admits an eigenvalue $\lambda = 99$ of multiplicity 3, the eigenspace being spanned by the functions

$$\sin x \sin(7y), \quad \sin(7x)\sin(5y) \quad \text{and} \quad \sin(9x)\sin(3y).$$

Using spherical coordinates on $S^2 \subset \ker(L - \lambda I)$, we obtain, by numerical calculations, 14 critical points. Two of the curves can be explicitly calculated; they correspond to solutions of the form $u(x,y) = \sin(x)v(y)$ and are represented by the equations

$$\frac{4}{\sqrt{\frac{1}{2}(\alpha-1)}} + \frac{3}{\sqrt{\frac{1}{2}(\beta-1)}} = 1 \quad \text{and} \quad \frac{3}{\sqrt{\frac{1}{2}(\alpha-1)}} + \frac{4}{\sqrt{\frac{1}{2}(\beta-1)}} = 1.$$

Table 3 gives the critical values of H and the deduced slopes of the Fučík curves emanating from the point $(99, 99)$.

4. Global structure of the Fučík spectrum: non-degenerate case

Hereafter, we extend the Fučík curves constructed in the previous section and also describe the local structure of the Fučík spectrum away from the diagonal. As before, J is an open bounded interval such that $J \cap \sigma_{\text{ess}}(L) = \emptyset$. These continuation results require a non-degeneracy condition introduced by Micheletti [18] and Pistoia [19] for elliptic equations. If this condition is satisfied at $(\alpha,\beta) \in J \times J$, then locally near (α,β) the spectrum is the union of a finite number of curves.

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Table 3.

critical values	slopes
± 0.168	-0.712 and -1.404
± 0.156	-0.731 and -1.369
± 0.148	-0.742 and -1.348
± 0.143	$-3/4$ and $-4/3$
± 0.077	-0.858 and -1.166
± 0.054	-0.898 and -1.114
± 0.0018	-0.996 and -1.004

This result follows again from the application of an implicit function theorem to a system equivalent to

$$Lu = \alpha u^+ - \beta u^-, \tag{4.1}$$

$$\|u\|^2 = 1. \tag{4.2}$$

The application of the implicit function theorem relies on the possibility of defining derivatives for the functions $u \mapsto u^+$ and $u \mapsto u^-$, in the sense of the following result of Solimini [23], in which $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$, $\Omega \subset \mathbb{R}^N$ being, as before, an open bounded set.

PROPOSITION 4.1. *If $1 < q < p$ and if $u \in L^p(\Omega)$ is such that $u \neq 0$ almost everywhere on Ω, given $\varepsilon > 0$, there exists a neighbourhood $U \subset L^p(\Omega)$ of u such that*

$$\|u_1^+ - u_2^+ - \chi_{\{u>0\}}(u_1 - u_2)\|_q \leqslant \varepsilon \|u_1 - u_2\|_p \quad \textit{for all } u_1, u_2 \in U.$$

In other words, if $1 < q < p$ and if $u \neq 0$ almost everywhere, the function $u \mapsto u^+$ has a strong Fréchet derivative at u, as a function from $L^p(\Omega)$ to $L^q(\Omega)$. This result will be used below, with $q = 2$ and $p > 2$. As already noted in the previous section, a difficulty lies in the fact that the set of points $u \in L^p(\Omega)$, at which derivatives are guaranteed to exist by proposition 4.1, will not be open. Therefore, a version of the implicit function theorem must be used which requires only the existence of a (strong) Fréchet derivative at one point. We will use the one presented in the appendix.

The non-degeneracy condition, where $(\alpha_0, \beta_0) \in \Sigma(L)$, can be written as follows:

$$\left.\begin{array}{l} \text{for any } u \neq 0 \text{ verifying } Lu = \alpha_0 u^+ - \beta_0 u^-, \\ \text{we have } u \neq 0 \text{ almost everywhere on } \Omega \text{ and} \\ \dim\ker(L - (\alpha_0 \chi_{\{u>0\}} + \beta_0 \chi_{\{u<0\}})I) = 1. \end{array}\right\} \tag{ND2}$$

Here, $\chi_{\{u>0\}}$ is the characteristic function of the set $\{t \in \Omega \mid u(t) > 0\}$, and $\chi_{\{u<0\}}$ is defined similarly. With condition (ND2), an implicit function theorem can be used to describe locally the structure of the Fučík spectrum.

THEOREM 4.2. *Let $L : \operatorname{dom} L \subset L^2(\Omega) \to L^2(\Omega)$ be self-adjoint, $J \subset \mathbb{R}$ be such that $J \cap \sigma_{\mathrm{ess}}(L) = \emptyset$ and $\alpha_0 \neq \beta_0$ be such that $(\alpha_0, \beta_0) \in (J \times J) \cap \Sigma(L)$. Assume that the equation $Lu = \lambda u$ has no solution of constant sign for $\lambda \in J$, that the non-degeneracy condition (ND2) holds and that $\operatorname{dom} L \subset L^p(\Omega)$ for some $p > 2$, the* *injection being continuous when* $\operatorname{dom} L$ *is equipped with the graph norm. Then, if* u_0 *is a solution of* $Lu_0 = \alpha_0 u_0^+ - \beta_0 u_0^-$, *with* $\|u_0\| = 1$, *there exists neighbourhoods* A *of* α_0, B *of* β_0, U *of* u_0 *and continuous functions* $\beta(\cdot) : A \to B : \alpha \mapsto \beta(\alpha)$, $u(\cdot) : A \to U : \alpha \mapsto u(\alpha)$ *such that*

(i) $\beta(\alpha_0) = \beta_0$, $u(\alpha_0) = u_0$,

(ii) $Lu(\alpha) = \alpha u^+(\alpha) - \beta(\alpha) u^-(\alpha)$, $\|u(\alpha)\| = 1$ *for* $\alpha \in A$,

(iii) $Lu = \alpha u^+ - \beta u^-$, $\|u\| = 1$, *with* $u \in U$, $\alpha \in A$, $\beta \in B \;\Rightarrow\; u = u(\alpha)$, $\beta = \beta(\alpha)$.

Moreover, $\beta(\cdot)$ *is differentiable at* α_0 *and*

$$\beta'(\alpha_0) = -\frac{\|u_0^+\|^2}{\|u_0^-\|^2}. \tag{4.3}$$

Proof. As indicated above, the idea of the proof is to apply the implicit function theorem of the appendix to a system equivalent to (4.1), (4.2). Since

$$Lu_0 = \alpha_0 u_0^+ - \beta_0 u_0^- = (\alpha_0 \chi_{\{u_0>0\}} + \beta_0 \chi_{\{u_0<0\}}) u_0,$$

we have, by (ND2),

$$\ker(L - (\alpha_0 \chi_{\{u_0>0\}} + \beta_0 \chi_{\{u_0<0\}}) I) = \mathbb{R} u_0.$$

We will denote by Q the orthogonal projection on that set, that is, $Qu = \langle u, u_0 \rangle u_0$. Let

$$K = ([L - (\alpha_0 \chi_{\{u_0>0\}} + \beta_0 \chi_{\{u_0<0\}}) I]_{|u_0^\perp})^{-1}.$$

As the injection $\operatorname{dom} L \to L^p(\Omega)$ is continuous, the operator K, considered as an operator from $L^2(\Omega)$ to $L^p(\Omega)$, is continuous. The system (4.1), (4.2) can be rewritten as

$$u = T(u, \alpha, \beta), \tag{4.4}$$

$$\|u\|^2 = 1, \tag{4.5}$$

where

$$\begin{aligned} T(u, \alpha, \beta) = \langle u, u_0 \rangle u_0 + K(I - Q)[&\alpha u^+ - \alpha_0 u_0^+ - \alpha_0 \chi_{\{u_0>0\}}(u - u_0) \\ &- \beta u^- + \beta_0 u_0^- - \beta_0 \chi_{\{u_0<0\}}(u - u_0)] \\ &+ [\alpha \langle u^+, u_0 \rangle - \alpha_0 \langle u, u_0^+ \rangle - \beta \langle u^-, u_0 \rangle + \beta_0 \langle u, u_0^- \rangle] u_0. \end{aligned}$$

The implicit function theorem will be used to solve locally (4.4), (4.5). For this purpose, we note that $T : L^p(\Omega) \times J \times J \to L^p(\Omega)$ is Lipschitzian with respect to α on bounded subsets of $L^p(\Omega) \times J \times J \to L^p(\Omega)$, and that the partial derivatives $(\partial T/\partial u)(u_0, \alpha, \beta_0)$, $(\partial T/\partial \beta)(u_0, \alpha, \beta_0)$, given by

$$\begin{aligned} \frac{\partial T}{\partial u}(u_0, \alpha, \beta_0)\xi = \langle \xi, u_0 \rangle u_0 + (\alpha - \alpha_0) K(I - Q)(\chi_{\{u_0>0\}} \xi) \\ + (\alpha - \alpha_0) \langle \xi, u_0^+ \rangle, u_0, \end{aligned} \tag{4.6}$$

$$\frac{\partial T}{\partial \beta}(u_0, \alpha, \beta_0) = -K(I - Q) u_0^- - \|u_0^-\|^2 u_0, \tag{4.7}$$

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are such that, for $\varepsilon > 0$ given, there exists neighbourhoods A_1 of α_0, B_1 of β_0 and U_1 of u_0 such that, for $\alpha \in A_1$, $\beta, \beta' \in B_1$ and $u, u' \in U_1$,

$$\left\| T(u,\alpha,\beta) - T(u',\alpha,\beta') - \frac{\partial T}{\partial u}(u_0,\alpha,\beta_0)(u-u') - \frac{\partial T}{\partial \beta}(u_0,\alpha,\beta_0)(\beta-\beta') \right\| \leqslant \varepsilon[\|u-u'\| + |\beta-\beta'|].$$

This result follows from proposition 4.1. In other words, the differentiability condition of the appendix is satisfied and the implicit function theorem A.1 can be used. It remains to check the invertibility of the derivative. More precisely, we must show that, given $z \in L^p(\Omega)$, $t \in \mathbb{R}$, the system

$$\xi - \frac{\partial T}{\partial u}(u_0,\alpha_0,\beta_0)\xi - \frac{\partial T}{\partial \beta}(u_0,\alpha_0,\beta_0)s = z, \tag{4.8}$$

$$\langle \xi, u_0 \rangle = t \tag{4.9}$$

has a unique solution (ξ, s). First, multiplying (4.8) by u_0 gives

$$s\|u_0^-\|^2 = \langle u_0, z\rangle,$$

leading to the value of s. Indeed, $\|u_0^-\|$ is different from 0, since we have assumed that the equation $Lu = \lambda u$ has no solution of constant sign for $\lambda \in J$. From (4.8), we then deduce, using (4.6), (4.7),

$$(L - (\alpha_0\chi_{\{u_0>0\}} + \beta_0\chi_{\{u_0<0\}})I)(\xi - z) = -s(I-Q)u_0^-,$$

from which $(I-Q)(\xi - z)$ is uniquely determined. Since $Q\xi$ is obtained from (4.9), a unique ξ is deduced from (4.8), (4.9).

To prove (4.3), we observe that, for $\alpha \in A$, with $x(\alpha) = Pu(\alpha)$, we have

$$h_0(x(\alpha),\alpha,\beta(\alpha)) = 0.$$

Since $\nabla_x h_0(x,\alpha,\beta) = 2c_0(x,\alpha,\beta)$, c_0 being continuous and x Lipschitzian in α by theorem A.1, we can write, with $x_0 = x(\alpha_0) = Pu_0$,

$$\begin{aligned} h_0(x_0,\alpha,\beta(\alpha)) - h_0(x_0,\alpha_0,\beta_0) &= -[h_0(x(\alpha),\alpha,\beta(\alpha)) - h_0(x_0,\alpha,\beta(\alpha))] \\ &= 2c_0(x(\alpha),\alpha,\beta(\alpha))(x(\alpha) - x_0) + o(|\alpha-\alpha_0|) \end{aligned}$$

for $\alpha \to \alpha_0$. But $c_0(x(\alpha),\alpha,\beta(\alpha)) = 0$, so that

$$h_0(x_0,\alpha,\beta(\alpha)) - h_0(x_0,\alpha_0,\beta_0) = o(|\alpha-\alpha_0|) \quad \text{for } \alpha \to \alpha_0.$$

An application of the mean-value theorem then yields

$$\frac{\partial h_0}{\partial \alpha}(x_0,\alpha_0,\beta(\alpha))(\alpha-\alpha_0) + \frac{\partial h_0}{\partial \beta}(x_0,\alpha_0,\tilde\beta)(\beta(\alpha)-\beta_0) = o(|\alpha-\alpha_0|) \quad \text{for } \alpha \to \alpha_0,$$

where $\tilde\beta \in [\beta_0, \beta(\alpha)]$. Dividing by $\alpha - \alpha_0$ and passing to the limit, we see that $\beta(\cdot)$ is differentiable at α_0 and that

$$\beta'(\alpha_0) = -\frac{(\partial h_0/\partial\alpha)(x_0,\alpha_0,\beta_0)}{(\partial h_0/\partial\beta)(x_0,\alpha_0,\beta_0)} = -\frac{\|u_0^+\|^2}{\|u_0^-\|^2}.$$

□

In § 2, given $(\alpha, \beta) \in \Sigma(L) \cap (J \times J)$, the non-trivial solutions u of

$$Lu = \alpha u^+ - \beta u^- \tag{4.10}$$

have been put in relation with the non-trivial solutions of $c_0(x, \alpha, \beta) = 0$. Under the hypotheses of theorem 4.2, the (normalized) solutions of (4.10) are isolated in the sense of (iii). Consequently, the solutions of $c_0(x, \alpha, \beta) = 0$, $\|x\| = 1$ are isolated; the number of solutions (of norm 1, for instance) must therefore be finite. This explains why, near (α_0, β_0), the Fučík spectrum consists of a finite number of curves. If (ND2) holds for any $(\alpha_0, \beta_0) \in \Sigma(L) \cap (J \times J)$ (with $\alpha_0 \neq \beta_0$), it can be shown, using continuation arguments, that the Fučík curves whose existence is proved locally in theorem 4.2 can be extended up to the boundary of $J \times J$ on one side, and up to a point (λ, λ), where $\lambda \in \sigma(L) \cap J$, on the other side. The proof makes use of the fact that if (α_n, β_n) belongs to the Fučík spectrum $\Sigma(L)$ for $n = 1, 2, \dots$, and if $\alpha_n \to \alpha^*$, $\beta_n \to \beta^*$ for $n \to \infty$, then (α^*, β^*) also belongs to $\Sigma(L)$. Indeed, by hypothesis, there exists $u_n \neq 0$ such that $Lu_n = \alpha_n u_n^+ - \beta_n u_n^-$. Keeping the notation of § 2, if x_n is the projection of u_n on X, we have $c_0(x_n, \alpha_n, \beta_n) = 0$. Since by lemma 2.1, $x_n = 0$ implies $u_n = 0$, we can take, without loss of generality, $\|x_n\| = 1$. Extracting a subsequence of $\{x_n\}$ converging to some

$$x^* \in \bigoplus_{\lambda \in \sigma(L) \cap J} \ker(L - \lambda I), \qquad \|x\| = 1,$$

it follows that $Lu_{x^*} = \alpha^* u_{x^*}^+ - \beta^* u_{x^*}^-$, showing that (α^*, β^*) belongs to the Fučík spectrum. This argument, combined with theorem 4.2, leads to the following statement.

THEOREM 4.3. *Assume that, for all $\lambda \in \sigma(L) \cap J$, the hypotheses of theorem 3.1 are satisfied for each $x_0 \in \ker(L - \lambda I)$ such that $P(|x_0|)$ is parallel to x_0. Assume also that the hypotheses of theorem 4.2 are satisfied for each $(\alpha_0, \beta_0) \in \Sigma(L) \cap J \times J$ with $\alpha_0 \neq \beta_0$. Then the Fučík spectrum within $J \times J$ is made of a finite number of curves, each one extending to the boundary of $J \times J$ and passing through a point (λ, λ) for some $\lambda \in \sigma(L)$. These curves are graphs of decreasing functions $\alpha \mapsto \beta(\alpha)$. Moreover, any such a curve can be associated with a critical point of the function*

$$H : \ker(L - \lambda I) \to \mathbb{R}, \qquad x \mapsto \langle |x|, x \rangle,$$

restricted to the unit sphere. That is, the curves locally constructed in § 3 and extended here are the only ones within $J \times J$.

Proof. Only the last statement is not yet proved. Let $(\alpha_n, \beta_n) \in \Sigma(L)$ such that $(\alpha_n, \beta_n) \stackrel{\neq}{\to} (\lambda, \lambda)$ along one of the Fučík curves. Let u_n a solution of the corresponding Fučík equation and x_n its projection onto $\ker(L - \lambda I)$. As $x_n \neq 0$, we can assume that $\|x_n\| = 1$ and, going if necessary to a subsequence, that $x_n \to x^* \in \ker(L - \lambda I)$. Thus we have

$$0 = c_0(x_n, \alpha_n, \beta_n) = -\tfrac{1}{2}(\alpha_n - \beta_n) P(|u_{x_n}|) + (\lambda - \tfrac{1}{2}(\alpha_n + \beta_n)) x_n$$

and, with the notation of § 3,

$$P(|u_{x_n}|) + \eta_n x_n = 0.$$

By lemma 2.1, $u_{x_n} \to u_{x^*} = x^*$, so that $P(|x^*|) = -\eta x^*$ for some $\eta \in (-1,1)$ ($\eta = -1$ or $\eta = 1$ would imply that x^* is of constant sign, which is excluded by (ND1)). This shows that x^* is a critical point of the function H, restricted to the unit sphere, and ends the proof. □

If the operator L has a compact resolvent, the interval J can be taken arbitrarily large. Hence, provided that the non-degeneracy conditions (ND1), (ND2) are satisfied globally, the Fučík curves provided by theorem 4.3 can be extended indefinitely. Moreover, if the eigenfunctions associated to an eigenvalue λ_0 are of constant sign, it can be shown that the Fučík spectrum does not intersect the sets $(-\infty,\lambda_0)\times(\lambda_0,+\infty)$ and $(\lambda_0,+\infty)\times(-\infty,\lambda_0)$. Consequently, if $\lambda > \lambda_0$ is another eigenvalue of L, the Fučík curves issued from (λ,λ) must have asymptotes $\alpha = \alpha^*$, $\beta = \beta^*$, with $\alpha^*, \beta^* \in [\lambda_0, \lambda]$. It is actually possible to prove (but the proof will be omitted here) that if the slope of a Fučík curve is p at (λ^*, λ^*), the values of α^*, β^* must lie within the interval $[\lambda_0, (\underline{\lambda} + \lambda|p|)/(1+|p|)]$, where $\underline{\lambda}$ is the point of $\sigma(L)$ nearest to λ on the left. Similar statements hold for $\lambda < \lambda_0$.

5. Fučík spectrum in $I \times I$ reduced to a curve

Let I be an open bounded interval such that $I \cap \sigma(L) = \lambda \in \sigma_{\mathrm{d}}(L)$. The Fučík spectrum turns out to be particularly simple when the following hypothesis holds:

$$\left.\begin{array}{l}\text{for any } (\alpha,\beta)\in I\times I, \text{ if there exists}\\ x\in\ker(L-\lambda I) \text{ such that } c_0(x,\alpha,\beta)=0,\\ \text{then } c_0(x,\alpha,\beta)=0 \text{ for all } x\in\ker(L-\lambda I).\end{array}\right\} \qquad (\mathcal{H})$$

Under this hypothesis, the sets F^- and F^+ defined in § 2 coincide, and no other point of the Fučík spectrum is contained in $I \times I$.

Condition $(\mathcal{H})$ appears, under a different form, in [6]. It is shown there that the condition is satisfied for periodic boundary-value problems for ordinary differential equations, when the operator L is autonomous. More precisely, the following result is presented in [6].

Lemma 5.1. *Let $H = L^2(0,2\pi)$ and $L : \operatorname{dom} L \subset H \to H$ be a self-adjoint linear ordinary differential operator of order $2N$ with constant coefficients, where*

$$\operatorname{dom} L = \{u \in H^{2N}(0,2\pi) \mid u(0) = u(2\pi), \dots, u^{(2N-1)}(0) = u^{(2N-1)}(2\pi)\}.$$

If $\dim\ker(L-\lambda I) = 2$, then $(\mathcal{H})$ holds.

Another situation where condition $(\mathcal{H})$ holds is provided by the example $L = -\Delta$, $\Omega = B(0,1) \subset \mathbb{R}^2$, under Dirichlet boundary conditions. Denoting by z_{n_i} the ith zero of the Bessel function J_n, it is well known that the numbers $z_{n_i}^2$ are eigenvalues of L, of multiplicity 2, for $n \geqslant 1$. Using the symmetry of rotation, a family of solutions of (1.3) is deduced from a particular solution, meaning that $(\mathcal{H})$ is verified. Hence $F^- = F^+$, an observation already made by Magalhães [15].

6. Degree computations

In order to solve the inhomogeneous equation (1.1), and also (1.2), we want to compute, for fixed $(\alpha,\beta)\in I\times I$, the Brouwer's degree of the map

$$c_0(\cdot,\alpha,\beta):\ker(L-\lambda I)\to\ker(L-\lambda I),$$

defined in § 2, with respect to sets containing 0. As before, I is an open interval such that $I\cap\sigma(L)=\lambda\in\sigma_{\mathrm{d}}(L)$. We assume that $(\alpha,\beta)\notin\Sigma(L)$ (non-resonance situation), so that $c_0(x,\alpha,\beta)\neq 0$ for $x\neq 0$. Let

$$f:S^{n-1}\subset\ker L-\lambda I\to S^{n-1}\subset\ker L-\lambda I:\qquad x\mapsto -\frac{c_0(x,\alpha,\beta)}{\|c_0(x,\alpha,\beta)\|},$$

where $n=\dim\ker(L-\lambda I)$. By the Lefschetz formula (see [2]),

$$\sum_{x|x=f(x)}L_x(f)=\sum_{p=0}^{\infty}(-1)^p\operatorname{tr}H_p(f,\mathbb{R}),$$

where $H_p(f,\mathbb{R})$ denotes the induced linear map on the p^{th}-homological space of S^{n-1} and $L_x(f)$ is the Lefschetz number of the fixed point x (also called the multiplicity of x). As, for the sphere S^{n-1}, only H^0 and H^{n-1} are non-trivial, this formula reduces to

$$\sum_{x|x=f(x)}L_x(f)=1+(-1)^{n-1}\deg(f).$$

Since $c_0(x,\alpha,\beta)$ is non-zero for $x\neq 0$, we have $\deg(f)=(-1)^n\deg(c_0(\cdot,\alpha,\beta))$, so that

$$\deg(c_0(\cdot,\alpha,\beta))=1-\sum_{x|f(x)=x}L_x(f).$$

Let $x\in S^{n-1}$ be a fixed point of f. Thus

$$c_0(x,\alpha,\beta)=-\|c_0(x,\alpha,\beta)\|x$$

and, by lemma 2.3,

$$\nabla_x h_0(x,\alpha,\beta)=-2\|c_0(x,\alpha,\beta)\|x.$$

This shows that x is a critical point of the function $h_0(\cdot,\alpha,\beta)$, restricted to the unit sphere in $\ker(L-\lambda I)$, with a negative critical value.

Let f_T be the vector field obtained by projecting f on S^{n-1}, that is,

$$f_T(x)=f(x)-\langle f(x),x\rangle x,$$

and $c_{0,T}$ the equivalent for c_0

$$c_{0,T}(x)=c_0(x,\alpha,\beta)-\langle c_0(x,\alpha,\beta),x\rangle x.$$

The Lefschetz number of a fixed point x of f is nothing but the topological index of the corresponding zero of the vector field $-f_T$ (see [14]). By a homotopy argument, it is clear that this index is the same as that of the vector field $c_{0,T}$ (they are everywhere parallel with a strictly positive coefficient of proportionality). We thus obtain the following result.

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THEOREM 6.1. *Let $(\alpha,\beta) \in (I \times I) \setminus \Sigma(L)$. Assume that the restriction of the function $h_0(\cdot,\alpha,\beta)$ to the unit sphere $S^{n-1} \subset \ker(L-\lambda I)$ (denoted $h_{0|S^{n-1}}$) has only isolated critical points. Then*

$$\deg(c_0(\cdot,\alpha,\beta),\mathcal{U},0) = 1 - \sum_{\substack{x\in S^{n-1}\\ \nabla(h_{0|S^{n-1}})(x)=0\\ h_0(x,\alpha,\beta)<0}} \operatorname{ind}_x(\nabla(h_{0|S^{n-1}})),$$

where $\mathcal{U}$ is any open bounded neighbourhood of $0 \in \ker(L-\lambda I)$.

Notice that working with the function $-f$ instead of f yields the equivalent formula

$$\deg(c_0(\cdot,\alpha,\beta),\mathcal{U},0) = (-1)^n - \sum_{\substack{x\in S^{n-1}\\ \nabla(h_{0|S^{n-1}})(x)=0\\ h_0(x,\alpha,\beta)>0}} \operatorname{ind}_x(\nabla(h_{0|S^{n-1}})),$$

which can also be deduced from the preceding one using the Hopf formula for the Euler characteristic.

When $\dim\ker(L-\lambda I) = 2$, the indices of the zeros of $\nabla h_{0|S^1}$ are 1 for a minimum point and -1 for a maximum one. Moreover, the number of zeros of $h_{0|S^1}$ is twice the difference between the number of negative minima and the number of negative maxima. The computation of the degree is then particularly simple, according to the following result.

THEOREM 6.2. *Let $(\alpha,\beta) \in (I \times I) \setminus \Sigma(L)$. Assume that $\dim\ker(L-\lambda I) = 2$ and that the restriction of the function $h_0(\cdot,\alpha,\beta)$ to the unit sphere $S^1 \subset \ker(L-\lambda I)$ has only isolated critical points and a number $2z$ of zeros. Then the Brouwer's degree of $c_0(\cdot,\alpha,\beta)$, with respect to any open bounded set $\mathcal{U}$ containing 0, is equal to $1-z$.*

Notice that these degree computations can also be performed for $(\alpha,\beta) \in J \times J \setminus \Sigma(L)$, where J is an open bounded interval such that $J \cap \sigma_{\mathrm{ess}}(L) = \emptyset$.

7. Global structure of the Fučík spectrum: general case

The Fučík spectrum can be considered as the boundary of the regions where the degree of $c_0(\cdot,\alpha,\beta)$ is constant. This observation allows us to obtain global results for the Fučík curves without assuming the non-degeneracy condition (ND2) of Micheletti [18] and Pistoia [19]. Indeed, let $\lambda \in J \cap \sigma(L)$ and d be a critical value of the function $h : x \mapsto \langle |x|, x\rangle$ defined on the unit sphere in $\ker(L-\lambda I)$. We assume that the non-degeneracy condition (ND1) is satisfied for each critical point x_0 with $h(x_0) = d$. The critical point(s) associated to d give rise to Fučík curves, locally defined according to theorem 3.1, which we denote by $C_1,\dots,C_m$. Take $r > 0$ small enough so that the open ball $\mathcal{U} := B_{(\lambda,\lambda)}(r)$ does not intersect any other point $(\tilde\lambda,\tilde\lambda)$ with $\tilde\lambda \in \sigma_{\mathrm{d}}(L)$. The connected component containing every C_i in $\Sigma(L)$ is denoted by $\mathcal{C}$ and $\mathcal{D} := \mathcal{C} \setminus \bigcup_{i=1}^m (C_i \cap \mathcal{U})$.

The following theorem does not require condition (ND2). Roughly speaking, it asserts that a component of the Fučík spectrum within $J \times J$ 'connects' a point of the diagonal to the boundary of $J \times J$ or to a (not necessarily distinct) point of the diagonal.

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THEOREM 7.1. *With $h : x \mapsto \langle |x|, x\rangle$ defined on the unit sphere in* $\ker(L - \lambda I)$, *assume that*

$$\sum_{\substack{x \mid \nabla h(x)=0 \\ h(x)=d}} \operatorname{ind}_x \nabla h \neq 0. \tag{7.1}$$

Then one of the following holds.

(i) $\overline{\mathcal{D}} \cap \partial(J \times J) \neq \emptyset$.

(ii) *There exists* $\lambda^* \in J \cap \sigma(L)$ *such that* $(\lambda^*, \lambda^*) \in \overline{\mathcal{D}}$.

Proof. The proof being rather long and cumbersome, we will only give a sketch of it. If none of the above statements is satisfied, then there exists $\varepsilon > 0$ sufficiently small such that if some (α, β) belongs to $\mathcal{D}_\varepsilon$, where $\mathcal{D}_\varepsilon$ is the boundary of the ε-neighbourhood of $\mathcal{D}$, and $(\alpha, \beta) \in \Sigma(L)$, then $(\alpha, \beta) \in C_i \cap \mathcal{U}$ for some $1 \leqslant i \leqslant m$ (see [20] for a related result). Condition (7.1) ensures that the degree of the function $c_0(\cdot, \alpha, \beta)$ is different, on one side of all the C_i, from the degree on the other side, at least locally (see the next section for a precise statement and proof). The technical part lies in showing that each component of $\mathcal{D}_\varepsilon$ can be parametrized as a closed curve; this is in fact only true for almost all small ε. The local parametrization is performed using the differentiability properties of the 'distance to a set' function (see [7]), Sard's lemma and the implicit function theorem. For almost all ε, this curve has a finite length due to the co-area formula and is closed. The contradiction then follows by performing a homotopy of c_0 along $\mathcal{D}_\varepsilon$, without crossing the Fučík spectrum. □

8. Existence results: non-resonance situation

When (α, β) does not belong to the Fučík spectrum (a case which can be considered as a non-resonance situation), theorems 6.1 and 6.2 provide existence conditions for the non-homogeneous equation (1.1). Indeed, if the degree of $c_0(\cdot, \alpha, \beta)$, with respect to open bounded sets containing 0, is different from 0, by continuity, the same will be true for the function $x \mapsto c(x, \varepsilon f, \alpha, \beta)$, provided that ε is small enough. Therefore, the equation

$$Lu = \alpha u^+ - \beta u^- + \varepsilon f,$$

and, consequently, also (1.1), will have at least one solution, for any $f \in L^2(\Omega)$.

On the other hand, according to lemma 2.1, we have, if $\alpha \neq \beta$,

$$\frac{2}{\beta - \alpha} c_0(x, \alpha, \beta) = P(|x|) - \frac{2\lambda - \alpha - \beta}{\alpha - \beta} x + O(\|(\alpha, \beta) - (\lambda, \lambda)\|) \quad \text{for } (\alpha, \beta) \to (\lambda, \lambda).$$

Notice that the coefficient in front of x above only depends on the slope of the line joining (λ, λ) to (α, β); if we write $\beta = \lambda + p(\alpha - \lambda)$, then

$$\frac{2\lambda - \alpha - \beta}{\alpha - \beta} = \frac{p+1}{p-1}.$$

The following is thus deduced from theorem 6.1.

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Table 4.

slopes of the Fučík curves at (λ, λ)	degree
-0.5	
	0
-0.75	
	-1
-0.7673	
	0
-1.3032	
	-1
-1.3333	
	0
-2	

Corollary 8.1. *Let $p \in \mathbb{R} \setminus \{1\}$ be such that $(p+1)/(p-1)$ is not a critical value of the function $h : S^{n-1} \subset \ker(L - \lambda I) \to \mathbb{R} : x \mapsto \langle |x|, x\rangle$. If*

$$\sum_{\substack{x \mid \nabla h(x)=0 \\ h(x)<(p+1)/(p-1)}} \operatorname{ind}_x \nabla h \neq 1, \tag{8.1}$$

there exists $\eta(p) > 0$ such that, if $|\alpha - \lambda| < \eta(p)$ and $\beta = \lambda + p(\alpha - \lambda)$, then (1.1) has at least one solution for any $f \in L^2(\Omega)$.

Notice that, when $p = -1$, the sum in (8.1) must be even, since $c_0(x, \alpha, \beta)$ is then close to the even function $\frac{1}{2}(\beta - \alpha)P(|x|)$.

When $\dim \ker(L - \lambda I) = 2$, we obtain the following result from theorem 6.2.

Corollary 8.2. *Let $p \in \mathbb{R} \setminus \{1\}$ be such that the function*

$$S^1 \subset \ker(L - \lambda I) \to \mathbb{R} : x \mapsto \langle |x|, x\rangle - \frac{p+1}{p-1}$$

has only simple zeros, the number of zeros being different from 2. There exists $\eta(p) > 0$ such that, if $|\alpha - \lambda| < \eta(p)$ and $\beta = \lambda + p(\alpha - \lambda)$, then (1.1) has at least one solution for any $f \in L^2(\Omega)$.

These two results allow us to compute, in particular examples, the degree of the function $c_0(\cdot, \alpha, \beta)$ in the regions between the Fučík curves around (λ, λ). This is what we have made in the following examples.

Example 8.3. For instance, coming back to the problem of example 3.2, defined by (3.6), (3.7), and looking at the graph of h, the degree of $c_0(\cdot, \alpha, \beta)$, with respect to open bounded sets containing 0, can be computed in the neighbourhood of (λ, λ). Table 4 indicates the values of degree obtained in the zones between the various Fučík curves.

For instance, for $p = -0.76$, the degree is -1. Therefore, for any function $f \in L^2((0, \pi) \times (0, \pi/\sqrt{6}))$, the problem

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = (55 + \alpha)u^+ - (55 - 0.76\alpha)u^- + f,$$
$$u_{|\partial\Omega} = 0$$

has at least one solution if $|\alpha|$ is sufficiently small.

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Table 5.

extremal values of h	slopes	degree
-0.16064	-0.7232	
		-1
$-1/13$	$-6/7$	
		-2
$1/13$	$-7/6$	
		-1
0.16064	-1.3828	

Table 6.

slopes of the Fučík curves at $(99, 99)$	degree	slopes of the Fučík curves at $(99, 99)$	degree
-0.712		-1.002	
	0		1
-0.732		-1.112	
	-1		0
-0.744		-1.165	
	-2		1
$-3/4$		$-4/3$	
	-1		2
-0.859		-1.345	
	0		1
-0.899		-1.367	
	-1		0
-0.998		-1.404	
	0		
-1.002			

Example 8.4. We consider problem (3.14), (3.15) of example 3.3, taking now $m = 6$, $n = 13$. Table 5 gives the extremal values of h, the corresponding values of the slopes at the Fučík curves at $(\lambda, \lambda) = (6084, 6084)$, as obtained by (3.5), and the value of the degree between the Fučík curves, deduced from the graph of h. Notice that the maximal value $0.160\,64$, as well as the minimal value $-0.160\,64$, is reached at two different points.

It is observed that, for this problem, equation (1.1) always has a solution if (α, β) does not belong to the Fučík spectrum (at least for (α, β) close to (λ, λ)).

Example 8.5. Coming back to the elliptic operator of example 3.4, we observe numerically that the critical points of the function H restricted to the unit sphere are all non-degenerate and as follows, in order of critical value: first three minima followed by two saddle points, then one minimum and one saddle point for the first seven ones. The last seven ones are deduced by symmetry: one saddle point then a maximum, two saddle points and finally three maxima. Using theorem 6.1, table 6, giving the degree values between the Fučík curves, is deduced.

Then the existence of a solution for (1.1) is ensured by corollary 8.1, in the regions between the Fučík curves where the degree is non-zero.

Adding compactness hypotheses, it is possible to treat similarly the equation with a supplementary nonlinear term,

$$Lu = \alpha u^{+} - \beta u^{-} + g(\cdot, u), \tag{8.2}$$

assuming $g(x, u)$ to have a sublinear growth with respect to u, for $u \to \infty$. Existence conditions for this equation are based on the following lemma, which relates the

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Brouwer's degree of $c_0(\cdot,\alpha,\beta)$ to the coincidence degree of $L-A$, where

$$A : L^2(\Omega) \to L^2(\Omega) : u \mapsto \alpha u^+ - \beta u^-,$$

relatively to L (for the definition and properties of the coincidence degree, see [17]). The coincidence degree with respect to a ball $B(0,R)$ of centre 0 and radius R is denoted by $D_L(L-A, B(0,R))$, the Brouwer's degree being denoted as before by $\deg(c_0(\cdot,\alpha,\beta), B(0,R), 0)$ (the ball appearing in this last expression is, of course, in $\ker(L-\lambda I)$).

LEMMA 8.6. *Assume that L has a compact resolvent and that (α,β) does not belong to the Fučík spectrum. Then, for $R>0$ sufficiently large,*

$$|D_L(L-A, B(0,R))| = |\deg(c_0(\cdot,\alpha,\beta), B(0,R), 0)|.$$

The above lemma is a consequence of results appearing in [17] (see corollary II.28 therein).

Using lemma 8.6 and taking the same hypotheses on the growth of g as in [21], it is an easy matter to obtain existence conditions for (8.2).

COROLLARY 8.7. *Assume that L has a compact resolvent and that g is a Carathéodory function on $\Omega\times\mathbb{R}$ satisfying*

$$|g(x,u)| \leqslant V(x)^{1-\sigma}|u|^\sigma + W(x),$$

where $0 \leqslant \sigma < 1$ and $V, W \in L^2(\Omega)$. Assume also that the hypotheses of corollary 8.1 are satisfied for a given p. Then (8.2) has at least one solution, provided that $|\alpha-\lambda|$ is small enough.

Appendix A.

Let X, Y, Z be Banach spaces, $x_0 \in X$, $y_0 \in Y$. Let U be a neighbourhood of x_0, V a neighbourhood of y_0, $F : U\times V \to Z$ a continuous function. Assuming $F(x_0,y_0)=0$, we want to use an implicit function theorem to solve the equation $F(x,y)=0$ with respect to y. As usual, we require that the derivative $(\partial F/\partial y)(x_0,y_0) : Y \to Z$ is a linear homeomorphism, but, in opposition to the usual hypotheses, we will not ask F to have continuous derivatives on a neighbourhood of (x_0,y_0). That hypothesis is replaced by the following form of 'strong' Fréchet differentiability:

$$\left.\begin{array}{l}\text{there exists a bounded linear map}\\ (\partial F/\partial y)(x_0,y_0) : Y \to Z \text{ such that, for any } \varepsilon>0\text{, there}\\ \text{exist neighbourhoods } U'\subset U \text{ of } x_0 \text{ and } V'\subset V \text{ of } y_0 \text{ with}\\ \left\|F(x,y)-F(x,y')-\dfrac{\partial F}{\partial y}(x_0,y_0)(y-y')\right\| \leqslant \varepsilon\|y-y'\|\\ \text{for } y,y'\in V', x\in U.\end{array}\right\} \tag{A}$$

On this basis, the following implicit function theorem can be written (see [4, proposition 4.3.1] or [13] for closely related statements).

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THEOREM A.1. *With the above notation, let* $F : U \times V \to Z : (x, y) \mapsto F(x, y)$ *be continuous with respect to* x *at* (x_0, y_0), *with* $F(x_0, y_0) = 0$. *Assume that (A) holds and that* $(\partial F/\partial y)(x_0, y_0) : Y \to Z$ *is a linear homeomorphism. Then there exists a mapping* $g : U_1 \to Y$, *defined on a neighbourhood* $U_1 \subset U$ *of* x_0, *continuous at* x_0, *such that* $g(x_0) = y_0$ *and* $F(x, g(x)) = 0\ \forall x \in U_1$. *Moreover, there exists* $V_1 \subset V$ *such that*

$$F(x, y) = 0 \quad \text{for } x \in U_1,\ y \in V_1 \quad \Rightarrow \quad y = g(x).$$

If F *is Lipschitzian with respect to* x *on* $U \times V$, g *is Lipschitzian on* U_1.

Theorem A.1 is applied here to functions involving the mappings

$$J \times L^p(\Omega) \to L^2(\Omega) : (\alpha, u) \mapsto \alpha u^+ \quad \text{and} \quad J \times L^p(\Omega) \to L^2(\Omega) : (\beta, u) \mapsto \beta u^-,$$

with $p > 2$. Due to the result of Solimini [23], it is easily seen that these mappings have strong Fréchet derivatives in the sense of (A).

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