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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Clinton Conley (Managing Editor), Mark van Atten, Benno van den Berg, Thomas Colcombet, Samuel Coskey, Bradd Hart, Bernard Linsky, Antonio Montalbán, Rahim Moosa, Christian Retoré, and Nam Trang. Authors and publishers are requested to send, for review, copies of books to *ASL*, *Department of Mathematics*, *University of Connecticut*, 341 *Mansfield Road*, *U*-1009, *Storrs*, *CT* 06269-1009, *USA*.

SCOTT S. CRAMER, *Inverse limit reflection and the structure of* $L(V_{\lambda+1})$. ***Journal of Mathematical Logic***, vol. 15 (2015), no. 1, p. 1550001 (38 pp.).

Let $I_0(\lambda)$ be the statement that there is an elementary embedding from $L(V_{\lambda+1})$ to $L(V_{\lambda+1})$ with critical point $<\lambda$, and let I_0 be $\exists\lambda I_0(\lambda)$. Axiom I_0 is at the highest level of the large cardinal hierarchy that is not known to be inconsistent with ZFC. It was first proposed by W. Hugh Woodin in the 80s to give a consistency upper bound for the axiom of determinacy (AD). Although the later work of Martin, Steel and Woodin significantly lowers the bound and Woodin's argument never appears in the literature, the theme behind the argument—the analogy between the structure of $L(V_{\lambda+1})$ under $I_0(\lambda)$ and the structure of $L(\mathbb{R})$ under Axiom of Determinacy—continues to attract researchers' interest. The early research along this line mostly can be found in the works of Richard Laver (more on rank-into-rank embeddings), e.g., a) *Implications between strong large cardinal axioms*. ***Annals of Pure and Applied Logic***, vol. 90 (1997), no. 13, p. 79–90; b) *Reflection of elementary embedding axioms on the* $L[V_{\lambda+1}]$ *hierarchy*. ***Annals of Pure and Applied Logic***, vol. 107 (2001), no. 13, p. 227–238, and the works of Hugh Woodin (mostly on the structure of $L(V_{\lambda+1})$, e.g., a) ***Notes on an*** AD-***like axiom***, Seminar notes taken by George Kafkoulis, July, 1990; b) *Suitable extender models II*: *Beyond* ω*-huge*. ***Journal of Mathematical Logic***, vol. 11 (2011), no. 2, p. 115–436. Scott Cramer took on this theme as the focus of his research, the article under review is part of the dissertation for his doctoral degree at the UC Berkeley.

The key tool in this article is the notion of an inverse limit, an embedding $\bar{j}: V_{\bar{\lambda}+1} \prec V_{\lambda+1}$ built out of an ω-sequence of embeddings $j_n : V_{\lambda+1} \prec V_{\lambda+1}$. Inverse limit was first investigated by Laver in the study of rank-into-rank embeddings for reflecting large cardinals (see the aforementioned Laver's second article). A natural question is to what extent the inverse limit $\bar{j}$ can be extended to $L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \prec L_\alpha(V_{\lambda+1})$, this is called the inverse limit reflection problem. After paving the preliminaries of I_0 theory in Section 2, Cramer establishes the basic theory for the inverse limit reflection in Section 3. $I_0(\lambda)$ does not imply inverse limit X-reflection (Definition 3.2) in general (Corollary 3.11), however Cramer shows that (strong) inverse limit reflection holds all the way up to the supremum of the lengths of prewellorderings on $V_{\lambda+1}$ in $L(V_{\lambda+1})$ (Theorem 3.7, 3.8). Furthermore, assuming the existence of an elementary embedding $L_\omega(V^\sharp_{\lambda+1}, V_{\lambda+1}) \prec L_\omega(V^\sharp_{\lambda+1}, V_{\lambda+1})$, he gives an example of $X \subset V_{\lambda+1}$ such that inverse limit X-reflection holds (at 0), and at the same time shows that I_0 is reflected, i.e., $I_0(\bar{\lambda})$ for some $\bar{\lambda} < \lambda$ (Theorem 3.9). The later result continues the sequence that I_1 reflects I_3 (by Martin), and I_0 reflects I_1 (by Woodin), $L_{\lambda^+ + \omega+1}(V_{\lambda+1}) \prec L_{\lambda^+ + \omega+1}(V_{\lambda+1})$ reflects to $L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1}) \prec L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1})$ for some $\bar{\lambda} < \lambda$ (by Laver). The next two sections are two applications of the inverse limit reflection to the structural properties of $L(V_{\lambda+1})$.

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The first one is regarding the club filter at λ^+, $\mathrm{Club}^{\lambda^+}$, which Woodin used to show that λ^+ is measurable in $L(V_{\lambda+1})$ – $\mathrm{Club}^{\lambda^+}$ is an ultrafilter on a stationary subset of $S^{\lambda^+}_{\omega} = \{\alpha < \lambda^+ \mid \mathrm{cof}(\alpha) = \omega\}$. Based on the analogy with the determinacy, Woodin proposed an axiom for the club filter, *Ultrafilter Axiom at* λ, which asserts that $I_0(\lambda)$ implies that $\mathrm{Club}^{\lambda^+}$ is an ultrafilter restricted to $S^{\lambda^+}_{\gamma}$ for every regular infinite cardinal $\gamma < \lambda$. Here we refer to the version in his paper, *The weak ultrafilter axiom*. ***Archive for Mathematical Logic***, vol. 55 (2016), p. 319–351. Woodin showed that it is relatively consistent with $I_0(\lambda)$ that Ultrafilter Axiom at λ fails at $\gamma > \omega$. As for the case $\gamma = \omega$, in the article under review, Cramer provides some evidences that at $S^{\lambda^+}_{\omega}$, $\mathrm{Club}^{\lambda^+}$ could still be an ultrafilter (Theorem 4.4, 4.10) in $L(V_{\lambda+1})$. Although unable to fully prove that, using inverse limits he shows that the so-called *weak club filter* (Definition 4.8) restricted to $S^{\lambda^+}_{\omega}$ is an ultrafilter in $L(V_{\lambda+1})$ (Corollary 4.9).

As the second application of inverse limit reflection, Cramer proves in Section 5 an I_0 analogue (Theorem 5.1) of the perfect set theorem (a consequence of AD), i.e., $I_0(\lambda)$ implies that every subset of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ is either of size at most λ or contains a perfect subsets of $V_{\lambda+1}$ (the basic open sets of this topology are sets of the form $O_{(a,\alpha)} = \{b \in V_{\lambda+1} \mid b \cap V_\alpha = a\}$, $\alpha < \lambda$ and $a \in V_{\lambda+1}$). This is the first regularity property theorem that is proved in the I_0 setting. Independently, Shi and Woodin partially proved the same result in *Axiom* I_0 *and higher degree theory*. ***Journal of Symbolic Logic***, vol. 80 (2015), no. 3, p. 970–1021. They did it by a forcing argument using Woodin's Generic Absoluteness, so they proved that the perfect set property for subsets of $V_{\lambda+1}$ goes up to the level at which the Generic Absoluteness holds. The key to Woodin's Generic Absoluteness Theorem is the notion of $U(j)$-representation, which was designed by Woodin in analogy to the weakly homogeneous Suslin tree representation for sets of reals. Prior to Cramer's results, only subsets of $V_{\lambda+1}$ in $L_\lambda(V_{\lambda+1})$ were known to have $U(j)$-representations. That sets a constraint to Shi and Woodin's argument.

In the last section (Section 6), Cramer proves that, assuming j witnesses $I_0(\lambda)$, the Tower Condition for $U(j)$ holds (Theorem 6.8), a criterion isolated by Woodin for showing the $U(j)$-representability of subsets of $V_{\lambda+1}$. Although the inverse limit reflection does not appear in Cramer's argument, the structure of inverse limit plays the central role. By Lemma 130 in Woodin's *Suitable Extender Models II*, Cramer's Tower Condition Theorem implies that the $U(j)$-representable subsets of $V_{\lambda+1}$ include at least those in $L_{\lambda^+}(V_{\lambda+1})$ (in fact a bit more), and hence extends Woodin's Generic Absoluteness Theorem past the level λ^+. In a later work, Cramer devised a notion called *j-Suslin representation*, with which he manages to propagate the Generic Absoluteness throughout the entire $L(V_{\lambda+1})$. That will be included in his upcoming book.

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