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On quasi-static contact problem with generalized Coulomb friction, normal compliance and damage†

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In this paper, we study a quasi-static frictional contact problem for a viscoelastic body with damage effect inside the body as well as normal compliance condition and multi-valued friction law on the contact boundary. The considered friction law generalizes Coulomb friction condition into multi-valued setting. The variational–hemi-variational formulation of the problem is derived and arguments of fixed point theory and surjectivity results for pseudo-monotone operators are applied, in order to prove the existence and uniqueness of solution.

Key words: viscoelastic body, sub-differential frictional condition of Coulomb type, damage, variational inequality, hemi-variational inequality

1 Introduction

In this paper, we study a mathematical model for a Kelvin–Voigt viscoelastic body, which comes into a frictional contact with a rigid foundation and we assume normal compliance condition on the contact boundary. The process is quasi-static as the volume forces and tractions are assumed to vary slowly in time and the acceleration can be neglected. We also assume that the damage inside the body may develop in the sense that micro-cracks and micro-cavities may open and grow as a result of the internal strains and stresses. This leads to the gradual or rapid decrease in the load-carrying capacity of the body, leading eventually to its breaking.

The unknowns of the model are the displacement field $\boldsymbol{u}$ governed by a multi-valued partial differential inclusion of the first order in time and the damage field β governed by an evolution variational inequality. The damage field β measures the decrease in the

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load-bearing capacity of the material; it is assumed to have values in the interval $[0, 1]$, if $\beta = 1$, then the material is in its full capacity and when $\beta = 0$ it is completely damaged. As the evolution of the displacement and damage may influence each other, the governing relations for both quantities are mutually coupled. The evolution variational inequality for damage that we use here, was introduced by Frémond (see for example [9, 10]) and has been recently extensively studied for springs (see [2, 5]), beams (see [1, 14]) and for three-dimensional deformable solids (see [4, 11, 15, 16, 20, 22]).

Contact conditions in the theory of (visco)-elasticity are usually expressed in the form of multi-valued laws. Such laws are particularly useful to model the phenomena associated with dry friction and normal contact. If the laws are monotone (such as the Coulomb of Tresca friction laws), then the associated problems are governed by variational inequalities. Sometimes, however, it is necessary to use non-monotone multi-valued laws. In such cases, a tool used in the modelling is the Clarke sub-differential which is a multi-function that generalizes the classical gradient to the class of locally Lipschitz functionals on Banach spaces. For more information about the multi-valued contact conditions and sub-differentials, see the monographs [6, 7, 17–19, 21].

The results of the present paper are closely related to [12] and [13] and can be viewed as the generalization of the existence and uniqueness results of both those articles. The difference between the present results and two mentioned papers lies in the choice of friction contact laws, which relate the friction force $\boldsymbol{\sigma}_\tau$ with the tangential slip rate $\dot{\boldsymbol{u}}_\tau$. In [12], the authors use the generalized Tresca type law on the contact boundary

$$-\boldsymbol{\sigma}_\tau(t) \in F(\dot{\boldsymbol{u}}_\tau(t)), \tag{1.1}$$

while in [13] the authors use the following Coulomb friction law

$$\begin{aligned} &\dot{\boldsymbol{u}}_\tau(t) = 0 \Rightarrow \|\boldsymbol{\sigma}_\tau(t)\|_{\mathbb{R}^d} \leqslant |\sigma_\nu(t)| \\ &\dot{\boldsymbol{u}}_\tau(t) \neq 0 \Rightarrow -\boldsymbol{\sigma}_\tau(t) = |\sigma_\nu(t)| \frac{\dot{\boldsymbol{u}}_\tau(t)}{\|\dot{\boldsymbol{u}}_\tau(t)\|_{\mathbb{R}^d}} \end{aligned} \tag{1.2}$$

where, additionally the dependence on the normal stress σ_ν is taken into account. Here, we replace the above conditions with the generalized Coulomb law of multi-valued type which encompasses (1.2) as a special case and involves the non-monotonicity effects as (1.1)

$$-\boldsymbol{\sigma}_\tau(t) \in |\sigma_\nu(t)| F(\dot{\boldsymbol{u}}_\tau(t)), \tag{1.3}$$

In the laws (1.1) and (1.3), F is a certain multi-function, typically $F(\mathbf{0})$ is a ball centred at zero, its radius representing a friction threshold, and for $\xi \neq 0$ $F(\xi)$ is single valued, with $\|F(\xi)\|_{\mathbb{R}^d}$ being less than the friction threshold, which represents the fact that the maximum static friction is typically greater than the kinetic friction. If we take $F(\xi) = \frac{\xi}{\|\xi\|_{\mathbb{R}^d}}$, then we get (1.2), where the kinetic friction is assumed to be always equal to the maximum static friction and, in consequence, the multi-function F is monotone.

In contrast to (1.1), in (1.3), the friction force $\boldsymbol{\sigma}_\tau$ depends on the normal contact stress σ_ν, which is equal to the foundation reaction force. This reflects the fact that the harder we press the body against the foundation, the higher the resultant reaction force is, and, in consequence, the larger is the friction. Moreover, if there is no contact between the body and the foundation, when we use the law (1.1), the non-physical frictional forces can still occur, while when we use the law (1.3), in such a case there will be $\sigma_\nu = 0$, and in consequence $\boldsymbol{\sigma}_\tau = 0$. Hence, while (1.1) remains a good approximation if the whole boundary on which the contact can potentially occur is actually in contact with the foundation, and the normal stresses are not too high, certainly (1.3) is more physically adequate.

Here, we need more restrictive assumptions on the problem data than those in [12]. Namely, the linear growth condition on the Clarke sub-differential of the friction potential (see $H(j_\tau)(c)$ in [12]) needs to be replaced with the boundedness of this sub-differential (see $H(j)(c)$ in the sequel). The second restriction we need to add here is the boundedness of the normal compliance function (see $H(p)(d)$ in the sequel), not present in [12].

Under these assumptions, using the fixed point technique, similarly to that of [12], we are able to prove the existence and uniqueness of solution for the associated problem.

The rest of the paper is structured as follows. The model and the mathematical problem are introduced in Section 2. Its weak formulation is described in Section 3, where the assumptions on the problem data are listed. The proofs of the existence of the unique solution of the variational-hemi-variational formulation of the model can be found in Section 4, and summarized in Theorem 4.1. The proof is done in steps in which auxiliary problems are introduced and solved by applying various fixed point arguments. Finally, in appendix, various mathematical notions and tools used throughout the article are recalled, and some details concerning the notation are provided.

2 The model

We consider a viscoelastic body occupying a domain $\Omega \subseteq \mathbb{R}^d$ (in applications $d = 2, 3$, but mathematically any natural number $d \geqslant 2$ can be used) with a Lipschitz boundary $\partial\Omega$. We assume that $\partial\Omega$ is divided into three mutually disjoint and relatively open sets Γ_D, Γ_C, and Γ_N such that $\overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C} = \partial\Omega$ and $\text{meas}_{d-1}(\Gamma_D) > 0$. The body is held fixed on Γ_D and surface tractions of density $\boldsymbol{f}_N$ act on Γ_N. The part of surface that can be in contact with the foundation is Γ_C and the gap given by g is measured along the outward normal.

Volume forces $\boldsymbol{f}_0$ act in Ω, and these and the tractions are assumed to vary slowly in time so we can neglect the acceleration in the system and the process is quasi-static. We describe the contact process with the normal compliance condition and a sub-differential Coulomb friction condition. We use the viscoelastic constitutive law with damage effect. The damage function β is governed by a parabolic differential inclusion and satisfies a homogeneous Neumann boundary condition.

We denote by $[0, T]$ the time interval of interest, with $T > 0$, and use the notation $Q = \Omega \times (0, T)$.
The model is given as follows.

Problem P : Find a displacement field $\boldsymbol{u} : Q \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : Q \to \mathbb{S}^d$ and a damage function $\beta : Q \to \mathbb{R}$, such that for all $t \in (0, T)$ we have

$$\boldsymbol{\sigma}(t) = \mathscr{A}(\varepsilon(\dot{\boldsymbol{u}})(t)) + \mathscr{G}(\varepsilon(\boldsymbol{u}(t)), \beta(t)) \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\dot{\beta}(t) - \kappa\Delta\beta(t) + \partial\psi_{[0,1]}(\beta(t)) \ni \phi(t, \varepsilon(\boldsymbol{u}(t)), \beta(t)) \quad \text{in} \quad \Omega, \tag{2.2}$$

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$$\text{Div } \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.3}$$

$$\frac{\partial \beta(t)}{\partial \boldsymbol{v}} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

$$\boldsymbol{u}(t) = 0 \quad \text{on} \quad \Gamma_D, \tag{2.5}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{v} = \boldsymbol{f}_N(t) \quad \text{on} \quad \Gamma_N, \tag{2.6}$$

$$-\sigma_\nu(t) = p(u_\nu(t) - g) \quad \text{on} \quad \Gamma_C, \tag{2.7}$$

$$-\boldsymbol{\sigma}_\tau(t) \in |\sigma_\nu(t)|\partial j(\dot{\boldsymbol{u}}_\tau(t)) \quad \text{on} \quad \Gamma_C, \tag{2.8}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \beta(0) = \beta_0 \quad \text{in} \quad \Omega. \tag{2.9}$$

Here, $\mathbb{S}^d$ denotes the space of second-order symmetric $d \times d$ matrices, $\boldsymbol{v}$ represents the unit outward normal on $\partial\Omega$, $\partial\beta/\partial\boldsymbol{v}$ is the normal derivative on $\partial\Omega$, Div denotes the divergence operator, σ_ν and $\boldsymbol{\sigma}_\tau$ stand for the normal and tangential traces of $\boldsymbol{\sigma}$, respectively, u_ν and $\dot{\boldsymbol{u}}_\tau$ are the normal and tangential components of displacement $\boldsymbol{u}$ and velocity $\dot{\boldsymbol{u}}$, respectively. The linearized strain tensor is given as the symmetric part of the displacement gradient $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)$ By $\partial\psi_{[0,1]}$ we denote the convex sub-differential of ψ, the indicator function of $[0, 1]$, and by ∂j we mean the Clarke sub-differential of j.

The viscoelastic constitutive law (2.1) depends on the velocity through the non-linear viscosity operator $\mathscr{A}$ and on the displacement through the non-linear mapping $\mathscr{G}$, which also includes effect of material damage. The damage process is described here exactly as in [12], namely, the evolution of the damage variable β is described by the parabolic inclusion (2.2) with the damage source function ϕ and boundary condition (2.4). Since we assume that the process is quasi-static, we use the equilibrium equation (2.3) to describe the evolution of the mechanical state of the body. Equations (2.5) and (2.6) represent the displacement and traction boundary conditions, respectively. The initial conditions for displacement and damage are given by (2.9). Relation (2.7) is the so called *normal compliance condition*. The function p models the relation between the normal distance from the foundation $u_\nu - g$ and the normal stress σ_ν. Usually, it is assumed that for non-positive values of $u_\nu - g$ the function p is equal to zero, which means that if we are away from the foundation there is no stress. If, in turn $u_\nu - g > 0$, meaning that the foundation is penetrated, then the function p typically has positive values, meaning that the reaction force is directed away from the foundation and depending on the penetration depth. Finally, the inclusion (2.8), which is the main novelty of this article in comparison to [12] is the generalized Coulomb friction law, in which the tangential stress depends linearly on the magnitude of the normal stress and, through the sub-differential frictional condition, on the slip rate. In particular, if there is no contact between the body and the foundation, i.e., $u_\nu \leqslant g$, assuming that the function p is equal to zero for the negative argument, the tangential stress is also equal to zero. In turn, if the foundation is penetrated, then the magnitude of the friction force increases together with the normal reaction.

3 Weak formulation

We assume the following on the problem data. The notation used in the assumptions, and throughout the following part of the article, is explained in details in the appendix.

$\underline{H(\mathscr{A})}$: The *viscosity operator* $\mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following:

(a) $\mathscr{A}(\cdot, \boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$;

(b) $\mathscr{A}(\boldsymbol{x}, \cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $\boldsymbol{x} \in \Omega$;

(c) there exist $a_0 \in L^2(\Omega)$, $a_0 \geqslant 0$ and $a_1 > 0$ such that for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$,

$$\|\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})\|_{\mathbb{S}^d} \leqslant a_0(\boldsymbol{x}) + a_1 \|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d};$$

(d) there exists $m_{\mathscr{A}} > 0$ such that for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$

$$(\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathscr{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d};$$

(e) $\mathscr{A}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0}$ for a.e. $\boldsymbol{x} \in \Omega$.

$\underline{H(\mathscr{G})}$: The *elasticity operator* $\mathscr{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ is such that

(a) $\mathscr{G}(\cdot, \boldsymbol{\varepsilon}, \beta)$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\beta \in \mathbb{R}$;

(b) there exists $L_{\mathscr{G}} > 0$ such that

$$\|\mathscr{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \beta_1) - \mathscr{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \beta_2)\|_{\mathbb{S}^d} \leqslant L_{\mathscr{G}}(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} + |\beta_1 - \beta_2|)$$

for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, $\beta_1, \beta_2 \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Omega$;

(c) $\mathscr{G}(\boldsymbol{x}, \boldsymbol{0}, 0) \in L^2(\Omega; \mathbb{S}^d)$ for a.e. $\boldsymbol{x} \in \Omega$.

$\underline{H(\phi)}$: The *damage source function* $\phi : Q \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies the following:

(a) $\phi(\cdot, \cdot, \boldsymbol{\varepsilon}, \beta)$ is measurable on Q for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\beta \in \mathbb{R}$;

(b) there exists $L_\phi > 0$ such that

$$|\phi(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1, \beta_1) - \phi(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_2, \beta_2)| \leqslant L_\phi(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} + |\beta_1 - \beta_2|)$$

for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, $\beta_1, \beta_2 \in \mathbb{R}$ and a.e. $(\boldsymbol{x}, t) \in Q$;

(c) $\phi(\boldsymbol{x}, \cdot, \boldsymbol{\varepsilon}, \beta)$ is continuous on $[0, T]$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\beta \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Omega$;

(d) $|\phi(\boldsymbol{x}, t, \boldsymbol{0}, 0)| \leqslant \overline{\phi}(\boldsymbol{x})$ for all $t \in [0, T]$ and almost all $\boldsymbol{x} \in \Omega$ where $\overline{\phi} \in L^2(\Omega)$.

$\underline{H(p)}$: The *normal compliance function* $p : \Gamma_C \times \mathbb{R} \to [0, \infty)$ is such that

(a) $p(\cdot, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$;

(b) there exists $L_\nu > 0$ such that for all $r_1, r_2 \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Gamma_C$

$$|p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \leqslant L_\nu |r_1 - r_2|;$$

(c) $p(\cdot, r) = 0$ for $r \leqslant 0$ on Γ_C;

(d) $p(\cdot, r) \leqslant c_p$ for all $r > 0$ on Γ_C with a constant $c_p > 0$.

$\underline{H(j)}$: The *friction potential* $j : \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following:

(a) $j(\cdot, \boldsymbol{\xi})$ is measurable on Γ_C for all $\boldsymbol{\xi} \in \mathbb{R}^d$;

(b) $j(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_C$;

(c) there exists $c_\tau > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and for a.e. $\boldsymbol{x} \in \Gamma_C$

$$\max_{\boldsymbol{\eta} \in \partial j(\boldsymbol{x}, \boldsymbol{\xi})} \|\boldsymbol{\eta}\|_{\mathbb{R}^d} \leqslant c_\tau;$$

(d) the one sided Lipschitz constant given by

$$L(\boldsymbol{x}, \partial j) = \inf_{\substack{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d, \boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2 \\ \boldsymbol{\eta}_1 \in \partial j(\boldsymbol{x}, \boldsymbol{\xi}_1) \\ \boldsymbol{\eta}_2 \in \partial j(\boldsymbol{x}, \boldsymbol{\xi}_2)}} \frac{(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \cdot (\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1)}{\|\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1\|^2_{\mathbb{R}^d}}$$

satisfies $L(\boldsymbol{x}, \partial j) \geqslant -L_\tau$ on Γ_C, for certain $L_\tau \geqslant 0$.

We remark that by $\partial j(\boldsymbol{x}, \xi)$ we mean the Clarke sub-differential of j taken with respect to the second variable ξ. Often, for the sake of the ease of notation, we do not write explicitly the dependence of various quantities on the variable $\boldsymbol{x}$, i.e. in place of $j(\boldsymbol{x}, \xi)$ we write only $j(\xi)$.

The *volume force density, surface traction density, gap function, initial functions* and *constants in the above hypotheses* are assumed to satisfy the following:

$\underline{H_0}$:

(a) $\boldsymbol{f}_0 \in C([0,T]; L^2(\Omega)^d)$;

(b) $\boldsymbol{f}_N \in C([0,T]; L^2(\Gamma_N)^d)$;

(c) $g \in L^\infty(\Gamma_C)$, $g \geqslant 0$;

(d) $\boldsymbol{u}_0 \in V$;

(e) $\beta_0 \in H^1(\Omega)$ is such that $0 \leqslant \beta_0 \leqslant 1$ a.e. in Ω;

(f) $c_p L_\tau \|\gamma_{\Gamma_C}\|^2 < m_{\mathscr{A}}$.

Remark 3.1 *Every convex function $j : \mathbb{R}^d \to \mathbb{R}$ satisfies assumptions $H(j)(b)$ and (d) with $L_\tau = 0$. Then obviously, also $H_0(f)$ holds.*

Remark 3.2 *The condition $H(j)(d)$ is equivalent to the fact that the functional $j(\boldsymbol{x}, \cdot) + L_\tau \frac{\|\cdot\|^2_{\mathbb{R}^d}}{2}$ is convex, or, equivalently, the multi-valued map of $\xi \to \partial j(\boldsymbol{x}, \xi) + L_\tau \xi$ is monotone.*

Remark 3.3 *The smallness assumption $H_0(f)$ is needed because of the approach based on the Banach fixed point theorem. This guarantees not only the existence, but also the uniqueness of the solution. It is an open problem, if without this assumption we have the solution existence only.*

We now derive the variational formulation of Problem **P**. The spaces V and $\mathscr{H}_1$ used in the weak formulation, are, together with their norms, defined in the appendix. We consider the function $\boldsymbol{f} : [0,T] \to V^*$, given by

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V^* \times V} = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_N(t), \gamma_{\Gamma_N} \boldsymbol{v})_{L^2(\Gamma_N)^d} \quad \text{for all } \boldsymbol{v} \in V,\ t \in [0,T], \tag{3.1}$$

and the set of admissible damage functions

$$\mathscr{K} = \{\zeta \in H^1(\Omega) : 0 \leqslant \zeta \leqslant 1 \text{ a.e. in } \Omega\}.$$

Assume that $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ are sufficiently smooth functions that solve (2.1)–(2.9), $\boldsymbol{v} \in V$, $\zeta \in \mathscr{K}$ and $t \in [0,T]$. First, we use the equilibrium equation (2.3) and the Green formula

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(A 2) to obtain

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)} = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + \int_{\partial\Omega} \boldsymbol{\sigma}(t)\boldsymbol{v} \cdot \boldsymbol{v}\, dS. \tag{3.2}$$

Taking into account the boundary conditions (2.5)–(2.8), in a standard way, we obtain

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{\sigma}(t)\boldsymbol{v} \cdot \boldsymbol{v}\, dS &= \int_{\Gamma_N} \boldsymbol{f}_N(t) \cdot \boldsymbol{v}\, dS \\ &\quad - \int_{\Gamma_C} p(u_\nu(t) - g)v_\nu\, dS - \int_{\Gamma_C} p(u_\nu(t) - g)\boldsymbol{\xi}(t) \cdot \boldsymbol{v}_\tau\, dS,\end{aligned} \tag{3.3}$$

for $\boldsymbol{\xi}(t) \in S^2_{\partial j(\dot{\boldsymbol{u}}_\tau(t))}$ (for a multi-function $F : \Gamma_C \to 2^{\mathbb{R}^d}$ we use the symbol S^2_F to denote all of its selections of class $L^2(\Gamma_C)^d$). Hence, using (3.3) and (3.1) in (3.2), we have

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)} + \int_{\Gamma_C} p(u_\nu(t) - g)v_\nu\, dS + \int_{\Gamma_C} p(u_\nu(t) - g)\boldsymbol{\xi}(t) \cdot \boldsymbol{v}_\tau\, dS = \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^* \times V}.$$

Next, using the definition of the sub-differential of the indicator function $\psi_{[0,1]}$ and integration by parts, we see that

$$0 \geqslant (\phi(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)) - \dot{\beta}(t),\ \zeta - \beta(t))_{L^2(\Omega)} - \kappa(\nabla\beta(t),\ \nabla\zeta - \nabla\beta(t))_{L^2(\Omega)^d}.$$

Collecting these relations and inequalities leads to the following weak formulation of Problem **P**.

Problem $\mathbf{P_V}$: Find $\boldsymbol{u} \in C^1([0,T];V)$, $\boldsymbol{\sigma} \in C([0,T];\mathscr{H}_1)$ and $\beta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that

$$\boldsymbol{\sigma}(t) = \mathscr{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))) + \mathscr{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)), \tag{3.4}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)} &+ \int_{\Gamma_C} p(u_\nu(t) - g)v_\nu\, dS \\ &+ \int_{\Gamma_C} p(u_\nu(t) - g)\boldsymbol{\xi}(t) \cdot \boldsymbol{v}_\tau\, dS = \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^* \times V} \\ &\qquad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T],\end{aligned} \tag{3.5}$$

$$\boldsymbol{\xi}(t) \in S^2_{\partial j(\dot{\boldsymbol{u}}_\tau(t))} \quad \text{for all} \quad t \in [0,T]; \tag{3.6}$$

$$\begin{aligned}(\dot{\beta}(t), \zeta - \beta(t))_{L^2(\Omega)} &+ \kappa(\nabla\beta(t), \nabla\zeta - \nabla\beta(t))_{L^2(\Omega)^d} \\ &\geqslant (\phi(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)),\ \zeta - \beta(t))_{L^2(\Omega)} \\ &\qquad \text{for all } \zeta \in \mathscr{K} \text{ and all } t \in [0,T],\end{aligned} \tag{3.7}$$

$$\beta(t) \in \mathscr{K} \quad \text{for all } t \in [0,T] \tag{3.8}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \beta(0) = \beta_0. \tag{3.9}$$

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4 Existence and uniqueness results

The main existence result of this paper is the following.

Theorem 4.1 *If hypotheses $H(\mathscr{A})$, $H(\mathscr{G})$, $H(\phi)$, $H(p)$, $H(j)$, H_0, and H_1 hold, then Problem $\boldsymbol{P}_V$ has a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$.*

The proof of the above theorem follows the lines of the proof of Theorem 5.1 in [12] with some modifications that we stress here. Before we pass to the proof, however, we formulate several auxiliary problems and we prove the existence and uniqueness of their solutions. First, let us assume that the elastic part of the stress $\boldsymbol{\eta} \in C([0,T]; L^2(\Omega; \mathbb{S}^d))$ and the damage source function $\theta \in C([0,T]; L^2(\Omega))$ are given, and consider the following two auxiliary problems.

Problem $\mathbf{P}_\theta$: Find $\beta_\theta \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ such that

$$(\dot\beta_\theta(t), \zeta - \beta_\theta(t))_{L^2(\Omega)} + \kappa(\nabla\beta_\theta(t), \nabla\zeta - \nabla\beta_\theta(t))_{L^2(\Omega)^d} \geqslant (\theta(t), \zeta - \beta_\theta(t))_{L^2(\Omega)} \quad \text{for all } \zeta \in \mathscr{K} \text{ and all } t \in [0,T] \tag{4.1}$$

$$\beta_\theta(t) \in \mathscr{K} \quad \text{for all } t \in [0,T] \tag{4.2}$$

$$\beta_\theta(0) = \beta_0. \tag{4.3}$$

Problem $\mathbf{P}^1_\eta$: Find $\boldsymbol{u}_\eta \in C^1([0,T]; V)$ and $\boldsymbol{\sigma}_\eta \in C([0,T]; \mathscr{H}_1)$ such that

$$\boldsymbol{\sigma}_\eta(t) = \mathscr{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t))) + \boldsymbol{\eta}(t), \tag{4.4}$$

$$(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)} + \int_{\Gamma_C} p(u_\nu(t) - g)v_\nu \, dS + \int_{\Gamma_C} p(u_\nu(t) - g)\boldsymbol{\xi}(t) \cdot \boldsymbol{v}_\tau \, dS = \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V^* \times V} \quad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T], \tag{4.5}$$

$$\boldsymbol{\xi}(t) \in S^2_{\partial j(\dot{\boldsymbol{u}}_\tau(t))} \quad \text{for all} \quad t \in [0,T], \tag{4.6}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.7}$$

We reformulate Problem $\mathbf{P}^1_\eta$ in terms of the velocity $\boldsymbol{w}_\eta = \dot{\boldsymbol{u}}_\eta$. Then,

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{w}_\eta(s)\, ds + \boldsymbol{u}_0 \quad \text{for all} \quad t \in [0,T]. \tag{4.8}$$

We also introduce some auxiliary operators and functions. The operator $A : V \to V^*$ is given by

$$\langle A(\boldsymbol{w}), \boldsymbol{v} \rangle_{V^* \times V} = (\mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{w})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)} \quad \text{for all} \quad \boldsymbol{v}, \boldsymbol{w} \in V,$$

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$\mathscr{R}: C([0,T];V) \to C([0,T];L^2(\Gamma_C))$ is given by

$$(\mathscr{R}\boldsymbol{w})(t) = p\left(\int_0^t w_\nu(s)\,ds + (u_0)_\nu - g\right) \quad \text{for all} \quad \boldsymbol{w} \in C([0,T];V) \quad \text{and} \quad t \in [0,T],$$

and the function $\widetilde{\boldsymbol{f}}: [0,T] \to V^*$ is given by

$$\langle \widetilde{\boldsymbol{f}}(t), \boldsymbol{v}\rangle_{V^*\times V} = \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} - (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)} \quad \text{for all} \quad \boldsymbol{v} \in V,\ t \in [0,T].$$

Using this notation, we can rewrite Problem $\mathbf{P}^1_\eta$ as follows.

Problem $\mathbf{P}^2_\eta$: Find $\boldsymbol{w}_\eta \in C([0,T];V)$ such that

$$\langle A(\boldsymbol{w}_\eta(t)), \boldsymbol{v}\rangle_{V^*\times V} + \int_{\Gamma_C} (\mathscr{R}\boldsymbol{w}_\eta)(t) v_\nu\, dS$$
$$+ \int_{\Gamma_C} (\mathscr{R}\boldsymbol{w}_\eta)(t)\boldsymbol{\xi}(t)\cdot \boldsymbol{v}_\tau\, dS = \langle \widetilde{\boldsymbol{f}}(t), \boldsymbol{v}\rangle_{V^*\times V} \tag{4.9}$$
$$\text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T], \tag{4.10}$$
$$\boldsymbol{\xi}(t) \in S^2_{\partial j((\boldsymbol{w}_\eta)_\tau(t))} \quad \text{for all} \quad t \in [0,T]. \tag{4.11}$$

For given $\boldsymbol{\mu} \in C([0,T];V)$, let $z_\mu = \mathscr{R}\boldsymbol{\mu}$. We define the function $\overline{\boldsymbol{f}}: [0,T] \to V$ by

$$\langle \overline{\boldsymbol{f}}(t), \boldsymbol{v}\rangle_{V^*\times V} = \langle \widetilde{\boldsymbol{f}}(t), \boldsymbol{v}\rangle_{V^*\times V} - \int_{\Gamma_C} z_\mu(t) v_\nu\, dS \quad \text{for all} \quad \boldsymbol{v} \in V,\ t \in [0,T],$$

and consider the following problem.

Problem $\mathbf{P}_{\eta\mu}$: Find $\boldsymbol{w}_{\eta\mu} \in C([0,T];V)$ such that

$$\langle A(\boldsymbol{w}_{\eta\mu}(t)), \boldsymbol{v}\rangle_{V^*\times V} + \int_{\Gamma_C} z_\mu(t)\boldsymbol{\xi}(t)\cdot \boldsymbol{v}_\tau\, d\Gamma = \langle \overline{\boldsymbol{f}}(t), \boldsymbol{v}\rangle_{V^*\times V}$$
$$\text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T], \tag{4.12}$$
$$\boldsymbol{\xi}(t) \in S^2_{\partial j((\boldsymbol{w}_{\eta\mu})_\tau(t))} \quad \text{for all} \quad t \in [0,T]. \tag{4.13}$$

Lemma 4.2 *Under the hypotheses of Theorem 4.1, Problem $\boldsymbol{P}_{\eta\mu}$ admits a unique solution $\boldsymbol{w}_{\eta\mu}$.*

Proof Fix $t \in [0,T]$. Consider the operator $B_t: V \to 2^{V^*}$ given by

$$B_t(\boldsymbol{v}) = \{\boldsymbol{\eta} \in V^* : \quad \text{there exists} \quad \boldsymbol{\xi} \in S^2_{\partial j(\boldsymbol{v}_\tau)}$$
$$\text{such that } \langle \boldsymbol{\eta}, \boldsymbol{w}\rangle_{V^*\times V} = \int_{\Gamma_C} z_\mu(t)\boldsymbol{\xi}\cdot \boldsymbol{w}_\tau\, dS \quad \text{for all} \quad \boldsymbol{w} \in V\}.$$

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We study the sum $A + B_t$. To this end, let us first remind the properties of the operator A established in the proof of Lemma 5.3 in [12].

First, A is bounded, and we have

$$\|A(\boldsymbol{u})\|_{V^*} \leqslant \sqrt{2}(\|a_0\|_{L^2(\Omega)} + a_1\|\boldsymbol{u}\|_V),$$

for all $\boldsymbol{u} \in V$. Next, the operator A is coercive in the sense that

$$\langle A(\boldsymbol{u}), \boldsymbol{u}\rangle_{V^*\times V} \geqslant m_{\mathscr{A}}\|\boldsymbol{u}\|_V^2, \tag{4.14}$$

for all $\boldsymbol{u} \in V$. Moreover, the operator A is monotone, and, finally, A is continuous, i.e., if the sequence $\{\boldsymbol{u}_n\}_{n\geqslant 1} \subset V$ is such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in V, then $A(\boldsymbol{u}_n) \to A(\boldsymbol{u})$ in V^*. Since the operator A is bounded, monotone and hemi-continuous, it is also pseudo-monotone (see [17, Theorem 3.69(i)]).

From Theorem 5.6.39 of [7], we know that the set $S^2_{\partial j(\boldsymbol{v}_\tau)}$ is non-empty for any $\boldsymbol{v} \in V$. Hence, as by $H(p)(c)$-(d) we have $z_\mu(t) \in L^\infty(\Gamma_C)$, the set $B_t(\boldsymbol{v})$ is non-empty for any $\boldsymbol{v} \in V$. The fact that $B_t(\boldsymbol{v})$ is convex for any $\boldsymbol{v} \in V$ follows in a straightforward way from the fact that $\partial j(\boldsymbol{x}, \boldsymbol{w})$ is a convex set in $\mathbb{R}^d$ for all $\boldsymbol{w} \in \mathbb{R}^d$ and a.e. $\boldsymbol{x} \in \Gamma_C$. We prove that $B_t(\boldsymbol{v})$ is closed in V^*. To this end, let $\boldsymbol{\eta}_n \in B_t(\boldsymbol{v})$ be such that $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$ in V^*. There exists $\boldsymbol{\xi}_n \in S^2_{\partial j(\boldsymbol{v}_\tau)}$ such that

$$\langle \boldsymbol{\eta}_n, \boldsymbol{w}\rangle_{V^*\times V} = \int_{\Gamma_C} z_\mu(t)\boldsymbol{\xi}_n \cdot \boldsymbol{w}_\tau \, dS \quad \text{for all} \quad \boldsymbol{w} \in V.$$

Since, by $H(j)(c)$, we have $\|\boldsymbol{\xi}_n\|_{L^\infty(\Gamma_C)^d} \leqslant c_\tau \text{meas}_{d-1}(\Gamma_C)$, it follows that $\boldsymbol{\xi}_n$ is bounded also in $L^2(\Gamma_C)^d$ and, for a subsequence, still denoted by the same notion,

$$\boldsymbol{\xi}_n \to \boldsymbol{\xi} \quad \text{weakly in} \quad L^2(\Gamma_C)^d$$

for certain $\boldsymbol{\xi} \in L^2(\Gamma_C)^d$. Since for $\boldsymbol{w} \in V$ we have $z_\mu(t)\boldsymbol{w}_\tau \in L^2(\Gamma_C)^d$, it holds $\int_{\Gamma_C} z_\mu(t)\boldsymbol{\xi}_n \cdot \boldsymbol{w}_\tau \, dS \to \int_{\Gamma_C} z_\mu(t)\boldsymbol{\xi} \cdot \boldsymbol{w}_\tau \, dS$, and hence

$$\int_{\Gamma_C} z_\mu(t)\boldsymbol{\xi} \cdot \boldsymbol{w}_\tau \, dS = \langle \boldsymbol{\eta}, \boldsymbol{w}\rangle_{V^*\times V} \quad \text{for all} \quad \boldsymbol{w} \in V.$$

Using Proposition A.10 from the fact that $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$ weakly in $L^2(\Gamma_C)^d$ and $\|\boldsymbol{\xi}_n(\boldsymbol{x})\|_{\mathbb{R}^d} \leqslant c_\tau$ it follows that

$$\boldsymbol{\xi}(\boldsymbol{x}) \in \overline{\text{conv}}\left(\limsup_{n\to+\infty} \{\boldsymbol{\xi}_n(\boldsymbol{x})\}\right) \quad \text{for a.e.} \quad \boldsymbol{x} \in \Gamma_C,$$

where $\limsup\limits_{n\to+\infty}$ is the Kuratowski–Painlevé upper limit of sets in the topology of $\mathbb{R}^d$. Furthermore, we have

$$\overline{\text{conv}}\left(\limsup_{n\to+\infty} \{\boldsymbol{\xi}_n(\boldsymbol{x})\}\right) \subset \overline{\text{conv}}\left(\limsup_{n\to+\infty} \partial j(\boldsymbol{x}, \boldsymbol{v}_\tau(\boldsymbol{x}))\right) = \partial j(\boldsymbol{x}, \boldsymbol{v}_\tau(\boldsymbol{x})),$$

for a.e. $\boldsymbol{x} \in \Gamma_C$, and hence $\boldsymbol{\xi} \in S^2_{\partial j(\boldsymbol{v}_\tau)}$, which completes the proof of closedness.

Next, we observe that by $H(p)(c)$-(d) and $H(j)(c)$ for any $\boldsymbol{\eta} \in B_t(\boldsymbol{v})$ we have

$$|\langle \boldsymbol{\eta}, \boldsymbol{w} \rangle_{V^* \times V}| \leqslant c_p c_\tau \sqrt{\text{meas}_{d-1}(\Gamma_C)} \|\gamma_{\Gamma_C} \boldsymbol{w}\|_{L^2(\Gamma_C)^d} \leqslant c_p c_\tau \sqrt{\text{meas}_{d-1}(\Gamma_C)} \|\gamma_{\Gamma_C}\| \|\boldsymbol{w}\|_V,$$

for $\boldsymbol{w} \in V$ and hence for any $\boldsymbol{\eta} \in B_t(\boldsymbol{v})$ we have

$$\|\boldsymbol{\eta}\|_{V^*} \leqslant c_p c_\tau \sqrt{\text{meas}_{d-1}(\Gamma_C)} \|\gamma_{\Gamma_C}\|, \tag{4.15}$$

so the multi-valued operator B_t is bounded.

Now we show that the operator B_t is generalized pseudo-monotone. Let $\{\boldsymbol{v}_n\}_{n\geqslant 1}$ be a sequence of V such that $\boldsymbol{v}_n \to \boldsymbol{v}$ weakly in V, let $\{\boldsymbol{v}_n^*\}_{n\geqslant 1}$ be a sequence of V^* such that $\boldsymbol{v}_n^* \to \boldsymbol{v}^*$ weakly in V^*, $\boldsymbol{v}_n^* \in B_t(\boldsymbol{v}_n)$ for all $n \geqslant 1$ and $\limsup_{n\to+\infty} \langle \boldsymbol{v}_n^*, \boldsymbol{v}_n - \boldsymbol{v} \rangle_{V^* \times V} \leqslant 0$. Since $\boldsymbol{v}_n \to \boldsymbol{v}$ weakly in V, from the compactness of the trace operator γ_{Γ_C} it follows that $(\boldsymbol{v}_n)_\tau \to \boldsymbol{v}_\tau$ strongly in $L^2(\Gamma_C)^d$, and, for a subsequence, still denoted with the same notion, we have

$$(\boldsymbol{v}_n)_\tau(\boldsymbol{x}) \to \boldsymbol{v}_\tau(\boldsymbol{x}) \quad \text{for a.e.} \quad \boldsymbol{x} \in \Gamma_C.$$

From the definition of the operator B_t, there exists the sequence $\boldsymbol{\xi}_n \in S^2_{\partial j((\boldsymbol{v}_n)_\tau)}$ such that

$$\langle \boldsymbol{v}_n^*, \boldsymbol{w} \rangle_{V^* \times V} = \int_{\Gamma_C} z_\mu(t) \boldsymbol{\xi}_n \cdot \boldsymbol{w}_\tau \, dS \quad \text{for all} \quad \boldsymbol{w} \in V.$$

Since, by $H(j)(c)$ we have $\|\boldsymbol{\xi}_n\|_{L^\infty(\Gamma_C)^d} \leqslant c_\tau \text{meas}_{d-1}(\Gamma_C)$, it follows that $\boldsymbol{\xi}_n$ is bounded in $L^2(\Gamma_C)^d$ and, passing to another subsequence if necessary, we have

$$\boldsymbol{\xi}_n \to \boldsymbol{\xi} \quad \text{weakly in} \quad L^2(\Gamma_C)^d$$

for certain $\boldsymbol{\xi} \in L^2(\Gamma_C)^d$. Since for $\boldsymbol{w} \in V$ we have $z_\mu(t)\boldsymbol{w}_\tau \in L^2(\Gamma_C)^d$, it holds

$$\int_{\Gamma_C} z_\mu(t) \boldsymbol{\xi}_n \cdot \boldsymbol{w}_\tau \, dS \to \int_{\Gamma_C} z_\mu(t) \boldsymbol{\xi} \cdot \boldsymbol{w}_\tau \, dS$$

and hence

$$\int_{\Gamma_C} z_\mu(t) \boldsymbol{\xi} \cdot \boldsymbol{w}_\tau \, dS = \langle \boldsymbol{v}^*, \boldsymbol{w} \rangle_{V^* \times V} \quad \text{for all} \quad \boldsymbol{w} \in V.$$

Using Proposition A.10 from the fact that $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$ weakly in $L^2(\Gamma_C)^d$ and $\|\boldsymbol{\xi}_n(\boldsymbol{x})\|_{\mathbb{R}^d} \leqslant c_\tau$ a.e. on Γ_C it follows that

$$\boldsymbol{\xi}(\boldsymbol{x}) \in \overline{\text{conv}}(\limsup_{n\to+\infty} \{\boldsymbol{\xi}_n(\boldsymbol{x})\}) \quad \text{for a.e.} \quad \boldsymbol{x} \in \Gamma_C.$$

Furthermore, we have

$$\overline{\text{conv}}(\limsup_{n\to+\infty} \{\boldsymbol{\xi}_n(\boldsymbol{x})\}) \subset \overline{\text{conv}}(\limsup_{n\to+\infty} \partial j(\boldsymbol{x}, (\boldsymbol{v}_n)_\tau(\boldsymbol{x}))),$$

for a.e. $\boldsymbol{x} \in \Gamma_C$. Since, by Proposition A.7 the multi-function $\partial j(\boldsymbol{x}, \cdot) : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ has a closed graph, we have

$$\limsup_{n\to+\infty} \partial j(\boldsymbol{x}, (\boldsymbol{v}_n)_\tau(\boldsymbol{x})) \subset \partial j(\boldsymbol{x}, \boldsymbol{v}_\tau(\boldsymbol{x})) \quad \text{for a.e.} \quad \boldsymbol{x} \in \Gamma_C.$$

Hence,

$$\xi(x) \in \partial j(x, v_\tau(x)) \quad \text{for a.e.} \quad x \in \Gamma_C,$$

and it follows that $\xi \in S^2_{\partial j(v_\tau)}$, which means that $v^* \in B_t(v)$. Moreover, we have

$$\langle v_n^*, v_n \rangle_{V^* \times V} = \int_{\Gamma_C} z_\mu(t) \xi_n \cdot v_{n\tau} \, dS \to \int_{\Gamma_C} z_\mu(t) \xi \cdot v_\tau \, dS = \langle v^*, v \rangle_{V^* \times V},$$

where, from the uniqueness of the limit, the convergence holds for the whole sequence. This proves that the operator B_t is generalized pseudo-monotone and therefore, by Proposition A.4 it is also pseudo-monotone.

Since the class of multi-valued pseudo-monotone operators is closed under addition of mappings (see e.g., [17, Proposition 3.59]), we deduce that $A + B_t$ is pseudo-monotone.

Finally, we establish the coercivity of the operator $A + B_t$. Using (4.15) and (4.14), for all $v \in V$, we have

$$\begin{aligned}\langle A(v) + B_t(v), v \rangle_{V^* \times V} &= \langle A(v), v \rangle_{V^* \times V} + \langle B_t(v), v \rangle_{V^* \times V} \\ &\geqslant m_{\mathscr{A}} \|v\|_V^2 - c_p c_\tau \sqrt{\text{meas}_{\,d-1}(\Gamma_C)} \|\gamma_{\Gamma_C}\| \|v\|_V,\end{aligned}$$

and the coercivity follows.

Since we have checked that under our hypotheses, the operator $A + B_t$ is bounded, pseudo-monotone and coercive, we can apply Theorem A.5 and deduce that $A + B_t : V \to 2^{V^*}$ is surjective and so for each $t \in [0, T]$ there exists $w_{\eta\mu}(t) \in V$ such that

$$A(w_{\eta\mu}(t)) + B_t(w_{\eta\mu}(t)) \ni \overline{f}(t)$$

and it follows that $w_{\eta\mu}(t)$ is a solution of Problem $\mathbf{P}_{\eta\mu}$.

Next, we show the uniqueness of $w_{\eta\mu}$. For a fixed $t \in [0, T]$, assume that we have $w_1(t), w_2(t) \in V$ two solutions of $\mathrm{P}_{\eta\mu}$. We write (4.12) for $w_1(t)$ and for $w_2(t)$, subtract two equalities and take $v = w_1(t) - w_2(t)$. We get

$$\begin{aligned}&\langle A(w_1(t)) - A(w_2(t)), w_1(t) - w_2(t) \rangle_{V^* \times V} \\ &\quad + \int_{\Gamma_C} z_\mu(t)(\xi_1(t) - \xi_2(t)) \cdot ((w_1)_\tau(t) - (w_2)_\tau(t)) \, dS = 0.\end{aligned}$$

Then, using hypotheses $H(\mathscr{A})(d)$, $H(j)(d)$, and $H(p)(c)$-(d) we obtain

$$m_{\mathscr{A}} \|w_1(t) - w_2(t)\|_V^2 \leqslant c_p L_\tau \|\gamma_{\Gamma_C}\|^2 \|w_1(t) - w_2(t)\|_V^2,$$

so from hypothesis $H_0(f)$, we deduce that $w_1(t) = w_2(t)$ for all $t \in [0, T]$.

To complete the proof of the lemma, we show that the mapping $[0, T] \ni t \to w_{\eta\mu}(t) \in V$ is continuous. Let $t_1, t_2 \in [0, T]$ and let us denote $\widehat{w}_i = w_{\eta\mu}(t_i)$, $\widehat{z}_i = z_\mu(t_i)$, $\widehat{f}_i = f(t_i)$, $\widehat{\eta}_i = \eta(t_i)$, and $\widehat{\xi}_i = \xi(t_i)$, for $i = 1, 2$. We write (4.12) for $t = t_1$ and $t = t_2$, subtract two resulting inequalities and take $\boldsymbol{v} = \widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2$, which yields

$$\langle A(\widehat{\boldsymbol{w}}_1) - A(\widehat{\boldsymbol{w}}_2),\ \widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\rangle_{V^*\times V} = \int_{\Gamma_C} (\widehat{z}_2 - \widehat{z}_1)((\widehat{w}_1)_\nu - (\widehat{w}_2)_\nu)\, dS$$
$$+ \int_{\Gamma_C} \left(\widehat{z}_2\widehat{\boldsymbol{\xi}}_2 - \widehat{z}_2\widehat{\boldsymbol{\xi}}_1 + \widehat{z}_2\widehat{\boldsymbol{\xi}}_1 - \widehat{z}_1\widehat{\boldsymbol{\xi}}_1\right) \cdot ((\widehat{\boldsymbol{w}}_1)_\tau - (\widehat{\boldsymbol{w}}_2)_\tau)\, dS$$
$$+ \langle \widehat{\boldsymbol{f}}_1 - \widehat{\boldsymbol{f}}_2,\ \widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\rangle_{V^*\times V} + \langle \widehat{\boldsymbol{\eta}}_2 - \widehat{\boldsymbol{\eta}}_1,\ \boldsymbol{\varepsilon}(\widehat{\boldsymbol{w}}_1) - \boldsymbol{\varepsilon}(\widehat{\boldsymbol{w}}_2)\rangle_{L^2(\Omega;\mathbb{S}^d)}.$$

Using hypotheses $H(\mathscr{A})(d)$, $H(p)$, $H(j)(c)$-(d), we obtain

$$m_{\mathscr{A}}\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V^2 \leqslant \|\widehat{z}_1 - \widehat{z}_2\|_{L^2(\Gamma_C)}\|\gamma_{\Gamma_C}\|\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V$$
$$+ c_\tau\|\widehat{z}_1 - \widehat{z}_2\|_{L^2(\Gamma_C)}\|\gamma_{\Gamma_C}\|\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V + c_p L_\tau\|\gamma_{\Gamma_C}\|^2\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V^2$$
$$+ \|\widehat{\boldsymbol{f}}_1 - \widehat{\boldsymbol{f}}_2\|_{V^*}\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V + \|\widehat{\boldsymbol{\eta}}_1 - \widehat{\boldsymbol{\eta}}_2\|_{L^2(\Omega;\mathbb{S}^d)}\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V.$$

Using hypothesis $H_0(f)$, we get

$$\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V \leqslant c\left(\|\widehat{\boldsymbol{z}}_1 - \widehat{\boldsymbol{z}}_2\|_{L^2(\Gamma_C)} + \|\widehat{\boldsymbol{f}}_1 - \widehat{\boldsymbol{f}}_2\|_{V^*} + \|\widehat{\boldsymbol{\eta}}_1 - \widehat{\boldsymbol{\eta}}_2\|_{L^2(\Omega;\mathbb{S}^d)}\right)$$

From the continuity of $\boldsymbol{z}_\mu$, $\boldsymbol{f}$, $\boldsymbol{\eta}$ in appropriate spaces, it follows that the function $[0, T] \ni t \to \boldsymbol{w}_{\eta\mu}(t) \in V$ is continuous. This completes the proof of the lemma. □

Lemma 4.3 *Under the hypotheses of Theorem 4.1, Problem $\boldsymbol{P}_\eta^2$ has a unique solution $\boldsymbol{w}_\eta$.*

Proof Let $\Lambda_\eta : C([0, T]; V) \to C([0, T]; V)$ be the operator defined by

$$\Lambda_\eta \boldsymbol{\mu} = \boldsymbol{w}_{\eta\mu} \quad \text{for all} \quad \boldsymbol{\mu} \in C([0, T]; V),$$

where $\boldsymbol{w}_{\eta\mu}$ is the unique solution to Problem $\mathbf{P}_{\eta\mu}$ (see Proposition 4.2). We show that Λ_η has a unique fixed point $\boldsymbol{\mu}^* \in C([0, T]; V)$. Let $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in C([0, T]; V)$ and $z_i = z_{\mu_i} = \mathscr{R}\boldsymbol{\mu}_i \in C([0, T]; L^2(\Gamma_C))$ for $i = 1, 2$. Let $\boldsymbol{w}_i = \boldsymbol{w}_{\eta\mu_i}$ be the solutions to Problem $\mathbf{P}_{\eta\mu}$ for $\boldsymbol{\mu} = \boldsymbol{\mu}_i$ $(i = 1, 2)$. Then,

$$\|(\Lambda_\eta\boldsymbol{\mu}_1)(t) - (\Lambda_\eta\boldsymbol{\mu}_2)(t)\|_V = \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V \quad \text{for all} \quad t \in [0, T]. \tag{4.16}$$

Subtracting equation (4.10) written for $\boldsymbol{w}_2(t)$ and $\boldsymbol{w}_1(t)$, and taking the test function $\boldsymbol{v} = \boldsymbol{w}_2(t) - \boldsymbol{w}_1(t)$, we get

$$\langle A\boldsymbol{w}_1(t) - A\boldsymbol{w}_2(t), \boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\rangle_{V^*\times V}$$
$$= \int_{\Gamma_C} (z_1(t) - z_2(t))((w_2)_\nu(t) - (w_1)_\nu(t))\, dS$$
$$+ \int_{\Gamma_C} (\boldsymbol{\xi}_2(t)z_2(t) - \boldsymbol{\xi}_1(t)z_2(t) + \boldsymbol{\xi}_1(t)z_2(t) - \boldsymbol{\xi}_1(t)z_1(t)) \cdot ((\boldsymbol{w}_1)_\tau(t) - (\boldsymbol{w}_2)_\tau(t))\, dS,$$

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with $\xi_1(t) \in S^2_{\partial j((\boldsymbol{w}_1)_\tau(t))}$ and $\xi_2(t) \in S^2_{\partial j((\boldsymbol{w}_2)_\tau(t))}$ for all $t \in [0, T]$. Using hypotheses $H(\mathscr{A})(d)$, $H(p)$, $H(j)$, we obtain

$$m_{\mathscr{A}} \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V^2 \leqslant \|z_1(t) - z_2(t)\|_{L^2(\Gamma_C)} \|\gamma_{\Gamma_C}\| \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V$$
$$+ c_\tau \|z_1(t) - z_2(t)\|_{L^2(\Gamma_C)} \|\gamma_{\Gamma_C}\| \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V$$
$$+ c_p L_\tau \|\gamma_{\Gamma_C}\|^2 \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V^2,$$

thus

$$(m_{\mathscr{A}} - c_p L_\tau \|\gamma_{\Gamma_C}\|^2) \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V \leqslant (1 + c_\tau) \|\gamma_{\Gamma_C}\| \|z_1(t) - z_2(t)\|_{L^2(\Gamma_C)}. \tag{4.17}$$

Next, using the hypothesis $H(p)$, we have

$$\begin{aligned} \|z_1(t) - z_2(t)\|_{L^2(\Gamma_C)} &= \|(\mathscr{R}\boldsymbol{\mu}_1)(t) - (\mathscr{R}\boldsymbol{\mu}_2)(t)\|_{L^2(\Gamma_C)} \\ &= \left\| p\left(\int_0^t (\mu_1)_\nu(s)\, ds + (u_0)_\nu - g \right) - p\left(\int_0^t (\mu_2)_\nu(s)\, ds + (u_0)_\nu - g \right) \right\|_{L^2(\Gamma_C)} \\ &\leqslant L_\nu \left\| \int_0^t |(\mu_1)_\nu(s) - (\mu_2)_\nu(s)|\, ds \right\|_{L^2(\Gamma_C)} \\ &\leqslant L_\nu \|\gamma_{\Gamma_C}\| \int_0^t \|\boldsymbol{\mu}_1(s) - \boldsymbol{\mu}_2(s)\|_V\, ds. \end{aligned} \tag{4.18}$$

It follows from (4.16)–(4.18) and hypothesis $H_0(f)$, that

$$\|(\Lambda_\eta \boldsymbol{\mu}_1)(t) - (\Lambda_\eta \boldsymbol{\mu}_2)(t)\|_V \leqslant c \int_0^t \|\boldsymbol{\mu}_1(s) - \boldsymbol{\mu}_2(s)\|_V\, ds \quad \text{for all} \quad t \in [0, T],$$

where $c > 0$ is a constant dependent only on the problem data. Theorem A.9 asserts that the operator Λ_η has a unique fixed point $\boldsymbol{\mu}^* \in C([0, T]; V)$. Thus, we have $z_{\mu^*}(t) = (\mathscr{R}\boldsymbol{\mu}^*)(t)$, $\boldsymbol{w}_{\eta\mu^*}(t) = \boldsymbol{\mu}^*(t)$ for all $t \in [0, T]$. Writing (4.12) with $\boldsymbol{\mu} = \boldsymbol{\mu}^*$ we conclude that $\boldsymbol{\mu}^*$ is a solution of Problem $\mathbf{P}^2_\eta$ for certain $\boldsymbol{\xi} : [0, T] \to L^2(\Gamma_C)^d$ with $\boldsymbol{\xi}(t) \in S^2_{\partial j(\boldsymbol{\mu}^*_\tau(t))}$ for all $t \in [0, T]$. Since every solution of Problem $\mathbf{P}^2_\eta$ is be a fixed point of Λ_η, by the uniqueness of this fixed point, the solution of Problem $\mathbf{P}^2_\eta$ is unique, which completes the proof of Lemma 4.3. □

Proposition 4.4 *Under the hypotheses of Theorem 4.1, Problem* $\boldsymbol{P}^1_\eta$ *admits a unique solution* $(\boldsymbol{u}_\eta, \boldsymbol{\sigma}_\eta)$ *for every* $\boldsymbol{\eta} \in C([0, T]; L^2(\Omega; \mathbb{S}^d))$.

Proof Let $\boldsymbol{w}_\eta \in C([0, T]; V)$ be the unique solution of Problem $\mathbf{P}^2_\eta$. Defining

$$\boldsymbol{\sigma}_\eta(t) = \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{w}_\eta(t))) + \boldsymbol{\eta}(t) \quad \text{for all} \quad t \in [0, T],$$

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{w}_\eta(s)\, ds + \boldsymbol{u}_0 \quad \text{for all} \quad t \in [0, T],$$

it follows that $(\boldsymbol{u}_\eta, \boldsymbol{\sigma}_\eta)$ is the unique solution to Problem $\mathbf{P}^1_\eta$. This completes the proof of Proposition 4.4. □

Now, we establish the existence and uniqueness of solution to Problem $\mathbf{P}_\theta$ (with the fixed damage source function θ).

Proposition 4.5 *Under the hypotheses of Theorem 4.1, for every $\theta \in C([0,T];L^2(\Omega))$ and $\beta_0 \in \mathscr{K}$, Problem $\boldsymbol{P}_\theta$ admits a unique solution β_θ.*

Proof The proof follows from standard results for parabolic variational inequalities (see e.g., Barbu [3, p. 124]). □

It remains to prove Theorem 4.1.

Proof of Theorem 4.1 Let

$$\Lambda : C([0,T];L^2(\Omega;\mathbb{S}^d) \times L^2(\Omega)) \to C([0,T];L^2(\Omega;\mathbb{S}^d) \times L^2(\Omega))$$

be the operator given by

$$\Lambda(\boldsymbol{\eta},\theta) \;=\; \big(\mathscr{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta),\beta_\theta),\; \phi(\cdot,\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta),\beta_\theta)\big) \;\text{ for all }\; (\boldsymbol{\eta},\theta) \in C([0,T];L^2(\Omega;\mathbb{S}^d) \times L^2(\Omega))$$

where $\boldsymbol{u}_\eta$ is the unique solution of Problem $\mathbf{P}^1_\eta$ with the corresponding selection $\xi_\eta(t) \in S^2_{(\partial j(\dot{\boldsymbol{u}}_\eta)_\tau(t))}$ (Proposition 4.4) and β_θ is the unique solution to Problem $\mathbf{P}_\theta$ (Proposition 4.5). Note that Λ is well defined because by hypothesis $H(\mathscr{G})(b)$ we have $\mathscr{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta),\beta_\theta) \in C([0,T];L^2(\Omega;\mathbb{S}^d))$ and by hypotheses $H(\phi)(b)$ and (d) we have $\phi(\cdot,\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta),\beta_\theta) \in C([0,T];L^2(\Omega))$. We show that operator Λ has a unique fixed point $(\boldsymbol{\eta}^*,\beta^*) \in C([0,T];L^2(\Omega;\mathbb{S}^d) \times L^2(\Omega))$. To this end let $(\boldsymbol{\eta}_1,\beta_1),(\boldsymbol{\eta}_2,\beta_2) \in C([0,T];L^2(\Omega;\mathbb{S}^d) \times L^2(\Omega))$. We denote $\boldsymbol{u}_i = \boldsymbol{u}_{\eta_i}$, $\boldsymbol{w}_i = \dot{\boldsymbol{u}}_{\eta_i}$, $\xi_i = \xi_{\eta_i}$ $\beta_i = \beta_{\theta_i}$ for $i = 1,2$. Using hypotheses $H(\mathscr{G})(b)$ and $H(\phi)(b)$, we deduce that for all $t \in [0,T]$,

$$\begin{aligned}
&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{L^2(\Omega;\mathbb{S}^d)\times L^2(\Omega)} \\
&\quad= \|\mathscr{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t)),\beta_1(t)) - \mathscr{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_2(t)),\beta_2(t))\|_{L^2(\Omega;\mathbb{S}^d)} \\
&\qquad+\|\phi(t,\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t)),\beta_1(t)) - \phi(t,\boldsymbol{\varepsilon}(\boldsymbol{u}_2(t)),\beta_2(t))\|_{L^2(\Omega)} \\
&\quad\leqslant (L_{\mathscr{G}} + L_\phi)\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2(t))\|_{L^2(\Omega;\mathbb{S}^d)} + \|\beta_1(t) - \beta_2(t)\|_{L^2(\Omega)}\big) \\
&\quad= (L_{\mathscr{G}} + L_\phi)\big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \|\beta_1(t) - \beta_2(t)\|_{L^2(\Omega)}\big).
\end{aligned} \tag{4.19}$$

Since $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) = \boldsymbol{u}_0$, using (4.8), we get

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \;\leqslant\; \int_0^t \|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V\, ds \quad\text{for all}\quad t \in [0,T]. \tag{4.20}$$

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For $s \in [0,t]$, subtracting (4.5) for $\boldsymbol{w}_1(s)$ and $\boldsymbol{w}_2(s)$, and taking the test function $\boldsymbol{v} = \boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)$, we get

$$\begin{aligned}
&(\mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{w}_1(s))) - \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{w}_2(s))),\ \boldsymbol{\varepsilon}(\boldsymbol{w}_1(s)) - \boldsymbol{\varepsilon}(\boldsymbol{w}_2(s)))_{L^2(\Omega;\mathbb{S}^d)} \\
&\quad = \int_{\Gamma_C} (p((u_2)_\nu(s) - g) - p((u_1)_\nu(s) - g))\,((w_1)_\nu(s) - (w_2)_\nu(s))\,dS \\
&\qquad + \int_{\Gamma_C} \big(p((u_2)_\nu(t) - g)\boldsymbol{\xi}_2(t) - p((u_1)_\nu(t) - g)\boldsymbol{\xi}_2(t) \\
&\qquad\qquad + p((u_1)_\nu(t) - g)\boldsymbol{\xi}_2(t) - p((u_1)_\nu(t) - g)\boldsymbol{\xi}_1(t)\big) \cdot \big((\boldsymbol{w}_1)_\tau(s) - (\boldsymbol{w}_2)_\tau(s)\big)\,dS \\
&\qquad + (\boldsymbol{\eta}_2(s) - \boldsymbol{\eta}_1(s),\ \boldsymbol{\varepsilon}(\boldsymbol{w}_1(s)) - \boldsymbol{\varepsilon}(\boldsymbol{w}_2(s)))_{L^2(\Omega;\mathbb{S}^d)}.
\end{aligned}$$

Using assumptions $H(\mathscr{A})(d)$, $H(p)$, and $H(j)$, we obtain

$$\begin{aligned}
&(m_{\mathscr{A}} - c_p L_\tau \|\gamma_{\Gamma_C}\|^2)\|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V \\
&\qquad \leqslant L_\nu(1 + c_\tau)\|\gamma_{\Gamma_C}\|^2 \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V + \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{L^2(\Omega;\mathbb{S}^d)},
\end{aligned} \tag{4.21}$$

and hypothesis $H_0(f)$ implies

$$\|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V \leqslant c\big(\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V + \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{L^2(\Omega;\mathbb{S}^d)}\big) \tag{4.22}$$

for all $s \in [0,t]$. From (4.20), (4.22) and the Gronwall inequality, we obtain

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leqslant c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{L^2(\Omega;\mathbb{S}^d)}\,ds \quad \text{for all} \quad t \in [0,T]. \tag{4.23}$$

Writing the inequality (4.1) for $\beta_1(s)$ with $\zeta = \beta_2(s)$, then for $\beta_2(s)$ with $\zeta = \beta_1(s)$ and adding the resulting inequalities, we obtain

$$\begin{aligned}
&(\dot\beta_1(s) - \dot\beta_2(s), \beta_1(s) - \beta_2(s))_{L^2(\Omega)} + \kappa(\nabla\beta_1(s) - \nabla\beta_2(s), \nabla\beta_1(s) - \nabla\beta_2(s))_{L^2(\Omega)^d} \\
&\qquad \leqslant (\theta_1(s) - \theta_2(s), \beta_1(s) - \beta_2(s))_{L^2(\Omega)} \quad \text{for all} \quad s \in [0,T].
\end{aligned}$$

Integrating this inequality over $(0,t)$ for $t \in (0,T)$ and using integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\beta_1(0) - \beta_2(0)\|^2_{L^2(\Omega)} + \kappa \int_0^t \|\nabla\beta_1(s) - \nabla\beta_2(s)\|^2_{L^2(\Omega)^d}\,ds \\
&\qquad \leqslant \int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}\|\beta_1(s) - \beta_2(s)\|_{L^2(\Omega)}\,ds \quad \text{for all} \quad t \in [0,T].
\end{aligned}$$

Since $\beta_1(0) = \beta_2(0) = \beta_0$, we get

$$\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Omega)} \leqslant \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \|\beta_1(s) - \beta_2(s)\|^2_{L^2(\Omega)}\,ds,$$

for all $t \in (0,T)$. Using the Gronwall inequality yields

$$\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Omega)} \leqslant c \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)}\,ds \quad \text{for all} \quad t \in [0,T]. \tag{4.24}$$

Applying (4.23) and (4.24) in (4.19), we obtain

$$\begin{aligned}
&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{L^2(\Omega;\mathbb{S}^d)\times L^2(\Omega)}\\
&\quad = c\int_0^t \left(\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|^2_{L^2(\Omega;\mathbb{S}^d)}+\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\right)ds\\
&\quad \leqslant c\int_0^t \|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|^2_{L^2(\Omega;\mathbb{S}^d)\times L^2(\Omega)}\,ds.
\end{aligned}$$

It follows from Theorem A.9 that Λ has a unique fixed point $(\boldsymbol{\eta}^*,\theta^*)$.

We now establish the existence of a solution to Problem $\mathbf{P_V}$. Let $(\boldsymbol{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*})$ be the solution of Problem $\mathbf{P}^1_\eta$ for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ (Proposition 4.4) and let β_{θ^*} be the solution of Problem $\mathbf{P}_\theta$ for $\theta=\theta^*$ (Proposition 4.5). From the definition of Λ, we have that

$$\boldsymbol{\eta}^* = \mathscr{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta^*}),\beta_{\theta^*}) \quad\text{and}\quad \theta^* = \phi(\cdot,\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta^*}),\beta_{\theta^*}),$$

therefore, $(\boldsymbol{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*},\beta_{\theta^*})$ is a solution of Problem $\mathbf{P_V}$. The uniqueness of solution for problem $\mathbf{P_V}$ follows, exactly as in Theorem 5.1 in [12] from the uniqueness of fixed point of Λ, the fact that if $(\boldsymbol{u},\boldsymbol{\sigma},\beta)$ is a solution of Problem $\mathbf{P_V}$, then $(\boldsymbol{\eta},\theta) = (\mathscr{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\beta),\phi(\cdot,\boldsymbol{\varepsilon}(\boldsymbol{u}),\beta))$ is a fixed point of Λ, and the uniqueness of solutions to Problems $\mathbf{P}^1_\eta$ (Proposition 4.4) and $\mathbf{P}_\theta$ (Proposition 4.5). The theorem is proved. □

5 Conclusions

In this paper, we formulate a new quasi-static model of contact between a Kelvin–Voigt viscoelastic body and a penetrable foundation. The normal compliance law is used for the normal contract, while a generalized non-monotone version of the Coulomb law is used to model friction. The equation for the displacement is mutually coupled with the variational inequality describing the internal damage which takes place in the body. Using the reasoning based on the Banach fixed point argument, we prove the main result of this paper, Theorem 4.1, which establishes the existence and uniqueness of a weak solution. The paper generalizes on one hand the results of Gasiński, Ochal, and Shillor [12], where the simpler Tresca model is used instead of the Coulomb model, and on the other hand the results of Han, Shillor and Sofonea [13] as the friction in our case can depend on the slip rate.

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Appendix A

Here, we provide several definitions, notation, and results needed in the article. If X is a reflexive Banach space, we denote by X^* its topological dual and $\langle \cdot, \cdot \rangle_{X^* \times X}$ denotes the duality pairing of X and X^*. A mapping $A : X \to X^*$ is called *bounded* if A maps bounded sets of X into bounded sets of X^*. It is called *monotone* if $\langle Au - Az, u - z \rangle_{X^* \times X} \geqslant 0$ for all $u, z \in X$. Moreover, the operator $A : X \to X^*$ is called to be *pseudo-monotone* if it is bounded and if $u_n \to u$ weakly in X and $\limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle_{X^* \times X} \leqslant 0$ imply

$$\langle Au, u - v \rangle_{X^* \times X} \leqslant \liminf_{n \to +\infty} \langle Au_n, u_n - v \rangle_{X^* \times X} \quad \text{for all} \quad v \in X.$$

Equivalently, a mapping $A : X \to X^*$ is pseudo-monotone if and only if it is bounded and if $u_n \to u$ weakly in X and $\limsup_{n\to+\infty} \langle Au_n, u_n - u\rangle_{X^*\times X} \leqslant 0$ imply $\lim_{n\to+\infty} \langle Au_n, u_n - u\rangle_{X^*\times X} = 0$ and $Au_n \to Au$ weakly in X^*.

Definition A.1 *Let X be a reflexive Banach space and $A : D(A) \subset X \to 2^{X^*}$ be a multi-valued operator. We say that the operator A is* monotone *if $\langle u^* - v^*, u - v\rangle_{X^*\times X} \geqslant 0$ for all $u, v \in D(A)$, $u^* \in Au$, $v^* \in Av$.*

Definition A.2 *Let X be a reflexive Banach space. We say that a multi-valued operator $A : X \to 2^{X^*}$ is* pseudo-monotone *if*

(a) *for every $u \in X$, the set $Au \subset X^*$ is non-empty, closed and convex;*

(b) *A is upper semi-continuous from each finite dimensional subspace of X into X^* endowed with its weak topology,*

(c) *for every sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \to u$ weakly in X, $u_n^* \in Au_n$ for all $n \geqslant 1$, and $\limsup_{n\to+\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \leqslant 0$, we have that for every $v \in X$, there exists $u^*(v) \in Au$ such that*

$$\langle u^*(v), u - v\rangle_{X^*\times X} \leqslant \liminf_{n\to+\infty} \langle u_n^*, u_n - v\rangle_{X^*\times X}.$$

Definition A.3 *Let X be a reflexive Banach space. A multi-valued operator $A : X \to 2^{X^*}$ is* generalized pseudo-monotone *if for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \to u$ weakly in X, $u_n^* \in Au_n$ for $n \geqslant 1$, $u_n^* \to u^*$ weakly in X^* and $\limsup_{n\to+\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \leqslant 0$, we have $u^* \in Au$ and*

$$\lim_{n\to+\infty} \langle u_n^*, u_n\rangle_{X^*\times X} = \langle u^*, u\rangle_{X^*\times X}.$$

The following result relates the notions of pseudo-monotonicity and generalized pseudo-monotonicity (cf. [8, Proposition 1.3.66, p. 58–59]).

Proposition A.4 *Let X be a reflexive Banach space and $A : X \to 2^{X^*}$ a bounded generalized pseudo-monotone operator. If for each $u \in X$, Au is a non-empty, closed and convex subset of X^*, then A is pseudo-monotone.*

The following surjectivity can be found in [8, Theorem 1.3.70].

Theorem A.5 *If X is a reflexive Banach space, $T : X \to 2^{X^*}$ is a multi-valued pseudo--monotone operator which is coercive in the following sense:*

$$\langle u^*, u\rangle_{X^*\times X} \;\geqslant\; c(\|u\|_X)\|u\|_X \quad \text{for all} \quad u \in D(T) \quad \text{and} \quad u^* \in T(u), \tag{A 1}$$

where $c : \mathbb{R}_+ \to \mathbb{R}$ is a function such that $c(r) \to +\infty$ as $r \to +\infty$, then T is surjective.

Let us recall some basic tools from convex analysis and non-smooth analysis, cf. Clarke [6].

Definition A.6 *Let X be a Banach space and let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of φ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x;v)$, is defined by*

$$\varphi^0(x;v) = \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{\varphi(y+tv) - \varphi(y)}{t}$$

and the generalized gradient (sub-differential) of φ at x, denoted by $\partial\varphi(x)$, is a subset of a dual space X^* given by

$$\partial\varphi(x) = \{\, \zeta \in X^* : \varphi^0(x;v) \geqslant \langle \zeta, v\rangle_{X^* \times X} \text{ for all } v \in X \,\}.$$

Proposition A.7 *If $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function on a Banach space X, then for every $x \in X$ the generalized gradient $\partial\varphi(x)$ is a non-empty, convex, and weak* compact subset of X^*, and the graph of the generalized gradient $\partial\varphi$ is sequentially closed in $X \times (w^*\text{–}X^*)$-topology, i.e., if $\{x_n\} \subset X$ and $\{\zeta_n\} \subset X^*$ are sequences such that $\zeta_n \in \partial\varphi(x_n)$ and $x_n \to x$ in X, $\zeta_n \to \zeta$ weak* in X^*, then $\zeta \in \partial\varphi(x)$.*

Given a convex, lower semi-continuous (l.s.c.) function $\varphi : X \to (-\infty, +\infty]$ on a Banach space, we recall that φ is proper if it is not identically $+\infty$. The effective domain of φ is denoted by $\mathrm{dom}\,\varphi = \{x \in X : \varphi(x) < +\infty\}$.

Definition A.8 *Let X be a Banach space and let $\varphi : X \to (-\infty, +\infty]$ be a proper, l.s.c. and convex function. The mapping $\partial\varphi : X \to 2^{X^*}$ defined by*

$$\partial\varphi(x) = \{x^* \in X^* \;:\; \langle x^*, v - x\rangle_{X^* \times X} \leqslant \varphi(v) - \varphi(x) \text{ for all } v \in X\}$$

for $x \in X$ with $\varphi(x) < +\infty$ and by $\partial\varphi(x) = \emptyset$ for $x \in X$ with $\varphi(x) = +\infty$, is called the sub-differential *of φ. An element $x^* \in \partial\varphi(x)$ (if any) is called a* sub-gradient *of φ at x.*

The proof of the next fixed point theorem, which is be needed in the sequel, is similar to that presented in Migórski-Ochal-Sofonea [17, pp. 107–108].

Theorem A.9 *If X is a Banach space and $\Lambda : C([0,T];X) \to C([0,T];X)$ is an operator for which there exist $k \in \mathbb{N}_+$ and $c > 0$ such that*

$$\|(\Lambda u)(t) - (\Lambda v)(t)\|_X^k \;\leqslant\; c \int_0^t \|u(s) - v(s)\|_X^k ds \quad \text{for all } u, v \in C([0,T];X), t \in [0,T],$$

then Λ has a unique fixed point in $C([0,T];X)$.

We use the notion of Kuratowski–Painlevé upper limit of a sequence of sets $A_n \subset E$, where (E, τ) is a Hausdorff topological space, given by

$$\tau\text{-}\limsup_{n\to\infty} A_n = \{\, x \in \mathbb{R}^m \,:\, x = \lim_{k\to\infty} x_{n_k}, x_{n_k} \in A_{n_k}, 0 < n_1 < n_2 < \ldots < n_k < \ldots \}.$$

We recall the following result on point-wise behaviour of weakly convergence sequences (see Proposition 3.6 in [17])

Proposition A.10 *Let (Γ, Σ, μ) be a σ-finite measure space, and let $d \geqslant 1$ be a natural number. Let $u_n \to u$ weakly in $L^2(\Gamma)^d$ and $u_n(x) \in G(x)$ for μ-a.e. $x \in \Gamma$ and all $n \in \mathbb{N}$, where $G(x)$ is a non-empty, bounded set for μ-a.e. $x \in \Gamma$, then*

$$u(x) \in \overline{\text{conv}} \left(\limsup_{n\to\infty} \{u_n(x)\} \right),$$

where $\limsup_{n\to\infty}$ stands for the Kuratowski–Painlevé upper limit of sets.

The indices i and j always run between 1 and d and the summation convention over repeated indices is used. Also, an index following a comma indicates a partial derivative.

In $\mathbb{R}^d$ by $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$, we denote the inner product and by $\|\boldsymbol{v}\|_{\mathbb{R}^d} = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ the euclidean norm. On $\mathbb{S}^d$, we use the inner product $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma}_{ij} \boldsymbol{\tau}_{ij}$ and the associated norm $\|\boldsymbol{\tau}\|^2_{\mathbb{S}^d} = \boldsymbol{\tau} \cdot \boldsymbol{\tau}$.

On the Hilbert space $L^2(\Omega; \mathbb{S}^d)$, we consider the scalar product $(\boldsymbol{\sigma}, \boldsymbol{\tau})_{L^2(\Omega;\mathbb{S}^d)} = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, d\boldsymbol{x}$ and the associated norm $\|\cdot\|_{L^2(\Omega;\mathbb{S}^d)}$. Defining the *deformation* operator $\boldsymbol{\varepsilon} \colon H^1(\Omega)^d \to L^2(\Omega; \mathbb{S}^d)$ by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}}),$$

we can equip the Hilbert space $H^1(\Omega)^d$ with the scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_{H^1(\Omega)^d} = (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)}.$$

The associated norm $\|\cdot\|_{H^1(\Omega)^d}$ is equivalent to the standard H^1 norm. Next, we introduce the space $\mathscr{H}_1 = \{\boldsymbol{\tau} \in L^2(\Omega; \mathbb{S}^d) \,:\, (\tau_{ij,j}) \in L^2(\Omega)^d\}$, which is a Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathscr{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{L^2(\Omega;\mathbb{S}^d)} + (\text{Div}\, \boldsymbol{\sigma}, \text{Div}\, \boldsymbol{\tau})_{L^2(\Omega)^d},$$

and the respective norm $\|\cdot\|_{\mathscr{H}_1}$. Here, $\text{Div} \colon \mathscr{H}_1 \to L^2(\Omega)^d$ is the *divergence* operator defined by $\text{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j})$. For an element $\boldsymbol{v} \in H^1(\Omega)^d$, we denote by $\boldsymbol{v}$ its trace on $\partial\Omega$ and by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ its *normal* and *tangential* components on the boundary. For an element $\boldsymbol{\sigma} \in \mathscr{H}_1$, σ_ν and $\boldsymbol{\sigma}_\tau$ denote the *normal* and *tangential* traces of $\boldsymbol{\sigma}$, namely

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

The following Green formula holds

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)} + (\text{Div}\, \boldsymbol{\sigma}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, dS, \tag{A 2}$$

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for all $\boldsymbol{v} \in H^1(\Omega)^d$ and $\boldsymbol{\sigma} \in \mathscr{H}_1$. By V we denote the closed subspace of $H^1(\Omega)^d$, given by

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega)^d \ : \ \boldsymbol{v} = \boldsymbol{0} \ \text{ a.e. on } \Gamma_D \right\}.$$

Since $\text{meas}_{d-1}(\Gamma_D) > 0$ and $\partial\Omega$ is Lipschitz, the Korn inequality holds

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \geqslant c\|\boldsymbol{v}\|_{H^1(\Omega)^d} \quad \text{for all} \quad \boldsymbol{v} \in V, \tag{A 3}$$

where, here and below, c represents a positive constant, which may change from line to line and may depend on the data. We define the norm on V by

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \quad \text{for all} \quad \boldsymbol{u} \in V. \tag{A 4}$$

It follows that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V. Moreover, for a measurable set $\Gamma \subset \partial\Omega$ we denote by $\gamma_\Gamma : V \to L^2(\Gamma)^d$ the trace operator, by $\|\gamma_\Gamma\|$ its norm in $\mathscr{L}(V; L^2(\Gamma)^d)$ and by $\gamma_\Gamma^* : L^2(\Gamma)^d \to V^*$ the adjoint operator to γ_Γ.

Finally, for a real Hilbert space $(X, \|\cdot\|_X)$ we use the standard notation for Bochner–Lebesgue spaces $L^p(0, T; X)$ (with $1 \leqslant p \leqslant +\infty$), Bochner–Sobolev spaces $H^k(0, T; X)$ (with $k \in \mathbb{N}$), space of vector-valued continuous functions $C([0, T]; X)$ and space of vector-valued continuously differentiable functions $C^1([0, T]; X)$. Moreover, if X_1 and X_2 are two real Hilbert spaces then $X_1 \times X_2$ denotes the product space endowed with the canonical inner product $\langle\cdot,\cdot\rangle_{X_1\times X_2}$ and norm $\|\cdot\|_{X_1\times X_2}$.