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# TRANSIENCE OF CONTINUOUS-TIME CONSERVATIVE RANDOM WALKS

SATYAKI BHATTACHARYA,* AND
STANISLAV VOLKOV ,* ** *Lund University*

**Abstract**

We consider two continuous-time generalizations of conservative random walks introduced in Englander and Volkov (2022), an orthogonal and a spherically symmetrical one; the latter model is also known as *random flights*. For both models, we show the transience of the walks when $d \geq 2$ and that the rate of direction changing follows a power law $t^{-\alpha}$, $0 < \alpha \leq 1$, or the law $(\ln t)^{-\beta}$ where $\beta > 2$.

*Keywords:* Random flight; non-time-homogeneous Markov chain; conservative random walk; transience; recurrence

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Secondary 60J75

## 1. Introduction

Conservative random walks (discrete time) were introduced in [8] as a time-inhomogeneous Markov chain $X_n$, $n = 1, 2, \ldots$, defined as a process on $\mathbb{Z}^d$ such that, for a given (non-random) sequence $p_1, p_2, \ldots$ where all $p_i \in (0, 1)$, the walk at time $n$ with probability $p_n$ randomly picks one of the $2d$ directions parallel to the axis, and otherwise continues moving in the direction it was going before. This walk can be viewed as a generalization of Gillis' random walk [9]. An interesting special case studied in [8] is when $p_n \to 0$, and in particular when $p_n \sim n^{-\alpha}$ where $\alpha \in (0, 1]$; note that the case $\alpha > 1$ is trivial as the walk would make only finitely many turns. The question of recurrence vs. transience of this walk was one of the main questions of that paper.

Similar processes have appeared in the literature under different names. Some of the earliest papers which mention a continuous process with memory of this type were probably [10, 12]. The term *persistent random* was used in [3, 4]; in these papers some very general criteria of recurrence vs. transience were investigated. A planar motion with just three directions was studied in [6]. A book on *Markov random flights* was recently published [14]. *Planar random motions with drifts* with four directions/speeds, switching at Poisson times, were studied in [17]. Applications of telegraph processes to option pricing can be found in [18]. Characteristic functions of correlated random walks were studied in [5].

The main difference between the conservative random walk and most of the models studied in the literature (except, perhaps, [19], which has a more applied focus) is that the underlying

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* Postal address: Centre for Mathematical Sciences, Lund University, Box 118 SE-22100, Lund, Sweden.
** Email address: stanislav.volkov@matstat.lu.se

process of direction switching is *time-inhomogeneous*, thus creating various new phenomena. The recurrence/transience of a discrete-space conservative random walk on $\mathbb{Z}^1$ was thoroughly studied in [7] (see also the references therein), and we believe that the continuous-time version in one dimension will have very similar features. The establishment of recurrence in one dimension is more or less equivalent to finding that the lim sup of the process is $+\infty$ and the lim inf of the process is $-\infty$, while in higher dimensions the situation is much more intricate. Hence, we concentrate on the case when the dimension of the space is at least 2, except for Theorem 3.1 which deals with the embedded process.

Below we formally introduce the two versions of a *continuous-time* conservative random walk on $\mathbb{R}^d$, $d \geq 1$.

### 1.1. Model A (orthogonal model)

Let $\lambda(t)$ be a non-negative function such that

$$\Lambda(T) = \int_0^T \lambda(t)\,\mathrm{d}t < \infty \quad \text{for all } T \geq 0; \qquad \lim_{T\to\infty} \Lambda(T) = \infty. \tag{1.1}$$

Let $\tau_1 < \tau_2 < \cdots$ be the consecutive points of an inhomogeneous Poisson point process (PPP) on $[0, \infty)$ with rate $\lambda(t)$, and $\tau_0 = 0$. Then the conditions in (1.1) guarantee that there will be finitely many $\tau_i$ in every finite interval, and that $\tau_n \to \infty$. Let $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2$ be an independent and identically distributed (i.i.d.) sequence of vectors, each of which has a uniform distribution on the set of $2d$ unit vectors $\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \ldots, \pm\mathbf{e}_d\}$ in $\mathbb{R}^d$.

The (orthogonal, continuous-time) conservative walk generated by the rate function $\lambda(\cdot)$ is a process $Z(t)$, $t \geq 0$, in $\mathbb{R}^d$, $d \geq 1$, such that $Z(0) = 0$ and at each time $\tau_k$, $k \geq 0$, the walk starts moving in the direction $\mathbf{f}_k$, and keeps moving in this direction until time $\tau_{k+1}$, when it updates its direction. Formally, we define $N(t) = \sup\{k \geq 0 \colon \tau_k \leq t\}$ as the number of points of the PPP by time $t$; then

$$Z(t) = \sum_{k=0}^{N(t)-1} (\tau_{k+1} - \tau_k)\mathbf{f}_k + (t - \tau_{N(t)})\mathbf{f}_{N(t)}.$$

We can also define the embedded process $W_n = Z(\tau_n)$ so that $W(0) = 0$ and, for $n \geq 1$, $W_n = \sum_{k=0}^{n-1} (\tau_{k+1} - \tau_k)\mathbf{f}_k$. The process $Z(t)$ can be viewed as a continuous equivalent of the conservative random walk introduced in [8].

### 1.2. Model B (von Mises–Fisher model)

This model is defined similarly to the previous one, except that now the random vectors $\mathbf{f}_k$, $k = 1, 2, \ldots$, have a uniform distribution on the $d$-dimensional unit sphere $\mathcal{S}^{d-1}$, often called the *von Mises–Fisher distribution*, instead of just on $2d$ unit vectors of $\mathbb{R}^d$.

Note that this model is similar to the 'random flights' model studied, e.g., in [16]; however, their results are only for a time-homogeneous Poisson process, unlike our case.

### 1.3. Aims of this paper

The results that we obtain in the current paper are somewhat different for the two models; however, since they share a lot of common features, certain statements will hold for both of them. The main goal is establishing transience vs. recurrence of the walks, defined as follows.

**Definition 1.1.** Let $\rho \geq 0$. We say that the walk Z(t) is $\rho$-recurrent if there is an infinite sequence of times $t_1 < t_2 < \cdots$, converging to infinity, such that $Z(t_i) \in [-\rho, \rho]^d$ for all $i = 1, 2, \ldots$

We say that the walk $Z(t)$ is transient if it is not $\rho$-recurrent for any $\rho > 0$, or, equivalently, $\lim_{t\to\infty} \|Z(t)\| = \infty$.

Recurrence and transience of the embedded process $W_n$ are defined analogously, with the exception that instead of $t_1, t_2, t_3, \ldots$ in the above definition, we have a strictly increasing sequence of positive *integers* $n_i$, $i = 1, 2, \ldots$

**Remark 1.1.** Note that a priori it is unclear if transience and recurrence are zero–one events; neither can we easily rule out the possibility of 'intermediate' situations (e.g. $\rho$-recurrence only for *some* $\rho$).

Our main results, which show transience for two types of rates, are presented in Theorems 3.1, 3.2, 3.3, and 4.1.

## 2. Preliminaries

Throughout the paper we use the following notation. We write $X \sim \mathrm{Poi}(\mu)$ when $X$ has a Poisson distribution with parameter $\mu > 0$. For any set $A$, $|A|$ denotes its cardinality. For $x \in \mathbb{R}^d$, $\|x\|$ denotes the usual Euclidean norm of $x$.

First, we state Kesten's generalization of the Kolmogorov–Rogozin inequality. Let $S_n = \xi_1 + \cdots + \xi_n$ where the $\xi_i$ are independent, and for any random variable $Y$ define $Q(Y; a) = \sup_x \mathbb{P}(Y \in [x, x + a])$.

**Lemma 2.1.** ([13].) *There exists $C > 0$ such that, for any real numbers $0 < a_1, \ldots, a_n \le 2L$,*

$$Q(S_n; L) \le \frac{CL \sum_{i=1}^n a_i^2 (1 - Q(\xi_i; a_i)) Q(\xi_i; a_i)}{\left[\sum_{i=1}^n a_i^2 (1 - Q(\xi_i; a_i))\right]^{3/2}}.$$

Second, if $D_1, \ldots, D_m$ is a sequence of independent events each with probability $p$, $\varepsilon > 0$, and $N_D(m) = \mathrm{card}(\{i \in \{1, \ldots, m\} \colon D_i \text{ occurs}\}) = \sum_{i=1}^m \mathbf{1}_{D_i}$, then

$$\mathbb{P}(|N_D(m) - pm| \ge \varepsilon m) \le 2\mathrm{e}^{-2\varepsilon^2 m} \tag{2.1}$$

by Hoeffding's inequality (see, e.g., [11]).

Suppose we have an inhomogeneous PPP with rate

$$\lambda(t) = \frac{1}{t^\alpha}, \qquad t > 0, \tag{2.2}$$

where $0 < \alpha < 1$ is constant; thus

$$\Lambda(T) = \int_0^T \lambda(t)\,\mathrm{d}t = \frac{T^{1-\alpha}}{1-\alpha}$$

and the conditions in (1.1) are fulfilled. Let $0 < \tau_1 < \tau_2 < \cdots$ denote the points of the PPP inincreasing order.

The following statement is probably known, but for the sake of completeness, we provide its short proof.

**Claim 2.1.** *Let $Z$ be a Poisson random variable with rate $\mu > 0$. Then*

$$\mathbb{P}\left(Z \ge \frac{3\mu}{2}\right) \le \mathrm{e}^{-((3\ln(3/2)-1)/2)\mu} = \mathrm{e}^{-0.108\ldots\mu},$$

$$\mathbb{P}\left(Z \le \frac{\mu}{2}\right) \le \mathrm{e}^{-((1-\ln 2)/2)\mu} = \mathrm{e}^{-0.153\ldots\mu}.$$

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*Proof.* By the Markov inequality, since $\mathbb{E}e^{uZ} = e^{\mu(e^u-1)}$, we have, for $u > 0$,

$$\mathbb{P}\left(Z \geq \frac{3\mu}{2}\right) \leq \mathbb{P}(e^{uZ} \geq e^{3\mu u/2}) \leq e^{-3\mu u/2}\mathbb{E}\, e^{uZ} = \exp\left\{-\mu\left[\frac{3u}{2} - e^u + 1\right]\right\}.$$

Setting $u = \ln(3/2)$ yields the first inequality in the claim.

For the second inequality, we use

$$\mathbb{P}\left(Z \leq \frac{\mu}{2}\right) \leq \mathbb{P}(e^{-uZ} \geq e^{-\mu u/2}) \leq e^{\mu u/2}\mathbb{E}\, e^{-uZ} = \exp\left\{-\mu\left[-\frac{u}{2} - e^{-u} + 1\right]\right\}.$$

Now let $u = \ln 2$. □

**Lemma 2.2.** *Suppose that the rate of the PPP is given by (2.2). For some $c_1 > c_0 > 0$, depending on $\alpha$ only,*

$$\mathbb{P}(\tau_k \leq c_0 k^{1/(1-\alpha)}) \leq e^{-k/15}; \qquad \mathbb{P}(\tau_k \geq c_1 k^{1/(1-\alpha)}) \leq e^{-k/15}.$$

*Proof.* Recall that $N(s)$ denotes the number of points of the PPP by time $s$. Then $N(s) \sim \mathrm{Poi}(\Lambda(s))$, and $\mathbb{P}(\tau_k \leq s) = \mathbb{P}(N(s) \geq k)$. Let $T_n = \Lambda^{(-1)}(n) = \sqrt[1-\alpha]{(1-\alpha)n}$. Noting that $N(T_n) \sim \mathrm{Poi}(n)$ for all $n$, we have

$$\mathbb{P}(\tau_k \leq T_{2k/3}) = \mathbb{P}(N(T_{2k/3}) \geq k) = \mathbb{P}\left(Z \geq \frac{3}{2}\mu\right) \leq e^{-0.072\,13\ldots k}$$

by Claim 2.1, with $Z \sim \mathrm{Poi}(\mu)$ where $\mu = 2k/3$. Similarly,

$$\mathbb{P}(\tau_k \geq T_{2k}) = \mathbb{P}(N(T_{2k}) \leq k) \leq \mathbb{P}\left(Z \leq \frac{1}{2}\mu\right) \leq e^{-0.306\,85\ldots k}$$

by Claim 2.1, with $Z \sim \mathrm{Poi}(\mu)$ where $\mu = 2k$. Note that $0.072\,13 > \frac{1}{15}$, $0.306\,85 > \frac{1}{15}$. Now the statement follows with $c_0 = \sqrt[1-\alpha]{2(1-\alpha)/3}$ and $c_1 = \sqrt[1-\alpha]{2(1-\alpha)}$. □

## 3. Analysis of Model A

**Theorem 3.1.** *Let $d = 1$, $\alpha \in \left(\frac{1}{3}, 1\right)$, and the rate be given by (2.2). Then the embedded walk $W_n$ is transient almost surely (a.s.).*

*Proof.* Assume without loss of generality that $n$ is even. We will show that for any $\rho > 0$ the walk $W_n$ visits $[-\rho, \rho]$ finitely often a.s. With probabilities close to 1, both the events

$$\mathcal{E}_1 = \left\{\tau_{n/2} \geq c_0(n/2)^{1/(1-\alpha)}\right\}, \qquad \mathcal{E}_2 = \left\{\tau_n \leq c_1 n^{1/(1-\alpha)}\right\} \tag{3.1}$$

occur; indeed,

$$\mathbb{P}(\mathcal{E}_1^c) \leq e^{-n/30}, \qquad \mathbb{P}(\mathcal{E}_2^c) \leq e^{-n/15} \tag{3.2}$$

by Lemma 2.2.

Since the rate $\lambda(t) = t^{-\alpha}$ of the Poisson process is monotonically decreasing, the random variables $\tau_i - \tau_{i-1}$, $n/2 \leq i \leq n$, under the condition stated in the event $\mathcal{E}_1$, are stochastically larger than i.i.d. exponential random variables $\zeta_i$ with rates equal to $[c_0(n/2)^{1/1-\alpha}]^\alpha = \tilde{c}_0 n^{-\alpha/(1-\alpha)}$ for some $\tilde{c}_0 > 0$. For $\zeta$, there exists $\beta = \beta(c_0, \alpha) > 0$ such that

$$\mathbb{P}\big(\zeta_i > \beta n^{\alpha/(1-\alpha)}\big) = \exp\big(-\tilde{c}_0 n^{-\alpha/(1-\alpha)} \cdot \beta n^{\alpha/(1-\alpha)}\big) = e^{-\tilde{c}_0\beta} = \frac{2}{3}.$$

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Let $I_n = \big\{i \in [n/2, n] \colon \ i \,\text{is even}, \ \tau_i - \tau_{i-2} > \beta n^{\alpha/(1-\alpha)}\big\}$, and note that $\text{card}(I_n) \le n/4$. Then, since $\tau_i - \tau_{i-2} > \tau_i - \tau_{i-1}$, by stochastic monotonicity and Hoeffding's inequality (2.1) with $m = n/4$, $p = \frac{2}{3}$, and $\varepsilon = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$,

$$\begin{aligned}\mathbb{P}\bigg(\text{card}(I_n) < \frac{1}{2}\cdot\frac{n}{4} \mid \mathcal{E}_1\bigg) &\le \mathbb{P}\bigg(\text{card}\Big(\big\{i \in [n/2, n] \colon \ i \,\text{is even}, \ \zeta_i > \beta n^{\alpha/(1-\alpha)}\big\}\Big) < \frac{n}{8}\bigg) \\ &\le 2\mathrm{e}^{-n/72}. \end{aligned} \tag{3.3}$$

Let $J_n = \{i \in I_n \colon \mathbf{f}_{i-1} = -\mathbf{f}_{i-2}\}$. Since the $\mathbf{f}_i$ are i.i.d. and independent of $\{\tau_1, \tau_2, \dots\}$, and $\mathbb{P}(\mathbf{f}_{i-1} = -\mathbf{f}_i) = \frac{1}{2}$, on the event $\{\text{card}(I_n) \ge n/8\}$ we have, by (2.1) with $m = \text{card}(I_n)$, $p = \frac{1}{2}$, and $\varepsilon = \frac{1}{34}$,

$$\begin{aligned}\mathbb{P}\bigg(\text{card}(J_n) < \frac{n}{17} \mid \text{card}(I_n) \ge \frac{n}{8}\bigg) &\le \mathbb{P}\bigg(\bigg|\text{card}(J_n) - \frac{\text{card}(I_n)}{2}\bigg| > \frac{\text{card}(I_n)}{34} \mid \text{card}(I_n) \ge \frac{n}{8}\bigg) \\ &\le 2\exp\bigg\{-2\cdot\frac{1}{34^2}\cdot\frac{n}{8}\bigg\} = 2\mathrm{e}^{-c_* n}, \end{aligned} \tag{3.4}$$

where $c_* = 1/4624$.

Our proof will rely on conditioning over the *even stopping times*, i.e. on the event $\mathcal{D} = \mathcal{D}_0 \cap \mathcal{D}_1 \cap \mathcal{E}_2$, where

$$\mathcal{D}_0 = \big\{\tau_{n/2} = t_{n/2}, \tau_{n/2+2} = t_{n/2+2}, \dots, \tau_n = t_n\big\}, \qquad \mathcal{D}_1 = \{\text{card}(J_n) \ge n/17\} \cap \mathcal{E}_1$$

for some strictly increasing sequence $0 < t_{n/2} < t_{n/2+2} < \cdots < t_n$. Note that

$$\begin{aligned}\mathbb{P}(\mathcal{D}_1^{\rm c}) &\le \mathbb{P}(\mathcal{E}_1^{\rm c}) + \mathbb{P}\bigg(\text{card}(J_n) < \frac{n}{17} \mid \mathcal{E}_1\bigg) \\ &\le \mathbb{P}(\mathcal{E}_1^{\rm c}) + \mathbb{P}\bigg(\text{card}(J_n) < \frac{n}{17} \mid \text{card}(I_n) \ge \frac{n}{8}, \mathcal{E}_1\bigg) + \mathbb{P}\bigg(\text{card}(I_n) < \frac{n}{8} \mid \mathcal{E}_1\bigg) \\ &\le \mathrm{e}^{-n/30} + 2\mathrm{e}^{-c_* n} + 2\mathrm{e}^{-n/72} \le 5\mathrm{e}^{-c_* n} \end{aligned} \tag{3.5}$$

by (3.2), (3.3), and (3.4).

We denote the $i$th step of the embedded walk by $X_i = W_i - W_{i-1}$, $i = 1, 2, \dots$, and

$$\xi_i := X_{i-1} + X_i = (\tau_{i-1} - \tau_{i-2})\mathbf{f}_{i-2} + (\tau_i - \tau_{i-1})\mathbf{f}_{i-1}.$$

Conditioned on $\mathcal{D}$, the random variables $\xi_2, \xi_4, \xi_6, \dots$ are then independent.

**Lemma 3.1.** *For $i \in J_n$, $\sup_{x\in\mathbb{R}} \mathbb{P}(\xi_i \in [x - \rho, x + \rho] \mid \mathcal{D}) \le c\rho/n^{\alpha/(1-\alpha)}$ for some $c = c(\alpha) > 0$.*

*Proof of Lemma 3.1.* Given $\tau_{i-2} = t_{i-2}$ and $\tau_i = t_i$, $\tau_{i-1}$ has the distribution of the only point of the PPP on $[t_{i-2}, t_i]$ with rate $\lambda(t)$ conditioned on the fact that there is exactly one point in this interval. Hence, the conditional density of $\tau_{i-1}$ is given by

$$f_{\tau_{i-1}|\mathcal{D}}(x) = \begin{cases} \dfrac{\lambda(t)}{\int_{t_{i-2}}^{t_i} \lambda(u)\,\mathrm{d}u} & \text{if } x \in [t_{i-2}, t_i], \\ 0 & \text{otherwise.} \end{cases}$$

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Assume without loss of generality that $X_{i-1} > 0 > X_i$ (recall that $i \in J_n$). Then

$$\xi_i = [\tau_{i-1} - \tau_{i-2}] - [\tau_i - \tau_{i-1}] = 2\tau_{i-1} - (t_{i-2} + t_i)$$

so that the maximum of the conditional density of $\xi_i$ equals one-half of the maximum of the conditional density of $\tau_{i-1}$. At the same time, since $\lambda$ is a decreasing function,

$$\sup_{x\in\mathbb{R}} f_{\tau_{i-1}|\mathcal{D}}(x) = \frac{\lambda(t_{i-2})}{\int_{t_{i-2}}^{t_i}\lambda(u)\,\mathrm{d}u} \le \frac{\lambda(t_{i-2})}{(t_i - t_{i-2})\lambda(t_i)} \le \frac{\lambda(t_{i-2})}{\beta n^{\alpha/(1-\alpha)}\lambda(t_i)} = \frac{1}{\beta n^{\alpha/(1-\alpha)}} \cdot \left(\frac{t_i}{t_{i-2}}\right)^{\alpha}$$
$$\le \frac{1}{\beta n^{\alpha/(1-\alpha)}} \cdot \left(\frac{\tau_n}{\tau_{n/2}}\right)^{\alpha} \le \frac{1}{\beta n^{\alpha/(1-\alpha)}} \cdot \left(\frac{c_1}{c_0 2^{-1/(1-\alpha)}}\right)^{\alpha}$$

since the events $\mathcal{E}_1$ and $\mathcal{E}_2$ occur. This implies the stated result with $c = \beta^{-1}(c_0^{-1} c_1 \sqrt[1-\alpha]{2})^{\alpha}$. □

Now we divide $W_n$ into two portions:

$$A = \sum_{i\in J_n} \xi_i, \qquad B = W_n - A = \sum_{i\in\{2,4,\ldots,n\}\setminus J_n} \xi_i.$$

**Lemma 3.2.** *For some $C = C(\alpha, \rho) > 0$,*

$$\sup_{x\in\mathbb{R}} \mathbb{P}(A \in [x-\rho, x+\rho] \mid \mathcal{D}) \le \frac{C}{n^{\alpha/(1-\alpha)+1/2}}.$$

*Proof of Lemma* 3.2. The result follows immediately from Lemma 2.1 with $a_i \equiv 2\rho = L$, using Lemma 3.1 and the fact that $\operatorname{card}(J_n) \ge n/17$. □

So,

$$\mathbb{P}(|W_n| \le \rho \mid \mathcal{D}) = \mathbb{P}(A + B \in [-\rho, \rho] \mid \mathcal{D}) = \int \mathbb{P}(A + b \in [-\rho, \rho] \mid \mathcal{D}) f_{B|\mathcal{D}}(b)\,\mathrm{d}b$$
$$\le \int \sup_x \mathbb{P}(A \in [x-\rho, x+\rho] \mid \mathcal{D}) f_{B|\mathcal{D}}(b)\,\mathrm{d}b \le \frac{C}{n^{\alpha/(1-\alpha)+1/2}},$$

where $f_{B|\mathcal{D}}(\cdot)$ is the density of $B$ conditional on $\mathcal{D}$.

Finally, using (3.2) and (3.5),

$$\mathbb{P}(W_n \in [-\rho, \rho]) \le \mathbb{P}(W_n \in [-\rho, \rho] \mid \mathcal{D}) + \mathbb{P}(\mathcal{D}_1^{\mathrm{c}}) + \mathbb{P}(\mathcal{E}_2^{\mathrm{c}}) \le \frac{C}{n^{\alpha/(1-\alpha)+1/2}} + 5\mathrm{e}^{-c_* n} + \mathrm{e}^{-n/15},$$

which is summable over $n$, so we can apply the Borel–Cantelli lemma to show that $\{|W_n| \le \rho\}$ occurs finitely often a.s. □

**Theorem 3.2.** *Let $d \ge 2$, $\alpha \in (0, 1)$, and the rate of the PPP be given by* (2.2). *Then Z(t) is transient a.s.*

**Remark 3.1.** This result also holds for $\alpha = 1$, and the proof is more or less identical to that of [8, Theorem 5.2], once we establish that, a.s., $\tau_n > \mathrm{e}^{cn}$ for some $c > 0$ and all large $n$; the latter follows from arguments similar to Lemma 2.2.

*Proof of Theorem* 3.2. We provide the proof only for the case $d = 2$ and $\rho = 1$; it can be easily generalized for all $d \ge 3$ and $\rho > 0$. Denote the coordinates of the embedded walk by

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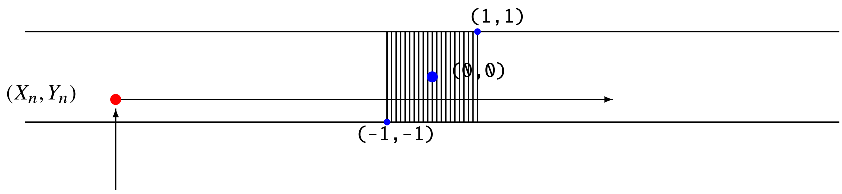


FIGURE 1. Model A: Recurrence of conservative random walk on $\mathbb{R}^2$.

$X_n$ and $Y_n$; thus $W_n = (X_n, Y_n) \in \mathbb{R}^2$. Fix some small $\varepsilon > 0$ and consider the event $\mathcal{R}_n = \{Z(t) \in [-1, 1]^2$ for some $t \in (\tau_n, \tau_{n+1}]\}$. We will show that events $\mathcal{R}_n$ occur finitely often a.s., thus ensuring the transience of $Z(t)$.

For the walk $Z(t)$ to hit $[-\rho, \rho]^2$ between times $\tau_n$ and $\tau_{n+1}$, we have to have either

- $|X_n| \le \rho$, $\mathbf{f}_n = -\text{sign}(Y_n)\mathbf{e}_2$, and $\tau_{n+1} - \tau_n \ge |Y_n| - 1$, or
- $|Y_n| \le \rho$, $\mathbf{f}_n = -\text{sign}(X_n)\mathbf{e}_1$, and $\tau_{n+1} - \tau_n \ge |X_n| - 1$

(see Figure 1.)

Using the fact that the $\mathbf{f}_n$ are independent of anything, we get

$$\mathbb{P}(\mathcal{R}_n) \le \frac{1}{4}\mathbb{P}(|X_n| \le \rho,\ \tau_{n+1} - \tau_n \ge |Y_n| - 1) + \frac{1}{4}\mathbb{P}(|Y_n| \le \rho,\ \tau_{n+1} - \tau_n \ge |X_n| - 1).$$

We will show that $\mathbb{P}(|X_n| \le \rho,\ |Y_n| \le \tau_{n+1} - \tau_n + 1)$ is summable in $n$ (and the same holds for the other summand by symmetry), hence by the Borel–Cantelli lemma that $\mathcal{R}_n$ occurs finitely often a.s.

**Lemma 3.3.** *Assume that $\varepsilon \in (0, 1)$. Then there exists $c_1^* = c_1^*(\varepsilon)$ such that, for all sufficiently large $n$, $\mathbb{P}\big(\tau_{n+1} - \tau_n + 1 \ge n^{\alpha/(1-\alpha)+\varepsilon}\big) \le 3\mathrm{e}^{-c_1^* n^{\varepsilon}}$.*

*Proof of Lemma 3.3.* For $s, t \ge 0$,

$$\mathbb{P}(\tau_{n+1} - \tau_n \ge s \mid \tau_n = t) = \mathrm{e}^{-[\Lambda(t+s)-\Lambda(t)]} = \exp\left\{-\frac{(t+s)^{1-\alpha} - t^{1-\alpha}}{1-\alpha}\right\},$$

so if $t < c_1 n^{1/(1-\alpha)}$, where $c_1$ is the constant from Lemma 2.2, and $s = n^{\alpha/(1-\alpha)+\varepsilon} - 1 = o(n^{1/(1-\alpha)})$, we have

$$\begin{aligned}\mathbb{P}(\tau_{n+1} - \tau_n \ge s \mid \tau_n = t) &\le \exp\left\{-\frac{(c_1 n^{1/(1-\alpha)} + s)^{1-\alpha} - c_1^{1-\alpha} n}{1-\alpha}\right\} \\ &\le \exp\left\{-\frac{n c_1^{1-\alpha}}{1-\alpha}\left[\left(1 + \frac{n^{\alpha/(1-\alpha)+\varepsilon} - 1}{c_1 n^{1/(1-\alpha)}}\right)^{1-\alpha} - 1\right]\right\} \\ &= \exp\left\{-\frac{n^{\varepsilon}(1+o(1))}{c_1^{\alpha}}\right\}.\end{aligned}$$

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Consequently, using Lemma 2.2,

$$\begin{aligned}\mathbb{P}\big(\tau_{n+1}-\tau_n \ge n^{\alpha/(1-\alpha)+\varepsilon}-1\big) &\le \mathbb{P}\big(\tau_{n+1}-\tau_n \ge n^{\alpha/(1-\alpha)+\varepsilon}-1 \mid \tau_n < c_1 n^{1/(1-\alpha)}\big)\\ &\quad + \mathbb{P}\big(\tau_n \ge c_1 n^{1/(1-\alpha)}\big)\\ &\le \mathrm{e}^{-c_1^* n^{\varepsilon}} + \mathrm{e}^{-n/15} \le 2\mathrm{e}^{-c_1^* n^{\varepsilon}} \end{aligned} \tag{3.6}$$

for some $c_1^* \in \big(0, \frac{1}{15}\big)$. □

Now we will modify the proof of Theorem 3.1 slightly to adapt to our needs. Let the events $\mathcal{E}_1$ and $\mathcal{E}_2$ be the same as in the proof of Theorem 3.1. We will also use the set $I_n$, but instead of $J_n$ we introduce the sets

$$J_n^1 = \{i \in I_n \colon \mathbf{f}_{i-1} = -\mathbf{f}_{i-2},\ \mathbf{f}_{i-1} \in \{\mathbf{e}_1, -\mathbf{e}_1\}\},$$
$$J_n^2 = \{i \in I_n \colon \mathbf{f}_{i-1} = -\mathbf{f}_{i-2},\ \mathbf{f}_{i-1} \in \{\mathbf{e}_2, -\mathbf{e}_2\}\}.$$

Similarly to the proof of Theorem 3.1, inequality (3.5), we immediately obtain that

$$\begin{aligned}\mathbb{P}(\mathrm{card}(J_n^1) \le n/34) &\le \mathbb{P}(\mathrm{card}(J_n^1) \le n/34 \mid \mathcal{E}_1) + \mathbb{P}(\mathcal{E}_1^{\mathrm{c}}) \le 6\mathrm{e}^{-c'_* n},\\ \mathbb{P}(\mathrm{card}(J_n^2) \le n/34) &\le \mathbb{P}(\mathrm{card}(J_n^2) \le n/34 \mid \mathcal{E}_1) + \mathbb{P}(\mathcal{E}_1^{\mathrm{c}}) \le 6\mathrm{e}^{-c'_* n}\end{aligned} \tag{3.7}$$

for some $c'_* > 0$.

Fix a deterministic sequence of unit vectors $\mathbf{g}_1, \ldots, \mathbf{g}_n$ such that each $\mathbf{g}_i \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$. We now also define the event

$$\begin{aligned}\mathcal{D} = \{\mathbf{f}_1 = \mathbf{g}_1, \ldots, \mathbf{f}_n = \mathbf{g}_n\} \cap \{\tau_k = t_k \text{ for all } k \le n \colon k \text{ is even or } \mathbf{f}_k \perp \mathbf{f}_{k+1}\} \cap\\ \left\{\mathrm{card}(J_n^1) \ge \frac{n}{34}\right\} \cap \left\{\mathrm{card}(J_n^2) \ge \frac{n}{34}\right\} \cap \mathcal{E}_1 \cap \mathcal{E}_2.\end{aligned}$$

Therefore, repeating the previous arguments for each of the horizontal and vertical components of $W_n$, we immediately obtain

$$\mathbb{P}(|X_n| \le 1 \mid \mathcal{D}) \le \frac{C}{n^{\alpha/(1-\alpha)+1/2}}, \qquad \sup_{x\in\mathbb{R}} \mathbb{P}(|Y_n - x| \le 1 \mid \mathcal{D}) \le \frac{C}{n^{\alpha/(1-\alpha)+1/2}}.$$

The second inequality implies that

$$\mathbb{P}(|Y_n| \le n^{\alpha/(1-\alpha)+\varepsilon} \mid \mathcal{D}) \le n^{\alpha/(1-\alpha)+\varepsilon} \cdot \frac{C}{n^{\alpha/(1-\alpha)+1/2}} = \frac{C}{n^{1/2-\varepsilon}}.$$

Now, by Lemma 3.3,

$$\begin{aligned}&\mathbb{P}(|X_n| \le 1, |Y_n| \le \tau_{n+1} - \tau_n + 1)\\ &\qquad \le \mathbb{P}\big(|X_n| \le 1, |Y_n| \le \tau_{n+1} - \tau_n + 1 \mid \tau_{n+1} - \tau_n + 1 \le n^{\alpha/(1-\alpha)+\varepsilon}\big)\\ &\qquad\quad + \mathbb{P}\big(\tau_{n+1} - \tau_n + 1 \ge n^{\alpha/(1-\alpha)+\varepsilon}\big)\\ &\qquad \le \mathbb{P}\big(|X_n| \le 1, |Y_n| \le n^{\alpha/(1-\alpha)+\varepsilon}\big) + 3\mathrm{e}^{-c_1^* n^{\varepsilon}}.\end{aligned}$$

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Observing that $X_n$ and $Y_n$ are actually independent given $\mathcal{D}$, we conclude that

$$\begin{aligned}\mathbb{P}\big(|X_n|\le 1,\,|Y_n|\le n^{\alpha/(1-\alpha)+\varepsilon}\big) &\le \mathbb{P}\big(|X_n|\le 1,\,|Y_n|\le n^{\alpha/(1-\alpha)+\varepsilon}\mid \mathcal{D}\big)\\ &\quad + \mathbb{P}\left(\operatorname{card}(J_n^1)<\frac{n}{34}\text{ or }\operatorname{card}(J_n^1)<\frac{n}{34}\right)+\mathbb{P}(\mathcal{E}_1^{\mathrm{c}})+\mathbb{P}(\mathcal{E}_2^{\mathrm{c}})\\ &\le \frac{C}{n^{\alpha/(1-\alpha)+1/2}}\cdot\frac{C}{n^{1/2-\varepsilon}}+12\mathrm{e}^{-c'_*n}+\mathrm{e}^{-n/15}+\mathrm{e}^{-n/15}\\ &= \frac{C^2+o(1)}{n^{1+[\alpha/(1-\alpha)-\varepsilon]}}\end{aligned}$$

by (3.2) and (3.7). Assuming $\varepsilon\in(0,\alpha/(1-\alpha))$, the right-hand side is summable, and thus $\mathbb{P}(|X_n|\le\rho,\,|Y_n|-1\le\tau_{n+1}-\tau_n)$ is also summable in $n$. □

**Theorem 3.3.** *Let $d\ge 2$, and the rate be given by*

$$\lambda(t)=\begin{cases}\dfrac{1}{(\ln t)^\beta}, & t\ge \mathrm{e};\\ 0, & \textit{otherwise}.\end{cases}\tag{3.8}$$

*Then $Z(t)$ is transient as long as $\beta>2$.*

*Proof.* The proof is analogous to the proof of Theorem 3.2; we will only indicate how that proof should be modified for this case. As before, we assume that $\rho=1$ and $d=2$, without loss of generality. First, we prove the following lemma.

**Lemma 3.4.** *Let $\Lambda(T)=\int_0^T\lambda(s)\,\mathrm{d}s=\int_{\mathrm{e}}^T\mathrm{d}s/(\ln s)^\beta$. Then*

$$\lim_{T\to\infty}\frac{\Lambda(T)}{T(\ln T)^{-\beta}}=1.$$

*Proof of Lemma 3.4.* Fix an $\varepsilon\in(0,1)$. Then

$$\Lambda(T)=\int_{\mathrm{e}}^T\frac{\mathrm{d}s}{(\ln s)^\beta}=\int_{\mathrm{e}}^{T^{1-\varepsilon}}\frac{\mathrm{d}s}{(\ln s)^\beta}+\int_{T^{1-\varepsilon}}^T\frac{\mathrm{d}s}{(\ln s)^\beta}<T^{1-\varepsilon}+\frac{T}{(\ln T)^\beta(1-\varepsilon)^\beta}.$$

At the same time, trivially, $\Lambda(T)>(T-\mathrm{e})/(\ln T)^\beta$. Now, the limit of the ratio of the upper and the lower bounds of $\Lambda(T)$ can be made arbitrarily close to 1 by choosing a small enough $\varepsilon$. Hence the statement of the lemma follows. □

The rest of the proof goes along the same lines as that of Theorem 3.2. First, note that for some $c>0$ the event $\{\tau_n\le cn(\ln n)^\beta\}$ occurs finitely often almost surely. Indeed,

$$\begin{aligned}\tilde\Lambda=\Lambda(cn(\ln n)^\beta)&=\frac{cn(\ln n)^\beta}{[\,\ln(cn(\ln n)^\beta)]^\beta}(1+o(1))=\frac{(\ln n)^\beta}{(1+o(1))(\ln n)^\beta}cn(1+o(1))\\ &=(1+o(1))cn\end{aligned}$$

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by Lemma 3.4, and thus

$$\begin{aligned}\mathbb{P}(\tau_n \le cn(\ln n)^\beta) &= \mathbb{P}(N(cn(\ln n)^\beta) \ge n)\\ &= \mathrm{e}^{-\tilde\Lambda}\left(\frac{\tilde\Lambda^n}{n!} + \frac{\tilde\Lambda^{n+1}}{(n+1)!} + \frac{\tilde\Lambda^{n+2}}{(n+2)!} + \cdots\right)\\ &= \mathrm{e}^{-\tilde\Lambda}\frac{\tilde\Lambda^n}{n!}\left(1 + \frac{\tilde\Lambda}{(n+1)} + \frac{\tilde\Lambda^2}{(n+1)(n+2)} + \cdots\right)\\ &\le \mathrm{e}^{-\tilde\Lambda}\frac{\tilde\Lambda^n}{n!}\left(1 + \frac{\tilde\Lambda}{1!} + \frac{\tilde\Lambda^2}{2!} + \cdots\right)\\ &= \frac{\tilde\Lambda^n}{n!} = \frac{[(1+o(1))cn]^n}{n!} = \mathcal{O}\left(\frac{[(1+o(1))c]^n n^n}{n^n \mathrm{e}^{-n}\sqrt{n}}\right).\end{aligned}$$

This quantity is summable as long as $c\mathrm{e} < 1$, and the statement follows from the Borel–Cantelli lemma.

Second, for any positive $\varepsilon$, the event $\{\tau_n \ge c^* n(\ln n)^\beta\}$, where $c^* = 1+\varepsilon$, occurs finitely often, almost surely. Indeed,

$$\bar\Lambda := \Lambda(c^* n(\ln n)^\beta) = \frac{c^* n(\ln n)^\beta}{[\ln(c^* n(\ln n)^\beta)]^\beta}(1+o(1)) = c^* n(1+o(1)) \ge (1+\varepsilon/2)n$$

for large enough $n$ by Lemma 3.4. Hence, since $\bar\Lambda > n$,

$$\begin{aligned}\mathbb{P}(\tau_n \ge c^* n(\ln n)^\beta) &= \mathbb{P}(N(c^* n(\ln n)^\beta) \le n)\\ &= \mathrm{e}^{-\bar\Lambda}\left(1 + \bar\Lambda + \frac{\bar\Lambda^2}{2!} + \cdots + \frac{\bar\Lambda^n}{n!}\right)\\ &\le \mathrm{e}^{-\bar\Lambda}(n+1)\frac{\bar\Lambda^n}{n!}\\ &= \mathrm{e}^{-\bar\Lambda}(n+1)\frac{\bar\Lambda^n}{n^n\mathrm{e}^{-n}\sqrt{2\pi n}}(1+o(1))\\ &= (1+o(1))\sqrt{\frac{n}{2\pi}}\exp\left\{-n\left[\frac{\bar\Lambda}{n} - 1 - \ln\frac{\bar\Lambda}{n}\right]\right\},\end{aligned}$$

which is summable in $n$, as $c^* = 1+\varepsilon$ implies that the expression in square brackets is strictly positive (this follows from the easy fact that $c - 1 - \ln c > 0$ for $c > 1$). Hence, $\{\tau_n \ge c^* n(\ln n)^\beta\}$ happens finitely often, almost surely, as stated.

Third, the event $\{\tau_{n+1} - \tau_n \ge 2(\ln n)^{1+\beta}\}$ occurs finitely often, almost surely. This holds because, for all sufficiently large $n$, $\tau_{n+1} \le c^*(n+1)(\ln(n+1))^\beta$, and hence, for $t \le \tau_{n+1}$,

$$\lambda(t) \ge \frac{1}{[\ln(c(n+1)(\ln(n+1))^\beta)]^\beta} = \frac{1+o(1)}{(\ln n)^\beta},$$

and thus $(\tau_{n+1} - \tau_n)$ is stochastically smaller than an exponential random variable $\mathcal{E}$ with parameter $(1+o(1))/(\ln n)^\beta$. So, for all sufficiently large $n$,

$$\mathbb{P}(\tau_{n+1} - \tau_n \ge 2(\ln n)^{1+\beta}) \le \mathbb{P}(\mathcal{E} \ge 2(\ln n)^{1+\beta}) = \frac{1}{n^{2-o(1)}}, \tag{3.9}$$

which is summable in $n$, and we can apply the Borel–Cantelli lemma.

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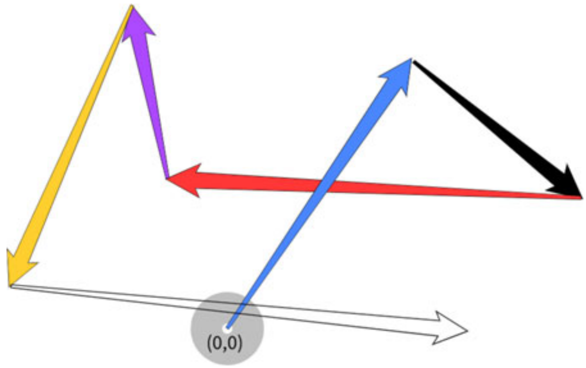


FIGURE 2. Model B: $\hat{W}_n$, the (projection of) a conservative random walk on $\mathbb{R}^2$.

Using the previous arguments for horizontal and vertical components, we obtain

$$\mathbb{P}(|X_n| \le 1) \le \frac{C}{\sqrt{n}(\ln n)^\beta}, \qquad \mathbb{P}(|Y_n| \le (\ln n)^{1+\beta}) \le (\ln n)^{1+\beta} \cdot \frac{C}{\sqrt{n}(\ln n)^\beta} = \frac{C \ln n}{\sqrt{n}}.$$

Again, similar to Theorem 3.2, the event $\{|X_n| \le \rho,\ |Y_n| \le 2(\ln n)^{1+\beta}\}$ has to happen infinitely often almost surely for $Z(t)$ to be recurrent.

Thus, it follows that

$$\begin{aligned}\mathbb{P}(Z(t) \text{ visits } [-1, 1]^2 \text{ for } t \in [\tau_n, \tau_{n+1}]) &\le \mathbb{P}(|X_n| \le 1,\ |Y_n| \le 2(\ln n)^{1+\beta}) + f(n)\\ &\le \frac{C^2}{n(\ln n)^{\beta-1}} + f(n) + g(n),\end{aligned}$$

where $f$ and $g$ are two summable functions over $n$, similar to Theorem 3.2. The right-hand side is summable over $n$ as long as $\beta > 2$. Hence, $Z(t)$ visits $[-1, 1]^2$ finitely often, almost surely. □

## 4. Analysis of Model B

Recall that in Model B, the vectors $\mathbf{f}_n$ are uniformly distributed over the unit sphere in $\mathbb{R}^d$.

Throughout this section, we again suppose that $\rho = 1$ and we show that the process is not 1-recurrent. The proof for general $\rho$ is analogous and is omitted.

Let $W_n = (X_n, Y_n, *, *, \ldots, *)$ be the embedded version of the process $Z(t) = (X(t), Y(t), *, *, \ldots, *)$; here, $X_n$ and $Y_n$ ($X(t)$ and $Y(t)$ respectively) stand for the process' first two coordinates. We denote the projection of $Z(t)$ ($W_n$ respectively) on the two-dimensional plane by $\hat{W}(t) = (X(t), Y(t))$ ($\hat{W}_n = (X_n, Y_n)$ respectively.) See Figure 2.

The following statement is quite intuitive.

**Lemma 4.1.** *Suppose $d \ge 3$, and let $\boldsymbol{f} = (f_1, f_2, \ldots, f_d)$ be a random vector uniformly distributed on the unit sphere $\mathcal{S}^{d-1}$ in $\mathbb{R}^d$. Then, for some $\gamma > 0$,*

$$\mathbb{P}\Big(\sqrt{f_1^2 + f_2^2} \ge \gamma\Big) \ge \frac{2}{3}.$$

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**Remark 4.1.** The statement is trivially true for the case $d = 2$ as well.

*Proof of Lemma* 4.1. We use the following well-known representation (see, e.g., [2, Section 2.5]) of $\mathbf{f}$:

$$\mathbf{f} = \left( \frac{\eta_1}{\|\eta\|}, \ldots, \frac{\eta_d}{\|\eta\|} \right),$$

where $\eta_i$, $i = 1, 2, \ldots$, are i.i.d. standard normal and $\|\eta\| = \sqrt{\eta_1^2 + \cdots + \eta_d^2}$. For some large enough $A > 0$,

$$\mathbb{P}\Big( \max_{i=3,\ldots,d} |\eta_i| < A \Big) = (\Phi(A) - \Phi(-A))^{d-2} \geq \sqrt{\frac{2}{3}},$$

where $\Phi(\cdot)$ is the distribution function of the standard normal random variable. Also, for some small enough $a \in (0, A)$, $\mathbb{P}( \max_{i=1,2} |\eta_i| > a) \geq \sqrt{\frac{2}{3}}$. On the intersection of these two independent events we have

$$f_1^2 + f_2^2 = \frac{\eta_1^2 + \eta_2^2}{(\eta_1^2 + \eta_2^2) + (\eta_3^2 + \cdots + \eta_d^2)} \geq \frac{a^2}{a^2 + (d-2)A^2} =: \gamma^2.$$

□

**Remark 4.2.** In fact, we can rigorously compute

$$\begin{aligned}
\mathbb{P}(f_1^2 + f_2^2 \geq \gamma^2) &= \mathbb{P}\left( \frac{\eta_1^2 + \eta_2^2}{\eta_1^2 + \eta_2^2 + \cdots + \eta_d^2} \geq \gamma^2 \right) \\
&= \mathbb{P}\left( \eta_1^2 + \eta_2^2 \geq \frac{\gamma^2}{1-\gamma^2} (\eta_3^2 + \cdots + \eta_d^2) \right) \\
&= \mathbb{P}\left( \chi^2(2) \geq \frac{\gamma^2}{1-\gamma^2} \chi^2(d-2) \right) \\
&= \iint_{x \geq (\gamma^2/(1-\gamma^2)) y \geq 0} \frac{\mathrm{e}^{-x/2}}{2} \cdot \frac{y^{d/2-2} \mathrm{e}^{-y/2}}{2^{d/2-1} \Gamma(d/2-1)} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^\infty \mathrm{e}^{-(y/2) \cdot (\gamma^2/(2(1-\gamma^2)))} \cdot \frac{y^{d/2-2} \mathrm{e}^{-y/2}}{2^{d/2-1} \Gamma(d/2-1)} \, \mathrm{d}y = (1-\gamma^2)^{d/2-1};
\end{aligned}$$

however, we do not really need this exact expression.

**Lemma 4.2.** *Suppose $d \geq 2$, and let $R_k = \{(x, y) \in \mathbb{R}^2 \colon k^2 \leq x^2 + y^2 \leq (k+1)^2\}$ be the ring of radius $k$ and width 1 centered at the origin. For some constant $C > 0$, possibly depending on $d$ and $\alpha$,*

$$\mathbb{P}\big(\hat{W}_n^{(1)} \in R_k\big) \leq \frac{Ck}{n^{1+(2\alpha/(1-\alpha))}}, \qquad \mathbb{P}\big(\hat{W}_n^{(2)} \in R_k\big) \leq \frac{Ck}{n(\ln n)^{2\beta}}$$

*for all large $n$, where $\hat{W}_t^{(1)}$ is the walk with rate* (2.2) *and $\hat{W}_t^{(2)}$ is the walk with rate* (3.8).

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*Proof.* Assume that the event $\mathcal{E}_1$ defined by (3.1) has occurred. We can write

$$\hat{W}_n = (X_n, Y_n) = \sum_{k=1}^{n} (\tau_k - \tau_{k-1})\tilde{\mathbf{f}}_k$$

where $\tilde{\mathbf{f}}_k = \ell_k[\mathbf{e_1} \cos(\phi_k) + \mathbf{e_2} \sin(\phi_k)]$, $\phi_k$, $k = 1, 2, \ldots,$ are uniformly distributed on $[-\pi, \pi]$, and $\ell_k$ is the length of the projection $\tilde{\mathbf{f}}_k$ of $\mathbf{f}_k$ on the two-dimensional plane. Note that the elements of the set $\{\ell_1, \eta_2, \eta_3, \ldots, \phi_1, \phi_2, \phi_3, \ldots\}$ are all independent. Also, define

$$\mathcal{D}_2 = \{\text{for at least half of the integers } i \in [n/2, n] \text{ we have } \ell_i \geq \gamma\},$$

where $\gamma$ is the constant from Lemma 4.1. Then $\mathbb{P}(\mathcal{D}_2^c) \leq 2e^{-n/36}$ by (2.1) and Lemma 4.1.

Let $\tilde{W}_n$ be the distribution of $\hat{W}_n \in \mathbb{R}^2$ *conditioned* on $\mathcal{E}_1 \cap \mathcal{D}_2$, and

$$\varphi_{\tilde{W}_n}(t) = \mathbb{E}\, e^{it\cdot\tilde{W}_n} = \mathbb{E} \exp\left\{ i \sum_{k=1}^{n} t \cdot \tilde{\mathbf{f}}_k(\tau_k - \tau_{k-1}) \right\}$$

be its characteristic function (here, $t \cdot \tilde{\mathbf{f}}_k = \ell_k(t_1 \cos(\phi_k) + t_2 \sin(\phi_k))$). We use the Lévy inversion formula, which allows us to compute the density of $\tilde{W}_n$, provided $|\varphi_{\tilde{W}_n}(t)|$ is integrable:

$$f_{\tilde{W}_n}(x, y) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{-i(t_1 x + t_2 y)} \varphi_{\tilde{W}_n}(t)\, dt_1\, dt_2 \leq \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \left|\varphi_{\tilde{W}_n}(t)\right| dt_1\, dt_2. \tag{4.1}$$

Let $\Delta_k = \tau_k - \tau_{k-1}$. Since $\phi_1, \ldots, \phi_n$ are i.i.d. Uniform$[-\pi, \pi]$ and independent of anything, we have

$$\varphi_{\tilde{W}_n}(t) = \mathbb{E}\left[ \prod_{k=1}^{n} \mathbb{E}\left(e^{i\Delta_k \ell_k (t_1 \cos\phi_k + t_2 \sin\phi_k)} \mid \tau_1, \ldots, \tau_n; \ell_1, \ldots, \ell_n\right) \right] = \mathbb{E}\left[ \prod_{k=1}^{n} J_0(\|t\| \Delta_k \ell_k) \right]$$

$$\Longrightarrow$$

$$|\varphi_{\tilde{W}_n}(t)| \leq \mathbb{E}\left[ \prod_{k=1}^{n} |J_0(\|t\| \Delta_k \ell_k)| \right],$$

where $\|t\| = \sqrt{t_1^2 + t_2^2}$, and $J_0(x) = \sum_{m=0}^{\infty} (-x^2/4)^m / m!^2$ is the Bessel $J_0$ function. Indeed, for any $x \in \mathbb{R}$, setting $\tilde{x} = x\sqrt{t_1^2 + t_2^2}$ and $\beta = \arctan(t_2/t_1)$, we get

$$\mathbb{E}\left[e^{ix(t_1 \cos\phi_k + t_2 \sin\phi_k)}\right] = \frac{1}{2\pi} \int_0^{2\pi} e^{ix(t_1 \cos\phi + t_2 \sin\phi)}\, d\phi = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tilde{x}(\cos\beta \cos\phi + \sin\beta \sin\phi)}$$

$$d\phi = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tilde{x}\cos(\phi+\beta)}\, d\phi = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tilde{x}\cos(\phi)}\, d\phi = \frac{1}{2\pi} \int_0^{2\pi} \cos(\tilde{x}\cos(\phi))\, d\phi = J_0(\tilde{x})$$

due to periodicity of $\cos(\cdot)$, the fact that the function $\sin(\cdot)$ is odd, and [1, (9.1.18)].

Let $\xi_k$ be the random variable with the distribution of $\Delta_k$ given $\tau_{k-1}$. Then, recalling that we are on the event $\mathcal{E}_1$, we get $\xi_k \succ \tilde{\xi}^{(1)}$ for $\hat{W}_n^{(1)}$ and $\xi_k \succ \tilde{\xi}^{(2)}$ for $\hat{W}_n^{(2)}$, where $\tilde{\xi}^{(1)}$

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($\tilde{\xi}^{(2)}$ respectively) is an exponential random variable with rate $1/a_1$ ($1/a_2$ respectively) with $a_1 = \tilde{c}_1 n^{\alpha/(1-\alpha)}$ and $a_2 = \tilde{c}_2(\ln n)^{\beta}$ for some constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$, and where '$\zeta_a \succ \zeta_b$' denotes that random variable $\zeta_a$ is stochastically larger than random variable $\zeta_b$. Consequently, setting $a = a_1$, $\tilde{\xi} = \tilde{\xi}^{(1)}$ or $a = a_2$, $\tilde{\xi} = \tilde{\xi}^{(2)}$, depending on which of the two models we are talking about, and $\mathcal{F}_k = \sigma(\tau_1, \ldots, \tau_k)$, we get

$$\begin{aligned}
\mathbb{E}[|J_0(\|t\|\Delta_k\ell_k)| \mid \mathcal{F}_{k-1}, \ell_k = \ell] &= \mathbb{E}\, |J_0(\|t\|\xi_k\ell)| \\
&\le \mathbb{E}\, G(\|t\|\xi_k\ell) \\
&\le \mathbb{E}\, G(\|t\|\tilde{\xi}\ell) \\
&= \int_0^\infty a^{-1}\mathrm{e}^{-y/a} G(\|t\|y\ell)\,\mathrm{d}y \\
&= \int_0^\infty \mathrm{e}^{-u} G(su)\,\mathrm{d}u \\
&= \int_0^\infty \frac{\mathrm{e}^{-u}}{\sqrt[4]{1+s^2u^2}}\,\mathrm{d}u =: h(s, \ell),
\end{aligned} \tag{4.2}$$

where $s = a\ell\|t\|$, since $|J_0(x)| \le G(x)$ by (A.1) and the facts that $G(\cdot)$ is a decreasing function and $\xi_k \succ \tilde{\xi}$.

We now estimate the function $h(t, \ell)$. Since $|J_0(x)| \le 1$, we trivially get $0 \le h(s, \ell) \le 1$. Additionally, for all $s > 0$,

$$h(s, \ell) \le \int_0^\infty \frac{\mathrm{e}^{-u}}{\sqrt{su}}\,\mathrm{d}u = \sqrt{\frac{\pi}{s}} = \sqrt{\frac{\pi}{a\ell\|t\|}}.$$

Let $n/2 \le j_1 < j_2 < \cdots < j_m \le n$ be the indices $i \in [n/2, n]$ for which $\ell_i \ge \gamma$. Since $|J_0(x)| \le 1$, we have, from (4.2),

$$\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^n |J_0(\|t\|\Delta_k\ell_k)|\right] &\le \mathbb{E}\left[\prod_{i=1}^m \left|J_0\left(\|t\|\Delta_{j_i}\ell_{j_i}\right)\right\|\right] \\
&= \mathbb{E}\left[\mathbb{E}\left(\left|J_0\left(\|t\|\Delta_{j_m}\ell_{j_m}\right)\right| \mid \mathcal{F}_{j_m-1}, \ell_{j_m}\right)\prod_{i=1}^{m-1}\left|J_0\left(\|t\|\Delta_{j_i}\ell_{j_i}\right)\right|\right] \\
&\le \sqrt{\frac{\pi}{a\gamma\|t\|}} \cdot \mathbb{E}\left[\prod_{i=1}^{m-1}\left|J_0\left(\|t\|\Delta_{j_i}\ell_{j_i}\right)\right|\right].
\end{aligned}$$

By iterating this argument for $i = j_{m-1}, j_{m-2}, \ldots, j_1$, we get

$$\left|\varphi_{\tilde{W}_n}(t)\right| \le \mathbb{E}\left[\prod_{k=1}^n |J_0(\|t\|\Delta_k\ell_k)|\right] \le \left(\sqrt{\frac{\pi}{a\gamma\|t\|}}\right)^{n/4}$$

(recall that $m \ge n/4$ on $\mathcal{D}_1$).

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Now consider two cases. For $\|t\| \geq \frac{2\pi}{a\gamma}$, part of the inversion formula gives

$$\begin{aligned}
\iint_{\|t\|>2\pi/a\gamma} \left|\varphi_{\tilde{W}_n}(t)\right| \, \mathrm{d}t_1 \, \mathrm{d}t_2 &\leq \iint_{\|t\|\geq 2\pi/a\gamma} \left(\frac{\pi}{\gamma\|t\|a}\right)^{n/8} \mathrm{d}t_1 \, \mathrm{d}t_2 \\
&= \left(\frac{\pi}{a\gamma}\right)^2 \int_0^{2\pi} \mathrm{d}\theta \int_2^\infty \frac{r\,\mathrm{d}r}{r^{n/8}} \\
&= \frac{2\pi^3}{a^2\gamma^2} \cdot \frac{r^{2-n/8}}{2-n/8}\bigg|_2^\infty = o(2^{-n/8}) \qquad (4.3)
\end{aligned}$$

by changing the variables $t_1 = (\pi/a\gamma)r\cos\theta$, $t_2 = (\pi/a\gamma)r\sin\theta$.

On the other hand, for $\|t\| \leq 2\pi/a\gamma$, when $\ell \geq \gamma$ and thus $\sigma := a\gamma\|t\| \leq s$,

$$\begin{aligned}
h(s, \ell) &= \int_0^\infty \frac{\mathrm{e}^{-u}\,\mathrm{d}u}{\sqrt[4]{1+s^2u^2}} \\
&\leq \int_0^\infty \frac{\mathrm{e}^{-u}\,\mathrm{d}u}{\sqrt[4]{1+\sigma^2u^2}} \\
&\leq \int_0^1 \frac{\mathrm{e}^{-u}\,\mathrm{d}u}{\sqrt[4]{1+\sigma^2u^2}} + \int_1^\infty \frac{\mathrm{e}^{-u}\,\mathrm{d}u}{\sqrt[4]{1+\sigma^2u^2}} \\
&\leq \int_0^1 \left(1 - \frac{\sigma^2u^2}{50}\right)\mathrm{e}^{-u}\,\mathrm{d}u + \int_1^\infty \mathrm{e}^{-u}\,\mathrm{d}u \\
&= -\int_0^1 \frac{\sigma^2u^2}{50}\mathrm{e}^{-u}\,\mathrm{d}u + \int_0^\infty \mathrm{e}^{-u}\,\mathrm{d}u \\
&= 1 - 0.0016\ldots\sigma^2 \leq \exp\left\{-\frac{\sigma^2}{700}\right\} = \exp\left\{-\frac{\gamma^2\|t\|^2a^2}{700}\right\},
\end{aligned}$$

since $(1+x^2)^{-1/4} \leq 1 - x^2/50$ for $0 \leq x \leq 7$, and $\sigma \leq 2\pi < 7$ by assumption. By iterating the same argument as before, we get

$$\varphi_{\tilde{W}_n} \leq h(s, \ell)^{n/4} \leq \exp\left\{-\frac{n\gamma^2\|t\|^2a^2}{2800}\right\}.$$

Consequently,

$$\begin{aligned}
\iint_{\|t\|\leq 2\pi/a\gamma} \left|\varphi_{\tilde{W}_n}(t)\right| \, \mathrm{d}t_1 \, \mathrm{d}t_2 &\leq \iint_{\|t\|\leq 2\pi/a\gamma} \exp\left\{-\frac{n\gamma^2\|t\|^2a^2}{2800}\right\} \mathrm{d}t_1\mathrm{d}t_2 \\
&\leq \frac{1400}{a^2\gamma^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^\infty \exp\left\{-\frac{nr^2}{2}\right\} r\,\mathrm{d}r = \frac{2800\pi}{na^2\gamma^2} \qquad (4.4)
\end{aligned}$$

by changing the variables $t_1 = (10\sqrt{14}/a\gamma)r\cos\theta$, $t_2 = (10\sqrt{14}/a\gamma)r\sin\theta$.

Finally, recalling that $a_1 \sim n^{\alpha/(1-\alpha)}$ for $\hat{W}_t^{(1)}$ and $a_2 \sim (\ln n)^\beta$ for $\hat{W}_t^{(2)}$, from (4.1), (4.3), and (4.4) we obtain

$$f_{\tilde{W}_n^{(1)}}(x, y) \leq O\left(\frac{1}{n^{1+(2\alpha/(1-\alpha))}}\right), \qquad f_{\tilde{W}_n^{(2)}}(x, y) \leq O\left(\frac{1}{n(\ln n)^{2\beta}}\right).$$

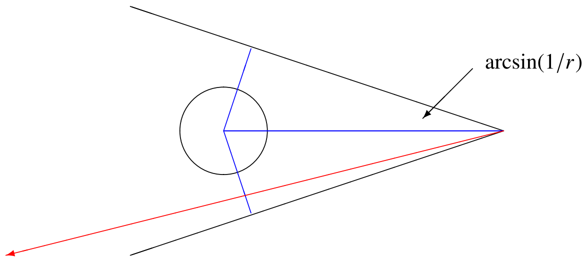


FIGURE 3. A ray from $\hat{W}_n$ passes through the unit circle.

Hence,

$$\begin{aligned}\mathbb{P}(\hat{W}_n \in R_k) &\le \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{D}_2^c) + \mathbb{P}(\tilde{W}_n \in R_k)\\ &= 4e^{-n/36} + \iint_{(x,y)\in R_k} f_{\tilde{W}_n}(x, y)\, dx\, dy\\ &= 2\pi k \times \begin{cases} O\left(\frac{1}{n^{1+(2\alpha/(1-\alpha))}}\right) & \text{for } W_n^{(1)},\\ O\left(\frac{1}{n(\ln n)^{2\beta}}\right) & \text{for } W_n^{(2)},\end{cases}\end{aligned}$$

since the area of $R_k$ is $\pi(2k+1)$. □

**Lemma 4.3.** *Suppose $d \ge 2$ and $\hat{W}_n = r > 1$. Then*

$$\mathbb{P}(\|\hat{W}(t)\| \le 1 \text{ for some } t \in [\tau_n, \tau_{n+1}]) \le \frac{1}{2r}.$$

*Proof.* First, note that $\hat{W}_n$ lies outside the unit circle on $\mathbb{R}^2$. The projection $\tilde{\mathbf{f}}_{n+1}$ of $\mathbf{f}_{n+1}$ on the first two coordinates' plane has an angle $\phi$ uniformly distributed over $[0, 2\pi]$. The probability in the statement of the lemma is monotone increasing in $\tau_{n+1}$, and hence is bounded above by

$$\mathbb{P}(\text{the infinite ray from } \hat{W}_n \text{ in the direction } \tilde{\mathbf{f}}_{n+1} \text{ passes through the unit circle}) = \frac{2 \arcsin(1/r)}{2\pi}$$

(see Figure 3.) Finally, $\arcsin(x) \le \pi x/2$ for $0 \le x \le 1$. □

**Theorem 4.1.** *Let $d \ge 2$ and the walk $Z(t)$ have either the rate (2.2) with $\alpha \in (0, 1)$ or the rate (3.8) with $\beta > 2$. Then $Z(t)$ is transient a.s.*

*Proof.* As mentioned before, we only show that the walk is not $\rho$-recurrent for $\rho = 1$. To do that, it will suffice that, a.s., there will be only finitely many $n$ such that the event $A_n = \{\|\hat{W}(t)\| \le 1 \text{ for some } t \in [\tau_n, \tau_{n+1}]\}$ occurs. Indeed, fix a positive integer $n$. At this time,

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$\hat{W}_n \in R_k$ for some $k \in \{0, 1, 2, \ldots\}$. According to Lemma 4.3,

$$\mathbb{P}(A_n \mid \hat{W}_n \in R_k) \le \begin{cases} 1 & \text{if } k = 0, \\ \dfrac{1}{2k} & \text{if } k \ge 1. \end{cases}$$

Fix some very small $\varepsilon > 0$. Then

$$\begin{aligned}\mathbb{P}(A_n) &\le \sum_{k=1}^{n^{\alpha/(1-\alpha)+\varepsilon}} \mathbb{P}\big(A_n \mid \hat{W}_n^{(1)} \in R_k\big)\mathbb{P}\big(\hat{W}_n^{(1)} \in R_k\big) + \mathbb{P}\big(A_n \mid \|\hat{W}_n^{(1)}\| \ge n^{\alpha/(1-\alpha)+\varepsilon}\big) \\ &\le \frac{Cn^{\alpha/(1-\alpha)+\varepsilon}}{n^{1+(2\alpha/(1-\alpha))}} + \mathbb{P}\big(\tau_{n+1} - \tau_n \ge n^{\alpha/(1-\alpha)+\varepsilon} - 1\big) \le \frac{C}{n^{1+(\alpha/(1-\alpha))-\varepsilon}} + 3\mathrm{e}^{-c_1^* n^{\varepsilon}}\end{aligned}$$

by (3.6) and Lemma 4.2. Hence, $\sum_n \mathbb{P}(A_n) < \infty$ and the result follows from the Borel–Cantelli lemma.

Similarly, in the other case,

$$\begin{aligned}\mathbb{P}(A_n) &\le \sum_{k=1}^{2(\ln n)^{1+\beta}} \mathbb{P}\big(A_n, \hat{W}_n^{(2)} \in R_k\big) + \mathbb{P}\big(A_n \mid \|\hat{W}_n^{(2)}\| \ge 2(\ln n)^{1+\beta}\big) \\ &\le \frac{2C(\ln n)^{1+\beta}}{n(\ln n)^{2\beta}} + \mathbb{P}(\tau_{n+1} - \tau_n \ge 2(\ln n)^{1+\beta} - 1) \le \frac{2C}{n(\ln n)^{\beta-1}} + \frac{1}{n^2}\end{aligned}$$

for all sufficiently large $n$ by (3.9) and Lemma 4.2. Hence, $\sum_n \mathbb{P}(A_n) < \infty$ for $\beta > 2$, and the result again follows from the Borel–Cantelli lemma. □

## Appendix A. Properties of the Bessel function $J_0$

Let

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{(m!)^2}$$

be the Bessel function of the first kind. The following statement might be known, but we could not find it in the literature.

**Claim A.1.** *For all* $x \ge 0$,

$$|J_0(x)| \le \frac{1}{\sqrt[4]{1+x^2}} =: G(x). \tag{A.1}$$

*Proof.* From [15, Theorem 1 and (1)] we get (with $\nu = 0$, $\mu = 3$)

$$J_0^2(x) \le \frac{4(4x^2-5)}{\pi((4x^2-3)^{3/2}-3)}, \quad x \ge 1.13.$$

This inequality also implies that

$$|J_0(x)| \le \frac{1}{\sqrt[4]{1+x^2}}, \quad x \ge 1.13.$$

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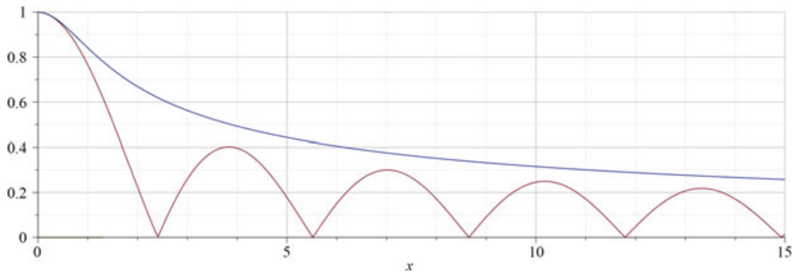


FIGURE 4. $|J_0(x)|$ (red) and its upper bound $G(x)$ (blue).

At the same time, for $0 \le x \le 2$ (using [1, 9.1.14]),

$$J_0^2(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(k!)^4} \left(\frac{x}{2}\right)^{2k} \le 1 - 2\left(\frac{x}{2}\right)^2 + \frac{3}{2}\left(\frac{x}{2}\right)^4 - \frac{5}{9}\left(\frac{x}{2}\right)^6 + \frac{35}{288}\left(\frac{x}{2}\right)^8,$$

and at the same time

$$\frac{1}{\sqrt{1+x^2}} \ge 1 - 2\left(\frac{x}{2}\right)^2 + \frac{3}{2}\left(\frac{x}{2}\right)^4 - \frac{5}{9}\left(\frac{x}{2}\right)^6 + \frac{35}{288}\left(\frac{x}{2}\right)^8,$$

yielding (A.1). □

Observe as well that $0 \le G(x) \le 1/\sqrt{x}$, and that $G$ is a decreasing function for $x \ge 0$; see Figure 4.

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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