Nagoya Math. J., **256** (2024), 970–1021
DOI 10.1017/nmj.2024.17

BIRATIONAL GEOMETRY OF SEXTIC DOUBLE SOLIDS WITH A COMPOUND A_n SINGULARITY

ERIK PAEMURRU

Abstract. Sextic double solids, double covers of $\mathbb{P}^3$ branched along a sextic surface, are the lowest degree Gorenstein terminal Fano 3-folds, hence are expected to behave very rigidly in terms of birational geometry. Smooth sextic double solids, and those which are $\mathbb{Q}$-factorial with ordinary double points, are known to be birationally rigid. In this paper, we study sextic double solids with an isolated compound A_n singularity. We prove a sharp bound $n \leq 8$, describe models for each n explicitly, and prove that sextic double solids with $n > 3$ are birationally nonrigid.

Contents

Received August 13, 2021. Revised October 2, 2023. Accepted May 16, 2024.
2020 Mathematics Subject Classification: Primary 14J45; Secondary 14J30, 14J17, 14E30, 14E05.
Keywords: Fano 3-folds, Sarkisov program, birational rigidity.
The author was supported by the Engineering and Physical Sciences Research Council (Grant No. 1820497) while in Loughborough University and the London Mathematical Society Early Career Fellowship (Grant No. ECF-1920-24) while in Imperial College London.

© *The Author(s), 2024. Published by Cambridge University Press on behalf of Foundation Nagoya Mathematical Journal.*

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

§1. Introduction

We work with projective varieties over $\mathbb{C}$. Classification of algebraic varieties is one of the fundamental goals in algebraic geometry. The Minimal Model Program says that every variety is birational to either a minimal model or a Mori fiber space. Two Mori fiber spaces are birational if they are connected by a sequence of Sarkisov links (see [17], [26]). In the extreme case, the Mori fiber space is *birationally rigid*, meaning that it is essentially the unique Mori fiber space in its birational class.

Examples of Mori fiber spaces include Fano varieties. The first birational rigidity result was in the seminal paper by Iskovskikh and Manin [28] for smooth quartic 3-folds in $\mathbb{P}^4$. A wealth of examples of birationally rigid varieties was given in [15], [19], by showing that every quasismooth member of the 95 families of Fano 3-folds that are hypersurfaces in weighted projective spaces is birationally rigid. One major consequence of birational rigidity is nonrationality. Birational rigidity remains an active area of research (see [3], [13], [16], [22], [23], [38], [45]).

Among smooth Fano 3-folds, the projective space has the highest degree (64), and sextic double solids, double covers of $\mathbb{P}^3$ branched along a sextic surface, have the least degree (2). In [29], it is proved that smooth sextic double solids are birationally rigid. It is interesting to see how this changes as we impose singularities on the variety. The paper [46] proved that sextic double solids stay birationally rigid if we impose an ordinary double point, meaning the 3-fold A_1 singularity $x_1^2+x_2^2+x_3^2+x_4^2$. A sextic double solid can have up to 65 singular points (see [6], [31], [51]), and for each $n \leq 65$, there exists a sextic double solid with exactly n ordinary double points and smooth otherwise (see [5], [12]). A sextic double solid with only ordinary double points is birationally rigid if and only if it is factorial, which is true, for example, if it has at most 14 ordinary double points (see [14, Th. B]).

The next natural question is to consider more complicated singularities in the Mori category. We study sextic double solids with an isolated *compound* A_n singularity, also called a cA_n singularity, meaning that the general section through the point is the Du Val A_n singularity $x_1x_2+x_3^{n+1}$. A cA_n singularity is locally analytically given by $x_1x_2+h(x_3,x_4)$ where the least degree among monomials in h is $n+1$. The first main result of the paper is describing sextic double solids with an isolated cA_n singularity.

THEOREM (see Theorem A). *If a sextic double solid has an isolated cA_n point, then $n \leq 8$.*

Moreover, in Theorem A, we explicitly parametrize all sextic double solids with an isolated cA_n singularity for every $n \leq 8$. These form 11 families, as there are four families for cA_7. Every family except for family 7.4 contains members that are Mori fiber spaces over a point.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Table 1. Birational models for general sextic double solids that are Mori fiber spaces with an isolated cA_n singularity

$$\begin{array}{ccccc} & & Y_0 & \dashrightarrow & Y_2 & & \\ & \overset{\varphi}{\swarrow} & & & & \overset{\psi}{\searrow} & \\ cA_n \in X_6 \subseteq \mathbb{P}(1^4,3) & & & & & & Z \end{array}$$

cA_n	Weighted blowup φ	$\dashrightarrow$	Weighted blowup or fibration ψ	Z
cA_4	$(3,2,1,1)$	10 Atiyah flops	$(\frac{1}{4},\frac{1}{4},\frac{3}{4})$	$\frac{1}{4}(1,1,3) \in Z_{5,6} \subseteq \mathbb{P}(1^3,2,3,4)$
cA_5	$(3,3,1,1)$	4 Atiyah flops	$(3,3,1,1)$	$cA_5 \in Z_6 \subseteq \mathbb{P}(1^4,3)$, $X \ncong Z$ if general
cA_6	$(4,3,1,1)$	2 Atiyah flops, then $(4,1,1,-2,-1;2)$-flip	$(3,1,1,1)$	$cA_3 \in Z_5 \subseteq \mathbb{P}(1^4,2)$
cA_7, 1	$(4,4,1,1)$	two $(4,1,1,-2,-1;2)$ flips	$(1,1,1,1)$	ODP $\in Z_{3,4} \subseteq \mathbb{P}(1^4,2^2)$
cA_7, 2	$(4,4,1,1)$	Atiyah flop, then two $(4,1,-1,-3)$-flips	$(3,3,2,1)$	$cA_2 \in Z_{2,4} \subseteq \mathbb{P}(1^5,2)$
cA_7, 3	$(4,4,1,1)$	2 Atiyah flops	dP_2-fibration	$\mathbb{P}^1$
cA_8	$(5,4,1,1)$	$(4,1,1,-2,-1;2)$-flip	$(3,2,2,1,5)$	$cD_4 \in Z_{3,3} \subseteq \mathbb{P}(1^5,2)$

We say a few words on bounding the number of cA_n singularities. It is clear that an isolated cA_n singularity has Milnor number at least n^2. Since the third Betti number of a smooth sextic double solid is 104 (see [30, Table 12.2]), an argument similar to [2, §3.2] shows that the total Milnor number of a sextic double solid which is a Mori fiber space is at most 104. This gives the bounds that a Mori fiber space sextic double solid can have up to 1 cA_8 singularity, or up to 2 cA_7 singularities, or up to 2 cA_6 singularities, …, or up to 26 cA_2 singularities. We do not expect these bounds to be sharp, as already for ordinary double points it gives an upper bound of 104, far from the actual 65. Using Theorem A, it is possible to construct sextic double solids with a cA_8 point, a cA_3 point, and two ordinary double points with both total Milnor and total Tjurina number at least 66.

The second main result is the following theorem.

THEOREM (see Theorem B and Proposition 5.6). *A general sextic double solid which is a Mori fiber space with an isolated cA_n singularity where $n \geq 4$ is not birationally rigid.*

Birational nonrigidity for a sextic double solid X is proved by describing a birational model, meaning a Mori fiber space $T \to S$ such that X and T are birational. We find the birational models by explicitly constructing a Sarkisov link for each family of sextic double solids, under the generality conditions described in Definition 5.1. Table 1 gives an overview of the Sarkisov links $X \leftarrow Y_0 \dashrightarrow Y_2 \to Z$ and the birational models, which are either fibrations $Y_2 \to Z$ or Fano varieties Z. In the latter case, $Y_2 \to Z$ is a divisorial contraction to the given singular point. The morphism $Y_0 \to X$ is a divisorial contraction with center the cA_n point. The birational maps $Y_0 \dashrightarrow Y_2$ are isomorphisms in codimension 1.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Note that when we say that a birational map $Y_0 \dashrightarrow Y_1$ is k Atiyah flops, then we mean that algebraically it is one flop, contracting k curves to k points and extracting k curves, and locally analytically around each of those points, it is an Atiyah flop. Similarly for flips. Also note that the Sarkisov link to a sextic double solid with a cA_4 singularity was already described in [43, §9, No. 9], starting from a general quasismooth complete intersection $X_{5,6} \subseteq \mathbb{P}(1,1,1,2,3,4)$.

We briefly describe the proof. The first step in the Sarkisov link starting from a Fano variety X is a divisorial contraction $Y \to X$. Kawakita described divisorial contractions to cA_n points locally analytically, showing that they are certain weighted blowups. To construct Sarkisov links, we need a global description. In Proposition 4.6 and Lemma 4.9, we show how to construct divisorial contractions to cA_n points algebraically on affine hypersurfaces, and use this in Section 5 to construct divisorial contractions $Y \to X$ for (projective) sextic double solids X. Using unprojection techniques (see [44] for a general theory of unprojection), we find an embedding of Y inside a toric variety T, such that the 2-ray link of T restricts to a Sarkisov link for X (following [4], [10]).

If we try the same methods as in the proof of Theorem B on sextic double solids with a cA_n singularity where $n \leq 3$, then we do not find any new birational models. More precisely, a $(3,1,1,1)$-Kawakita blowup of a cA_3 singularity on a general Mori fiber space sextic double solid initiates a Sarkisov link to itself $X \dashrightarrow X$. A $(2,2,1,1)$-Kawakita blowup for a cA_3 singularity, a $(2,1,1,1)$-Kawakita blowup for an $x_1x_2 + x_3^3 + x_4^3$ singularity, and the (usual) blowup for an ordinary double point on a general Mori fiber space sextic double solid initiate *bad links*, which end in either a nonterminal 3-fold or a K3-fibration. These are 2-ray links which are not Sarkisov links, where in the last step of the 2-ray game only K-trivial curves are contracted, leaving the Mori category. We expect that general Mori fiber space sextic double solids with a cA_3 singularity are birationally rigid, and with certain cA_2 or cA_1 singularities are birationally superrigid.

Organization of the paper

In Sections 2.1, 2.3, and 2.5, we give known results that we use, respectively, in Sections 3–5. In Section 3, we construct a parameter space of sextic double solids in Theorem A with an isolated cA_n singularity. In Section 4, we explain the relationship between algebraic and local analytic weighted blowups, and in Proposition 4.6 and the technical Lemma 4.9, we show how to construct divisorial contractions to cA_n points algebraically on affine hypersurfaces. In Section 5, we construct birational models for general sextic double solids which are Mori fiber spaces with an isolated cA_n singularity where $n \geq 4$, thereby showing that they are not birationally rigid. We treat the seven families separately.

§2. Preliminaries

An algebraic variety is an integral separated scheme of finite type over the complex numbers $\mathbb{C}$. When we say *morphism*, we mean a morphism over $\mathbb{C}$.

We study sextic double solids, which are double covers of the projective 3-space branched along a sextic surface. We use the following equivalent characterization.

DEFINITION 2.1. A *sextic double solid* is the variety given by the zero locus of an irreducible polynomial $w^2 + g$ in the weighted projective space $\mathbb{P}(1,1,1,1,3)$ with variables x, y, z, t, w, where $g \in \mathbb{C}[x,y,z,t]$ is homogeneous of degree 6.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

2.1 Singularity theory

We recall some results from the singularity theory of complex analytic spaces and terminal singularities.

We denote the variables on $\mathbb{C}^n$ by $\boldsymbol{x} = (x_1, \ldots, x_n)$, where n is a positive integer. Let $\mathbb{C}\{\boldsymbol{x}\}$ denote the convergent power series ring. The zero set of an ideal $I \subseteq \mathbb{C}\{\boldsymbol{x}\}$ is denoted by $\mathbb{V}(I)$, where I is either an ideal of regular functions or holomorphic functions, depending on the context. Given a point $P \in \mathbb{V}(I)$, the pair $(\mathbb{V}(I), P)$ denotes the (possibly reducible or non-reduced) complex space subgerm of $(\mathbb{C}^n, P)$ given by I. Given a regular or holomorphic function f on a variety or a complex space X, denote the nonzero locus of f by X_f. Given positive integer weights $\boldsymbol{w} = (w_1, \ldots, w_n)$ for $\boldsymbol{x}$, we can write a nonzero polynomial or power series f as a sum of its weighted homogeneous parts f_i. Then, the *weight* of f, denoted $\mathrm{wt}(f)$, is the least nonnegative integer d such that $f_d \neq 0$. We define $\mathrm{wt}(0) = \infty$. If $\boldsymbol{w} = (1, \ldots, 1)$, then d is called the *multiplicity*, denoted $\mathrm{mult}(f)$. A *hypersurface singularity* is a complex analytic space germ (not necessarily irreducible or reduced) that is isomorphic to $(\mathbb{V}(f), \mathbf{0})$ for some $f \in \mathbb{C}\{\boldsymbol{x}\}$. A singularity is *isolated* if it has a smooth analytic punctured neighborhood.

Definition 2.2 [25, Def. 2.9]. Let $f, g \in \mathbb{C}\{\boldsymbol{x}\}$.

(a) We say f and g are *right equivalent* if there exists a biholomorphic map germ $\varphi\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that $g = f \circ \varphi$.
(b) We say f and g are *contact equivalent* if there exist a biholomorphic map germ $\varphi\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ and a unit $u \in \mathbb{C}\{x_1, \ldots, x_n\}$ such that $g = u(f \circ \varphi)$.

Remark 2.3 [25, Rem. 2.9.1(3)]. Two convergent power series $f, g \in \mathbb{C}\{\boldsymbol{x}\}$ are contact equivalent if and only if the complex analytic space germs $(\mathbb{V}(f), \mathbf{0})$ and $(\mathbb{V}(g), \mathbf{0})$ are isomorphic.

We use the following proposition in Section 3 to parametrize sextic double solids with a cA_1 singularity.

Proposition 2.4 [25, Rem. 2.50.1]. *Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$ be two contact equivalent power series with zero constant term. Then their multiplicity m (as defined above) is the same and, furthermore, f_m and g_m are the same up to an invertible linear change of coordinates.*

We use the following proposition in Section 3 to construct sextic double solids with a cA_n singularity where $n \geq 2$, as well as in Section 4 to describe weighted blowups of cA_n points.

Proposition 2.5. *Let $F = x_1^2 + \cdots + x_k^2 + f$ and $G = x_1^2 + \cdots + x_k^2 + g$, where f and g are convergent power series in $\mathbb{C}\{x_{k+1}, \ldots, x_n\}$ with zero constant term. Then F and G are contact (resp. right) equivalent if and only if f and g are contact (resp. right) equivalent.*

Proof. By a result of Mather and Yau [40] (see also [25, Th. 2.26]), f and g are contact equivalent if and only if the Tjurina algebras T_f and T_g are isomorphic. A simple computation shows that $T_f \cong T_F$ and $T_g \cong T_G$, which proves the proposition for contact equivalence.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

The proof for right equivalence is similar. Namely, we use a statement analogous to [40]: two elements $h,k \in \mathbb{C}\{\boldsymbol{x}\}$ with zero constant term are right equivalent if and only if the Milnor algebras M_h and M_k are isomorphic as algebras over the ring $\mathbb{C}\{t\}$, where t acts on M_h (resp. M_k) by multiplying by h (resp. k) (see [25, Th. 2.28]). ∎

Reid defined in [47, Def. 2.1] that a *compound Du Val singularity* is a three-dimensional singularity where a hypersurface section is a Du Val singularity, also called a surface ADE singularity. The singularity is denoted cA_n, cD_n, or $ce:n$, respectively, if the general hyperplane section is an A_n, D_n, or E_n singularity, respectively. Reid showed in [48, Th. 0.6] that a three-dimensional hypersurface singularity is terminal if and only if it is an isolated compound Du Val singularity.

In this paper, we focus on the most general class of compound Du Val singularities, namely cA_n singularities. Since a surface A_n singularity is given by $x^2+y^2+z^{n+1}$, we have the following corollary.

Corollary 2.6. *Let n be a positive integer. A singularity is of type cA_n if and only if it is isomorphic to the complex analytic space subgerm $(\mathbb{V}(x_1^2+x_2^2+g),\mathbf{0})$ of $(\mathbb{C}^4,\mathbf{0})$ with variables x_1,x_2,x_3,x_4 for some convergent power series $g \in \mathbb{C}\{x_3,x_4\}$ of multiplicity $n+1$.*

For a proof of Corollary 2.6, see [35, Th. 2.8].

The simplest example of a cA_1 singularity is the *ordinary double point*, given by $x^2+y^2+z^2+t^2$.

Remark 2.7. Terminal sextic double solids have only isolated hypersurface singularities, therefore only cA_n, cD_n, and $ce:n$ singularities. Sextic double solids are Gorenstein, since by [24, Cor. 21.19] every variety with local complete intersection singularities is Gorenstein.

2.2 $\mathbb{Q}$-factoriality

Definition 2.8. A Weil divisor D on a normal algebraic variety is $\mathbb{Q}$-*Cartier* if a positive integer multiple of D is Cartier. A normal algebraic variety X is *factorial* (resp. $\mathbb{Q}$-*factorial*), if every Weil divisor on X is Cartier (resp. $\mathbb{Q}$-Cartier).

Definition 2.9. A *Fano variety* is a normal projective algebraic variety with an ample $\mathbb{Q}$-Cartier anti-canonical divisor.

To prove factoriality of certain singular sextic double solids, we use the following proposition by Namikawa.

Proposition 2.10 [42, Prop. 2]. *Let X be a Fano 3-fold with Gorenstein terminal singularities and D its general effective anti-canonical divisor. Then, the natural homomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(D)$ is an injection.*

Remark 2.11. The proof of [42, Prop. 2] contains a few typos that do not affect the result:

(1) The sentence “Since $\operatorname{Pic}(X) \cong \operatorname{Pic}(U)$, we have shown that…” should be replaced with “Since $\operatorname{Pic}(X)$ injects into $\operatorname{Pic}(U)$, we have shown that….” The isomorphism of $\operatorname{Pic}(X)$ and $\operatorname{Pic}(U)$ would imply that every X that is smooth along D is factorial, which is not true. To see that $\operatorname{Pic}(X)$ injects into $\operatorname{Pic}(U)$ for every Zariski open set U

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

containing D, note that since the complement of U in X is of codimension at least 2, the class groups $\mathrm{Cl}(X)$ and $\mathrm{Cl}(U)$ are isomorphic. We have a map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$, since every Weil divisor which is Cartier on X is Cartier on U. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is injective.

(2) The sentence "Thus, the complement $X - U$ is of codimension 2 in X" should be replaced with "Thus, the complement $X - U$ is of codimension at least 2 in X."

(3) The sentence "There is a Zariski open subset U of $W\ldots$" should be replaced with "There is a Zariski open subset U of $X\ldots$."

We remind that a terminal variety is log terminal (see [37, Def. 2.34]). The Picard number of log terminal sextic double solids is 1 by the following proposition.

PROPOSITION 2.12. *Let X be a log terminal complete intersection Fano variety of dimension $n \geq 3$ in a weighted projective space $\mathbb{P}$. Then the Picard number of X is 1.*

Proof. By [30, Prop. 2.1.2], we have natural isomorphisms $\mathrm{Pic}(\mathbb{P}) \cong H^2(\mathbb{P}^{\mathrm{an}}_{\mathrm{top}}, \mathbb{Z})$ and $\mathrm{Pic}(X) \cong H^2(X^{\mathrm{an}}_{\mathrm{top}}, \mathbb{Z})$, where $\mathbb{P}^{\mathrm{an}}_{\mathrm{top}}$, respectively $X^{\mathrm{an}}_{\mathrm{top}}$ denotes the underlying topological space of the analytification of $\mathbb{P}$, respectively X. By [41, Proposition 1.4], the restriction map $H^i(\mathbb{P}^{\mathrm{an}}_{\mathrm{top}}, \mathbb{C}) \to H^i(X^{\mathrm{an}}_{\mathrm{top}}, \mathbb{C})$ is an isomorphism for is an isomorphism for $i < n$. By [53, Corollary 1], X and $\mathbb{P}$ are simply connected. The proposition now follows from universal coefficient theorems. ∎

To show that some sextic double solids are not $\mathbb{Q}$-factorial, we use the lemma below.

LEMMA 2.13. *Let X be a projective variety of Picard number one. Let D be a non-zero effective $\mathbb{Q}$-Cartier divisor and C a closed curve in X. Then $D \cdot C > 0$.*

Proof. Replacing D by a suitable multiple, it suffices to consider the case where D is Cartier. There are no non-zero effective principal divisors on a normal projective variety. Therefore, since X has Picard number one, either D or $-D$ is ample. Since D intersects some closed integral curve positively, D is ample by Kleiman's criterion. Again by Kleiman's criterion, D intersects C positively. ∎

2.3 Weighted blowups

We remind the definition of weighted blowups, Definition 2.15.

DEFINITION 2.14. Let $\varphi\colon Y \to X$ and $\varphi'\colon Y' \to X'$ be birational morphisms of varieties (or bimeromorphic holomorphisms of complex analytic spaces). We say that an isomorphism $X \to X'$ *lifts* if there exists an isomorphism $Y \cong Y'$ such that the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ X & \longrightarrow & X' \end{array}$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

commutes. We say that φ and φ' are *equivalent* if there exists an isomorphism $X \cong X'$ that lifts. We say φ and φ' are *locally equivalent* if there exist isomorphic open subsets $U \subseteq X$ and $U' \subseteq X'$ containing the centers of the morphisms φ and φ' such that the restrictions $\varphi|_{\varphi^{-1}U} \colon \varphi^{-1}U \to U$ and $\varphi'|_{\varphi'^{-1}U'} \colon \varphi'^{-1}U' \to U'$ are equivalent.

If we consider the complex analytic space corresponding to a variety or when we wish to emphasize that we are working in the category of complex analytic spaces, we sometimes say *analytically equivalent* or *locally analytically equivalent*.

Definition 2.15. Let n be a positive integer, and let $\boldsymbol{w} = (w_1, \dots, w_n)$ be positive integers, called the weights of the blowup. Define a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by $\lambda \cdot (u, x_1, \dots, x_n) = (\lambda^{-1}u, \lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n)$ and define T by the geometric quotient $(\mathbb{C}^{n+1} \setminus \mathbb{V}(x_1, \dots, x_n))/\mathbb{C}^*$ (or its analytification). Then the map $\varphi \colon T \to \mathbb{C}^n$, $[u, x_1, \dots, x_n] \mapsto (u^{w_1}x_1, \dots, u^{w_n}x_n)$ is called the *$\boldsymbol{w}$-blowup of* $\mathbb{C}^n$. If $Z \subseteq \mathbb{C}^n$ is a closed subvariety (or a closed complex subspace $Z \subseteq D$ where $D \subseteq \mathbb{C}^n$ is open) and $\tilde{Z}$ is the closure of $\varphi^{-1}(Z \setminus \{\mathbf{0}\})$ in T (in $\varphi^{-1}D$), then the restriction $\varphi|_{\tilde{Z}} \colon \tilde{Z} \to Z$ is called the *$\boldsymbol{w}$-blowup of Z*. Let $\rho \colon Y \to X$ be a surjective birational morphism of varieties (or a surjective bimeromorphic holomorphism of complex spaces). Given an open subset $U \subseteq X$ containing the center of ρ and an isomorphism $U \cong Z \subseteq \mathbb{C}^n$ taking a point $P \in X$ to the origin $\mathbf{0}$, the map ρ is called the *$\boldsymbol{w}$-blowup of X at P* if the isomorphism $U \cong Z$ lifts to $\rho^{-1}U \to \tilde{Z}$.

Remark 2.16.

(a) A weighted blowup crucially depends on both the isomorphism $U \cong X'$ and a choice of coordinates $x_1, \dots, x_n$, even though it is not explicit in the notation.
(b) Replacing $\boldsymbol{w}$ by $(w_1/g, \dots, w_n/g)$ in Definition 2.15, where g is the greatest common divisor of $w_1, \dots, w_n$, gives an isomorphic blowup over X.
(c) By [21, Th. 5.1.11], the weighted blowup of an affine space in Definition 2.15 coincides with the toric description of subdividing a cone in [36, Prop.–Def. 10.3].

We give alternative definitions of weighted blowup in Definitions 2.17 and 2.18 that we use in Corollary 4.4.

Definition 2.17. Let $n \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$. Let $X = \operatorname{Spec} \mathbb{C}[\boldsymbol{x}]/I$ be an affine variety. Define the $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra

$$R_X = \mathbb{C}\left[\left\{t^d \bar{x}_i \;\middle|\; i \in \{1, \dots, n\}, d \in \{0, \dots, w_i\}\right\}\right],$$

where t denotes the grading and $\bar{x}_i \in \mathbb{C}[\boldsymbol{x}]/I$ denotes the image of $x_i \in \mathbb{C}[\boldsymbol{x}]$. Define the morphism $\operatorname{Proj} R_X \to X$.

Definition 2.18. Let $n \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$. Let $D \subseteq \mathbb{C}^n$ be an open subset. Let $X \subseteq D$ be a closed complex analytic space. For every open $V \subseteq D$, we denote the image of $f \in \mathcal{O}_{\mathbb{C}^n}(V)$ in $\mathcal{O}_X(X \cap V)$ by $\bar{f}$. Define the finitely presented $\mathbb{Z}_{\geq 0}$-graded $\mathcal{O}_X$-algebra $\mathcal{B}_X$ to be the sheafification of the presheaf $\mathcal{A}_X$ given by

$$\mathcal{A}_X(U) = \mathcal{O}_X(U)\left[\left\{t^d \bar{x}_i \;\middle|\; i \in \{1, \dots, n\}, d \in \{0, \dots, w_i\}\right\}\right],$$

where $U \subseteq X$ is open and t denotes the grading. By [11, Prop. II.3.19], we have a morphism $\operatorname{Projan} \mathcal{B}_X \to X$, where Projan is the analytic homogeneous spectrum.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Lemma 2.19. *The morphisms in Definitions 2.17 and 2.18 are* $\boldsymbol{w}$*-blowups.*

Proof. First, we show that Definition 2.17 is the $\boldsymbol{w}$-blowup when X is the affine space $\mathbb{A}^n = \operatorname{Spec}\mathbb{C}[x_1,\ldots,x_n]$. Let $S = \mathbb{C}[u,\boldsymbol{x}]$ be the $\mathbb{Z}$-graded $\mathbb{C}$-algebra with grading $(-1, \operatorname{wt} x_1, \ldots, \operatorname{wt} x_n)$ for $u, \boldsymbol{x}$. Let $S_{\geq 0}$ be the nonnegatively graded part of S. By definition of the geometric quotient, the weighted blowup of $\mathbb{A}^n$ is given by $\operatorname{Proj} S_{\geq 0} \to \mathbb{A}^n$. The $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra isomorphism

$$S_{\geq 0} \to R_{\mathbb{A}^n}$$
$$u \mapsto t^{-1}, \quad x_i \mapsto t^{\operatorname{wt} x_i} x_i$$

induces an isomorphism $\operatorname{Proj} R \to \operatorname{Proj} S_{\geq 0}$ over $\mathbb{A}^n$.

We show that Definition 2.17 is the $\boldsymbol{w}$-blowup for any X. Define $N = n \cdot \operatorname{lcm}(w_1,\ldots,w_n)$. If $M = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ is any monomial such that $\sum a_i w_i > N$, then M is divisible by $x_k^{N/(nw_k)}$ for some k. It follows that the Nth Veronese subring $R_X^{(N)}$ of R_X is generated by its degree 1 part $(R_X)_N$. Therefore, $\operatorname{Proj} R_X$ is isomorphic over X to $\operatorname{Bl}_{(R_X)_N} X$, where $\operatorname{Bl}_{(R_X)_N} X \to X$ is blowup of X along $(R_X)_N$. Since the intersection of $\operatorname{Spec}(R_{\mathbb{A}^n})_N$ and X is $\operatorname{Spec}(R_X)_N$, we find that $\operatorname{Bl}_{(R_X)_N} X$ is the strict transform of X under the blowup of $\mathbb{A}^n$ along $(R_{\mathbb{A}^n})_N$, which coincides with the closure of the inverse image of $X \setminus \mathbb{V}(x_1,\ldots,x_n)$ in $\operatorname{Bl}_{(R_{\mathbb{A}^n})_N} \mathbb{A}^n$.

We show that Definition 2.18 is the $\boldsymbol{w}$-blowup. We similarly prove that $\operatorname{Projan} \mathcal{B}_X$ is the closure of the inverse image of $X \setminus \{\mathbf{0}\}$ in $\operatorname{Projan} \mathcal{B}_D$. Now, it suffices to note that the analytification of $\operatorname{Proj} R_{\mathbb{A}^n} \to \mathbb{A}^n$ is $\operatorname{Projan} \mathcal{B}_{\mathbb{C}^n} \to \mathbb{C}^n$. ∎

In Corollary 4.4, we give a simple criterion for a local biholomorphism to lift to weighted blowups.

2.4 Divisorial contractions

The first step in a Sarkisov link from a Fano variety is a divisorial contraction.

Definition 2.20. A *divisorial contraction* is a proper birational morphism $\varphi\colon Y \to X$ between normal varieties with terminal singularities such that

(1) the exceptional locus of φ is a prime divisor and
(2) $-K_Y$ is φ-ample.

Kawakita [32] described divisorial contractions with center a cA_n point by weighted blowups. Notational differences from [32, Th. 1.13] are that below we have left out the description for cA_1 singularities and an exceptional case for cA_2. Also, we have written out the converse statement more explicitly (that being a Kawakita blowup implies that it is a divisorial contraction).

Theorem 2.21 [32, Th. 1.13]. *Let P be a cA_n point where $n \geq 3$ of a variety X with terminal singularities. Let $\varphi\colon Y \to X$ be a morphism of varieties such that the restriction $\varphi|_{Y\setminus E}\colon Y \setminus E \to X \setminus \{P\}$ is an isomorphism, where the closed subvariety E is given by $\varphi^{-1}\{P\}$. If φ is a divisorial contraction, then φ is locally analytically equivalent to the $(r_1, r_2, a, 1)$-blowup of $\mathbb{V}(x_1x_2 + g(x_3,x_4)) \subseteq \mathbb{C}^4$ at $\mathbf{0}$ with variables x_1, x_2, x_3, x_4 where*

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

(1) *a divides r_1+r_2 and is coprime to both r_1 and r_2,*
(2) *g has weight r_1+r_2, and*
(3) *the monomial $x_3^{(r_1+r_2)/a}$ appears in g with nonzero coefficient.*

Moreover, any φ which is locally analytically equivalent to a weighted blowup as above is a divisorial contraction, even for $n=2$.

Any weighted blowup that is locally analytically equivalent to φ in Theorem 2.21 for $n\geq 2$ is called a $(r_1,r_2,a,1)$-*Kawakita blowup*, or simply a Kawakita blowup.

2.5 Sarkisov links

One of the possible outcomes of the minimal model program is a Mori fiber space.

Definition 2.22. A *Mori fiber space* is a morphism of normal projective varieties $\varphi\colon X\to S$ with connected fibers such that

(1) X is $\mathbb{Q}$-factorial and has terminal singularities,
(2) the anti-canonical class $-K_X$ is φ-ample,
(3) X/S has relative Picard number 1, and
(4) $\dim S<\dim X$.

If $\dim S>0$, then we say φ is a *strict* Mori fiber space.

The main examples of Mori fiber spaces we see in this paper are Fano 3-folds that are projective, $\mathbb{Q}$-factorial, with terminal singularities and Picard number 1, considered as a morphism over a point.

Any birational map between two Mori fiber spaces is a composition of Sarkisov links (see [17] or [26]). Below, we describe the two possible types of Sarkisov links starting from a Fano variety.

Definition 2.23. A *Sarkisov link* of type I (resp. II) between a Fano variety X and a strict Mori fiber space $Y_k\to Z$ (resp. Fano variety Z) is a diagram of the form

$$\begin{array}{ccccc} & Y_0 & \dashrightarrow \dots \dashrightarrow & Y_k & \\ \swarrow^{\varphi} & & & & \searrow^{\psi} \\ X & & & & Z \end{array}$$

where X, Y_0, …, Y_k, Z are normal, projective, and $\mathbb{Q}$-factorial, the varieties X, Y_0, …, Y_k have terminal singularities, Z has terminal singularities if it three-dimensional, X has Picard number 1, $\varphi\colon Y_0\to X$ is a divisorial contraction, $Y_0\dashrightarrow\cdots\dashrightarrow Y_k$ is a sequence of anti-flips, flops, and flips, and $\psi\colon Y_k\to Z$ is a strict Mori fiber space (resp. divisorial contraction). If we do not require the varieties $X,Y_0,\dots Y_k$ (resp. $X,Y_0,\dots Y_k,Z$) to be terminal and we do not require $-K_{Y_0}$ to be φ-ample and we do not require $-K_{Y_k}$ to be ψ-ample but all the other properties hold, then the diagram above is called a *2-ray link* [10, Def. 2.1].

Definition 2.24. A Fano 3-fold X that is a Mori fiber space is *birationally rigid* if for any Mori fiber space $Y\to S$ such that X and Y are birational, we have that S is a point and X and Y are isomorphic.

In Section 5, we show that a general sextic double solid X with a cA_n singularity with $n\geq 4$ which is a Mori fiber space is not birationally rigid. We show this by explicitly

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

constructing a Sarkisov link between X and another Mori fiber space. We find the Sarkisov link by restricting from a toric 2-ray link, as described in Construction 2.25.

See [20] for the definition of Cox rings for toric varieties (where it is called the *homogeneous coordinate ring*) and [27, Def. 2.6] for the definition of Cox rings for Mori dream spaces. Note that isomorphic varieties can have different Cox rings. By [20, Th. 3.7], closed subschemes of a toric variety T with only cyclic quotient singularities are given by homogeneous ideals in the Cox ring $\operatorname{Cox} T$, which is a polynomial ring.

Construction 2.25. Let X be a Fano variety embedded in a weighted projective space $\mathbb{P}$, where X is a Mori fiber space, and let $Y_0 \to X$ be a divisorial contraction from a projective $\mathbb{Q}$-factorial variety Y. By [2, Lem. 2.9], the divisorial contraction $Y_0 \to X$ can be part of a Sarkisov link only if Y_0 is a Mori dream space.

By [27, Prop. 2.11], we can embed a Mori dream space Y_0 into a projective toric variety T_0 with cyclic quotient singularities such that the Mori chambers of Y_0 are unions of finitely many Mori chambers of T_0. Moreover, we can embed Y_0 in such a way that Y_0 is given by a homogeneous ideal I_Y in $\operatorname{Cox} T_0$, and the toric 2-ray link

$$\begin{array}{ccccccc} & T_0 & \dashrightarrow & T_1 & \dashrightarrow \cdots \dashrightarrow & T_r & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow \\ \mathbb{P} & & \mathcal{W}_0 & & \cdots & & S_T \end{array}$$

restricts to a 2-ray link

$$\begin{array}{ccccccc} & Y_0 & \dashrightarrow & Y_1 & \dashrightarrow \cdots \dashrightarrow & Y_r & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow \\ X & & W_0 & & \cdots & & S, \end{array}$$

where each $Y_i \subseteq T_i$ is given by the same ideal $I_Y \subseteq \operatorname{Cox} T_0 = \cdots = \operatorname{Cox} T_r$, and $W_i \subseteq \mathcal{W}_i$ is given by the ideal $I_Y \cap \mathbb{C}[\nu_0, \dots, \nu_s]$, where $\mathcal{W}_i$ is given by $\operatorname{Proj} \mathbb{C}[\nu_0, \dots, \nu_s]$ for some polynomials $\nu_j \in \operatorname{Cox} T_0$ that depend on i (see [4, Rem. 4]). In this case, $\operatorname{Cox}(T_0)/I_Y$ is a Cox ring for Y_0 and we say that I_Y *2-ray follows* T_0. In contrast to [4, Def. 3.5], we emphasize the ideal I_Y, since there could be other ideals I satisfying $\mathbb{V}(I_Y) = \mathbb{V}(I)$ such that the toric 2-ray link restricts to a 2-ray link for I_Y but not for I.

Note that some of the small birational maps $T_i \dashrightarrow T_{i+1}$ may restrict to isomorphisms $Y_i \to Y_{i+1}$. If all the varieties Y_i are terminal and the anti-canonical divisor $-K_{Y_0}$ of Y_0 is inside the interior $\operatorname{int}(\operatorname{Mov} Y_0)$ of the movable cone, then the 2-ray link for Y_0 is a Sarkisov link (see [2, Lem. 2.9]), otherwise it is called a *bad link*.

In Section 5, where X is a sextic double solid and the center of $Y_0 \to X$ is a cA_n point, we use a projective version of Corollary 4.10 to construct the divisorial contraction $Y_0 \to X$, which is the restriction of a toric weighted blowup $\bar{T}_0 \to \mathbb{P}$. This gives us an embedding $Y_0 \to \mathbb{V}(I_{\bar{Y}}) \subseteq \bar{T}_0$ where $I_{\bar{Y}}$ might not 2-ray follow $\bar{T}_0$. We use unprojection to modify $\bar{T}_0$ to find an embedding $Y_0 \to \mathbb{V}(I_Y) \subseteq T_0$ such that I_Y 2-ray follows T_0. See [49, §2.1] for a simple example of unprojection, and §§5.2, 5.5, 5.6, and 5.8 for applications of unprojection.

To explain the notation we use for 2-ray links, we do an example in detail, namely the 2-ray link for the ambient space of the sextic double solid with a cA_4 singularity in Section 5.2.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

EXAMPLE 2.26 (2-ray link for $\mathbb{P}(1,1,1,1,3,5)$). Denote the variables on $\mathbb{P}(1,1,1,1,3,5)$ by x,y,z,t,α,ξ. We perform the weighted blowup $T_0 \to \mathbb{P}(1,1,1,1,3,5)$ with weights $(1,1,2,3,6)$ for variables y,z,t,α,ξ, where the center is the point $P_x = [1,0,0,0,0,0]$.

We define T_0 as a geometric quotient. By a slight abuse of notation, we denote the variables on $\mathbb{C}^7$ by u,x,y,z,α,ξ,t, repeating the symbols for $\mathbb{P}(1,1,1,1,3,5)$. Define a $(\mathbb{C}^*)^2$-action on $\mathbb{C}^7$ for all $(\lambda,\mu) \in (\mathbb{C}^*)^2$ by

$$(\lambda,\mu)\cdot(u,x,y,z,\alpha,\xi,t) = (\mu^{-1}u, \lambda x, \lambda\mu y, \lambda\mu z, \lambda^3\mu^3\alpha, \lambda^5\mu^6\xi, \lambda\mu^2 t).$$

Define the irrelevant ideal $I_0 = (u,x)\cap(y,z,\alpha,\xi,t)$, and define T_0 by the geometric quotient $\mathbb{C}^7 \setminus \mathbb{V}(I_0)/(\mathbb{C}^*)^2$. We use the notation

$$T_0\colon \left(\begin{array}{cc|ccccc} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & 0 & 1 & 1 & 3 & 6 & 2 \end{array}\right).$$

to describe this construction of T_0. Note that we order the variables $u,x,\dots,t$ such that the corresponding rays $\binom{0}{-1}$, $\binom{1}{0}$, $\dots$, $\binom{1}{2}$are ordered anticlockwise around the origin. The vertical bar indicates that the irrelevant ideal is $(u,x)\cap(y,z,\alpha,\xi,t)$. The Cox ring of T_0 is given by $\operatorname{Cox} T_0 = \mathbb{C}[u,x,y,z,\alpha,\xi,t]$. The weighted blowup $T_0 \to \mathbb{P}(1,1,1,1,3,5)$ is given by

$$[u,x,y,z,\alpha,\xi,t] \mapsto [x, uy, uz, u^2t, u^3\alpha, u^6\xi]. \tag{2.1}$$

We describe the cones of the toric variety T_0 (Figure 1). By [27], T_0 is a Mori dream space. The Picard group of T_0 is generated by $\mathbb{V}(u)$, the reduced exceptional divisor, and $\mathbb{V}(x)$, the strict transform of a plane not passing through P_x, which have bidegree $\binom{0}{-1}$ and $\binom{1}{0}$, respectively. The variety T_0 is $\mathbb{Q}$-factorial, and any two divisors with the same bidegree are linearly equivalent. As in [10, §4.1.3], the effective cone $\operatorname{Eff}(T_0)$ is given by $\langle \mathbb{V}(u), \mathbb{V}(x)\rangle$, a cone in the group $N^1(T_0)$ of divisors of T_0 up to numerical equivalence with coefficients in $\mathbb{R}$. As in [4, §3.2], the movable cone $\mathrm{Mov(T_0)}$ is $\langle \mathbb{V}(x), \mathbb{V}(\xi)\rangle$, and it is divided into the nef cone $\operatorname{Nef}(T_0) = \langle \mathbb{V}(x), \mathbb{V}(y)\rangle$ of T_0 and $\langle \mathbb{V}(y), \mathbb{V}(\xi)\rangle$, which is the pullback of the nef cone of the small $\mathbb{Q}$-factorial modification T_1 of T_0. The cones $\langle \mathbb{V}(x), \mathbb{V}(y)\rangle$ and $\langle \mathbb{V}(y), \mathbb{V}(\xi)\rangle$ are called *Mori chambers*. The variety T_1 is defined by

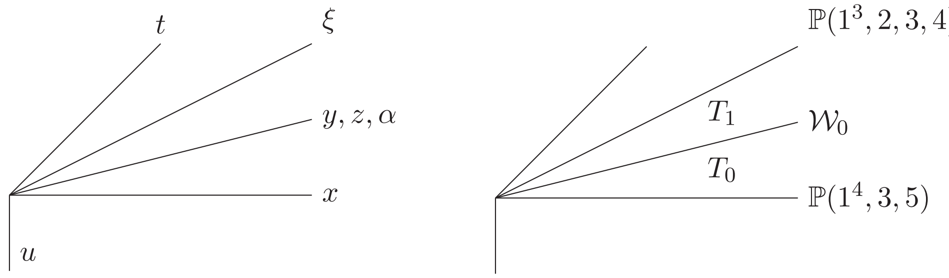
$$T_1\colon \left(\begin{array}{ccccc|cc} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & 0 & 1 & 1 & 3 & 6 & 2 \end{array}\right).$$

Here, T_1 is the geometric quotient $(\mathbb{C}^7 \setminus I_1)/(\mathbb{C}^*)^2$, where the irrelevant ideal I_1 is given by $(u,x,y,z,\alpha)\cap(\xi,t)$, which is indicated by the position of the vertical bar in the action matrix. The Cox ring of T_1 is equal to the Cox ring of T_0, namely $\operatorname{Cox} T_1 = \mathbb{C}[u,x,y,z,\alpha,\xi,t]$.

The weighted blowup morphism $T_0 \to \mathbb{P}(1,1,1,1,3,5)$ can be read off from the action-matrix of T_0. Consider the ray given by $\mathbb{V}(x)$ in $N^1(T_0)$. The union of the linear systems $|\binom{n}{0}|$ where $n \geq 0$ has a $\mathbb{C}$-algebra basis $x, uy, uz, u^2t, u^3\alpha, u^6\xi$. So, the ample model (see [7, Def. 3.6.5]) of the divisor class $\mathbb{V}(x)$ is the morphism

$$T_0 \to \operatorname{Proj} \bigoplus_{n\geq 0} H^0(T_0, \mathcal{O}_{T_0}(n\tbinom{1}{0}))) = \operatorname{Proj} \mathbb{C}[x,uy,uz,u^2t,u^3\alpha,u^6\xi] = \mathbb{P}(1,1,1,1,3,5)$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press



(a) Rays in $N^1(T_0)$ (b) Ample models of T_0

Figure 1.
Cones of T_0.

given by

$$[u,x,y,z,\alpha,\xi,t] \mapsto [x,uy,uz,u^2t,u^3\alpha,u^6\xi],$$

which is precisely the weighted blowup $T_0 \to \mathbb{P}(1,1,1,1,3,5)$ given in Equation (2.1).

As in [10, §2.1], there are two projective morphisms of relative Picard number 1 from T_0 up to isomorphisms, corresponding to the ample models of divisors in the two edges of the nef cone of T_0. The ample model of any divisor in the interior of the nef cone of T_0 gives an embedding of T_0 into a weighted projective space. The ample model of $\mathbb{V}(y) \in N^1(T_0)$ is given by

$$\begin{aligned} T_0 &\to \operatorname{Proj}\mathbb{C}[y,z,\alpha,u\xi,ut,x\xi,xt] \subseteq \mathbb{P}(1,1,3,5,1,6,2) \\ [u,x,y,z,\alpha,\xi,t] &\mapsto [y,z,\alpha,u\xi,ut,x\xi,xt]. \end{aligned}$$

Denoting $\mathcal{W}_0 = \operatorname{Proj}\mathbb{C}[y,z,\alpha,u\xi,ut,x\xi,xt]$, we see that the morphism $T_0 \to \mathcal{W}_0$ contracts $\mathbb{V}(\xi,t)$ to the surface $\mathbb{P}(1,1,3) \subseteq \mathcal{W}_0$ and is an isomorphism elsewhere. The ample model of $\mathbb{V}(y) \in N^1(T_1)$ is given similarly by

$$T_1 \to \operatorname{Proj}\mathbb{C}[y,z,\alpha,u\xi,ut,x\xi,xt] = \mathcal{W}_0,$$

contracting $\mathbb{V}(u,x)$ to $\mathbb{P}(1,1,3)$. This induces a birational map $T_0 \dashrightarrow T_1$, a small $\mathbb{Q}$-factorial modification, given by

$$[u,x,y,z,\alpha,\xi,t] \mapsto [u,x,y,z,\alpha,\xi,t].$$

Note that this is the identity map on the affine space $\mathbb{A}^7$, but it is not an isomorphism between T_0 and T_1 since the irrelevant ideals are different. The diagram $T_0 \to \mathcal{W}_0 \leftarrow T_1$ is a flop.

Note that multiplying the action matrix of T_0 or T_1 with a matrix in $\mathrm{GL}(2,\mathbb{Q})$ is equivalent to choosing a different basis for the group $(\mathbb{C}^*)^2$, so the geometric quotients T_0 and T_1 stay the same (see [1, Lem. 2.4]). If we multiply with a matrix with negative determinant, then we change the order of the rays in $N^1(T_0)$ from anticlockwise to clockwise.

Similarly, there are only two projective morphisms of relative Picard number 1 from T_1: the contraction $T_1 \to \mathcal{W}_0$ and the ample model of $\mathbb{V}(\xi)$. We multiply the action matrix of T_1 by the matrix $\left(\begin{smallmatrix} 6 & -5 \\ 2 & -1 \end{smallmatrix}\right)$ with determinant 4 to find

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

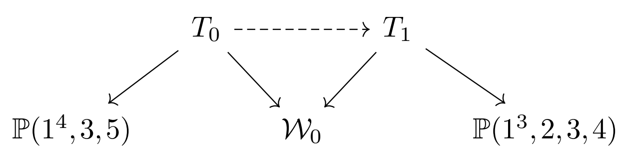
$$T_1\colon \left(\begin{array}{ccccc|cc} u & x & y & z & \alpha & \xi & t \\ 5 & 6 & 1 & 1 & 3 & 0 & -4 \\ 1 & 2 & 1 & 1 & 3 & 4 & 0 \end{array}\right).$$

The ample model of $\mathbb{V}(\xi)$ is given by

$$T_1 \to \mathbb{P}(1,1,1,2,3,4)$$
$$[u,x,y,z,\alpha,\xi,t] \mapsto \left[t^{\frac{5}{4}}u, t^{\frac{1}{4}}y, t^{\frac{1}{4}}z, t^{\frac{3}{2}}x, t^{\frac{3}{4}}\alpha, \xi\right].$$

Note that this is a morphism of varieties despite having fractional powers (see [9]).

The 2-ray link that we have found for $\mathbb{P}(1,1,1,1,3,5)$ is summarized by the diagram below.



For more examples on toric 2-ray links, see [10, §4].

§3. Constructing sextic double solids with a cA_n singularity

In this section, we give a bound $n \le 8$ for an isolated cA_n singularity on a sextic double solid, and we explicitly describe all sextic double solids that contain an isolated cA_n singularity where $n \le 8$. The main tool we use for this is the splitting lemma from singularity theory, first introduced in [50], which is used for separating the quadratic terms and the higher-order terms of a power series.

3.1 Splitting lemma from singularity theory

The splitting lemma below is taken from [25, Th. 2.47], with a slight modification in notation. Specifically, we write $v(x+p)$ instead of $x+g$, where v is a unit in the power series ring and p does not depend on x, as we use this form in Section 5 for constructing birational models.

Theorem 3.1 (Splitting lemma). *Let m be a positive integer, and let $\boldsymbol{y}$ denote variables $(y_1,\dots,y_m)$. Let $f \in \mathbb{C}\{x,\boldsymbol{y}\}$ be a convergent power series of multiplicity 2, with degree two part of the form $x^2 +$ (terms in $\boldsymbol{y}$). Then, there exist unique $v \in \mathbb{C}[[x,\boldsymbol{y}]]$ and $p,h \in \mathbb{C}[[\boldsymbol{y}]]$, where v is a unit and the multiplicity of p is at least 2, such that*

$$f = (v(x+p))^2 + h.$$

Moreover, the power series h,p, and v are absolutely convergent around the origin, and the multiplicity of h is at least 2. It follows immediately that f is right equivalent to x^2+h.

Proof. It is proved in [25, Th. 2.47] that there exist unique $g \in \mathbb{C}[[x,\boldsymbol{y}]]$ and $h \in \mathbb{C}[[\boldsymbol{y}]]$, where the multiplicity of g is at least 2, such that $f = (x+g)^2 + h$. Moreover, it is proved that the power series g and h are absolutely convergent around the origin, and the multiplicity of h is at least 2.

By the Weierstrass preparation theorem (see [25, Th. 1.6]), there exist a unique unit $v \in \mathbb{C}\{x,\boldsymbol{y}\}$ and a unique $p \in \mathbb{C}\{\boldsymbol{y}\}$ such that $x+g = v(x+p)$. ∎

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Below, we give explicit recurrent formulas for g,h,p,v of the splitting lemma in terms of the coefficients of f.

Proposition 3.2 (Explicit splitting lemma). *Below, we use the same notation as in the splitting lemma Theorem 3.1 and its proof. Denote*

$$f=\sum_{i,d\geq 0} x^i f_{i,d}, \qquad g=\sum_{i,d\geq 0} x^i g_{i,d}, \qquad h=\sum_{d\geq 0} h_d, \qquad p=\sum_{d\geq 0} p_d, \qquad v=\sum_{i,d\geq 0} x^i v_{i,d},$$

where $f_{i,d},g_{i,d},h_d,p_d,v_{i,d}\in\mathbb{C}[\boldsymbol{y}]$ are homogeneous of degree d. Then,

$$g_{1,0}=0,$$

$$g_{i,d}=\frac{1}{2}\left(f_{i+1,d}-\sum_{k=0}^{d}\sum_{j=\max(0,2-k)}^{\min(i+1,i+d-k-1)} g_{j,k}g_{i+1-j,d-k}\right), \quad \textit{if } (i,d)\neq(1,0), \tag{3.1}$$

$$h_d=f_{0,d}-\sum_{j=2}^{d-2} g_{0,j}g_{0,d-j}, \tag{3.2}$$

$$p_d=g_{0,d}-\sum_{j=2}^{d-1} v_{0,d-j}p_j, \tag{3.3}$$

$$v_{0,0}=1,$$

$$v_{i,d}=g_{i+1,d}-\sum_{j=2}^{d}(v_{i+1,d-j}p_j), \quad \textit{if } (i,d)\neq(0,0). \tag{3.4}$$

Proof. Taking the degree d part of the coefficient of x^{i+1} in $f=(x+g)^2+h$ where $i\geq 0$, we find Equation (3.1). Taking all degree d terms of $f=(x+g)^2+h$ that are not divisible by x, we find Equation (3.2). Taking the degree d part of the coefficient of x^{i+1} in $x+g=v(x+p)$ where $i\geq 0$, we find Equation (3.4), and taking all degree d terms not divisible by x, we find Equation (3.3). ∎

Example 3.3. Using the notation of Proposition 3.2, the first few homogeneous parts of h are given in terms of coefficients of f by

$$\begin{aligned}
h_2&=f_{0,2},\\
h_3&=f_{0,3},\\
h_4&=f_{0,4}-\frac{f_{1,2}^2}{4},\\
h_5&=f_{0,5}-\frac{f_{1,2}^2f_{2,1}}{4}-\frac{f_{1,2}f_{1,3}}{2},\\
h_6&=f_{0,6}-\frac{f_{1,2}^3f_{3,0}}{8}+\frac{f_{1,2}^2f_{2,2}}{4}-\frac{f_{1,2}^2f_{2,1}^2}{4}+\frac{f_{1,2}f_{1,3}f_{2,1}}{2}-\frac{f_{1,2}f_{1,4}}{2}-\frac{f_{1,3}^2}{4}.
\end{aligned}$$

3.2 Parameter spaces of sextic double solids

We apply the explicit splitting lemma (Proposition 3.2) to describe the equation of a sextic double solid $X\subseteq\mathbb{P}(1,1,1,1,3)$ that has a singular point at $P_x=[1,0,0,0,0]$.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

NOTATION 3.4. Let X be the subscheme of $\mathbb{P}(1,1,1,1,3)$, with variables x,y,z,t,w, defined by f, where

$$\begin{aligned} f = -w^2 &+ x^4(t^2+Q_2) \\ &+ x^3(4t^3a_0+4t^2a_1+2ta_2+a_3) \\ &+ x^2(2t^4b_0+2t^3b_1+2t^2b_2+2tb_3+b_4) \\ &+ x(2t^5c_0+2t^4c_1+2t^3c_2+2t^2c_3+2tc_4+c_5) \\ &+ t^6d_0+2t^5d_1+t^4d_2+2t^3d_3+t^2d_4+2td_5+d_6, \end{aligned} \tag{3.5}$$

where the polynomials $a_j,b_j,c_j,d_j \in \mathbb{C}[y,z]$ and $Q_j \in \mathbb{C}[y,z,t]$ are homogeneous of degree j.

We define the following 11 technical conditions, where $i \in \{1,2,3,4\}$:

(1) (This condition is always true).
(2) $Q_2 = 0$.
(3) Condition (2) holds and $a_3 = 0$.
(4) Condition (3) holds and $b_4 = a_2^2$.
(5) Condition (4) holds and $c_5 = 2a_2b_3 - 4a_1a_2^2$.
(6) Condition (5) holds and $d_6 = 2a_2c_4 + b_3^2 - 8a_1a_2b_3 - 2a_2^2b_2 + 4a_0a_2^3 + 16a_1^2a_2^2$.
(7.i) Condition (6) holds and there exist polynomials $q,r,s,e \in \mathbb{C}[y,z]$ that are, respectively, homogeneous of degrees $i-1$, $3-i$, $4-i$, $i+1$, where 0 is considered to be the only polynomial homogeneous of degree -1, such that

$$\begin{aligned} a_2 &= qr, \\ b_3 &= qs + 4a_1qr, \\ c_4 &= 2a_1qs - 6a_0q^2r^2 + 8a_1^2qr + er, \\ d_5 &= 2b_2qs - 8a_1^2qs - es - b_1q^2r^2 + c_3qr. \end{aligned}$$

(8) Condition (7.1) holds and there exist a constant $A_0 \in \mathbb{C}$ and a polynomial $B_1 \in \mathbb{C}[y,z]$ homogeneous of degree 1 such that

$$\begin{aligned} e_2 &= 4A_0r_2 + b_2 - 6a_1^2, \\ c_3 &= 6a_0s_3 - 4A_0s_3 + 4a_0a_1r_2 - 8A_0a_1r_2 + B_1r_2 + 2a_1b_2 - 4a_1^3, \\ d_4 &= -2s_3B_1 + 16r_2^2A_0^2 - 8b_2r_2A_0 + 16a_1^2r_2A_0 + 4b_1s_3 \\ &\quad - 8a_0a_1s_3 - 2b_0r_2^2 + 2c_2r_2 + b_2^2 - 4a_1^2b_2 + 4a_1^4. \end{aligned}$$

Note that zero is homogeneous of every nonnegative degree, so, for example, in Condition (7.1), the term e can be zero.

Next, define the set of 11 rational indices

$$\mathrm{Inds} := \{1, 2, 3, 4, 5, 6, 7.1, 7.2, 7.3, 7.4, 8\}.$$

Let $\lfloor k \rfloor$ denote the greatest integer not greater than k. For every $k \in \mathrm{Inds}$, let R_k denote the $\mathbb{C}$-algebra freely generated by the coefficients of the polynomials

- Q_2, a_i, b_i, c_i, d_i if $k \le 6$,
- $a_i, b_i, c_i, d_i, q, r, s, e$ if $k \in \{7.1, 7.2, 7.3, 7.4\}$, and
- $a_i, b_i, c_i, d_i, q, r, s, e, A_0, B_1$ if $k = 8$,

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Table 2. Dimension of the space of sextic double solids with an isolated $cA_{\lfloor k \rfloor}$

k	1	2	3	4	5	6	7.1, 7.2, 7.3, 7.4	8
$\dim \operatorname{Spec} R_k$	80	74	70	65	59	52	45	36
Expected moduli space dim	67	64	60	55	49	42	34	25

where we consider the coefficients to be variables satisfying Condition (k). Define

$$F_k = \operatorname{Spec}(R_k[x,y,z,t,w]/(f)),$$

where $f \in R_k[x,y,z,t,w]$ is the polynomial in Equation (3.5). Let *family* k denote the set of fibers of $F_k \to \operatorname{Spec} R_k$ over closed points. We say that a *general* sextic double solid in family k satisfies a property if the property is satisfied by all the fibers of $F_k \to \operatorname{Spec} R_k$ over the closed points of some Zariski open dense set in $\operatorname{Spec} R_k$. We say that an *analytically very general* sextic double solid in family k satisfies a property if there is a Zariski open dense subset U of $\operatorname{Spec} R_k$ such that the property is satisfied by all the fibers of $F_k \to \operatorname{Spec} R_k$ over the closed points of U that are in the complement of some countable union of closed analytic proper subsets.

Remark 3.5.

(a) The following are equivalent in Notation 3.4:
  (i) X is a sextic double solid,
  (ii) X is a variety,
  (iii) f is irreducible, and
  (iv) $f + w^2$ is not the square of a polynomial in $\mathbb{C}[x,y,z,t]$.
  Note that if $(\mathbb{V}(f), \mathbf{0})$ is a cA_n singularity for some n, then f is irreducible.

(b) Every closed point of $\operatorname{Spec} R_k$ bijectively corresponds to a choice of complex coefficients of
  - Q_2, a_i, b_i, c_i, d_i if $k \le 6$,
  - $a_i, b_i, c_i, d_i, q, r, s, e$ if $k \in \{7.1, 7.2, 7.3, 7.4\}$, and
  - $a_i, b_i, c_i, d_i, q, r, s, e, A_0, B_1$ if $k = 8$,

  so determines a unique polynomial $f \in \mathbb{C}[x,y,z,t,w]$. For every closed point $P \in \operatorname{Spec} R_k$ such that f is irreducible, the fiber of $F_k \to \operatorname{Spec} R_k$ over P is a sextic double solid.

(c) The varieties $\operatorname{Spec} R_k$ are affine spaces, and their dimensions are given in Table 2. The affine spaces $\operatorname{Spec} R_{7.1}$, $\operatorname{Spec} R_{7.2}$, $\operatorname{Spec} R_{7.3}$, and $\operatorname{Spec} R_{7.4}$ all have the same dimension.

(d) Let $k \in \mathrm{Inds}$, and let $f \in \mathbb{C}[x,y,z,t,w]$ in Notation 3.4 satisfy Condition (k). The graded $\mathbb{C}$-algebra automorphisms σ of $\mathbb{C}[x,y,z,t,w]$, which fix the point $P_x = [1,0,0,0,0]$ and take f to another polynomial $\sigma(f)$ of the form in Notation 3.4 satisfying Condition (k), are given by

$$\begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix} \mapsto \begin{pmatrix} \alpha & R_3 & \\ & M_3 & \\ & & \pm 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix}$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

when $k = 1$, and given by

$$\begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix} \mapsto \begin{pmatrix} \alpha & R_2 & \beta & \\ & M_2 & C_2 & \\ & & \alpha^{-2} & \\ & & & \pm 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix}$$

when $k \geq 2$, where $M_i \in \mathrm{GL}(i,\mathbb{C})$ are matrices, $R_i \in \mathbb{C}^i$ are row vectors, $C_2 \in \mathbb{C}^2$ is a column vector, and $\alpha \in \mathbb{C}^*, \beta \in \mathbb{C}$ are scalars. These automorphisms form an algebraic group which is of dimension 13 if $k = 1$ and of dimension 10 if $k \geq 2$.

If $k > 7$, then we also have the $\mathbb{C}^*$-action

$$\lambda \cdot q := \lambda q, \quad \lambda \cdot r := \lambda^{-1} r, \quad \lambda \cdot s := \lambda^{-1} s, \quad \lambda \cdot e := \lambda e,$$

which leaves f invariant.

If a coarse moduli space of sextic double solids with an isolated $cA_{\lfloor k \rfloor}$ singularity exists, then we expect its dimension to differ from $\dim \operatorname{Spec} R_k$ by 13 if $k = 1$, by 10 if $2 \leq k \leq 6$, and by 11 if $k > 7$. The moduli space of smooth sextic double solids has dimension 68. Table 2 shows the expected moduli space dimensions.

(e) If X has an isolated singularity at P_x, then by using the $\mathbb{C}^*$-action described in (d) for $k > 7$ and Proposition 3.8, we can set $q = 1$, $r = 1$, and $s = 1$, respectively, for families 7.1, 7.3, and 7.4.

We state the main theorem of this section, describing sextic double solids with an isolated cA_n singularity.

Theorem A. *For every positive integer n, both of the following hold:*

(a) *If a sextic double solid has an isolated cA_n singularity, then $n \leq 8$.*
(b) *Every sextic double solid with an isolated cA_n singularity P is isomorphic to a variety X in Notation 3.4 satisfying Condition (l) for some $l \in \mathrm{Inds}$ such that $\lfloor l \rfloor = n$, with the isomorphism sending P to $P_x = [1,0,0,0,0]$.*

Furthermore, for every $k \in \mathrm{Inds}$, all of the following hold:

(c) *If $k \geq 2$, then every scheme X in Notation 3.4 satisfying Condition (k) has either a (possibly non-isolated) cA_m singularity or the singularity $(\mathbb{V}(x_1^2 + x_2^2), \mathbf{0}) \subseteq (\mathbb{C}^4, \mathbf{0})$ at P_x, where $m \geq \lfloor k \rfloor$ and $\mathbb{C}^4$ has variables x_1, x_2, x_3, x_4.*
(d) *A general sextic double solid in family k is smooth outside a $cA_{\lfloor k \rfloor}$ singularity at P_x.*
(e) *An analytically very general sextic double solid in family k is factorial, except for $k = 7.4$. No terminal variety in family 7.4 is $\mathbb{Q}$-factorial.*

Remark 3.6.

(a) By Proposition 2.12, all log terminal sextic double solids have Picard number 1. Therefore, by Theorem A(d) and (e), an analytically very general sextic double solid in each family $k \in \mathrm{Inds} \setminus \{7.4\}$ is a Mori fiber space over a point.
(b) Let $k \in \mathrm{Inds} \setminus \{1, 8\}$. Let X satisfy Condition (k) but not Condition (l) for any $l \in \mathrm{Inds}$ satisfying $\lfloor l \rfloor = \lfloor k \rfloor + 1$. The proof of Theorem A(b) implies that if one of the following holds:

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

- $k < 6$,
- P_x is an isolated singularity, or
- $\lfloor k \rfloor = 7$, r and s are coprime, and q and e are coprime,

  then X has a $cA_{\lfloor k \rfloor}$ singularity at P_x.

3.3 Bound $n \leq 8$ for an isolated cA_n singularity

In this section, we prove Theorem A(a) and (c), showing that the parameter spaces in Notation 3.4 describe sextic double solids with a cA_n singularity. In addition, we prove Theorem A(b), namely the bound $n \leq 8$ for an isolated cA_n singularity. The bound $n \leq 8$ for an isolated cA_n singularity is proved by explicitly describing a curve of singularities for $n > 9$.

First, we state a few lemmas needed for the proof.

LEMMA 3.7. *If X in Notation 3.4 satisfies Condition* (6) *and P_x is an isolated singularity of X, then $a_2 \neq 0$ or $b_3 \neq 0$.*

Proof. If Condition (6) holds and $a_2 = b_3 = 0$, then $a_3 = b_4 = c_5 = d_6 = 0$. Let C be the curve defined by the ideal $(t, w, 2xc_4 + 2d_5)$. Note that C contains P_x. Taking partial derivatives, we see that every point of C is a singular point of X. ∎

The following proposition is useful when using Notation 3.4.

PROPOSITION 3.8. *If X in Notation 3.4 satisfies Condition* (k) *and P_x is an isolated singularity of X where $k > 7$, then q and e are coprime and r and s are coprime as polynomials in $\mathbb{C}[y,z]$.*

Proof. Let $D \in \mathbb{C}[y,z]$ be a common prime divisor of r and s or a common prime divisor of q and e. Then D divides a_2, b_3, c_4, d_5, and D^2 divides a_3, b_4, c_5, d_6. Let C be the curve defined by the ideal (D, t, w). Note that C contains P_x. Taking partial derivatives, we see that X is singular at every point of C ∎

LEMMA 3.9. *Let $r, s \in \mathbb{C}[y,z]$ have no common prime divisors, and let $q \in \mathbb{C}[y,z]$ be nonzero. Let $h_n \in \mathbb{C}[y,z]$ be of the form $h_n = q^\alpha(r^\beta C_r - s^\gamma C_s)$ where $C_r, C_s \in \mathbb{C}[y,z]$ and α, β, γ are nonnegative integers. Then*

$$h_n = 0 \iff \text{there exists } C \in \mathbb{C}[y,z] \text{ such that } C_r = s^\gamma C \text{ and } C_s = r^\beta C.$$

Proof. Obvious. ∎

Proof of Theorem A(b). First, we prove that every sextic double solid $Y \subseteq \mathbb{P}(1,1,1,1,3)$ with a singular point P (not necessarily of type cA_n) is isomorphic to some X in Notation 3.4, with the isomorphism sending P to $P_x = [1,0,0,0,0]$. For this, it suffices to note that Notation 3.4 describes all sextic double solids with a singular point at P_x, and that we can move any point of Y to P_x using an automorphism of $\mathbb{P}(1,1,1,1,3)$. This proves the case $n = 1$. For the rest of the proof, X is given by some f in Notation 3.4 with a (not necessarily isolated) cA_n singularity at P_x and n is at least 2.

Let X^{an} denote the analytification of X. By Propositions 2.4 and 2.5 and Corollary 2.6, after applying a suitable linear invertible coordinate change on y, z, t, Condition (2) holds. This proves the case $n = 2$. For the rest of the proof, Condition (2) holds and n is at least 3.

Let

$$X_x = \operatorname{Spec}(\mathbb{C}[y,z,t,w]/(f(1,y,z,t,w)))$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

denote the affine open of X given by inverting x. Let $g \in \mathbb{C}\{y,z,t\}$ and $h \in \mathbb{C}\{y,z\}$ be the unique convergent power series of multiplicity at least 2 such that

$$f(1,y,z,t,w) = -w^2 + (t+g)^2 + h.$$

Since by assumption (X^{an}, P_x) is a cA_n singularity, Propositions 2.4 and 2.5 and Corollary 2.6 imply that $h_2 = \cdots = h_n = 0$, where $h_j \in \mathbb{C}[x_3,x_4]$ is the homogeneous degree j part of h.

Using the explicit splitting lemma (Proposition 3.2), it is straightforward to compute that $h_2 = \cdots = h_n = 0$ is equivalent to satisfying Condition (n) when $n \leq 6$, even if P_x is not an isolated singularity. This proves the cases $n \in \{3,\ldots,6\}$. For the rest of the proof, Condition (6) holds, (X^{an}, P_x) is an isolated cA_n singularity, and n is at least 7.

By Lemma 3.7, $a_2 \neq 0$ or $b_3 \neq 0$. Define q to be a homogeneous greatest common divisor of a_2 and b_3. Define r and $s \in \mathbb{C}[y,z]$ to be the unique homogeneous polynomials such that

$$a_2 = qr,$$
$$b_3 = qs + 4a_1qr.$$

Then r and s are coprime. Using the explicit splitting lemma (Proposition 3.2), we compute that

$$h_7 = q(r(-12a_0q^2rs + 4b_2qs - 2b_1q^2r^2 + 2c_3qr - 2d_5) - s(2c_4 - 4a_1qs)).$$

Using Lemma 3.9, the equations $h_2 = \cdots = h_7 = 0$ imply the existence of a polynomial $e \in \mathbb{C}[y,z]$ such that

$$c_4 = 2a_1qs - 6a_0q^2r^2 + 8a_1^2qr + er,$$
$$d_5 = 2b_2qs - 8a_1^2qs - es - b_1q^2r^2 + c_3qr.$$

Therefore, $h_2 = \cdots = h_7 = 0$ implies Condition $(7.i)$, where i is defined by

$$i := \deg\gcd(a_2,b_3) + 1,$$

where $\deg\gcd(a_2,b_3)$ is the degree of a greatest common divisor of $a_2 \in \mathbb{C}[y,z]$ and $b_3 \in \mathbb{C}[y,z]$. This proves $n = 7$.

Next, we show that if $h_2 = \cdots = h_8 = 0$ and one of Conditions (7.2)–(7.4) holds, then r and s have a common prime divisor or q and e have a common prime divisor, which contradicts Proposition 3.8. In Condition (7.2), we calculate that $h_8 + e^2r^2$ is divisible by q, giving $r = Cq$ for some $C \in \mathbb{C}$. Substituting into h_8, we compute that $h_8 - 2qes^2$ is divisible by q^2. Therefore, q and s have a common prime divisor, giving that r and s have a common prime divisor, a contradiction. Conditions (7.3) and (7.4) are similar.

Hence, if $h_2 = \cdots = h_8 = 0$, then Condition (7.1) holds. Using the explicit splitting lemma, we calculate h_8, and using Lemma 3.9, we can show that $h_2 = \cdots = h_8 = 0$ implies Condition (8). ∎

Proof of Theorem A(a). Assume that X is a sextic double solid with an isolated cA_n singularity where $n \geq 9$. Using the notation in the proof of Theorem A(b), we find that Condition (8) holds and $h_2 = \cdots = h_9 = 0$. Using the explicit splitting lemma, we compute h_9, and using Lemma 3.9, we find that there exists $B_0 \in \mathbb{C}$ such that

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

$$\begin{aligned}
A_0 &= a_0,\\
B_1 &= b_1,\\
d_3 &= -s_3B_0 + 2b_0s_3 - 2a_0^2s_3 + c_1r_2 - 4a_0b_1r_2\\
&\quad + 16a_0^2a_1r_2 + b_1b_2 - 4a_0a_1b_2 - 2a_1^2b_1 + 8a_0a_1^3,\\
c_2 &= r_2B_0 - 6a_0^2r_2 + 2a_0b_2 + 2a_1b_1 - 12a_0a_1^2.
\end{aligned}$$

Substituting into f gives

$$\begin{aligned}
&x^3a_3 + x^2b_4 + xc_5 + d_6 = (s_3 + 2a_1r_2 + xr_2)^2,\\
&x^3a_2 + x^2b_3 + xc_4 + d_5 = (s_3 + 2a_1r_2 + xr_2)(-2a_0r_2 + b_2 - 2a_1^2 + 2xa_1 + x^2).
\end{aligned}$$

Define the curve C by the ideal $(w, t, s_3 + 2a_1r_2 + xr_2)$. Taking partial derivatives, we find that X is singular at every point of C, a contradiction. ∎

Proof of Theorem A(c). Let X^{an} denote the analytification of X. Using the explicit splitting lemma (Proposition 3.2), we can compute that the complex space germ (X^{an}, P_x) is isomorphic to $(\mathbb{V}(-w^2 + t^2 + h))$, where $h \in \mathbb{C}[y,z]$ is zero or has multiplicity at least $\lfloor k \rfloor + 1$. By Propositions 2.4 and 2.5 and Corollary 2.6, X has either a (possibly non-isolated) cA_m singularity or the singularity $(\mathbb{V}(x_1^2 + x_2^2), \mathbf{0}) \subseteq (\mathbb{C}^4, \mathbf{0})$ at P_x, where $m \geq \lfloor k \rfloor$ and $\mathbb{C}^4$ has variables x_1, x_2, x_3, x_4. ∎

3.4 Smoothness outside the isolated cA_n point

In this section, we prove Theorem A(d) using dimension count arguments, showing that a general sextic double solid with an isolated cA_n singularity is smooth outside the cA_n point.

LEMMA 3.10. *For every $k \in \mathrm{Inds}$, a general member of family k in Notation 3.4 is smooth at every point with t-coordinate nonzero.*

Proof. Let $\widehat{\mathcal{A}}_k$ denote the set of closed points Q of $\operatorname{Spec} R_k$ such that the fiber of $F_k \to \operatorname{Spec} R_k$ over Q has a singular point at $P_t = [0,0,0,1,0]$. We find

$$f(P_t) = d_0, \quad \frac{\partial f}{\partial x}(P_t) = 2c_0, \quad \frac{\partial f}{\partial y}(P_t) = 2\frac{\partial d_1}{\partial y}, \quad \frac{\partial f}{\partial z}(P_t) = 2\frac{\partial d_1}{\partial z}, \quad \frac{\partial f}{\partial t}(P_t) = 6d_0.$$

By the Jacobian criterion ([39, Exer. 4.2.10]), $\widehat{\mathcal{A}}_k$ is the set of closed points of

$$\mathcal{A}_k = \mathbb{V}_{\operatorname{Spec} R_k}\left(d_0, c_0, \frac{\partial d_1}{\partial y}, \frac{\partial d_1}{\partial z}\right).$$

We see that $\dim \mathcal{A}_k = \dim \operatorname{Spec} R_k - 4$.

The $\mathbb{C}$-algebra automorphism $x \mapsto x + \alpha_x t$, $y \mapsto y + \alpha_y t$, $z \mapsto z + \alpha_z t$ of $\mathbb{C}[x,y,z,t,w]$ defines a morphism

$$\pi_{\mathcal{A}_k} \colon \operatorname{Spec} \mathcal{A}_k \times \operatorname{Spec} \mathbb{C}[\alpha_x, \alpha_y, \alpha_z] \to \operatorname{Spec} R_k$$

with closed image. The set of closed points Q of $\operatorname{Spec} R_k$, where the fiber of $F_k \to \operatorname{Spec} R_k$ over Q has a singular point with t-coordinate nonzero, is precisely the set of closed points of the image of $\pi_{\mathcal{A}_k}$. The image of $\pi_{\mathcal{A}_k}$ has codimension at least 1. ∎

LEMMA 3.11. *For every $k \in \mathrm{Inds}$, a general member of family k in Notation 3.4 is smooth at every point different from P_x that has t-coordinate zero.*

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Proof. Let $P = [0,\beta,\gamma,0,0] \in \mathbb{P}(1,1,1,1,3)$, where $(\beta,\gamma) \in \mathbb{C}^2 \setminus \{(0,0)\}$. We find

$$f(P) = d_6(P),\ \frac{\partial f}{\partial x}(P) = c_5(P),\ \frac{\partial f}{\partial y}(P) = \frac{\partial d_6}{\partial y}(P),\ \frac{\partial f}{\partial z}(P) = \frac{\partial d_6}{\partial z}(P),\ \frac{\partial f}{\partial t}(P) = 2d_5(P).$$

Define the linear polynomial $l = \gamma y - \beta z$. By the Jacobian criterion ([39, Exer. 4.2.10]), P is a singular point of X if and only if the following divisibility constraint is satisfied:

$$l \text{ divides } c_5 \text{ and } d_5 \text{ and } l^2 \text{ divides } d_6. \tag{3.6}$$

The set of closed points $Q \in \operatorname{Spec} R_k$, where the fiber of $F_k \to \operatorname{Spec} R_k$ over Q is singular at P for some $(\beta,\gamma) \in \mathbb{C}^2 \setminus \{(0,0)\}$, coincides with the set of closed points of a closed subset $\mathcal{B}_k$ of $\operatorname{Spec} R_k$. We show that $\dim \mathcal{B}_k$ is at most $\dim \operatorname{Spec} R_k - 2$.

- If $k \leq 4$, then the 19 coefficients of c_5, d_5, and d_6 are algebraically independent in R_k. By the divisibility constraint (3.6), $\dim \mathcal{B}_k = \dim \operatorname{Spec} R_k - 3$.
- If $k = 5$, then the 20 coefficients of a_2, b_3, d_5, and d_6 are algebraically independent in R_k. We have $c_5 = a_2(2b_3 - 4a_1a_2)$. If l divides c_5, then l divides a_2 or l divides $b_3 - 2a_1a_2$. By the divisibility constraint (3.6), in both cases we have three less degrees of freedom. More formally, $\mathcal{B}_k$ is the union of the images of two morphisms, both having codimension exactly 3 in $\operatorname{Spec} R_k$. Therefore, $\dim \mathcal{B}_k = \dim \operatorname{Spec} R_k - 3$.
- If $k = 6$, then the 23 coefficients of a_2, b_3, c_4, d_5 are algebraically independent in R_k. We have $c_5 = a_2(2b_3 - 4a_1a_2)$ and $d_6 = a_2 \cdot (2c_4 + G) + b_3^2$ for a polynomial $G \in \mathbb{C}[y,z]$ homogeneous of degree 4 which does not contain c_4.

  If l divides a_2, then using the divisibility constraint (3.6), we find that l divides b_3. Now, l^2 divides a_2 or l divides $2c_4 + G$. So, there are three less degrees of freedom in choosing a_2, b_3, c_4, and d_5.

  If l does not divide a_2, then l divides $b_3 - 2a_1a_2$, so $b_3 = 2a_1a_2 + Ql$ for some homogeneous quadratic form $Q \in \mathbb{C}[y,z]$. From $l \mid d_6$, we find that l divides $c_4 - a_2b_2 + 2a_0a_2^2 + 2a_1^2a_2$, so $c_4 = Cl + a_2b_2 - 2a_0a_2^2 - 2a_1^2a_2$ for some homogeneous cubic form $C \in \mathbb{C}[y,z]$. From $l^2 \mid d_6$, we find that l divides $C - 4Qa_1$. Therefore, after fixing a_0, a_1, a_2, and b_2, there are at least two less degrees of freedom in choosing b_3, c_4, and d_5.

  In both cases, we see that $\dim \mathcal{B}_k \leq \dim \operatorname{Spec} R_k - 2$.
- If $\lfloor k \rfloor = 7$, then

$$\begin{aligned}
c_5 &= 4q^2r(2s + a_1r),\\
d_5 &= -es + q(2b_2s - a_1^2s - 4b_1qr^2 + c_3r),\\
d_6 &= 4q(er^2 + q(s^2 + a_1rs - 8a_0qr^3 - b_2r^2 + a_1^2r^2)).
\end{aligned}$$

  Let us consider f for a closed point in $\mathcal{B}_k$. If $l \mid q$, then since q and e are coprime, we have $l \mid r$ and $l \mid s$, a contradiction. If $l \mid r$, then since $l \mid d_6$, we find $l \mid s$, a contradiction. Therefore, l divides neither q nor r.

  So, l divides $2s + a_1r$. Using $l^2 \mid d_6$, we see that l^2 divides $-32a_0q^2r - 4b_2q + 3a_1^2q + 4e$. After fixing a_0, a_1, b_2, q, and r, we see that there are at least two less degrees of freedom in choosing s and e. So, we have $\dim \mathcal{B}_k \leq \dim \operatorname{Spec} R_k - 2$.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

- If $k = 8$, then

$$c_5 = 2r_2(s_3 + 2a_1r_2),$$
$$d_5 = r_2(r_2B_1 - 8s_3A_0 - 8a_1r_2A_0 + 6a_0s_3 - b_1r_2 + 4a_0a_1r_2 + 2a_1b_2 - 4a_1^3) + s_3(b_2 - 2a_1^2),$$
$$d_6 = r_2(8r_2^2A_0 + 4a_1s_3 - 8a_0r_2^2 + 4a_1^2r_2) + s_3^2.$$

We consider f for a closed point in $\mathcal{B}_k$. If $l \mid r_2$, then $l \mid s_3$, a contradiction. So, l divides $s_3 + 2a_1r_2$. Since l divides d_6, we have $l \mid r_2^3(A_0 - a_0)$. So, $A_0 = a_0$. Since l divides d_5, we see that $l \mid r_2^2(B_1 - b_1)$. We find that the coefficients of f have at least two less degrees of freedom, namely $A_0 = a_0$, and the polynomials $B_1 - b_1$ and $s_3 + 2a_1r_2$ have a common prime divisor. So, we have $\dim \mathcal{B}_k \le \dim \operatorname{Spec} R_k - 2$.

The $\mathbb{C}$-algebra automorphism $x \mapsto x + \alpha y$ of $\mathbb{C}[x,y,z,t,w]$ defines a morphism

$$\pi_{\mathcal{B}_k,1}\colon \operatorname{Spec}\mathcal{B}_k \times \operatorname{Spec}\mathbb{C}[\alpha] \to \operatorname{Spec} R_k,$$

and the $\mathbb{C}$-algebra automorphism $x \mapsto x + \alpha z$ of $\mathbb{C}[x,y,z,t,w]$ defines a morphism

$$\pi_{\mathcal{B}_k,2}\colon \operatorname{Spec}\mathcal{B}_k \times \operatorname{Spec}\mathbb{C}[\alpha] \to \operatorname{Spec} R_k.$$

Every closed point Q of $\operatorname{Spec} R_k$, where the fiber of $F_k \to \operatorname{Spec} R_k$ over Q has a singular point different from P_x with t-coordinate zero, belongs to the image of $\pi_{\mathcal{B}_k,1}$ or $\pi_{\mathcal{B}_k,2}$. The union of the images of $\pi_{\mathcal{B}_k,1}$ and $\pi_{\mathcal{B}_k,2}$ has codimension at least 1. ∎

Proof of Theorem A(d). It follows from Lemmas 3.10 and 3.11 that a general sextic double solid in family k has exactly one singular point, namely the point P_x. The singularity of X at P_x is of type $cA_{\lfloor k \rfloor}$ if the homogeneous part $h_{\lfloor k \rfloor + 1} \in \mathbb{C}[y,z]$ of h is nonzero, where h is as in the proof of Theorem A(b). Since this is an open condition, a general sextic double solid in family k has a $cA_{\lfloor k \rfloor}$ singularity at P_x. ∎

3.5 Factoriality

LEMMA 3.12 [33, Lemma 5.1]. *A terminal Gorenstein Fano 3-fold is factorial if and only if it is $\mathbb{Q}$-factorial.*

LEMMA 3.13. *There are no $\mathbb{Q}$-factorial log terminal sextic double solids in family* 7.4.

Proof. Let X be a log terminal variety in family 7.4. The Cartier divisor $\mathbb{V}_X(t)$ is the sum of the two prime divisors $D_1 = \mathbb{V}(t, q - w)$ and $D_2 = \mathbb{V}(t, q + w)$. Let $l \varepsilon \mathbb{C}[y,z]$ be a non-zero linear form that does not divide q. Define the curve $C = V(q + w, x, l)$. If D_1 is $\mathbb{Q}$-Cartier, then $D_1 \cdot C = 0$, which contradicts Proposition 2.12 and lemma 2.13. Therefore, neither D_1 nor D_2 is $\mathbb{Q}$-Cartier. ∎

Our proof of factoriality relies on the following corollary of Proposition 2.10.

COROLLARY 3.14. *Let X be a Gorenstein terminal Fano 3-fold which is smooth along its general effective anti-canonical divisor D. Then the natural homomorphism* $\operatorname{Cl}(X) \to \operatorname{Pic}(D)$ *from the class group of X is injective.*

Proof. Let U be any Zariski open set in the smooth locus of X that contains D. By Remark 2.11(1), we have an isomorphism of class groups $\operatorname{Cl}(X) \cong \operatorname{Cl}(U)$. Since U is smooth, we have an isomorphism $\operatorname{Cl}(U) \cong \operatorname{Pic}(U)$. It follows from the proof of [42, Prop. 2] that we can choose a small enough U such that $\operatorname{Pic}(U)$ injects into $\operatorname{Pic}(D)$. ∎

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

COROLLARY 3.15. *Let X be a terminal Gorenstein Fano 3-fold and D a smooth effective anti-canonical divisor such that X is smooth along D and D has Picard number* 1*. Then X is factorial.*

Proof. By adjunction, every smooth anti-canonical divisor of a Fano variety is a K3 surface. A very general projective K3 surface has Picard number 1. Therefore, X is smooth along an analytically very general anti-canonical divisor with Picard number 1. By Corollary 3.14, X is $\mathbb{Q}$-factorial. By Lemma 3.12, X is factorial. ∎

LEMMA 3.16. *For every $k \in \text{Inds} \setminus \{7.4\}$ and for an analytically very general sextic double solid X in family k, the subvariety $\mathbb{V}_X(x)$ is smooth and has Picard number* 1*.*

Proof. Let S_k be the $\mathbb{C}$-algebra freely generated by the 28 coefficients, considered as variables, of polynomials $g \in \mathbb{C}[y,z,t]$ homogeneous of degree 6. By Remark 3.5(b), closed points P of $\operatorname{Spec} R_k$ bijectively correspond to polynomials $f_P \in \mathbb{C}[x,y,z,t,w]$ in Notation 3.4. Let $\theta\colon \mathbb{C}[x,y,z,t,w] \to \mathbb{C}[y,z,t]$ be the homomorphism $x \mapsto 0$, $w \mapsto 0$. Let $\pi_k\colon \operatorname{Spec} R_k \to \operatorname{Spec} S_k$ be the morphism of affine spaces given on closed points by $f_P \mapsto \theta(f_P)$. The $\mathbb{C}$-algebra automorphisms $t \mapsto \alpha y + \beta z + t$ of $\mathbb{C}[y,z,t]$ induce a morphism $\tau\colon \operatorname{Spec} S_k \times \mathbb{A}^2 \to \operatorname{Spec} S_k$. Define ρ_k to be the composition

$$\rho_k := \tau \circ (\pi_k \times \mathrm{id}_{\mathbb{A}^2})\colon \operatorname{Spec} R_k \times \mathbb{A}^2 \to \operatorname{Spec} S_k.$$

We can compute that the rank of the Jacobian matrix of ρ_k at some specified point is 28 for all $k \in \text{Inds} \setminus \{7.4\}$. It follows that ρ_k is a dominant morphism of affine spaces for all $k \in \text{Inds} \setminus \{7.4\}$.

The closed points Q of $\operatorname{Spec} S_k$ bijectively correspond to polynomials $g_Q \in \mathbb{C}[y,z,t]$ homogeneous of degree 6, and therefore also to subschemes Z_Q of $\mathbb{P}(1,1,1,3)$ with variables y,z,t,w given by $-w^2 + g_Q$. Smooth schemes $Z_Q \subseteq \mathbb{P}(1,1,1,3)$ are K3 surfaces that are called *sextic double planes*. It is known that a very general projective K3 surface has Picard number 1. It follows that an analytically very general sextic double solid X in family $k \in \text{Inds} \setminus \{7.4\}$ satisfies that $\mathbb{V}_X(x)$ has Picard number 1. ∎

Proof of Theorem A(e). By Theorem A(d), a general sextic double solid in family k is terminal and is smooth along the anti-canonical divisor $\mathbb{V}(x)$. By Corollary 3.15 and Lemma 3.16, an analytically very general sextic double solid in family $k \neq 7.4$ is factorial. ∎

REMARK 3.17. In some cases, we can prove that it suffices if the sextic double solid is only *general* in Theorem A(e) as opposed to *analytically very general*:

(a) A general sextic double solid in family 1 has only one singularity and that singularity is an ordinary double point. Every sextic double solid which is smooth outside an ordinary double point is factorial and has Picard number 1 (see [14, Th. B]).

(b) A general sextic double solid in family 4 is factorial, since in Section 5.2 we construct a Sarkisov link to a complete intersection $Z_{5,6} \subseteq \mathbb{P}(1,1,1,2,3,4)$ which is $\mathbb{Q}$-factorial if it is general.

3.6 Other cA_n singularities

Although the primary interest is in isolated cA_n singularities since these are terminal, it is also possible to study non-isolated singularities with the same methods.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

We describe uncountably many examples of sextic double solids with a non-isolated cA_n singularity for all $9 \le n \le 11$.

PROPOSITION 3.18. *Let $8 \le n \le 11$. Let r_2 and s_3 be coprime, and let q_0 be nonzero. Let X in Notation 3.4 satisfy Condition (n) but not satisfy Condition $(n+1)$, where Conditions (9)–(12) are defined below:*

(9) *Condition (8) of Notation 3.4 is satisfied, and there exists $B_0 \in \mathbb{C}$ such that*

$$\begin{aligned} A_0 &= a_0, \\ B_1 &= b_1, \\ d_3 &= -s_3B_0 + 2b_0s_3 - 2a_0^2s_3 + c_1r_2 - 4a_0b_1r_2 \\ &\quad + 16a_0^2a_1r_2 + b_1b_2 - 4a_0a_1b_2 - 2a_1^2b_1 + 8a_0a_1^3, \\ c_2 &= r_2B_0 - 6a_0^2r_2 + 2a_0b_2 + 2a_1b_1 - 12a_0a_1^2. \end{aligned}$$

(10) *Condition (9) is satisfied and*

$$\begin{aligned} B_0 &= b_0, \\ d_2 &= 2c_0r_2 - 8a_0b_0r_2 + 16a_0^3r_2 + 2b_0b_2 - 4a_0^2b_2 + b_1^2 - 8a_0a_1b_1 - 4a_1^2b_0 + 24a_0^2a_1^2, \\ c_1 &= 2a_0b_1 + 2a_1b_0 - 12a_0^2a_1, \end{aligned}$$

(11) *Condition (10) is satisfied and*

$$\begin{aligned} c_0 &= 2a_0b_0 - 4a_0^3, \\ d_1 &= b_0b_1 - 2a_0^2b_1 - 4a_0a_1b_0 + 8a_0^3a_1, \end{aligned}$$

(12) *Condition (11) is satisfied and $d_0 = b_0^2 - 4a_0^2b_0 + 4a_0^4$.*

Then P_x is a cA_n singularity of X. Moreover, if $n \ge 9$, then the singularity is non-isolated.

Proof. Use the explicit splitting lemma (Proposition 3.2) and repeatedly apply Lemma 3.9 similarly to the proof of Theorem A(b). ∎

REMARK 3.19.

(1) By the proof of Theorem A(a), if X in Proposition 3.18 satisfies Condition (9), then X is singular along the curve $C\colon \mathbb{V}(t, w, s_3 + 2a_1r_2 + xr_2)$ passing through P_x. We can compute that at a general point of C, the singularity is locally analytically $\mathbb{C}^1 \times \text{ODP}$, that is, it is isomorphic to the germ $(Z, \mathbf{0})$ where Z is $\mathbb{V}(x_1^2 + x_2^2 + x_3^2) \subseteq \mathbb{C}^4$ with variables x_1, x_2, x_3, x_4.
(2) Translating the point $P_t = [0,0,0,1,0]$ to $[1,0,0,0,0]$, we can find conditions similar to Notation 3.4 for having a cA_n singularity at $P_t \in X$, which can be used to construct general sextic double solids with two cA_n singularities. The following is a simple example with cA_5 singularities at P_x and at P_t:

$$\mathbb{V}(-w^2 + x^4t^2 + x^2t^4 + y^6 + z^6) \subseteq \mathbb{P}(1,1,1,1,3).$$

§4. Divisorial contractions with center a cA_n point

In this section, we discuss weighted blowups from both algebraic and local analytic points of view. In Proposition 4.6, we show that to check whether a weighted blowup is a Kawakita blowup (see Theorem 2.21), it suffices to compute the weight of the defining power series.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Using this, in the technical Lemma 4.9, we show how to algebraically construct Kawakita blowups of cA_n points on affine hypersurfaces.

4.1 Weight-respecting maps

Let n and m be positive integers. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ denote the coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Choose positive integer weights for $\boldsymbol{x}$ and $\boldsymbol{y}$.

DEFINITION 4.1. Let $X \subseteq \mathbb{C}^n$ and $X' \subseteq \mathbb{C}^m$ be complex analytic spaces. We say that a biholomorphic map $\psi\colon X \to X'$ taking $\mathbf{0}$ to $\mathbf{0}$ is *weight-respecting* if denoting its inverse by θ, we can locally analytically around the origins write $\psi = (\psi_1, \ldots, \psi_m)$ and $\theta = (\theta_1, \ldots, \theta_n)$ where for all i and j, the power series $\psi_j \in \mathbb{C}\{\boldsymbol{x}\}$ and $\theta_i \in \mathbb{C}\{\boldsymbol{y}\}$ satisfy $\operatorname{wt}(\psi_j) \geq \operatorname{wt}(y_j)$ and $\operatorname{wt}(\theta_i) \geq \operatorname{wt}(x_i)$.

It is known that a biholomorphic map taking the origin to the origin lifts to a unique biholomorphic map of the blown-up spaces under the usual weights $(1, \ldots, 1)$ (see, e.g., [25, Rem. 3.17.1(4)]). It is easy to come up with examples where a biholomorphic map does not lift under weighted blowups. We give one example below.

EXAMPLE 4.2. Let $X \subseteq \mathbb{C}^3$ be the complex analytic space given by $\mathbb{V}(f)$ where

$$f = x_2^2 x_3 + x_1^3 + a x_1 x_3^2 + b x_3^3$$

for some $a, b \in \mathbb{C}^*$. Define $X' \subseteq \mathbb{C}^3$ by $\mathbb{V}(f')$ where $f' = f(x_1, x_2, -x_2 + x_3)$. Choose weights $(1,1,2)$ for (x_1, x_2, x_3). Then, X and X' are biholomorphic and $\operatorname{wt} f = \operatorname{wt} f'$, but the weighted blowups of X and X' are not locally analytically equivalent.

Proof. Let $\psi\colon X \to X'$ be any local biholomorphism taking the origin to the origin. Composing with a suitable weight-respecting biholomorphic map and using Corollary 4.4, it suffices to consider the case where ψ is a linear biholomorphism. Since the elliptic curve defined by f in $\mathbb{P}^2$ with variables x_1, x_2, x_3 has only two automorphisms, there are only four possibilities for a linear biholomorphism $X \to X'$, namely $(x_1, x_2, x_3) \mapsto (x_1, \pm x_2, \pm x_2 + x_3)$.

Let $Y \to X$ and $Y' \to X'$ be the $(1,1,2)$-blowups of X and X', respectively. Then Y is given by $\mathbb{V}(g)$ where

$$g(u, x_1, x_2, x_3) = u x_2^2 x_3 + x_1^3 + a u^2 x_1 x_3^2 + b u^3 x_3^3.$$

Denoting the points of Y and Y' by $[u, x_1, x_2, x_3]$, the lifted map $\psi_Y : Y \to Y'$ is given by $[u, x_1, x_2, x_3] \mapsto [u, x_1, \pm x_2, \pm x_2/u + x_3]$, which is not holomorphic on the exceptional locus $\mathbb{V}(u)$. ∎

On the other hand, a weight-respecting coordinate change does lift to weighted blowups (see Corollary 4.4).

LEMMA 4.3. *Let $X \subseteq \mathbb{C}^n$ and $X' \subseteq \mathbb{C}^m$ be complex analytic spaces strictly containing the origins and $\psi\colon X \to X'$ a biholomorphism. Then ψ is weight-respecting if and only if ψ induces an isomorphism of the $\mathbb{Z}_{\geq 0}$-graded $\mathcal{O}_{X'}$-algebras $\mathcal{B}_{X'}$ and $\psi_* \mathcal{B}_X$ of Definition 2.18.*

Proof. "$\Longrightarrow$." The induced morphism $\mathcal{B}_{X'} \to \psi_* \mathcal{B}_X$ is given by

$$\mathcal{B}_{X'}(U) \to \mathcal{B}_X(\psi^{-1} U)$$
$$t^d \bar{y}_j \mapsto t^d \bar{\psi}_j,$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

where $U \subseteq X'$ is open. Since $\operatorname{wt}\psi_j \geq \operatorname{wt} y_j$, the morphism $\mathcal{B}_{X'} \to \psi_* \mathcal{B}_X$ is well-defined. Similarly, we define a morphism $\mathcal{B}_{X'} \leftarrow \psi_* \mathcal{B}_X$, which is its inverse.

"$\Longleftarrow$." Let $t^{\operatorname{wt} y_j} \Psi_j$ be the image of $t^{\operatorname{wt} y_j} y_j$ under $\mathcal{B}_{X'}(X') \to \psi_* \mathcal{B}_X(X)$. There exists $\psi_j \in \mathbb{C}\{\boldsymbol{x}\}$ such that $\operatorname{wt}\psi_j \geq \operatorname{wt} y_j$ and $\bar{\psi}_j = \Psi_j$. Similarly, we can find θ_i, showing that ψ is weight-respecting. ∎

Corollary 4.4. *A weight-respecting biholomorphism ψ from $X \subseteq \mathbb{C}^n$ to $X' \subseteq \mathbb{C}^m$ lifts to the weighted blown-up spaces.*

4.2 Kawakita blowup in analytic neighborhoods

In the following, we focus on Kawakita blowups (see Theorem 2.21). Unlike Example 4.2, for cA_n singularities, having the correct weight for the defining power series is enough for the local analytic equivalence of weighted blowups.

Notation 4.5. We choose positive integer weights $\boldsymbol{w} = (r_1, r_2, a, 1)$ for variables $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ on $\mathbb{C}^4$ and define $n = (r_1 + r_2)/a - 1$ such that

- a divides $r_1 + r_2$ and is coprime to both r_1 and r_2,
- $r_1 \geq r_2$, and
- $n \geq 2$.

Proposition 4.6. *Using Notation 4.5, let $f \in \mathbb{C}\{\boldsymbol{x}\}$ be such that $\mathbb{V}(f)$ has an isolated cA_n singularity at the origin and f has weight $r_1 + r_2$. Then, the $\boldsymbol{w}$-blowup of $\mathbb{V}(f) \subseteq \mathbb{C}^4$ is a $\boldsymbol{w}$-Kawakita blowup.*

Proof. First, we remind that the terms *homogeneous*, *degree*, and *multiplicity* are with respect to the standard weights $(1, \ldots, 1)$. Let the *quadratic part* of f denote the homogeneous part of f of degree 2. After a suitable invertible linear weight-respecting coordinate change, the quadratic part of f is $x_1 x_2$.

We find that $f = x_1 x_2 + x_1 G + H$, where $G \in \mathbb{C}\{x_1, \ldots, x_4\}$ has weight at least r_2 and multiplicity $m \geq 2$, and $H \in \mathbb{C}\{x_2, x_3, x_4\}$. The coordinate change $x_2 \mapsto x_2 - G_m$, where G_m is the homogeneous degree m part of G, takes f to $x_1 x_2 + x_1 G' + H'$, where G' has multiplicity at least $m+1$. By induction, this defines the unique formal power series $K \in \mathbb{C}[[x_1, \ldots, x_4]]$ of multiplicity at least 2 and weight at least r_2 such that the transformation $x_2 \mapsto x_2 + K$ takes f to the form $x_1 x_2 + H''$ where $H'' \in \mathbb{C}[[x_2, x_3, x_4]]$. Similarly, we transform f into $x_1 x_2 + h$ where $h \in \mathbb{C}[[x_3, x_4]]$, using $x_1 \mapsto x_1 + L$ where $L \in \mathbb{C}[[x_2, x_3, x_4]]$.

We show how to find a convergent weight-respecting coordinate change which changes f to $x_1 x_2 + h$. Instead of the coordinate changes $x_2 \mapsto x_2 + K$, $x_1 \mapsto x_1 + L$, which might not be convergent, we do a coordinate change Θ_N with truncated power series $K_{\leq N}$ and $L_{\leq N}$ of homogeneous parts of K and L of degree at most N. The coordinate change $\Psi \colon x_1 \mapsto x_1 + i x_2$, $x_2 \mapsto x_1 - i x_2$ takes $x_1 x_2$ into $x_1^2 + x_2^2$. Now we use the splitting lemma, which gives a convergent coordinate change Φ_N which respects the weighting when N is large enough, to give f the form $x_1^2 + x_2^2 + h(x_3, x_4)$ where h converges. Applying Ψ^{-1}, we get $x_1 x_2 + h$. Note that the coordinate changes Ψ and Ψ^{-1} might not respect the weighting $\boldsymbol{w}$, but the total coordinate change $\Psi^{-1} \circ \Phi_N \circ \Psi \circ \Theta_N$ is weight-respecting if N is large enough.

Since the singularity is cA_n where $n = (r_1 + r_2)/a - 1$, h must contain a monomial of degree $(r_1 + r_2)/a$. Since $x_1 x_2 + h$ has weight $r_1 + r_2$, if $a > 1$, then the coefficient of $x_3^{(r_1+r_2)/a}$ in h

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

is nonzero. If $a=1$, then after a suitable invertible linear coordinate change on $\mathbb{C}\{x_3,x_4\}$, the coefficient of $x_3^{(r_1+r_2)/a}$ in h is nonzero.

We found that we can transform f into the form x_1x_2+h where the coefficient of $x_3^{(r_1+r_2)/a}$ in h is nonzero, by using only weight-respecting coordinate changes. By Corollary 4.4, the weighted blowup of f is locally analytically equivalent to the weighted blowup of x_1x_2+h, which is precisely a Kawakita blowup. ∎

Given a variety X with an isolated cA_n point P, we show that any two $\boldsymbol{w}$-Kawakita blowups $Y\to X$ and $Y'\to X$ of the point P are locally analytically equivalent. Note that they need not be globally algebraically equivalent. For example, [18, Rem. 2.4] describes two different $(2,1,1,1)$-Kawakita blowups of a cA_2 singularity on a quartic 3-fold.

PROPOSITION 4.7. *Any two $\boldsymbol{w}$-Kawakita blowups of locally biholomorphic singularities are locally analytically equivalent.*

Proof. Let $f=x_1x_2+g(x_3,x_4)$ and $f'=x_1x_2+g'(x_3,x_4)$ be contact equivalent, where $g,g'\in\mathbb{C}\{x_3,x_4\}$ have weight r_1+r_2 and $x_3^{(r_1+r_2)/a}$ appears in both g and in g' with nonzero coefficient. It suffices to show that there exist a weight-respecting map from $\mathbb{V}(f)$ to $\mathbb{V}(f')$.

Since f and f' are contact equivalent, there exist a unit $u\in\mathbb{C}\{\boldsymbol{x}\}$ and a local biholomorphism $\psi\colon(\mathbb{C}^4,\mathbf{0})\to(\mathbb{C}^4,\mathbf{0})$ such that $f'=u(f\circ\psi)$. Note that f' and $f\circ\psi$ have the same weight r_1+r_2, and $x_3^{(r_1+r_2)/a}$ appears in $f\circ\psi$ with nonzero coefficient. It suffices to show that there exist a weight-respecting map from $\mathbb{V}(f)$ to $\mathbb{V}(f\circ\psi)$.

Using arguments similar to the proof of Proposition 4.6, we can find a weight-respecting biholomorphic map germ $\theta\colon(\mathbb{C}^4,\mathbf{0})\to(\mathbb{C}^4,\mathbf{0})$ such that $f\circ\psi\circ\theta$ is of the form x_1x_2+g'' where $g''\in\mathbb{C}\{x_3,x_4\}$ contains $x^{(r_1+r_2)/a}$ and has weight r_1+r_2. It suffices to show that there exist a weight-respecting map from $\mathbb{V}(f)$ to $\mathbb{V}(f\circ\psi\circ\theta)$.

By Proposition 2.5, g and g'' are right equivalent, meaning there exists an automorphism Φ of $\mathbb{C}\{x_3,x_4\}$ such that $\Phi(g)=g''$. Since $x_3^{(r_1+r_2)/a}$ has nonzero coefficient in both g and g'', and both g and g'' have weight r_1+r_2, the image of x_3 has weight a under both Φ and Φ^{-1}. Define the biholomorphic map germ $\varphi\colon(\mathbb{V}(f\circ\psi\circ\theta),\mathbf{0})\to(\mathbb{V}(f),\mathbf{0})$ by $\boldsymbol{x}\mapsto(x_1,x_2,\Phi(x_3),\Phi(x_4))$. By Corollary 4.4, the $\boldsymbol{w}$-blowups of $\mathbb{V}(f\circ\psi\circ\theta)\subseteq\mathbb{C}^4$ and $\mathbb{V}(f)\subseteq\mathbb{C}^4$ are locally analytically equivalent. ∎

4.3 Kawakita blowups on affine hypersurfaces

In this section, we see how to construct weighted blowups for affine hypersurfaces with a cA_n singularity where $n\geq 2$ such that locally analytically they are Kawakita blowups.

Most cA_n singularities do not admit $(r_1,r_2,a,1)$-Kawakita blowups where $a\geq 2$. Below, we define the *type* of an isolated cA_n singularity, which for $n\geq 2$ is equal to the highest integer a such that it admits some $(r_1,r_2,a,1)$-Kawakita blowup locally analytically. General sextic double solids with an isolated cA_n singularity have a type $1cA_n$ singularity.

DEFINITION 4.8. Let (X,P) be the complex analytic space germ of an isolated cA_n singularity. Let a be the largest integer such that (X,P) is isomorphic to some germ $(\mathbb{V}(x_1x_2+g),\mathbf{0})$ where $g\in\mathbb{C}\{x_3,x_4\}$ has weight $a(n+1)$ under the weighting $(a,1)$ for (x_3,x_4). Then, we say that the cA_n singularity is of **type** a.

It is not obvious how to globally algebraically construct a Kawakita blowup for a variety with a cA_n singularity. We show this for affine hypersurfaces in the technical Lemma 4.9.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

We use a projectivization of Corollary 4.10 in Section 5 for constructing Kawakita blowups of sextic double solids.

We describe the notation for Lemma 4.9. Choose positive integers n, r_1, r_2, and a as in Notation 4.5. Let $F \in \mathbb{C}[x_1,x_2,x_3,x_4]$ have multiplicity at least 3, and let

$$f = -x_1^2 + x_2^2 + F$$

be such that $\mathbb{V}(f) \subseteq \mathbb{C}^4$ has terminal singularities and has a cA_n singularity of type at least a at the origin. Let q, w be the power series when splitting with respect to x_1 (Theorem 3.1), and p, v be the power series when splitting with respect to x_2, that is,

$$f = -((x_1+q)w)^2 + ((x_2+p)v)^2 + h, \tag{4.1}$$

where $q \in \mathbb{C}\{x_2,x_3,x_4\}$ and $p \in \mathbb{C}\{x_3,x_4\}$ both have multiplicity at least 2, and $w \in \mathbb{C}\{x_1,x_2,x_3,x_4\}$ and $v \in \mathbb{C}\{x_2,x_3,x_4\}$ are units, and $h \in \mathbb{C}\{x_3,x_4\}$ has multiplicity at least 3. If $a > 1$, then perform a coordinate change on x_3, x_4 for f such that h has weight $r_1 + r_2$.

Now choose weights

$$\boldsymbol{w} = \operatorname{wt}(\alpha,\beta,x_3,x_4) = (r_1, r_2, a, 1)$$

for the variables α, β, x_3, x_4 on $\mathbb{C}^4$ and

$$\boldsymbol{w}' = \operatorname{wt}(\alpha,\beta,x_1,x_2,x_3,x_4) = (r_1,\ r_2,\ m,\ \min(r_2, \operatorname{mult} p),\ a,\ 1)$$

for the variables $\alpha, \beta, x_1, x_2, x_3, x_4$ on $\mathbb{C}^6$, where $m = \min(r_2, \operatorname{mult} q)$. Writing a power series $s \in \mathbb{C}\{x_1,x_2,x_3,x_4\}$ as a sum of its $\boldsymbol{w}'$-weighted homogeneous parts $s = \sum_{i=0}^{\infty} s_i$, let $s_{<k}$ denote $\sum_{i<k} s_i$ and $s_{\geq k}$ denote $\sum_{i \geq k} s_i$. Define the ideal

$$I = (f,\ -\alpha + (x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2},\ -\beta + x_2 + p_{<r_2})$$

of $\mathbb{C}[\alpha,\beta,x_1,x_2,x_3,x_4]$, where $v_{<r_1-r_2}$ is defined to be 1 when $r_1 = r_2$ and where $w_{<r_1-m}$ is defined to be 1 when $r_1 = m$. Note that the affine varieties $\mathbb{V}(f) \subseteq \mathbb{C}^4$ and $\mathbb{V}(I) \subseteq \mathbb{C}^6$ are isomorphic.

LEMMA 4.9. *Using the notation above, the $\boldsymbol{w}'$-blowup of $\mathbb{V}(I)$ is a $\boldsymbol{w}$-Kawakita blowup.*

Proof. The morphism

$$\begin{aligned} &\varphi\colon \mathbb{C}^4 \to \mathbb{C}^4 \\ &(x_1,x_2,x_3,x_4) \mapsto ((x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2},\ x_2 + p_{<r_2},\ x_3,\ x_4) \end{aligned}$$

has a local analytic inverse φ^{-1}, given by

$$\begin{aligned} &\varphi^{-1}\colon (\mathbb{C}^4,\mathbf{0}) \to (\mathbb{C}^4,\mathbf{0}) \\ &(\alpha,\beta,x_3,x_4) \mapsto ((\alpha - (\beta - p_{<r_2} + p_{<r_1})v')u - q',\ \beta - p_{<r_2},\ x_3,\ x_4), \end{aligned}$$

where $u \in \mathbb{C}\{\alpha,\beta,x_3,x_4\}$ is a unit, $v' = v_{<r_1-r_2}(\beta - p_{<r_2}, x_3, x_4)$ and $q' = q_{<r_1}(\beta - p_{<r_2}, x_3, x_4)$. Define the map germ

$$\begin{aligned} &\psi\colon (\mathbb{C}^4,\mathbf{0}) \to (\mathbb{C}^6,\mathbf{0}) \\ &(\alpha,\beta,x_3,x_4) \mapsto (\alpha,\beta,\varphi^{-1}(\alpha,\beta,x_3,x_4)). \end{aligned}$$

The restriction of ψ to $\mathbb{V}(I) \to \mathbb{V}(f \circ \psi)$ is a weight-respecting local biholomorphism, whose inverse is a projection. Therefore, the $\boldsymbol{w}$-blowup of $\mathbb{V}(f \circ \psi)$ is equivalent to the $\boldsymbol{w}'$-blowup

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

of $\mathbb{V}(I)$. If the $\boldsymbol{w}$-weight of $f \circ \psi$ is $r_1 + r_2$, then by Proposition 4.6, the $\boldsymbol{w}$-blowup of $\mathbb{V}(f \circ \psi)$ is the $\boldsymbol{w}$-Kawakita blowup map germ. Using Equation (4.1), it suffices to show that

$$\operatorname{wt}[((x_1+q)w + (x_2+p)v) \circ \psi] = r_1, \tag{4.2}$$

$$\operatorname{wt}[(-(x_1+q)w + (x_2+p)v) \circ \psi] = r_2. \tag{4.3}$$

Since ψ is weight-respecting, we have

$$\begin{aligned} \operatorname{wt}[(x_1+q)w_{\geq r_1 - m} \circ \psi)] &\geq r_1, \\ \operatorname{wt}[q_{\geq r_1} w_{< r_1 - m} \circ \psi)] &\geq r_1, \\ \operatorname{wt}[(x_2+p)v_{\geq r_1 - r_2} \circ \psi] &\geq r_1, \\ \operatorname{wt}[p_{\geq r_1} v_{< r_1 - r_2} \circ \psi] &\geq r_1. \end{aligned}$$

Since $((x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2}) \circ \psi = \alpha$, this proves Equation (4.2). Using, in addition, that $\operatorname{wt}[(x_2 + p_{<r_1})v_{<r_1-r_2} \circ \psi] = r_2$, Equation (4.3) follows. ∎

COROLLARY 4.10. *Using the notation above, if $F \in \mathbb{C}[x_2,x_3,x_4]$, or equivalently, if $q = 0$ and $w = 1$, then define the ideal $J \subseteq \mathbb{C}[\alpha,\beta,x_2,x_3,x_4]$ by*

$$J = (-(\alpha - (x_2 + p_{<r_1})v_{<r_1-r_2})^2 + x_2^2 + F,\ -\beta + x_2 + p_{<r_2}), \tag{4.4}$$

where $v_{<r_1-r_2}$ is defined to be 1 if $r_1 = r_2$. Then, $\mathbb{V}(J)$ and $\mathbb{V}(f)$ are isomorphic affine varieties, and the $(r_1, r_2, \min(r_2, \operatorname{mult} p), a, 1)$-blowup of $\mathbb{V}(J)$ is a $\mathbf{w}$-Kawakita blowup. If in addition $r_1 = r_2$, then define the ideal $J' \subseteq \mathbb{C}[x_1,\beta,x_2,x_3,x_4]$ by

$$J' = (f, -\beta + x_2 + p_{<r_2}). \tag{4.5}$$

Then, $\mathbb{V}(J')$ and $\mathbb{V}(f)$ are isomorphic affine varieties, and the $(r_1, r_2, \min(r_2, \operatorname{mult} p), a, 1)$-blowup of $\mathbb{V}(J')$ is a $\mathbf{w}$-Kawakita blowup.

Proof. The isomorphism between $\mathbb{V}(I)$ and $\mathbb{V}(J)$ is a projection, with inverse given by $x_1 \mapsto \alpha - (\beta - p_{<r_2} + p_{<r_1})v_{<r_1-r_2}$, which is weight-respecting. If $r_1 = r_2$, the isomorphism between $\mathbb{V}(J)$ and $\mathbb{V}(J')$ is given by $x_1 \mapsto \alpha - \beta$, which is weight-respecting. ∎

The power series p, v, q, w can be expressed in terms of the coefficients of F using the explicit splitting lemma, Proposition 3.2.

§5. Birational models of sextic double solids

In this section, we prove Theorem B on birational nonrigidity of certain sextic double solids. First, we give the generality conditions that we use.

DEFINITION 5.1. Let X be a sextic double solid in Notation 3.4. Let $\mathbb{P}(1,1,3)$ have variables y, z, w, and let $\mathbb{P}^1$ have variables y, z. Define the following generality conditions, depending on the family that X lies in:

(4) $\mathbb{V}(2wa_2 + c_5, w^2 - d_6) \subseteq \mathbb{P}(1,1,3)$ is 10 distinct points,
(5) $\mathbb{V}(a_2, -w^2 + d_6) \subseteq \mathbb{P}(1,1,3)$ is 4 distinct points,
(6) $c_4 - 2a_1b_3 - a_2b_2 + 2a_0a_2^2 + 6a_1^2a_2 \in \mathbb{C}[y,z]$ is nonzero, and $\mathbb{V}(a_2) \subseteq \mathbb{P}^1$ is two distinct points, and for both of these points P, one of $b_3(P)$, $c_4(P)$ or $d_5(P)$ is nonzero,
(7.1) $\mathbb{V}(-e_2 + 4a_0r_2 + b_2 - 6a_1^2) \subseteq \mathbb{P}^1$ is two distinct points,
(7.2) r_1 and q_1 are coprime in $\mathbb{C}[y,z]$,

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

(7.3) $q_2 \in \mathbb{C}[y,z]$ is not a square,
(8) $a_0 \neq A_0$.

THEOREM B. *Every terminal $\mathbb{Q}$-factorial sextic double solid with a cA_n singularity with $n \geq 4$ that satisfies the generality conditions in Definition 5.1 has a Sarkisov link starting with a weighted blowup with center the cA_n point.*

We treat each of the seven families separately. We use the notation in Construction 2.25 and Example 2.26 for the 2-ray links. We write the cA_4 case in more detail. Below, when we say that a birational map is m Atiyah flops, then we mean that the base of the flop is m points, above each we are contracting a curve and extracting a curve, and locally analytically above each of the points it is an Atiyah flop. Similarly for flips. Below, for a morphism $\Phi\colon T_0 \to \mathbb{P}$, we let $\Phi^*\colon \operatorname{Cox}\mathbb{P} \to \operatorname{Cox} T_0$ denote a corresponding $\mathbb{C}$-algebra homomorphism of Cox rings (this is described explicitly in the proof of Proposition 5.4).

5.1 Singularities after divisorial contraction

The non-Gorenstein singularities on Y for an ordinary type divisorial contraction $Y \to X$ with center a cA_n singularity can be easily found using the result by Kawakita, Theorem 2.21. On the other hand, the structure or the number of Gorenstein singularities is unclear. We show in Proposition 5.3 that if X in one of the 11 families is general, then Y has no Gorenstein singularities. We do not give the generality conditions of Proposition 5.3 explicitly. We do not need Proposition 5.3 for proving Theorem B.

LEMMA 5.2. *Let $a,b \in \mathbb{C}[y,z]$ be nonzero homogeneous polynomials with $\deg a \geq \deg b$ such that for every homogeneous polynomial $c \in \mathbb{C}[y,z]$ of degree $\deg a - \deg b$, the polynomial $a + bc$ is divisible by the square of a linear form. Then a and b are both divisible by the square of the same linear form.*

Proof. Suffices to prove that for polynomials $f,g \in \mathbb{C}[x]$, if $f + \lambda g$ has a repeated root for infinitely $\lambda \in \mathbb{C}$, then f and g have a common repeated root. Dividing f and g by suitable linear polynomials, it suffices to consider the case where every common root of f and g is a common repeated root of f and g.

If the set

$$A = \{\alpha \in \mathbb{C} \mid \alpha \text{ is a repeated root of } f + \lambda_\alpha g \text{ for some } \lambda_\alpha \in \mathbb{C}\}$$

is finite, then there exist $\alpha \in \mathbb{C}$ and $\lambda_1 \neq \lambda_2$ such that α is a repeated root of both $f + \lambda_1 g$ and $f + \lambda_2 g$. It follows that α is a repeated root of both f and g.

Without loss of generality, both f and g are nonconstant. Subtracting $g \cdot \dfrac{\mathrm{d}(f+\lambda g)}{\mathrm{d}x}$ from $\dfrac{\mathrm{d}g}{\mathrm{d}x} \cdot (f + \lambda g)$, we find that a repeated root of $f + \lambda g$ is necessarily a root of $f\dfrac{\mathrm{d}g}{\mathrm{d}x} - g\dfrac{\mathrm{d}f}{\mathrm{d}x}$. If $f\dfrac{\mathrm{d}g}{\mathrm{d}x} = g\dfrac{\mathrm{d}f}{\mathrm{d}x}$, then a prime factor of g is a prime factor of f. If $f\dfrac{\mathrm{d}g}{\mathrm{d}x} \neq g\dfrac{\mathrm{d}f}{\mathrm{d}x}$, then the set A is finite. In both cases, f and g have a common repeated root. ∎

PROPOSITION 5.3. *Let X be a member of family $k \in \mathrm{Inds}$ of Notation 3.4 which is smooth outside a $cA_{\lfloor k \rfloor}$ singularity at $P_x = [1,0,0,0,0]$. Let $Y \to X$ be a divisorial contraction with center P_x, which is an $(r_1, r_2, 1, 1)$-Kawakita blowup. If X is general, then Y has a quotient*

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

singularity $1/r_1(1,1,r_1-1)$ *if* $r_1>1$ *and a quotient singularity* $1/r_2(1,1,r_2-1)$ *if* $r_2>1$ *and is smooth elsewhere.*

Proof. We show that Y has only up to two quotient singularities on the exceptional divisor and is smooth elsewhere. Since $Y \to X$ is an $(r_1,r_2,1,1)$-Kawakita blowup, we can consider the local analytic coordinate system around P_x where X is given by $wt+h(y,z)$ where $h \in \mathbb{C}\{y,z\}$ has multiplicity $n+1$. The variety Y is locally analytically around the exceptional divisor given by $wt+\frac{1}{u^{n+1}}h(uy,uz)$ inside the geometric quotient $(\mathbb{C}^5 \setminus \mathbb{V}(w,t,y,z))/\mathbb{C}^*$ where the $\mathbb{C}^*$-action is given by $\lambda \cdot (u,w,t,y,z) = (\lambda^{-1}u, \lambda^{r_1}w, \lambda^{r_2}t, \lambda y, \lambda z)$. The singular locus of Y is given by

$$\operatorname{Sing} Y = \mathbb{V}\left(u,w,t,h_{n+1},\frac{\partial h_{n+1}}{\partial y},\frac{\partial h_{n+1}}{\partial z},h_{n+2}\right) \cup \{P_w\}_{\text{if } r_1>1} \cup \{P_t\}_{\text{if } r_2>1},$$

where h_i denotes the homogeneous degree i part of h, and P_w and P_t are the points $[0,1,0,0,0]$ and $[0,0,1,0,0]$, respectively. We see that Y is singular outside of P_w and P_t if and only if there exists a homogeneous linear form $L \in \mathbb{C}[y,z]$ such that L^2 divides h_{n+1} and L divides h_{n+2}. For all $k \in \text{Inds}$, exactly one of the following holds:

- $Y \setminus \{P_w,P_t\}$ is smooth for a general X in family k, or
- for all X in family k, there exists a homogeneous linear form $L \in \mathbb{C}[y,z]$ such that L^2 divides h_{n+1} and L divides h_{n+2}.

We write the proof for family 8 in detail, the proofs for the other 10 families are similar. Using the explicit splitting lemma (Proposition 3.2), we compute that

$$h_9 = Q - 2d_3r_2^3 = 8(a_0-A_0)s_3^3 + r_2R,$$

where $Q,R \in \mathbb{C}[y,z]$ are homogeneous of degrees 9 and 7, respectively, and Q does not contain the polynomial d_3. Assume that for all X in family 8, there exists L such that L^2 divides h_9. Using Lemma 5.2 with $(a,b,c) = (Q, r_2^3, -2d_3)$, we find that a prime divisor of r_2 divides h_9. Therefore, a general member X of family 8 satisfies that r_2 and s_3 have a common prime divisor, contradicting Theorem A(d) and Proposition 3.8. So, for a general X in family 8, $Y \setminus \{P_w,P_t\}$ is smooth. ∎

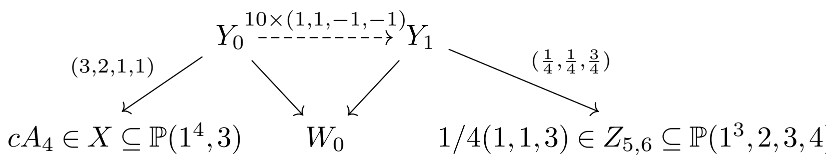
5.2 cA_4 model

Note that Okada described a Sarkisov link starting from a general complete intersection $Z_{5,6} \subseteq \mathbb{P}(1,1,1,2,3,4)$ to a sextic double solid (see entry No. 9 of the table in [43, §9]). We show the converse:

PROPOSITION 5.4. *A sextic double solid with a* cA_4 *singularity satisfying Definition 5.1 has a Sarkisov link to a complete intersection* $Z_{5,6} \subseteq \mathbb{P}(1,1,1,2,3,4)$*, starting with a* $(3,2,1,1)$*-blowup of the* cA_4 *point, then* 10 *Atiyah flops, and finally a Kawamata divisorial contraction (see [34]) to a terminal quotient* $1/4(1,1,3)$ *point. Under further generality conditions (Proposition 5.3),* Z *is quasismooth.*

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Proof. We exhibit the diagram below.



The corresponding diagram for the ambient toric spaces is given in detail in Example 2.26.

By Theorem A, every sextic double solid $\hat{X}$ with an isolated cA_4 singularity can be given by

$$\hat{X}\colon \mathbb{V}(\hat{f}) \subseteq \mathbb{P}(1,1,1,1,3)$$

with variables x,y,z,t,w where

$$\hat{f} = -w^2 + x^4t^2 + 2x^3ta_2 + x^3t^2A_1 + x^2a_2^2 + x^2tB_3 + xC_5 + D_6,$$

where $a_2 \in \mathbb{C}[y,z]$ is homogeneous of degree 2, and $A_i, B_i, C_i, D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i.

Below, we perform the following constructions:

(1) We define a weighted projective space $\mathbb{P} = \mathbb{P}(1,1,1,1,3,5)$.
(2) We define a subvariety X of $\mathbb{P}$ by explicitly describing a homogeneous ideal.
(3) We show that X and $\hat{X}$ are isomorphic by constructing an explicit isomorphism.
(4) We construct a toric variety T_0.
(5) We define a morphism $\Phi\colon T_0 \to \mathbb{P}$.
(6) We construct a subvariety Y_0 of T_0 by explicitly describing a bihomogeneous ideal I_Y of the Cox ring of T_0.
(7) We restrict the morphism Φ to Y_0 and check that its image is X.

Although computational, the above steps are completely elementary. The reason for these constructions is that, as we prove below, the morphism $Y_0 \to X$ is the $(3,2,1,1)$-Kawakita blowup and I_Y 2-ray follows T_0.

The reader might have the philosophical question of how the author found the varieties $\mathbb{P}, X, T_0$ and Y_0, described below, and why they are defined exactly as they are. In Remark 5.5, we describe the methods we used to arrive at the construction of $\mathbb{P}, X, T_0$ and Y_0. Note that the choices involved in (1), (2), (4), and (6) above are somewhat arbitrary. Namely, there exist other varieties $\mathbb{P}, X, T_0$ and Y_0 such that $Y_0 \to X$ is the $(3,2,1,1)$-Kawakita blowup and I_Y 2-ray follows T_0.

We start by constructing X. Define the bidegree $(5,6)$ complete intersection X, isomorphic to $\hat{X}$, by

$$X\colon \mathbb{V}(f, -x\xi + \alpha^2 - D_6) \subseteq \mathbb{P}(1,1,1,1,3,5)$$

with variables x,y,z,t,α,ξ, where

$$f = -\xi + 2\alpha a_2 + 2\alpha xt + x^2t^2A_1 + xtB_3 + C_5.$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

The isomorphism is given by

$$\begin{gathered}\hat{X} \to X \\ [x,y,z,t,w] \mapsto \left[x,y,z,t,\alpha',2\alpha' a_2 + 2\alpha' xt + x^2t^2A_1 + xtB_3 + C_5\right],\end{gathered}$$

where $\alpha' = w + x^2t + xa_2$, with inverse

$$[x,y,z,t,\alpha,\xi] \mapsto [x,y,z,t,\alpha - x^2t - xa_2].$$

We describe the divisorial contraction $\varphi\colon Y_0 \to X$. Define the toric variety

$$T_0\colon \left(\begin{array}{cc|ccccc} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & 0 & 1 & 1 & 3 & 6 & 2 \end{array}\right),$$

as in Example 2.26. Let Φ be the ample model of $\mathbb{V}(x)$, that is,

$$\begin{gathered}\Phi\colon T_0 \to \mathbb{P}(1,1,1,1,3,5) \\ [u,x,y,z,\alpha,\xi,t] \mapsto [x,uy,uz,u^2t,u^3\alpha,u^6\xi].\end{gathered}$$

Let Y_0 be the strict transform of X. Let Φ^* denote the corresponding $\mathbb{C}$-algebra homomorphism, namely

$$\begin{gathered}\Phi^*\colon \mathbb{C}[x,y,z,t,\alpha,\xi] \to \mathbb{C}[u,x,y,z,\alpha,\xi,t] \\ \Phi^*\colon x \mapsto x,\ y \mapsto uy,\ z \mapsto uz,\ t \mapsto u^2t,\ \alpha \mapsto u^3\alpha,\ \xi \mapsto u^6\xi.\end{gathered}$$

Define

$$A_Y = A_1(y,z,ut), \qquad B_Y = B_3(y,z,ut), \qquad C_Y = C_5(y,z,ut), \qquad D_Y = D_6(y,z,ut)$$

and define the polynomial $g = \Phi^* f/u^5$, that is,

$$g = -u\xi + 2\alpha a_2 + 2\alpha xt + x^2t^2A_Y + xtB_Y + C_Y.$$

Then, Y_0 is given by

$$Y_0\colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (g, -x\xi + \alpha^2 - D_Y).$$

We will see later that I_Y 2-ray follows T_0. Note that there exist other ideals that define the same variety $Y_0 \subseteq T_0$ (see [20, Cor. 3.9]), but where the ideal might not 2-ray follow T_0. Also note that we have not (and do not need to) prove that the ideal I_Y is saturated with respect to u, although in general, saturating might help in finding the ideal that 2-ray follows T_0. The morphism $Y_0 \to X$ is the restriction of $T_0 \to \mathbb{P}(1,1,1,1,3,5)$. Locally, $(Y_0)_x \to X_x$ is the $(3,2,1,1)$-blowup of $\mathbb{V}(f') \subseteq \mathbb{C}^4$ with variables α,t,y,z, where

$$f' = -\alpha^2 + 2\alpha a_2 + 2\alpha t + t^2A_1 + tB_3 + C_5 + D_6.$$

Since $\operatorname{wt} f' = 5$, by Proposition 4.6, $(Y_0)_x \to X_x$ is a $(3,2,1,1)$-Kawakita blowup.

The first diagram in the 2-ray game for Y_0 is 10 Atiyah flops, under Definition 5.1. We describe the diagram $Y_0 \to W_0 \leftarrow Y_1$ globally. Multiplying the action matrix of T_0 by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, define

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

$$T_1\colon \left(\begin{array}{ccccc|cc} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

Define Y_1 by $\mathbb{V}(I_Y) \subseteq T_1$. Define the morphisms $Y_0 \to W_0$ and $Y_1 \to W_0$ as the ample models of $\mathbb{V}(y)$. The exceptional locus of $Y_0 \to W_0$ is $E_0^- = \mathbb{V}(\xi,t) \subseteq Y_0$, the exceptional locus of $Y_1 \to W_0$ is $E_1^+ = \mathbb{V}(u,x) \subseteq Y_1$, and the base of the flop is

$$\{P_i\} = \mathbb{V}(2\alpha a_2 + C_5(y,z,0), \alpha^2 - D_6(y,z,0)) \subseteq \mathbb{P}(1,1,3) \subseteq W_0,$$

where $\mathbb{P}(1,1,3)$ has variables y,z,α. If a_2, $C_5(y,z,0)$ and $D_5(y,z,0)$ are general enough, that is, if Definition 5.1 is satisfied, then the base of the flop is 10 points $\{P_i\}_{1\le i\le 10}$, and both E_0^- and E_1^+ are 10 disjoint curves mapping to $\{P_i\}_{1\le i\le 10}$.

We show that locally analytically, the diagram $Y_0 \to W_0 \leftarrow Y_1$ is 10 Atiyah flops. Let $P \in W_0$ be any point in the base of the flop. Then, P has y or z coordinate nonzero. We consider the case where the y-coordinate is nonzero, the other case is similar. Since the base of the flop is 10 points, the point P is smooth in $\mathbb{P}(1,1,3)$. By the implicit function theorem, we can locally analytically equivariantly express α and z in terms of the variables u,x,ξ,t on the patches $(Y_0)_y$, $(W_0)_y$, and $(Y_1)_y$. So, the flop $Y_0 \to W_0 \leftarrow Y_1$ is locally analytically a $(1,1,-1,-1)$-flop, the so-called Atiyah flop, around P.

The last morphism $Y_1 \to Z$ in the link for X is a divisorial contraction. Multiplying the action matrix of T_0 by the matrix $\left(\begin{smallmatrix} 6 & -5 \\ 2 & -1\end{smallmatrix}\right)$ with determinant 4, we see that

$$T_1 \cong \left(\begin{array}{ccccc|cc} u & x & y & z & \alpha & \xi & t \\ 5 & 6 & 1 & 1 & 3 & 0 & -4 \\ 1 & 2 & 1 & 1 & 3 & 4 & 0 \end{array}\right).$$

Let $Y_1 \to Z$ be the ample model of $\frac{1}{4}\mathbb{V}(\xi)$, that is,

$$\begin{gathered} Y_1 \to Z \\ [u,x,y,z,\alpha,\xi,t] \mapsto \left[t^{\frac{5}{4}}u, t^{\frac{1}{4}}y, t^{\frac{1}{4}}z, t^{\frac{3}{2}}x, t^{\frac{3}{4}}\alpha, \xi\right]. \end{gathered}$$

Then Z is the bidegree $(5,6)$ complete intersection

$$Z\colon \mathbb{V}(h, -x\xi + \alpha^2 - D_6(y,z,u)) \subseteq \mathbb{P}(1,1,1,2,3,4)$$

with variables u,y,z,x,α,ξ, where the h is given by applying the $\mathbb{C}$-algebra homomorphism $t \mapsto 1$ to g. The morphism $Y_1 \to Z$ contracts the exceptional divisor $\mathbb{V}(t) \subseteq Y_1$ to the point $P_\xi = [0,0,0,0,0,1]$. On the quasiprojective patch $(Y_1)_\xi$, we can express u and x locally analytically equivariantly in terms of y,z,α,t. So, the morphism $Y_1 \to Z$ is locally analytically the Kawamata weighted blowdown (see [34]) to the terminal quotient singular point P_ξ of type $1/4(1,1,3)$. ∎

REMARK 5.5. We explain below how we found the variety X used in Proposition 5.4. We start with the variety $\hat{X}$, given by Theorem A. Note that it is not possible to assign weights to the coordinates of $\mathbb{P}(1,1,1,1,3)$ such that the corresponding weighted blowup of $\hat{X}$ would be a $(3,2,1,1)$-Kawakita blowup. To amend this, we first replace the variety $\hat{X}$ by a variety $\bar{X}$ such that choosing the weights appropriately, the weighted blowup of $\bar{X}$ is the $(3,2,1,1)$-Kawakita blowup. So far the process is algorithmic. Unfortunately, as we see below, the constructed ideal $I_{\bar{Y}}$ does not 2-ray follow the ambient toric variety $\bar{T}_0$. Using

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

the technique known as *unprojection*, we construct another toric variety T_0 and a subvariety Y_0 given by an ideal I_Y. This time we are lucky, since as the proof of Proposition 5.4 shows, the ideal I_Y 2-ray follows T_0. Note that the variety Y_0 has higher codimension in T_0 than $\bar{Y}_0$ had in $\bar{T}_0$. We give details below.

We perform the coordinate change $\hat{X} \to \bar{X}$ given in Equation (4.4) of Corollary 4.10, with $(r_1, r_2, a, 1) = (3, 2, 1, 1)$, $p_2 = a_2$ and $v_0 = 1$. We see that $\hat{X}$ is isomorphic to

$$\bar{X}\colon \mathbb{V}(\bar{f}) \subseteq \mathbb{P}(1,1,1,1,3)$$

with variables x, y, z, t, α, where

$$\bar{f} = \alpha(-\alpha + 2x^2 t + 2x a_2) + x^3 t^2 A_1 + x^2 t B_3 + x C_5 + D_6.$$

We construct a $(3,2,1,1)$-Kawakita blowup $\bar{Y}_0 \to \bar{X}$. Define the toric variety $\bar{T}_0$ by

$$\bar{T}_0\colon \left(\begin{array}{cc|cccc} u & x & y & z & \alpha & t \\ 0 & 1 & 1 & 1 & 3 & 1 \\ -1 & 0 & 1 & 1 & 3 & 2 \end{array}\right).$$

In other words, $\bar{T}_0$ is given by the geometric quotient

$$\bar{T}_0 = \frac{\mathbb{C}^6 \setminus \mathbb{V}((u,x) \cap (y,z,\alpha,t))}{(\mathbb{C}^*)^2}.$$

Let $\bar{\Phi}$ be the ample model of $\mathbb{V}(x)$, and let $\bar{Y}_0 \subseteq \bar{T}_0$ be the strict transform of $\bar{X}$. By Corollary 4.10, $\bar{Y}_0 \to \bar{X}$ is a $(3,2,1,1)$-Kawakita blowup. Alternatively, define $\bar{Y}_0$ by $\mathbb{V}(\bar{g}) \subseteq T_0$ where

$$\bar{g} = \alpha(-u\alpha + 2x^2 t + 2x a_2) + x^3 t^2 A_Y + x^2 t B_Y + x C_Y + u D_Y$$

and use Proposition 4.6 on the patch $(\bar{Y}_0)_x \to \bar{X}_x$ to show that $\bar{Y}_0 \to \bar{X}$ is a $(3,2,1,1)$-Kawakita blowup, similarly to Proposition 5.4.

We show that $I_{\bar{Y}}$ does not 2-ray follow $\bar{T}_0$. We describe the next (and the final) map in the 2-ray game for $\bar{T}_0$. Acting by the matrix $\left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$, we can write $\bar{T}_0$ by

$$\bar{T}_0 \cong \left(\begin{array}{cc|cccc} u & x & y & z & \alpha & t \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 2 & 1 & 1 & 3 & 0 \end{array}\right).$$

The ample model of the divisor $\mathbb{V}(y)$ is the weighted blowup

$$\begin{aligned} \bar{T}_0 &\to \mathbb{P}(1,1,1,2,3) \\ [u,x,y,z,\alpha,t] &\mapsto [y,z,ut,xt,\alpha], \end{aligned}$$

where the center is the surface $\mathbb{P}(1,1,3)$ given by $\mathbb{V}(u,x) \subseteq \mathbb{P}(1,1,1,2,3)$ with variables y, z, u, x, α. Above every point in $\mathbb{P}(1,1,3)$, the fiber is $\mathbb{P}^1$. Define

$$\bar{Z}\colon \mathbb{V}(\bar{h}) \subseteq \mathbb{P}(1,1,1,2,3),$$

where

$$\bar{h} = \alpha(-u\alpha + 2x^2 + 2x a_2) + x^3 A_Z + x^2 B_Z + x C_Z + u D_Z,$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

where

$$A_Z = A_1(y,z,u), \qquad B_Z = B_3(y,z,u), \qquad C_Z = C_5(y,z,u), \qquad D_Z = D_6(y,z,u).$$

We show that when restricting the weighted blowup to $\bar{Y}_0 \to \bar{Z}$, the exceptional locus is one-dimensional. After restricting to $\bar{Y}_0$, the exceptional divisor $\mathbb{V}(t)$ becomes $\mathbb{V}(t, x(2\alpha a_2 + C_5(y,z,0)) + u(-\alpha^2 + D_6(y,z,0)))$. By Definition 5.1, there are exactly 10 points $P_1, \dots, P_{10} \in \mathbb{P}(1,1,3) \subseteq \bar{Z}$ such that $2\alpha a_2 + C_5(y,z,0)$ and $-\alpha^2 + D_6(y,z,0)$ have a common solution. Above each of those points, the fiber is $\mathbb{P}^1$. Above any other point, the fiber is just one point. Therefore, the morphism $\bar{Y}_0 \to \bar{Z}$ contracts 10 curves onto 10 points, and is an isomorphism elsewhere. This shows that $\bar{Y}_0$ does not 2-ray follow $\bar{T}_0$, since $\bar{Z}$ is not $\mathbb{Q}$-factorial and a 2-ray link ends with either a fibration or a divisorial contraction.

The problem with the previous embedding was that $\bar{g}$ belonged to the irrelevant ideal (u,x). We *unproject* the divisor $\mathbb{V}(u,x)$, to embed $\bar{Y}_0$ into a toric variety T_0 such that Y_0 2-ray follows T_0. The varieties $Y_0 \subseteq T_0$ are defined as in the proof of Proposition 5.4. We see that $\bar{Y}_0$ is isomorphic to Y_0 through the map

$$[u,x,y,z,\alpha,t] \mapsto \left[u,x,y,z,\alpha,\frac{\alpha^2 - D_Y}{x},t\right].$$

The map is a morphism, since we have the equality

$$\frac{\alpha^2 - D_Y}{x} = \frac{2\alpha a_2 + 2\alpha x t + x^2 t^2 A_Y + x t B_Y + C_Y}{u}$$

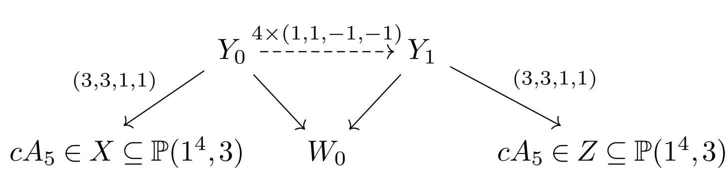
in the field of fractions of $\mathbb{C}[u,x,y,z,\alpha,t]$, and x or u is nonzero at every point of T_0. For more details on this kind of *unprojection*, see [49, §2] or [44, §2.3].

The coordinate change $\bar{Y}_0 \to Y_0$ induces a coordinate change $\bar{X} \to X$, where X is defined as in the proof of Proposition 5.4.

5.3 cA_5 model

Proposition 5.6. *A sextic double solid X which is a Mori fiber space with a cA_5 singularity satisfying Definition 5.1 has a Sarkisov link to a sextic double solid Z with a cA_5 singularity, starting with a $(3,3,1,1)$-blowup of the cA_5 point in X, then four Atiyah flops, and finally a $(3,3,1,1)$-blowdown to a cA_5 point. If in addition c_4 is general after fixing a_i, b_i, and d_6 in Notation 3.4, then X and Z are not isomorphic. Under further generality conditions (Proposition 5.3), both X and Z are smooth outside the cA_5 point.*

Proof. We exhibit the diagram below.



We construct X and a $(3,3,1,1)$-Kawakita blowup $Y_0 \to X$. Using Theorem A, and performing the coordinate change in Equation (4.5) of Corollary 4.10 (with $p_2 = a_2$), we can write a sextic double solid X with a cA_5 singularity by

$$X\colon \mathbb{V}(f, -\beta + xt + a_2) \subseteq \mathbb{P}(1,1,1,1,2,3),$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

with variables x,y,z,t,β,w where

$$f = -w^2 + x\beta(2b_3 - 4\beta a_1 + 8xta_1 + x\beta) + 4x^3t^3a_0 + x^2t^2B_2 + xtC_4 + D_6,$$

where B_i, C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degrees i. Define T_0 by

$$T_0\colon \left(\begin{array}{cc|ccccc} u & x & y & z & w & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 3 & 3 & 2 \end{array}\right).$$

Let $\Phi\colon T_0 \to \mathbb{P}(1,1,1,1,2,3)$ be the ample model of $\mathbb{V}(x)$, $Y_0 \subseteq T_0$ the strict transform of X, and $Y_0 \to X$ the restriction of Φ. Then, Y_0 is given by

$$Y_0\colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/u^6, -u\beta + xt + a_2),$$

and $Y_0 \to X$ is a $(3,3,1,1)$-Kawakita blowup.

We show that the first map in the 2-ray game for Y_0 is a flop, locally analytically 4 Atiyah flops, under Definition 5.1. Acting by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, we find

$$T_0 \cong \left(\begin{array}{cc|ccccc} u & x & y & z & w & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

The base of the flop in $\mathbb{P}(1,1,3) \subseteq W_0$ is given by $\mathbb{V}(a_2, -w^2 + D_6(y,z,0)) \subseteq \mathbb{P}(1,1,3)$. If a_2 and $D_6(y,z,0)$ are general, that is, Definition 5.1 is satisfied, then this is exactly four points. In this case, any such point P is a smooth point in $\mathbb{P}(1,1,3)$. Consider the case where the y-coordinate of P is nonzero, the case where z is nonzero is similar. Locally analytically equivariantly, we can express z and w in terms of u,x,β,t in Y_0, W_0, and Y_1. So, the diagram $Y_0 \to W_0 \leftarrow Y_1$ is locally analytically four Atiyah flops.

The last map in the 2-ray game of Y_0 is a weighted blowdown $Y_1 \to Z$. After acting by $\left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$ on the initial matrix of T_0, we find that T_1 is given by

$$T_1\colon \left(\begin{array}{ccccc|cc} u & x & y & z & w & \beta & t \\ 2 & 3 & 1 & 1 & 3 & 0 & -1 \\ 1 & 2 & 1 & 1 & 3 & 1 & 0 \end{array}\right).$$

We see that $Z \subseteq \mathbb{P}(1,1,1,1,2,3)$ with variables β,u,y,z,x,w is given by the ideal

$$I_Z = (h, -u\beta + x + a_2),$$

where h is given by sending t to 1 in $\Phi^* f/u^6$, namely

$$h = -w^2 + x\beta(2b_3 - 4u\beta a_1 + 8xa_1 + x\beta) + 4x^3a_0 + x^2B_Z + xC_Z + D_Z$$

and

$$B_Z = B_2(y,z,u), \qquad C_Z = C_4(y,z,u), \qquad D_Z = D_6(y,z,u).$$

Substituting $x = u\beta - a_2$ into h, we find that Z is a sextic double solid. Applying the explicit splitting lemma (Proposition 3.2), we find that the complex analytic space germ (Z, P_β) is isomorphic to $(\mathbb{V}(h_{\text{ana}}), \mathbf{0}) \subseteq (\mathbb{C}^4, \mathbf{0})$ with variables w,u,y,z, where

$$h_{\text{ana}} = -w^2 + u^2 + d_6 - (b_3 - 2a_1a_2)^2 + (\text{h.o.t in } y,z),$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

where (h.o.t in y,z) stands for higher-order terms in the variables y,z. So, $P_\beta \in Z$ is a cA_5 singularity. On the patch where β is nonzero, we can substitute $u = xt + a_2$, so the morphism $(Y_1)_\beta \to Z_\beta$ is a weighted blowup of a hypersurface given by a weight 6 polynomial. By Proposition 4.6, $Y_1 \to Z$ is a $(3,3,1,1)$-Kawakita blowup.

We show that X and Z are not isomorphic when $a_2 \neq 0$ and c_4 is general, using a dimension counting argument similar to [25, Th. 2.55]. Using the explicit splitting lemma, we find that the complex analytic space germ (X, P_x) is isomorphic to $(\mathbb{V}(f_{\text{ana}}), \mathbf{0}) \subseteq (\mathbb{C}^4, \mathbf{0})$ with variables w,t,y,z where

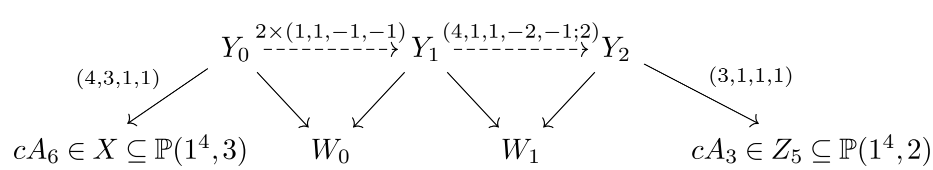
$$f_{\text{ana}} = -w^2 + t^2 + d_6 - 2a_2c_4 + 2a_2^2b_2 - 4a_0a_2^3 - (b_3 - 4a_1a_2)^2 + (\text{h.o.t in } y,z).$$

If X and Z are isomorphic, then this implies that the complex analytic space germs (X, P_x) and (Z, P_β) are isomorphic, implying by Propositions 2.4 and 2.5 that the degree 6 parts of $f_{\text{ana}}(0,0,y,z)$ and $h_{\text{ana}}(0,0,y,z)$ are the same up to an invertible linear coordinate change on y,z. Fixing a_0, a_1, a_2, b_2, b_3, and d_6, we see that $h_{\text{ana}}(0,0,y,z)$ is fixed, but $f_{\text{ana}}(0,0,y,z)$ has 5 degrees of freedom. Since there are only 4 degrees of freedom in picking an element of $\mathrm{GL}(2,\mathbb{C})$, the polynomials $f_{\text{ana}}(0,0,y,z)$ and $h_{\text{ana}}(0,0,y,z)$ are not related by an invertible linear coordinate change when c_4 is general. This shows that if X is general, then the varieties X and Z are not isomorphic. ∎

5.4 cA_6 model

Proposition 5.7. *A sextic double solid that is a Mori space with a cA_6 singularity satisfying Definition 5.1 has a Sarkisov link to a hypersurface $Z_5 \subseteq \mathbb{P}(1,1,1,1,2)$ with a cA_3 singularity, starting with a $(4,3,1,1)$-blowup of the cA_6 point, then two $(1,1,-1,-1)$-flops, then a $(4,1,1,-2,-1;2)$-flip, and finally a $(2,2,1,1)$-blowdown to a cA_3 point. Under further generality conditions, the singular locus of Z consists of three points, namely the cA_3 point, the $1/2(1,1,1)$ quotient singularity, and an ordinary double point.*

Proof. We exhibit the diagram below.



We construct X and a $(4,3,1,1)$-Kawakita blowup $Y_0 \to X$. Using Theorem A and Corollary 4.10 with $p_2 = a_2$ and $p_3 = b_3 - 4a_1a_2$, we can write a sextic double solid X with a cA_6 singularity by

$$X\colon \mathbb{V}(f, -\beta + xt + a_2) \subseteq \mathbb{P}(1,1,1,1,2,3),$$

with variables x,y,z,t,β,w where

$$\begin{aligned} f = {} & \alpha(-\alpha + 2(b_3 - 4\beta a_1 + 4xta_1 + x\beta)) \\ & + 2\beta(c_4 - \beta b_2 + 2xtb_2 + 2x\beta a_1 + 2\beta^2 a_0 - 6xt\beta a_0 + 6x^2t^2a_0) \\ & + x^2t^3B_1 + xt^2C_3 + tD_5, \end{aligned}$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

where B_i, C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i. Define T_0 by

$$T_0\colon \left(\begin{array}{cc|ccccc} u & x & y & z & \alpha & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 3 & 2 \end{array}\right).$$

Let $\Phi\colon T_0 \to \mathbb{P}(1,1,1,1,2,3)$ be the ample model of $\mathbb{V}(x)$, $Y_0 \subseteq T_0$ the strict transform of X, and $Y_0 \to X$ the restriction of Φ. Then, Y_0 is given by

$$Y_0\colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/u^7, -u\beta + xt + a_2),$$

and $Y_0 \to X$ is a $(4,3,1,1)$-Kawakita blowup.

We show that the first diagram $Y_0 \to W_0 \leftarrow Y_1$ in the 2-ray game for Y_0 is locally analytically two Atiyah flops under Definition 5.1, namely that $\mathbb{V}(a_2) \subseteq \mathbb{P}^1$ with variables y,z consists of exactly two points, and for both of the points P, one of $b_3(P)$, $c_4(P)$ or $d_5(P)$ is nonzero, where $D_5 = t^5 d_0 + 2t^4 d_1 + t^3 d_2 + 2t^2 d_3 + t d_4 + 2d_5$. Acting by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ -1 & 1 \end{smallmatrix}\right)$, we find

$$T_0\colon \left(\begin{array}{cc|ccccc} u & x & y & z & \alpha & \beta & t \\ 3 & 4 & 1 & 1 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

Under the above condition, after a suitable linear change of coordinates on y,z, we find that $a_2 = yz$. Let $P = \mathbb{V}(z) \in \mathbb{P}^1 \subseteq W_0$, the case where $P = \mathbb{V}(y)$ is similar. On the patch where y is nonzero, we can substitute $z = u\beta - xt$. The contracted locus is $\mathbb{P}^1 \cong \mathbb{V}(\alpha,\beta,t) \subseteq (Y_0)_y$, and the extracted locus is $\mathbb{V}(u,x) = \mathbb{V}(u,x,\alpha b_3(1,0) + \beta c_4(1,0) + t d_5(1,0)) \subseteq (Y_1)_y$. By Definition 5.1, we can express one of α,β,t equivariantly locally analytically in the other variables. So, the flop diagram $Y_0 \to W_0 \leftarrow Y_1$ is locally analytically a $(1,1,-1,-1)$-flop above both of the points.

We show that the next diagram in the 2-ray game of Y_0 is a $(4,1,1,-2,-1;2)$-flip (this is case (1) in [8, Th. 8]). The toric variety T_1 is given by

$$T_1\colon \left(\begin{array}{cccc|ccc} u & x & y & z & \alpha & \beta & t \\ 3 & 4 & 1 & 1 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

The base of the flip is $P_\alpha = [0,0,0,0,1,0,0]$. On the patch where α is nonzero, we can express u locally analytically and equivariantly in terms of x,y,z,β,t. After substitution, the ideal is principal, with generator $f' = -\beta \cdot (2x + \cdots) + xt + a_2$. Under Definition 5.1, a_2 has a nonzero coefficient in f', so the flip diagram corresponds to case (1) in [8, Th. 8]. The flips contracts a curve containing a $1/4(1,1,3)$ singularity and extracts a curve containing a $1/2(1,1,1)$ singularity and an ordinary double point. The ordinary double point on Y_2 is at $[u_0,0,0,0,2,1,1]$ for some $u_0 \in \mathbb{C}$.

We show that the last map in the 2-ray game of Y_0 is a weighted blowup $Y_2 \to Z$, where Z is isomorphic to a hypersurface $Z_5 \subseteq \mathbb{P}(1,1,1,1,2)$ with variables u,y,z,β,α. Acting by the matrix $\left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$ on the initial action-matrix of T_0, we find that T_2 is given by

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

$$T_2\colon \begin{pmatrix} & u & x & y & z & \alpha & \beta & t \\ & 2 & 3 & 1 & 1 & 1 & 0 & -1 \\ & 1 & 2 & 1 & 1 & 2 & 1 & 0 \end{pmatrix}.$$

Define the bidegree $(5,2)$ complete intersection $Z\colon \mathbb{V}(h, a_2 - u\beta + x) \subseteq \mathbb{P}(1,1,1,1,2,2)$ with variables u,y,z,β,x,α, where

$$\begin{aligned} h = {} & \alpha(-u\alpha + 2(b_3 - 4u\beta a_1 + 4xa_1 + x\beta)) \\ & + 2\beta(c_4 - u\beta b_2 + 2xb_2 + 2x\beta a_1 + 2u^2\beta^2 a_0 - 6ux\beta a_0 + 6x^2 a_0) \\ & + x^2 B_Z + xC_Z + D_Z, \end{aligned}$$

where

$$B_Z = B_1(y,z,u), \qquad C_Z = C_3(y,z,u), \qquad D_Z = D_5(y,z,u).$$

The morphism $Y_2 \to Z$ given by the ample model of $\mathbb{V}(\beta)$ is a weighted blowdown with center P_β and exceptional locus $\mathbb{V}(t)$. Substituting

$$x = u\beta - a_2 \tag{5.1}$$

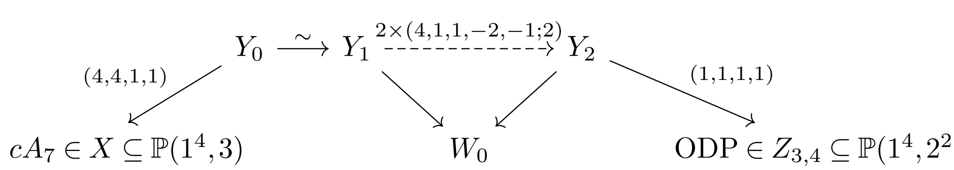
into h, we find that Z is isomorphic to a hypersurface $Z_5 \subseteq \mathbb{P}(1,1,1,1,2)$ with variables u,y,z,β,α. The substitution (5.1) does not lift onto Y_2. Instead, on the patch Z_β, we can substitute $u = (a_2 + x)/\beta$. This substitution lifts to $(Y_2)_\beta$. By Definition 5.1, $P_\beta \in Z$ is a cA_3 singularity and the hypersurface Z_β is given by a weight 4 polynomial. By Proposition 4.6, $(Y_2)_\beta \to Z_\beta$ is a $(3,1,1,1)$-Kawakita blowup.

Note that Z has an ordinary double point at $[u_0, 0, 0, 1, 2]$ for some $u_0 \in \mathbb{C}$. ∎

5.5 cA_7 family 7.1 model

PROPOSITION 5.8. *A Mori fiber space sextic double solid with a cA_7 singularity in family 7.1 satisfying Definition 5.1 has a Sarkisov link to $Z_{3,4} \subseteq \mathbb{P}(1,1,1,1,2,2)$ with an ordinary double point, starting with a $(4,4,1,1)$-blowup of the cA_7 point, then two $(4,1,1,-2,-1;2)$-flips, and finally a blowdown (with standard weights $(1,1,1,1)$) to an ordinary double point. Under further generality conditions, Z has exactly five singular points, namely two $1/2(1,1,1)$ singularities and three ordinary double points.*

Proof. We exhibit the diagram below.



We construct X and a $(4,4,1,1)$-Kawakita blowup $Y_0 \to X$. We can write a sextic double solid X with an isolated cA_7 singularity in family 7.1 by

$$X\colon \mathbb{V}(f, \beta - xt - r_2, \gamma - x\beta - s_3) \subseteq \mathbb{P}(1,1,1,1,2,3,3)$$

with variables x,y,z,t,β,γ,w, where

$$\begin{aligned} f = {} & -w^2 + \gamma^2 - 2t\gamma e_2 + 2\beta^2 e_2 + 2t\beta c_3 + 4t\gamma b_2 - 2\beta^2 b_2 - 2t\beta^2 b_1 + 4xt^2\beta b_1 \\ & + 2x^2t^4 b_0 - 16t\gamma a_1^2 + 16\beta^2 a_1^2 + 4\beta\gamma a_1 - 8\beta^3 a_0 + 12xt\beta^2 a_0 + xt^3 C_2 + t^2 D_4, \end{aligned}$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

where $C_i,\ D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i. Define T_0 by

$$T_0\colon \left(\begin{array}{cc|cccccc} u & x & y & z & w & \gamma & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 3 & 2 \end{array}\right).$$

Define Y_0 by

$$Y_0\colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/u^8,\, u\beta - r_2 - xt,\, u\gamma - s_3 - x\beta).$$

The ample model of $\mathbb{V}(x) \subseteq Y_0$ is a $(4,4,1,1)$-Kawakita blowup $Y_0 \to X$.

We show that the diagram $Y_0 \to W_0 \leftarrow Y_1$ induces an isomorphism $Y_0 \to Y_1$. Acting by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ -1 & 1 \end{smallmatrix}\right)$, we find

$$T_0 \cong \left(\begin{array}{cc|cccccc} u & x & y & z & w & \gamma & \beta & t \\ 3 & 4 & 1 & 1 & 0 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

Define T_1 (resp. T_2) with the same action as T_0 but with irrelevant ideal $(u,x,y,z) \cap (w,\gamma,\beta,t)$ (resp. $(u,x,y,z,w,\gamma) \cap (\beta,t)$). Define $Y_1 \subseteq T_1$ and $Y_2 \subseteq T_2$ by the same ideal I_Y. The base of the flop $T_0 \to \mathcal{W}_0 \leftarrow T_1$ restricts to $\mathbb{V}(r_2,s_3) \subseteq \mathbb{P}^1 \subseteq W_0$, which is empty. Therefore, $Y_0 \to W_0$ and $W_0 \leftarrow Y_1$ are isomorphisms.

We show that the next diagram $Y_1 \to W_1 \leftarrow Y_2$ in the 2-ray game of Y_0 is locally analytically two $(4,1,1,-2,-1;2)$-flips. The only monomials in $\Phi^* f/u^8$ that are not in (u,x,y,z,β,t) are $-w^2$ and γ^2. Therefore, the base of the flip is two points, $[1,1]$ and $[-1,1] \in \mathbb{P}^1$ with variables w and γ inside W_1. We make a change of coordinates $w' = w - \gamma$, respectively $w' = w + \gamma$, for the flip above $[1,1]$, respectively $[-1,1]$. On the patch where γ is nonzero, we can substitute $u = s_3 + x\beta$ in $\Phi^* f/u^8$, and express w' locally analytically and equivariantly above $[1,1]$, respectively $[-1,1]$, in terms of x,y,z,β,t. After projecting away the variables u and w', we are left with the principal ideal $(\beta s_3 - r_2 + x\beta^2 - xt)$. Since it contains both r_2 and xt, by case (1) in [8, Th. 8], it is a terminal $(4,1,1,-2,-1;2)$-flip above both $[1,1]$ and $[-1,1]$. The flip contracts two curves, both containing a $1/4(1,1,3)$ singularity, and extracts two curves, both containing a $1/2(1,1,1)$ singularity and a cA_1 singularity. The cA_1 points are both ordinary double points if r_2 is not a square of a linear form, and are both 3-fold A_2 singularities (given by $x_1^2 + x_2^2 + x_3^2 + x_4^3$) otherwise. On Y_2, the cA_1 singularities are at $[0,0,0,0,1,1,1,1]$ and $[0,0,0,0,-1,1,1,1]$.

We show that the last map in the link for X is a divisorial contraction $Y_2 \to Z'$. Acting by the matrix $\left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$ on the initial action-matrix of T_0, we see that

$$T_2 \cong \left(\begin{array}{cccccc|cc} u & x & y & z & w & \gamma & \beta & t \\ 2 & 3 & 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & 2 & 2 & 1 & 0 \end{array}\right).$$

Define $Z' \subseteq \mathbb{P}(1,1,1,1,2,2,2)$ with variables u,y,z,β,w,γ,x by the ideal $I_{Z'}$, where $I_{Z'}$ is the image of the ideal I_Y under the homomorphism $t \mapsto 1$. Let $Y_2 \to Z'$ be the ample model of $\mathbb{V}(\beta)$. On the affine patch Z'_β, we can express u and x locally analytically and equivariantly in terms of y,z,w,γ,β,t. This coordinate change lifts to Y_2. By Definition 5.1, we can compute that $P_\beta \in Z'$ is an ordinary double point, and $Y_2 \to Z'$ is locally analytically the (usual) blowup with center P_β.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

The variety Z' is isomorphic to a complete intersection $Z_{3,4} \subseteq \mathbb{P}(1^4, 2^2)$, by projecting away from x. The variety Z is given by

$$Z_{3,4}\colon \mathbb{V}(-s_3 + \beta r_2 + u\gamma - u\beta^2, h) \subseteq \mathbb{P}(1,1,1,1,2,2)$$

with variables $u, y, z, \beta, w, \gamma$, where

$$h = -w^2 + \gamma^2 + 2b_0 r_2^2 - 4\beta b_1 r_2 - 4u\beta b_0 r_2 - 12\beta^2 a_0 r_2 - 2\gamma e_2 + 2\beta^2 e_2 + 2\beta c_3 + 4\gamma b_2 \\ - 2\beta^2 b_2 + 2u\beta^2 b_1 + 2u^2\beta^2 b_0 - 16\gamma a_1^2 + 16\beta^2 a_1^2 + 4\beta\gamma a_1 + 4u\beta^3 a_0 + (u\beta - r_2)C_Z + D_Z,$$

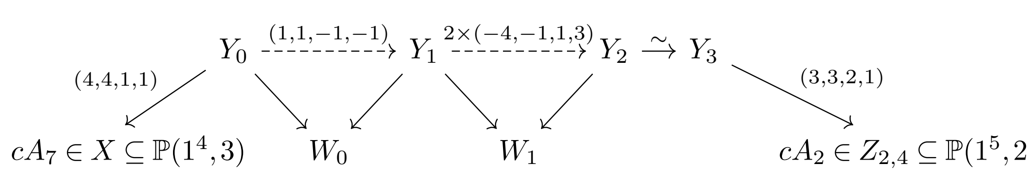
where $C_Z = C_2(y,z,u)$ and $D_Z = D_4(y,z,u)$. The variety Z has two cA_1 singularities at $[0,0,0,1,1,1]$ and $[0,0,0,1,-1,1]$. ∎

Remark 5.9. We explain how we found the variety X. Using $p_2 = r_2$ and $p_3 = s_3$, we can write a sextic double solid with an isolated cA_7 in family 7.1 by $\bar{X}\colon \mathbb{V}(\bar{f}, x^2 t + x r_2 + s_3 - \bar{\gamma})$ inside $\mathbb{P}(1,1,1,1,3,3)$ with variables $x, y, z, t, w, \bar{\gamma}$, where $\bar{f}$ is given as in Theorem A. The $(1,1,4,4,2)$-blowup $\bar{Y}_0 \to \bar{X}$ for variables $y, z, w, \bar{\gamma}, t$ is a $(4,4,1,1)$-Kawakita blowup, but the 2-ray game of $\bar{Y}_0$ does not follow the ambient toric variety $\bar{T}_0$. Namely, the toric anti-flip $\bar{T}_0 \to \bar{\mathcal{W}}_0 \leftarrow \bar{T}_1$ restricts to $\bar{Y}_0 \to \bar{W}_0 \leftarrow \bar{Y}_1$, where $\bar{Y}_0 \to \bar{W}_0$ is an isomorphism and $\bar{W}_0 \leftarrow \bar{Y}_1$ extracts $\mathbb{P}^2$, a divisor on $\bar{Y}_1$. The reason why $\bar{Y}_0$ was not the correct variety is that one of the generators of the ideal of $\bar{Y}_0$ is $\bar{g}_1 = x^2 t + x r_2 + u s_3 - u\bar{\gamma}$, which is inside the irrelevant ideal (u, x). We find the correct variety Y_0 by *unprojecting* $\bar{g}_1 = 0$ with respect to u, x. By *unprojection*, we mean the coordinate change $\bar{Y}_0 \to Y_0$, an isomorphism. See [49, §2] or [44, §2.3] for more details on this type of unprojection. This coordinate change induces the coordinate change $\bar{X} \to X$, where X is given as in the proof of Proposition 5.8.

5.6 cA_7 family 7.2 model

Proposition 5.10. *A Mori fiber space sextic double solid with a cA_7 singularity in family 7.2 satisfying Definition 5.1 has a Sarkisov link to a complete intersection $Z_{2,4} \subseteq \mathbb{P}(1,1,1,1,1,2)$ with a cA_2 singularity, starting with a $(4,4,1,1)$-blowup of the cA_7 point, followed by one Atiyah flop, then two $(4,1,-1,-3)$-flips, and finally a $(3,3,2,1)$-blowdown to a cA_2 point. Under further generality conditions, the variety Z is smooth outside the cA_2 point.*

Proof. We exhibit the diagram below.



We describe the sextic double solid X. Define $X \subseteq \mathbb{P}(1,1,1,1,2,3,3,3)$ with variables x, y, z, t, β, w, γ, ξ by the ideal

$$I_X = (f - 2e_3\xi,\ \beta - q_1 r_1 - xt,\ \gamma - q_1 s_2 - x\beta,\ -\xi + t s_2 - \beta r_1), \tag{5.2}$$

where

$$f = -w^2 + \gamma^2 + 2t\beta c_3 + 4t\gamma b_2 - 2\beta^2 b_2 - 2t\beta^2 b_1 + 4xt^2\beta b_1 + 2x^2 t^4 b_0 \\ - 16t\gamma a_1^2 + 16\beta^2 a_1^2 + 4\beta\gamma a_1 - 8\beta^3 a_0 + 12xt\beta^2 a_0 + xt^3 C_2 + t^2 D_4,$$

where C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

We describe the weighted blowup $Y_0 \to X$, restriction of $\Phi\colon T_0 \to \mathbb{P}(1,1,1,1,2,3,3,3)$. Define T_0 by

$$T_0\colon \left(\begin{array}{cc|ccccccc} u & x & y & z & w & \gamma & \beta & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 3 & 2 & 3 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 3 & 5 & 2 \end{array}\right).$$

Define $Y_0 \subseteq T_0$ by the ideal I_Y with the six generators

$$g - 2e_3\xi, \quad u\beta - q_1r_1 - xt, \quad u\gamma - q_1s_2 - x\beta,$$
$$-u\xi + ts_2 - \beta r_1, \quad -x\xi + \beta s_2 - \gamma r_1, \quad -q_1\xi + t\gamma - \beta^2,$$

where $g = \Phi^* f/u^8$. On the affine patch X_x, we can express β, t, and ξ in terms of w,γ,y,z, to get a hypersurface in $\mathbb{C}^4$ given by $f_{\mathrm{hyp}} \in \mathbb{C}[w,\gamma,y,z]$. Note that these coordinate changes lift to $(Y_0)_x$. Since f_{hyp} has weight 8, by Proposition 4.6, $Y_0 \to X$ is a $(4,4,1,1)$-Kawakita blowup.

We show that the first diagram $Y_0 \to W_0 \leftarrow Y_1$ in the 2-ray game of Y_0 is an Atiyah flop, provided that r_1 and q_1 are coprime in $\mathbb{C}[y,z]$. Acting by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ -1 & 1 \end{smallmatrix}\right)$ on the action-matrix of T_0, define T_1 by

$$T_1\colon \left(\begin{array}{cccc|ccccc} u & x & y & z & w & \gamma & \beta & \xi & t \\ 3 & 4 & 1 & 1 & 0 & 0 & -1 & -3 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

Define $Y_1 \subseteq T_1$ by the ideal I_Y. The base of the flop is $\mathbb{V}(q_1) \subseteq \mathbb{P}^1$ with variables y,z, which is one point. Perform a suitable invertible linear coordinate change on y,z such that $q_1 = z$ and $r_1 = y$. Since $u\beta - q_1r_1 - xt$ is in I_Y, we can substitute $z = u\beta - xt$ on the patch where y is nonzero. The coefficients of β in $-u\xi + ts_2 - \beta y \in I_Y$ and γ in $-x\xi + \beta s_2 - \gamma y \in I_Y$ are nonzero on the patch where y is nonzero. Therefore, we can locally analytically equivariantly express β and γ in terms of u,x,w,t. After substituting z,β,γ, we find that the diagram $Y_0 \to W_0 \leftarrow Y_1$ is locally analytically the Atiyah flop.

The next diagram in the 2-ray game of Y_0 is the flip $Y_1 \to W_1 \leftarrow Y_2$. The base of the flip is $\mathbb{V}(\gamma^2 - w^2) \subseteq \mathbb{P}^1$ with variables w,γ, which is two points $[1,1]$ and $[-1,1]$. We consider the point $P = [1,1]$, the flip for the other point is similar. Perform a coordinate change $w' = w - \gamma$. On the patch where γ is nonzero, we find $u = q_1s_2 + x\beta$ and $t = q_1\xi + \beta^2$. Writing $q_1 = z$ and $r_1 = y$ as before, we find $y = -x\xi + \beta s_2$. We are left with the principal ideal in $\mathbb{C}[x,z,w',\beta,\xi]$ generated by $-w'(2+w')+$terms not involving w'. So, we can locally analytically equivariantly express w' in terms of x,z,β,ξ. So, the diagram $Y_1 \to W_1 \leftarrow Y_2$ is locally analytically two $(-4,-1,1,3)$-flips.

The next diagram in the toric 2-ray game $T_2 \to \mathcal{W}_2 \leftarrow T_3$ restricts to isomorphisms $Y_2 \to W_2 \leftarrow Y_3$. The reason is that the base of the toric flip P_β restricts to an empty set in W_2, since I_Y contains the polynomial $t\gamma - \beta^2 - q\xi$.

We show that the last diagram in the 2-ray game of Y_0 is a divisorial contraction $Y_3 \to Z$. Multiplying the action-matrix of T_1 by $\left(\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right)$, we see that T_3 is given by

$$T_3\colon \left(\begin{array}{ccccccc|cc} u & x & y & z & w & \gamma & \beta & \xi & t \\ 3 & 5 & 2 & 2 & 3 & 3 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \end{array}\right).$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Consider the variety $Z \subseteq \mathbb{P}(1,1,1,1,1,2,2,2)$ with variables $\xi,u,y,z,\beta,x,w,\gamma$ where $Y_3 \to Z$ is the ample model of $\mathbb{V}(\xi)$. On the patch Z_ξ, we can substitute $u = s_2 - \beta r_1$, $x = \beta s_2 - \gamma r_1$ and $z = \gamma - \beta^2$, and compute that Z_ξ is a hypersurface given by a weight 6 polynomial, with a cA_2 singularity at $P_\xi \in Z_\xi$, of type at least 2 (see Definition 4.8). These substitutions lift to $(Y_3)_\xi$, showing that $Y_3 \to Z$ is a $(3,3,2,1)$-Kawakita blowup with center P_ξ. If the coefficients are general, namely when

$$-2e_\beta + 8\beta^4 a_0 r_\beta - 2\beta^2 b_\beta + 12\beta^2 a_\beta^2 \in \mathbb{C}[y,\beta]$$

is not a full square, where $r_\beta = r_1(y,-\beta^2)$, $e_\beta = e_3(y,-\beta^2)$, $a_\beta = a_1(y,-\beta^2)$, and $b_\beta = b_2(y,-\beta^2)$, then the point P_ξ is exactly of type 2.

The variety Z is isomorphic to a complete intersection $Z_{2,4} \subseteq \mathbb{P}(1,1,1,1,1,2)$ with variables u,y,z,β,ξ,w. We see this by substituting $x = u\beta - q_1 r_1$ and $\gamma = q_1\xi + \beta^2$. We find that Z is isomorphic to $Z_{2,4}\colon \mathbb{V}(-u\xi + s_2 - \beta r_1, h)$, where

$$\begin{aligned} h = &-w^2 + \xi^2 q_1^2 - 2e_3\xi + \beta^4 + 2b_0 q_1^2 r_1^2 - 4\beta b_1 q_1 r_1 - 4u\beta b_0 q_1 r_1 - 12\beta^2 a_0 q_1 r_1 + 4\xi b_2 q_1 \\ &- 16\xi a_1^2 q_1 + 4\beta\xi a_1 q_1 + 2\beta^2\xi q_1 + 2\beta c_3 + 2\beta^2 b_2 + 2u\beta^2 b_1 + 2u^2\beta^2 b_0 + 4\beta^3 a_1 + 4u\beta^3 a_0 \\ &+ (u\beta - q_1 r_1)C_Z + D_Z, \end{aligned}$$

where $C_Z = C_2(y,z,u)$ and $D_Z = D_4(y,z,u)$. ∎

REMARK 5.11. We explain below how we found the embedding of X. Using Theorem A and the coordinate change in cA_7 family 7.1, we can write a sextic double solid $\bar{X}$ with an isolated cA_7 in family 7.2 by

$$\bar{X}\colon \mathbb{V}(f - 2e_3(ts_2 - \beta r_1), \beta - xt - q_1 r_1, \gamma - x\beta - q_1 s_2) \subseteq \mathbb{P}(1,1,1,1,2,3,3)$$

with variables x,y,z,t,β,γ,w.

We construct a $(4,4,1,1)$-Kawakita blowup $\bar{Y}_0 \to \bar{X}$. Define $\bar{T}_0$ by

$$\bar{T}_0\colon \left(\begin{array}{cc|cccccc} u & x & y & z & w & \gamma & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 3 & 2 \end{array}\right).$$

Let $T_0 \to \mathbb{P}(1,1,1,1,2,3,3)$ be the ample model of $\mathbb{V}(x)$ and $Y_0 \subseteq T_0$ the strict transform of X. Then $\bar{Y}_0$ is given by the ideal $I_{\bar{Y}} = (\bar{g}_1,\ldots,\bar{g}_5)$, where

$$\begin{aligned} &\bar{g}_1 = ug + 2e_3(\beta r_1 - ts_2), \quad \bar{g}_2 = u\beta - q_1 r_1 - xt, \quad \bar{g}_3 = u\gamma - q_1 s_2 - x\beta, \\ &\bar{g}_4 = xg + 2e_3(\gamma r_1 - \beta s_2), \quad \bar{g}_5 = q_1 g + 2e_3(\beta^2 - t\gamma)). \end{aligned}$$

The morphism $\bar{Y}_0 \to \bar{X}$ is a $(4,4,1,1)$-Kawakita blowup, as can be checked on the patch $(\bar{Y}_0)_x \to \bar{X}_x$.

Note that we do not prove that $I_{\bar{Y}}$ is saturated with respect to u. In fact, the saturation will not be I_Y if we do not use assume some generality conditions, similarly to cA_6 and cA_7 family 7.1. As a heuristic argument to see why $I_{\bar{Y}}$ might be saturated in the general case (*general* meaning a Zariski open dense set of the parameter space), we can use computer algebra software, pretend that a_i, b_i, c_i, d_i, q_1, r_1, s_2, e_3 are algebraically independent variables of a polynomial ring over $\mathbb{Q}$ or $\mathbb{Z}_p$ for a large prime p, and calculate that the saturation in that case indeed equals the ideal $I_{\bar{Y}}$.

Similarly to the diagram $Y_0 \to W_0 \leftarrow Y_1$ in the proof of Proposition 5.10, the diagram $\bar{Y}_0 \to \bar{W}_0 \leftarrow \bar{Y}_1$ is an Atiyah flop, provided r_1 and q_1 are coprime.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

We show that $I_{\bar{Y}}$ does not 2-ray follow $\bar{T}_0$, namely that the diagram $\bar{Y}_1 \to \bar{W}_1 \leftarrow Y_2$ contracts a curve and extracts a divisor. Acting by the matrix $\left(\begin{smallmatrix}4 & -3\\ -1 & 1\end{smallmatrix}\right)$ on the action matrix of $\bar{T}_0$, define $\bar{T}_1$ by

$$\bar{T}_1\colon \begin{array}{c} \begin{matrix} u & x & y & z & w & \gamma & \beta & t \end{matrix} \\ \left(\begin{array}{cccc|cccc} 3 & 4 & 1 & 1 & 0 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right), \end{array}$$

and define $\bar{Y}_1 \subseteq \bar{T}_1$ by the zeros of $I_{\bar{Y}}$. We consider the toric flip $\bar{T}_1 \to \bar{\mathcal{W}}_1 \leftarrow \bar{T}_2$ and restrict it to $\bar{Y}_1 \to \bar{W}_1 \leftarrow \bar{Y}_2$. Since $I_{\bar{Y}}$ is the zero ideal when restricting to $\mathbb{V}(u,x,y,z,\beta,t)$, the base $\mathbb{P}^1 \subseteq \bar{\mathcal{W}}_1$ of the toric flip restricts to $\mathbb{P}^1 \subseteq \bar{W}_1$ with variables w,γ. The morphism $\bar{Y}_1 \to \bar{W}_1$ contracts a curve $\mathbb{P}^1$ to both of the points $[1,1]$ and $[1,-1]$ in the base $\mathbb{P}^1 \subseteq \bar{W}_1$ and is an isomorphism elsewhere. The morphism $\bar{W}_1 \leftarrow \bar{Y}_2$ extracts a curve $\mathbb{P}^1$ for every point in the base $\mathbb{P}^1 \subseteq \bar{W}_1$, so extracts a divisor on $\bar{Y}_2$. The diagram $\bar{Y}_1 \to \bar{W}_1 \leftarrow \bar{Y}_2$ is not a step in the 2-ray game of $\bar{Y}_0$, so $I_{\bar{Y}}$ does not 2-ray follow $\bar{T}_0$. The reason for this was that the ideal $I_{\bar{Y}}$ is contained in (u,x,y,z), so the surface $\mathbb{V}(u,x,y,z) \subseteq \bar{T}_2$ exists on $\bar{Y}_2$, but does not exist on $\bar{T}_1$.

We *unproject* $\bar{g}_1 = \bar{g}_4 = \bar{g}_5 = 0$ with respect to u,x,y,z in $\bar{Y}_1 \subseteq \bar{T}_1$, to find a variety $Y_1 \subseteq T_1$. We explain below what we mean by this. We can write the system of equations $\bar{g}_1 = \bar{g}_4 = \bar{g}_5 = 0$ in the matrix form

$$\begin{pmatrix} g & 0 & 0 & \beta r_1 - ts_2 \\ 0 & g & 0 & \gamma r_1 - \beta s_2 \\ 0 & 0 & g & \beta^2 - t\gamma \end{pmatrix} \begin{pmatrix} u \\ x \\ q_1 \\ 2e_3 \end{pmatrix} = \mathbf{0}.$$

If the above equations hold, then we have

$$\frac{\left|\begin{pmatrix} 0 & 0 & \beta r_1 - ts_2 \\ g & 0 & \gamma r_1 - \beta s_2 \\ 0 & g & \beta^2 - t\gamma \end{pmatrix}\right|}{u} = \frac{\left|\begin{pmatrix} g & 0 & \beta r_1 - ts_2 \\ 0 & 0 & \gamma r_1 - \beta s_2 \\ 0 & g & \beta^2 - t\gamma \end{pmatrix}\right|}{-x} = \frac{\left|\begin{pmatrix} g & 0 & \beta r_1 - ts_2 \\ 0 & g & \gamma r_1 - \beta s_2 \\ 0 & 0 & \beta^2 - t\gamma \end{pmatrix}\right|}{q_1} = \frac{\left|\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix}\right|}{-2e_3}.$$

Calculating the determinants and dividing by $-g^2$, we find the equalities

$$\frac{ts_2 - \beta r_1}{u} = \frac{\beta s_2 - \gamma r_1}{x} = \frac{t\gamma - \beta^2}{q_1} = \frac{g}{2e_3}, \tag{5.3}$$

between elements of the field of fractions of $\mathbb{C}[u,x,y,z,w,\gamma,\beta,t]/I_{\bar{Y}}$. Using Equation (5.3), we see that the morphism $\bar{Y}_1 \to Y_1$ given by

$$[u,x,y,z,w,\gamma,\beta,t] \mapsto [u,x,y,z,w,\gamma,\beta,\frac{ts_2 - \beta r_1}{u},t]$$

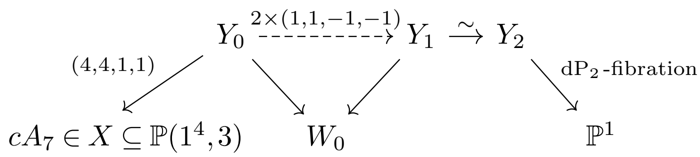
is an isomorphism, where Y_1 is described in the proof of Proposition 5.10.

The coordinate change $\bar{Y}_1 \to Y_1$ induces an isomorphism $\bar{X} \to X$, giving the variety X.

5.7 cA_7 family 7.3 model

PROPOSITION 5.12. *A Mori fiber space sextic double solid with a cA_7 singularity in family 7.3 satisfying Definition 5.1 has a Sarkisov link to a degree 2 del Pezzo fibration, starting with a $(4,4,1,1)$-blowup of the cA_7 point and followed by two Atiyah flops.*

Proof. We exhibit the diagram below.



First, we define X and a $(4,4,1,1)$-Kawakita blowup $Y_0 \to X$. Any sextic double solid with an isolated cA_7 family 7.3 can be given by a bidegree $(6,2)$ complete intersection

$$X\colon \mathbb{V}(f, -\xi + ts_1 - q_2 - xt) \subseteq \mathbb{P}(1,1,1,1,2,3)$$

with variables x,y,z,t,ξ,w, where

$$\begin{aligned} f = &-w^2 + x^2\xi^2 - 2\xi e_4 + \xi^2(s_1^2 + 4a_1s_1 + 2xs_1 - 2b_2 + 16a_1^2 + 4xa_1 + 8\xi a_0) \\ &+ t(ts_1^4 + 4ta_1s_1^3 - 8t^2a_0s_1^3 - 2\xi s_1^3 + 2tb_2s_1^2 - 2t^2b_1s_1^2 - 8\xi a_1s_1^2 + 24t\xi a_0s_1^2 \\ &+ 12xt^2a_0s_1^2 - 2x\xi s_1^2 + 2tc_3s_1 + 4t\xi b_1s_1 + 4xt^2b_1s_1 - 16\xi a_1^2s_1 - 4x\xi a_1s_1 \\ &- 24\xi^2a_0s_1 - 24xt\xi a_0s_1 - 2\xi c_3 - 4x\xi b_2 - 2\xi^2b_1 - 4xt\xi b_1 + 2x^2t^3b_0 + 16x\xi a_1^2 \\ &+ 12x\xi^2a_0 + xt^2C_2 + tD_4), \end{aligned}$$

where C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i. Define

$$T_0\colon \left(\begin{array}{cc|ccccc} u & x & y & z & w & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 2 \end{array}\right).$$

Define $\Phi\colon T_0 \to \mathbb{P}(1,1,1,1,2,3)$ by the ample model of $\mathbb{V}(x)$, and define Y_0 as the strict transform of X. Then, Y_0 is given by

$$Y_0\colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^*f/u^8, -u^2\xi + uts_1 - q_2 - xt),$$

Using Proposition 4.6, we see that $Y_0 \to X$ is a $(4,4,1,1)$-Kawakita blowup.

We describe the flop $Y_0 \to W_0 \leftarrow Y_1$. Multiplying the action matrix of T_0 by $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, we find

$$T_0 \cong \left(\begin{array}{cc|ccccc} u & x & y & z & w & \xi & t \\ 1 & 1 & 0 & 0 & -1 & -2 & -1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 2 \end{array}\right).$$

The base of the flop is given by $\mathbb{V}(q_2) \subseteq \mathbb{P}^1 \subseteq W_0$. After a suitable coordinate change on y,z, we find $q_2 = yz$. Consider the flop over $\mathbb{V}(y)$, the flop over the other point is similar. Since q_2 and e_4 have no common divisor, on the patch where z is nonzero, we can express y and ξ locally analytically equivariantly in terms of u,x,t,w. So, $Y_0 \to W_0 \leftarrow Y_1$ is locally analytically two Atiyah flops.

The morphisms $Y_1 \to W_1 \leftarrow Y_2$ are isomorphisms, since w^2 has a nonzero coefficient in Φ^*f/u^8.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

We show that Y_2 is a degree 2 del Pezzo fibration. Multiplying the original action matrix of T_0 by the matrix $\left(\begin{smallmatrix}1 & 0\\ 2 & -1\end{smallmatrix}\right)$ with determinant -1, we find

$$T_2\colon \begin{array}{c} \\ \left(\begin{array}{ccccc|cc} u & x & y & z & w & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 & 0 & 0 \end{array}\right). \end{array}$$

The ample model of $\mathbb{V}(t)$ is

$$\begin{aligned} Y_2 &\to \mathbb{P}(2,1)\\ [u,x,y,z,w,\xi,t] &\mapsto [\xi,t]. \end{aligned}$$

Since $\mathbb{P}(2,1)$ is isomorphic to $\mathbb{P}^1$, we see that Y_2 is a fibration onto $\mathbb{P}^1$. On the patch $(Y_2)_t$, we can substitute $x = us_1 - q_2 - u^2\xi$, to find that the general fiber is a weighted degree 4 hypersurface in $\mathbb{P}(1,1,1,2)$, so a degree 2 del Pezzo surface. ∎

5.8 cA_8 model

PROPOSITION 5.13. *A Mori fiber space sextic double solid with a cA_8 singularity satisfying Definition 5.1 has a Sarkisov link to a complete intersection $Z_{3,3} \subseteq \mathbb{P}(1,1,1,1,1,2)$ with a cD_4 singularity, starting with a $(5,4,1,1)$-blowup of the cA_8 point, followed by a $(4,1,1,-1,-2;2)$-flip, and finally a $(3,2,2,1,5)$-blowdown to the cD_4 singularity. Under further generality conditions, the singular locus of Z consists of three points, namely the cD_4 point, the $1/2(1,1,1)$ singularity, and an ordinary double point.*

Proof. We exhibit the diagram below.

$$\begin{array}{ccccccc} & & Y_0 \xrightarrow{\sim} Y_1 & \overset{(4,1,1,-1,-2;2)}{\dashrightarrow} & Y_2 \xrightarrow{\sim} Y_3 & & \\ & \overset{(1,1,4,5)}{\swarrow} & \searrow & & \swarrow & \overset{(3,2,2,1,5)}{\searrow} & \\ cA_8 \in X_6 \subseteq \mathbb{P}(1^4,3) & & & W_1 & & & cD_4 \in Z_{3,3} \subseteq \mathbb{P}(1^5,2) \end{array}$$

First, we describe X and the weighted blowup $Y_0 \to X$. A sextic double solid with a cA_8 singularity can be given by a multidegree $(6,2,3)$ complete intersection

$$X\colon \mathbb{V}(f,\, \beta - xt - r_2,\, \gamma - x\beta - s_3) \subseteq \mathbb{P}(1,1,1,1,2,3,3),$$

with variables x,y,z,t,β,γ,ξ where

$$\begin{aligned} f = {} & 8\beta^3(A_0 - a_0) + \xi(-\xi + 2\gamma - 8tA_0r_2 + 2tb_2 - 4ta_1^2 + 4\beta a_1)\\ & + t(-16t\beta A_0^2 r_2 + 2t\beta c_2 + 4t\gamma b_1 - 2\beta^2 b_1 - 2t\beta^2 b_0 + 4xt^2\beta b_0 - 8t\gamma a_0 a_1 + 8\beta^2 a_0 a_1\\ & + 12\beta\gamma a_0 - 2t\gamma B_1 + 2\beta^2 B_1 + 16t\beta^2 A_0^2 - 16xt^2\beta A_0^2 - 8\beta\gamma A_0 + xt^3C_1 + t^2D_3), \end{aligned}$$

where C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i. Note that $B_1 \in \mathbb{C}[y,z]$. Define

$$T_0\colon \left(\begin{array}{cc|cccccc} u & x & y & z & \gamma & \beta & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 3 & 1 \\ -1 & 0 & 1 & 1 & 4 & 3 & 5 & 2 \end{array}\right).$$

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Let $\Phi\colon T_0 \to \mathbb{P}(1,1,1,1,2,3,3)$ be the ample model of $\mathbb{V}(x)$, and let $Y_0 \subseteq T_0$ be the strict transform of X. Then Y_0 is given by

$$Y_0\colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = \left(\Phi^* f/u^9, u\beta - xt - r_2, u\gamma - x\beta - s_3\right),$$

and $Y_0 \to X$ is a $(5,4,1,1)$-Kawakita blowup.

The first diagram in the 2-ray game of T_0 restricts to a isomorphisms $Y_0 \to W_0 \leftarrow Y_1$, since r_2 and s_3 are coprime.

The second diagram in the 2-ray game of T_0 restricts to a $(4,1,1,-1,-2;2)$-flip $Y_1 \to W_1 \leftarrow Y_2$. Define the toric variety T_1 by multiplying the action matrix of T_0 by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ 3 & -2 \end{smallmatrix}\right)$,

$$T_1\colon \begin{pmatrix} \overset{u}{3} & \overset{x}{4} & \overset{y}{1} & \overset{z}{1} & \vline & \overset{\gamma}{0} & \overset{\beta}{-1} & \overset{\xi}{-3} & \overset{t}{-2} \\ 2 & 3 & 1 & 1 & \vline & 1 & 0 & -1 & -1 \end{pmatrix}.$$

On the patch where γ is nonzero, we have $u = x\beta + s_3$ and we can write ξ locally analytically equivariantly in terms of x, y, z, β, t. We are left with the hypersurface given by $x\beta^2 + \beta s_3 - xt - r_2$ in $\mathbb{C}^5$ with variables x, y, z, β, t with weights $(4,1,1,-1,-2)$. The polynomial contains xt and r_2, so this corresponds to case (1) in [8, Th. 8], a $(4,1,1,-1,-2;2)$-flip. Similarly to Proposition 5.8, the flip contracts a curve containing a $1/4(1,1,3)$ singularity, and extracts a curve containing a $1/2(1,1,1)$ singularity and a cA_1 singularity, which is an ordinary double point if r_2 is not a square and is a 3-fold A_2 singularity otherwise. The cA_1 singularity on Y_2 is at $[0,0,0,0,1,1,-2a_0,1]$.

The third diagram in the 2-ray game of T_0 restricts to isomorphisms $Y_2 \to W_2 \leftarrow Y_3$, under Definition 5.1, namely that $a_0 \neq A_0$. On the patch where β is nonzero, the base of the toric flip restricts to $\mathbb{V}(A_0 - a_0, u, x, y, z, \gamma, \xi, t) \subseteq W_2$.

We describe the weighted blowdown $Y_3 \to Z$. Multiplying the action matrix of T_0 by the matrix $\left(\begin{smallmatrix} 5 & -3 \\ 2 & -1 \end{smallmatrix}\right)$, the toric variety T_3 is given by

$$T_3\colon \begin{pmatrix} \overset{u}{3} & \overset{x}{5} & \overset{y}{2} & \overset{z}{2} & \overset{\gamma}{3} & \overset{\beta}{1} & \vline & \overset{\xi}{0} & \overset{t}{-1} \\ 1 & 2 & 1 & 1 & 2 & 1 & \vline & 1 & 0 \end{pmatrix}.$$

The ample model of $\mathbb{V}(\xi)$ is $Y_3 \to Z$ where Z is the tridegree $(3,2,3)$ complete intersection

$$Z\colon \mathbb{V}(h, u\beta - x - r_2, u\gamma - x\beta - s_3) \subseteq \mathbb{P}(1,1,1,1,1,2,2)$$

with variables $u, y, z, \beta, \xi, x, \gamma$, where

$$\begin{aligned} h = {} & 8\beta^3(A_0 - a_0) + \xi(-u\xi + 2\gamma - 8A_0 r_2 + 2b_2 - 4a_1^2 + 4\beta a_1) \\ & - 16\beta A_0^2 r_2 + 2\beta c_2 + 4\gamma b_1 - 2\beta^2 b_1 - 2u\beta^2 b_0 + 4x\beta b_0 - 8\gamma a_0 a_1 + 8\beta^2 a_0 a_1 \\ & + 12\beta\gamma a_0 - 2\gamma B_1 + 2\beta^2 B_1 + 16u\beta^2 A_0^2 - 16x\beta A_0^2 - 8\beta\gamma A_0 + xC_Z + D_Z, \end{aligned}$$

where $C_Z = C_1(y,z,u)$ and $D_Z = D_3(y,z,u)$. Substituting $x = u\beta - r_2$, we see that Z is isomorphic to a complete intersection of bidegree $(3,3)$ in $\mathbb{P}(1^5, 2)$ with variables $u, y, z, \beta, \xi, \gamma$. The variety Z has a cA_1 singularity at $[0,0,0,1,-2a_0,1]$. We can compute that the point $P_\xi \in Z$ is a cD_4 point, by showing the complex analytic space germ (Z, P_ξ)

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

is isomorphic to $(\mathbb{V}(u^2+2\beta r_2-s_3+\text{h.o.t}),\mathbf{0})\subseteq(\mathbb{C}^4,\mathbf{0})$ with variables u,β,y,z, where h.o.t are higher-order terms in y,z,β. We can compute that $Y_3\to Z$ is the divisorial contraction to a cD_4 point described in [52, Th. 2.3]. ∎

Acknowledgment. The author would like to thank Jihun Park for discussions on factoriality.

References

[1] H. Ahmadinezhad, *On pliability of del Pezzo fibrations and Cox rings*, J. Reine Angew. Math. **723** (2017), 101–125.

[2] H. Ahmadinezhad and A.-S. Kaloghiros, *Non-rigid quartic 3-folds*, Compos. Math. **152** (2016), no. 5, 955–983.

[3] H. Ahmadinezhad and T. Okada, *Birationally rigid Pfaffian Fano 3-folds*, Algebr. Geom. **5** (2018), no. 2, 160–199.

[4] H. Ahmadinezhad and F. Zucconi, *Mori dream spaces and birational rigidity of Fano 3-folds*, Adv. Math. **292** (2016), 410–445.

[5] W. Barth, *Two projective surfaces with many nodes, admitting the symmetries of the icosahedron*, J. Algebraic Geom. **5** (1996), no. 1, 173–186.

[6] A. B. Basset, *The maximum number of double points on a surface*, Nature. **73** (1906), no. 1889, 246.

[7] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[8] G. Brown, *Flips arising as quotients of hypersurfaces*. Math. Proc. Camb. Philos. Soc. **127** (1999), no. 1, 13–31.

[9] G. Brown and J. Buczyński, *Maps of toric varieties in Cox coordinates*. Fundam. Math. **222** (2013), no. 3, 213–267.

[10] G. Brown and F. Zucconi, *Graded rings of rank 2 Sarkisov links*, Nagoya Math. J. **197** (2010), 1–44.

[11] F. Campana, G. Dethloff, H. Grauert, Th. Petemell and R. Remmert, *Several complex variables VII: Sheaf-theoretical methods in complex analysis*, Encyclopaedia of Mathematical Sciences, Vol. 74, Springer, Berlin, 1994. A reprint of *Current problems in mathematics. Fundamental directions* (in Russian), Vol. 74, Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow.

[12] F. Catanese and G. Ceresa, *Constructing sextic surfaces with a given number d of nodes*, J. Pure Appl. Algebra. **23** (1982), no. 1, 1–12.

[13] I. Cheltsov and M. Grinenko, *Birational rigidity is not an open property*, Bull. Korean Math. Soc. **54** (2017), no. 5, 1485–1526.

[14] I. Cheltsov and J. Park, "Sextic double solids" in F. Bogomolov and Y. Tschinkel (eds.), *Cohomological and geometric approaches to rationality problems: New perspectives*, Birkhäuser, Boston, 2010, pp. 75–132.

[15] I. Cheltsov and J. Park, *Birationally rigid Fano threefold hypersurfaces*, Memoirs of the American Mathematical Society, Vol. 246(1167), American Mathematical Society, Providence, RI, 2017, v + 117.

[16] I. Cheltsov and C. Shramov, *Finite collineation groups and birational rigidity*, Sel. Math. New Ser. **25** (2019), no. 5, Article No. 71.

[17] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254.

[18] A. Corti and M. Mella, *Birational geometry of terminal quartic 3-folds. I*, Amer. J. Math. **126** (2004), no. 4, 739–761.

[19] A. Corti, A. Pukhlikov and M. Reid, "Fano 3-fold hypersurfaces" in *Explicit birational geometry of 3-folds*, London Mathematical Society Lecture Note Series, Vol. 281, Cambridge University Press, Cambridge, 2000, pp. 175–258.

[20] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50.

[21] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, Vol. 124, American Mathematical Society, Providence, RI, 2011.

[22] T. de Fernex, *Birational rigidity of singular Fano hypersurfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5). **17** (2017), no. 3, 911–929.

[23] T. Eckl and A. Pukhlikov, *Effective birational rigidity of Fano double hypersurfaces*, Arnold Math. J. **4** (2018), nos. 3–4, 505–521.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

[24] D. Eisenbud, *Commutative algebra: With a view toward algebraic geometry*, Graduate Texts in Mathematics, Springer, New York, 1995.
[25] G.-M. Greuel, C. Lossen and E. Shustin, *Introduction to singularities and deformations*, Springer Monographs in Mathematics, Springer, Berlin, 2007.
[26] C. D. Hacon and J. McKernan, *The Sarkisov program*, J. Algebraic Geom. **22** (2013), no. 2, 389–405.
[27] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), no. 1, 331–348.
[28] V. A. Iskovskih and J. I. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sb. (N.S.). **86** (1971), no. 128, 140–166.
[29] V. A. Iskovskikh, *Birational automorphisms of three-dimensional algebraic varieties*, J. Sov. Math. **13** (1980), no. 6, 815–868.
[30] V. A. Iskovskikh and Y. G. Prokhorov, *Algebraic geometry V: Fano varieties*, Springer, Berlin–Heidelberg, 1999.
[31] D. B. Jaffe and D. Ruberman, *A sextic surface cannot have* 66 *nodes*, J. Algebraic Geom. **6** (1997), no. 1, 151–168.
[32] M. Kawakita, *General elephants of three-fold divisorial contractions*, J. Amer. Math. Soc. **16** (2003), no. 2, 331–362.
[33] Y. Kawamata, *Crepant blowing-up of* 3*-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. Math. **127** (1988), no. 1, 93–163.
[34] Y. Kawamata, "Divisorial contractions to 3-dimensional terminal quotient singularities" in *Higher-dimensional complex varieties (Trento, 1994)*, de Gruyter, Berlin, 1996, pp. 241–246.
[35] J. Kollár, *Real algebraic threefolds. I. Terminal singularities*, Collect. Math. **49** (1998), nos. (2–3, 335–360. Dedicated to the memory of Fernando Serrano.
[36] J. Kollár and S. Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc. **5** (1992), no. 3, 533–703.
[37] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Tracts in Mathematics, Vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
[38] I. Krylov, *Birational geometry of del Pezzo fibrations with terminal quotient singularities*, J. Lond. Math. Soc. (2). **97** (2018), no. 2, 222–246.
[39] Q. Liu, *Algebraic geometry and arithmetic curves*, Translated by Reinie Erné, Oxford Graduate Texts in Mathematics, Vol. 6, Oxford University Press, Oxford, 2006.
[40] J. N. Mather and S. S. T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. **69** (1982), no. 2, 243–251.
[41] A. R. Mavlyutov, *Cohomology of complete intersections in toric varieties*, Pacific J. Math. **191** (1999), no. 1, 133–144.
[42] Y. Namikawa, *Smoothing Fano* 3*-folds*, J. Algebraic Geom. **6** (1997), no. 2, 307–324.
[43] T. Okada, *Birational Mori fiber structures of* $\mathbb{Q}$*-Fano 3-fold weighted complete intersections*, Proc. Lond. Math. Soc. (3). **109** (2014), no. 6, 1549–1600.
[44] S. A. Papadakis and M. Reid, *Kustin–Miller unprojection without complexes*, J. Algebraic Geom. **13** (2004), no. 3, 563–577.
[45] Y. G. Prokhorov, *The rationality problem for conic bundles*, Uspekhi Mat. Nauk. **73** (2018), no. 3(441), 3–88.
[46] A. V. Pukhlikov, *Birational automorphisms of double spaces with singularities*, J. Math. Sci. (New York). **85** (1997), no. 4, 2128–2141.
[47] M. Reid, "Canonical 3-folds" in *Journées de géometrie algébrique d'Angers, Juillet 1979/algebraic geometry, Angers, 1979*, Sijthoff & Noordhoff, Alphen aan den Rijn, Germantown, MD, 1980, pp. 273–310.
[48] M. Reid, "Minimal models of canonical 3-folds" in *Algebraic varieties and analytic varieties (Tokyo, 1981)*, Advanced Studies in Pure Mathematics, Vol. 1, North-Holland, Amsterdam, 1983, pp. 131–180.
[49] M. Reid, "Graded rings and birational geometry" in K. Ohno (ed.), *Proceedings of algebraic geometry symposium (Kinosaki, October 2000)*, pp. 1–72. https://repository.kulib.kyoto-u.ac.jp/dspace/handle/2433/214728.
[50] R. Thom, *Stabilité structurelle et morphogénèse: Essai d'une théorie générale des modèles*, Mathematical Physics Monograph Series, W. A. Benjamin, Inc., Reading, MA, 1972.
[51] J. Wahl, *Nodes on sextic hypersurfaces in* $\mathbb{P}^3$, J. Differ. Geom. **48** (1998), no. 3, 439–444.
[52] Y. Yamamoto, *Divisorial contractions to cDV points with discrepancy greater than 1*, Kyoto J. Math. **58** (2018), no. 3, 529–567.
[53] Q. Zhang, *Rational connectedness of log* $\mathbb{Q}$*-Fano varieties*, J. Reine Angew. Math. **590** (2006), 131–142.

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press

Erik Paemurru
Mathematatik und Informatik
Universität des Saarlandes
Saarbrücken 66123
Germany
paemurru@math.uni-sb.de

https://doi.org/10.1017/nmj.2024.17 Published online by Cambridge University Press