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BOOK REVIEWS

NUMMELIN, E. *General irreducible Markov chains and non-negative operators* (Cambridge Tracts in Mathematics 83, Cambridge University Press, 1984), 156 pp. £20.

A major theme of modern probability theory is the study of random processes; systems that evolve in time according to definite probabilistic laws. The concept of the *Markov property* pervades this study; a process is *Markov* if, in order to make predictions of its future behaviour and knowing only its present state, one need not attempt to discover its past history. Markov processes are useful approximations in the description of many systems arising in applications; they arise in the theories of queues, storage systems, random particle dynamics, biological growth, human and animal populations, and in many other cases.

Preliminary classification of a Markov process is based first on whether it is observed over a continuous period of time (continuous-time) or at isolated instants (discrete-time), and second on whether its state space S is finite, countable, or uncountable. The restriction to discrete-time simplifies matters considerably and is appropriate in many applications; we then call the process a *Markov chain* though this terminology is not universal. If furthermore the statistics of the Markov chain are stationary in time then the stochastic dynamics of the process are summarised in the *transition kernel* $K(x,A)$. This gives the probability that the system will be in the set A at time $n+1$ given that its state at time n is x. The kernel corresponds to a non-negative operator defined on function spaces over S. So the probability theory of Markov chains is entwined with analysis of non-negative operators.

Evidently X can be decomposed by determining the subsets of S that are invariant under the stochastic dynamics as summarised by the kernel K. If X does not admit such a decomposition it is *irreducible*; a general theory must give a thorough discussion of irreducible Markov chains.

If S is countable then K is represented by a stochastic matrix and behaviour of X is determined by countably-infinite matrix theory. Alternatively its behaviour can be analysed by the elegant and intuitively appealing *coupling technique* (due to Doeblin, 1938). A second Markov chain Y is constructed on S using the same kernel K. The movement of Y is independent of X up to the first time T that Y equals X, and afterwards Y and X move as one. Many results on the asymptotic behaviour of X flow directly from analysis of the coupling time T.

The coupling technique is deservedly popular in current research. It is subtle and powerful and yet accessible enough to find a place in recent undergraduate texts.

The simple coupling above breaks down if S is uncountable; in general X and Y almost surely never meet. However it is possible to construct pairs of processes that will couple if an inequality holds such as

$$K(x,A) > \lambda\mu(A) \text{ for all } x \text{ in } B \text{ and all measurable sets } A.$$

Here λ is a positive real, B is a measurable subset of S, and μ is a probability measure on S. Using this inequality one may construct two Markov chains X and Y that initially evolve independently but couple with probability λ whenever X and Y are simultaneously in B. This approach (if generalised) can be used systematically to represent general state space Markov chains as simpler countable state space chains. Indeed results then flow from the special branch of random processes studied by renewal theory. So coupling techniques reduce yet another bastion of probability!

The monograph under review uses coupling to expound the general theory of Markov chains. It concentrates on the analytical details and uses a compressed style; applications appear only in summary form. Discussion is restricted to general questions such as convergence of transition probabilities and kernels, and recurrence. For a short text this is a good decision; the probabilist-technician will find this a helpful manual about a very useful tool. But as with all manuals it is better used than read. The reader will find that the abstractions and general treatment come to life when confronted by specific application to particular Markov chains.

One application of these ideas (not covered in the book) is found in stochastic differential geometry. Consider a Brownian motion on a compact Riemannian manifold. Many workers have investigated the way in which the Brownian path wraps itself round the manifold, in homotopy or in homology. Various approaches have been adopted; one of these (exploited to great effect by Lyons and Sullivan, 1984) uses a coupling similar to the one described above. The Brownian motion is sampled whenever it visits a particular small sphere; the Markov chain is made up of the position of the Brownian motion on the sphere together with the homotopy type of its most recent excursion through the whole manifold (to make this precise one needs the notion of the local time on the sphere of the Brownian motion). The result is *almost* a random walk on the homotopy group of the manifold. However to obtain a Markov property one must employ information on where the Brownian motion last hit the small sphere. This moves the problem into the province of Markov chains on uncountable state space. It should be plain from the brief description here how coupling methods can be applied. One can establish estimates on harmonic measure and thus obtain bounds (of the form given above) on the transition kernel.

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BERNDT, BRUCE C. *Ramanujan's Notebooks*, Part I (Springer-Verlag, 1985), x+357 pp. DM 188.

During the years before he came to Cambridge in 1914 the distinguished Indian mathematician Srinivasa Ramanujan filled three notebooks with mathematical formulae and theorems, most of them being stated without proof. The third and shortest of the notebooks consists of approximately 100 pages of unorganized material. The second, consisting of 252 pages divided into 21 chapters, is a revised and enlarged edition of the first notebook and is thus the fullest and most interesting of the three. At the time when G. H. Hardy and B. M. Wilson were preparing Ramanujan's collected papers for publication by the Cambridge University Press it had been intended to publish the notebooks as an appendix. However, this proposal was abandoned, largely for financial reasons, but possibly also because Hardy realized (as a result of his excessive labour in preparing chapter 12 of the first notebook for publication) that to accomplish the necessary editorial work satisfactorily was a mammoth task likely to extend over many years.

By the mid twenties, however, G. N. Watson of Birmingham had begun, in collaboration with a colleague, to write a number of papers on "Theorems stated by Ramanujan" and, for this reason, Hardy gradually over the years passed on to him all his Ramanujan material. Watson and B. M. Wilson divided the 21 chapters of Notebook 2 between them, Wilson taking chapters 2–13 and Watson chapters 14–21; chapter 1, on magic squares, is of less interest, having been written when Ramanujan was a schoolboy, and containing nothing new. Both men did a great deal of work on their assigned chapters, but Wilson died suddenly in 1935 in Dundee, where he had moved from Liverpool to succeed Professor Steggall. His material was passed to Watson, whose output of work on the notebooks gradually declined and had ceased altogether by the beginning of the second world war; nevertheless he retained his interest in Ramanujan until his death in 1965. The considerable dossier of his papers found in his house after his death was passed to Trinity College, Cambridge, of which he, Hardy and Ramanujan had all been Fellows. There it lay, comparatively unnoticed, until Professor Berndt assumed the arduous task of continuing the work begun by Watson and Wilson and editing the notebook for publication. In so doing he has done a tremendous service to the mathematical world by his careful and critical appraisal of Ramanujan's