CANONICAL CONNECTIONS AND PONTRJAGIN CLASSES

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In the previous paper [7], we have studied the relationship between the Riemannian connection of an n-dimensional Riemannian space M imbedded into the $(n+k)$-dimensional Euclidean space R^{n+k} and the canonical connection in the bundle $P_{n,k} = O(n+k)/\{1\} \times O(k)$ over the Grassmann manifold $M_{n,k} = O(n+k)/O(n) \times O(k)$.

In the first half of the present paper, the relationship between the canonical connections in bundles $P_{n,k}$, $P^*_{n,k} = O(n+k)/O(n) \times \{1\}$, $O(n)$ over $M_{n,k}$ and the invariant Riemannian connection on $M_{n,k}$ will be discussed. We obtain the holonomy groups of these canonical connections.

In the second half of the paper, we shall study the Pontrjagin classes of manifolds, using exclusively differential forms. To facilitate the calculation, we introduce two types of characteristic cocycles which are closely related to the Pontrjagin cocycles. The duality theorem for the Pontrjagin classes, which has been proved by Wu Wen-Tsun using the cellular subdivision of the Grassmann manifold $M_{n,k}$ [13], [4], [5], is proved here very easily using the theory of symmetric functions. Our result gives a little bit more precise informations than that of Wu Wen-Tsun, in the sense that we express the duality theorem as a relation between the Pontrjagin *cocycles* (in stead of *classes*) and the normal Pontrjagin *cocycles*. This may not be interesting for topologists, but may have some value from the differential geometrical point of view. We prove also that the normal Pontrjagin *cocycles* of a Riemannian space depend only on the Riemannian connection, not on the way how to imbed it. Finally we show that the normal Pontrjagin *classes* of a manifold M depend only on the differentiable structure of M.

The second half of the paper (§4–§8) can be read independently of the

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first half.

§ 1. Bundles associated with homogeneous spaces

Let G be a Lie group acting on a manifold M on the left as a differentiable transformation group and let H be the isotropic subgroup of G at $x_0 \in M$. Clearly H is a closed subgroup of G, hence a Lie group. Then M is identified with G/H in a natural fashion. Since H acts on G on the right naturally, G is a principal fibre bundle over $G/H = M$ with group H and with the natural projection $p : G \to G/H$. We shall study the relation between this bundle G over M and the bundle of frames over M.

Every element g of G induces a transformation δg of $T(M)$, which maps $T_{x_0}(M)$ onto $T_{gx_0}(M)$ isomorphically. We take $T_{x_0}(M)$ as a standard n-dimensional vector space R^n ($n = \dim M$). Then $\delta g : R^n \to T_{gx_0}(M)$ is a frame at $gx_0 \in M$, which we shall denote by u_g. Let P be the bundle of frames over M (with group $GL(n, R)$) with projection π. Then we have the following commutative diagram.

$$p(g) = \pi(u_g) \qquad \begin{array}{ccc} G & \xrightarrow{h} & P \\ p\downarrow & & \downarrow\pi \\ M & \xrightarrow{1} & M \end{array}$$

where $h(g) = u_g$.

The following proposition is well known.

PROPOSITION 1. *h is a one-to-one mapping of G into P, if G acts effectively on M and if there exists an affine connection on M which is invariant by G.*

Under the assumption of Prop. 1, we can consider G as the bundle obtained from P by reducing the structure group $GL(n, R)$ to H. And an infinitesimal connection in the bundle G can be considered as an affine connection on M [11].

Consider now a slightly more general case: the case where G is *almost effective*, i.e., the subgroup N consisting of all elements in G which leave M fixed pointwise is discrete. Note that N is a normal subgroup of G contained in H. Now, G/N is a principal fibre bundle over M with group H/N, whose projection will be denoted by p'. The natural map ψ of G onto G/N is a bundle homomorphism with discrete kernel, which may be called a semi-isomorphism. The semi-isomorphism ψ induces a one-one correspondence between the set of connections in G and the set of connections in G/N [8]. Explicitly, to each connection form ω on G/N, there corresponds the connection form $\psi^*(\omega)$ on G.

Now suppose that G is almost effective on M and that there exists an affine connection on M invariant by G. From Prop. 1 and the above argument, it follows that every connection in the bundle G can be considered as an affine connection on M.

We shall apply the above procedure to the case where the orthogonal group $O(n+k)$ in $n+k$ variables acts on the Grassmann manifold $M_{n,k}=O(n+k)/O(n)\times O(k)$ in a natural manner. It is easy to see that $O(n+k)$ is almost effective on $M_{n,k}$ (only I and $-I$ act trivially on $M_{n,k}$). Since the isotropic subgroup $O(n)\times O(k)$ is compact, there exists an invariant affine connection on $M_{n,k}$ [10]. Therefore we can consider every connection in the bundle $O(n+k)$ over $M_{n,k}$ as an affine connection on $M_{n,k}$. We shall define the canonical connection in the bundle $O(n+k)$ which corresponds to the invariant Riemannian connection on $M_{n,k}$.

Let $\mathfrak{o}(n+k)$, $\mathfrak{o}(n)$ and $\mathfrak{o}(k)$ be the Lie algebras of $O(n+k)$, $O(n)$ and $O(k)$ respectively. Let $\mathfrak{m}_{n,k}$ be the orthogonal complement to the subspace $\mathfrak{o}(n)+\mathfrak{o}(k)$ in $\mathfrak{o}(n+k)$ with respect to the Killing-Cartan bilinear form on $\mathfrak{o}(n+k)$. Then

$$\mathfrak{o}(n+k)=\mathfrak{o}(n)+\mathfrak{o}(k)+\mathfrak{m}_{n,k}$$
$$ad(s)\cdot\mathfrak{m}_{n,k}\subseteqq\mathfrak{m}_{n,k} \qquad \text{for all } s\in O(n)\times O(k)$$
$$[\mathfrak{m}_{n,k},\ \mathfrak{m}_{n,k}]\subseteqq\mathfrak{o}(n)+\mathfrak{o}(k).$$

Observe that the first two conditions say that $M_{n,k}$ is reductive in the sense of Nomizu [10] and the last one tells that it is moreover symmetric in the sense of E. Cartan [2], [10].

Let θ be the $\mathfrak{o}(n+k)$-valued left invariant linear differential form on $O(n+k)$ defined by

$$\theta(\bar{s})=s^{-1}\bar{s} \qquad \bar{s}\in T_s(O(n+k)).$$

Let ω, η and ξ be the $\mathfrak{o}(n)$-, $\mathfrak{o}(k)$- and $\mathfrak{m}_{n,k}$-components of the form θ respectively:

$$\theta=\omega+\eta+\zeta$$

It is easy to see that the form $\omega+\eta$ defines a connection in the bundle $O(n+k)$

over $M_{n,k}$, which we shall call the *canonical connection in the principal fibre bundle* $O(n+k)$.

From the fact that $M_{n,k}$ is a symmetric space, it follows easily that the canonical connection in $O(n+k)$ corresponds to the invariant affine connection of the 2nd kind [10] on $M_{n,k}$, which is nothing but the invariant Riemannian connection on $M_{n,k}$. In the next section, we shall give an explicit correspondence between the canonical connection in $O(n+k)$ and the invariant Riemannian connection on $M_{n,k}$.

§ 2. Invariant Riemannian connection on $M_{n,k}$

The Lie algebra $\mathfrak{o}(n+k)$ is the set of skew-symmetric matrices of degree $n+k$ and the spaces $\mathfrak{o}(n)$, $\mathfrak{o}(k)$ and $\mathfrak{m}_{n,k}$ are the sets of matrices of the following types respectively:

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}, \quad \begin{pmatrix} 0 & Z \\ -Z^t & 0 \end{pmatrix},$$

where X and Y are skew-symmetric matrices of degree n and k respectively and Z is a matrix of (n, k)-type.

Let x_0 be the point of $M_{n,k}$ which is left fixed by $O(n)\times O(k)$. Then the tangent space $T_{x_0}(M_{n,k})$ can be identified naturally with $\mathfrak{m}_{n,k}$, hence a matrix Z can be considered as an element of $T_{x_0}(M_{n,k})$ and conversely. Let s and s' be elements of $O(n)\times\{1\}$ and $\{1\}\times O(k)$ respectively. Then the linear transformation of $T_{x_0}(M_{n,k})$ induced by $s\times s'\in O(n)\times O(k)$ corresponds to the following matrix transformation of $\mathfrak{m}_{n,k}$:

$$\mathfrak{m}_{n,k}\ni z \longrightarrow (s\times s')\cdot z\cdot (s\times s')^{-1}\in \mathfrak{m}_{n,k}.$$

Let A and B be the matrices of degree n and k respectively which corresponds to s and s'. Then

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & Z \\ -Z^t & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} 0 & AZB^t \\ -(AZB^t)^t & 0 \end{pmatrix}.$$

Therefore the linear transformation $s\times s'$ of $T_{x_0}(M_{n,k})$ is expressed, in terms of matrices, as follows:

$$Z \longrightarrow \bar{Z} \equiv AZB^t.$$

If

$$Z = (z_r^i), \qquad \bar{Z} = (\bar{z}_r^i), \qquad A = (a_j^i), \qquad B = (b_r^q),$$

where $i, j = 1, \ldots, n$ and $q, r = 1, \ldots, k$, then

$$\bar{z}_r^i = \sum a_j^i b_b^r z_q^j.$$

Consider A and B as linear transformations of the vector spaces V_n and V_k with bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_k$ respectively and identify the tensor product $V_n \otimes V_k$ with the space of matrices Z as follows:

$$e_i \otimes f_r \longleftrightarrow Z_r^i,$$

where Z_r^i is the matrix (z_r^i) of (k, n)-type with

$$z_q^j = 1 \quad \textit{if} \quad j = i \quad \text{and} \quad q = r$$
$$z_q^j = 0 \quad \text{otherwise.}$$

Then the above transformation $Z \to AZB^t$ can be identified with the tensor product (the Kronecker product) of A and B. We have shown the

Proposition 2. *Let P be the bundle of frames over $M_{n,k}$ and h the natural semi-isomorphism of $O(n+k)$ into P (see §1). Let τ be the natural homomorphism of $O(n) \times O(k)$ onto $O(n) \otimes O(k)$; then we have the following commutative diagram:*

$$\begin{array}{ccc} O(n+k) \times (O(n) \times (O(k)) & \xrightarrow{\varphi} & O(n+k) \\ \downarrow h \times \tau & & \downarrow h \\ P \times GL(nk, R) & \xrightarrow{\rho} & P \end{array}$$

where φ is the group multiplication in $O(n+k)$ and ρ is the right multiplication by the structure group $GL(nk, R)$.

Let $P^* = h(O(n+k))$. Then P^* is a principal fibre bundle over $M_{n,k}$ with the structure group $O(n) \otimes O(k)$ and the above proposition can be stated as follows.

Proposition 2'. *A pair of mappings h and τ gives a semi-isomorphism of the bundle $O(n+k)$ onto the bundle P^*, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} O(n+k) \times (O(n) \times O(k)) & \xrightarrow{\varphi} & O(n+k) \\ \downarrow h \times \tau & & \downarrow h \\ P^* \times (O(n) \otimes O(k)) & \xrightarrow{\rho} & P^* \end{array}$$

Let ω^* (resp. η^*) be the $\mathfrak{o}(n)$-valued (resp. $\mathfrak{o}(k)$-valued) linear differential form on P^* corresponding to the form ω (resp. η) on $O(n+k)$ (see § 1). The forms ω^* and η^* can be considered as skew-symmetric matrix valued differential forms on P^*. Let I_n (resp. I_k) be the unit matrix of degree n (resp. k). Then $\omega^* \otimes I_k + I_n \otimes \eta^*$ is an $\mathfrak{o}(nk)$-valued linear differential form on P^*. From Prop. 2′, we derive the

PROPOSITION 3. *The following diagram is commutative*:

$$\begin{array}{ccc} T(O(n+k)) & \xrightarrow{\omega+\eta} & \mathfrak{o}(n)+\mathfrak{o}(k) \\ \downarrow \delta h & & \downarrow \delta\tau \\ T(P^*) & \xrightarrow{\omega^*\otimes I_k + I_n\otimes\eta^*} & \mathfrak{o}(n)\otimes I_k + I_n \otimes \mathfrak{o}(k)\,; \end{array}$$

that is, the form $\omega^* \otimes I_k + I_n \otimes \eta^*$ defines the connection in P^* which corresponds to the canonical connection in the bundle $O(n+k)$.

Remark. The differential $\delta\tau$ of the group homomorphism τ of $O(n)\times O(k)$ onto $O(n)\otimes O(k)$ induces an algebra isomorphism of $\mathfrak{o}(n)+\mathfrak{o}(k)$ onto $\mathfrak{o}(n)\otimes I_k + I_n \otimes \mathfrak{o}(k)$.

The above defined connection in P^* is evidently an invariant affine connection on $M_{n,k}$ whose homogeneous holonomy group is a subgroup of $O(n)\otimes O(k)$. Let ζ^* be the form on P^* corresponding to the $\mathfrak{m}_{n,k}$-valued form ζ on $O(n+k)$. Then the form of *soudure* on P [7], restricted on P^*, is ζ^* (under the identification of R^{nk} with $\mathfrak{m}_{n,k}$). Then the Maurer-Cartan equation on $O(n+k)$ gives

$$d\zeta^i_r = -\sum \omega^i_j \wedge \zeta^j_r - \sum \eta^r_q \wedge \zeta^i_q.$$

If we make use of matrix notations, this can be written as follows:

$$d\zeta^* = -(\omega^* \otimes I_k + I_n \otimes \eta^*)\zeta^*,$$

which shows that the connection in P^* has no torsion. In general, if an affine connection on a Riemannian manifold has no torsion and its homogeneous holonomy group is contained in the orthogonal group, then it is the Riemannian connection. Hence we have shown explicitly the

THEOREM 1. *The canonical connection in the principal fibre bundle $O(n+k)$ over $M_{n,k}$ corresponds naturally to the invariant Riemannian connection on $M_{n,k}$.*

§ 3. Product of two connections

Let P_1 and P_2 be principal fibre bundles over the same manifold M and with groups G_1 and G_2 respectively. The direct product $P_1 \times P_2$ can be considered in a natural manner as a principal fibre bundle over $M \times M$ with group $G_1 \times G_2$. The part of $P_1 \times P_2$ over the diagonal $\Delta(M \times M) \equiv \{(x, x) \in M \times M\}$ is a principal fibre bundle over M $(\cong \Delta(M \times M))$ with group $G_1 \times G_2$, which we shall denote by $P_1 \circ P_2$. Let ω_i be the $\mathfrak{g}_i$-valued linear differential form on P_i defining a connection in P_i for $i = 1, 2$, where $\mathfrak{g}_i$ is the Lie algebra of G_i. Then $\mu_1^*(\omega_1) + \mu_2^*(\omega_2)$ defines a connection in $P_1 \times P_2$, where μ_i is the natural projection of $P_1 \times P_2$ onto P_i. Its restriction on $P_1 \circ P_2$ defines a connection in $P_1 \circ P_2$, which we call the *product of the connections* ω_1 *and* ω_2.

Let c be a curve in M starting from x_0 and ending at x_1 and let u_i be any point of P_i such that $\pi_i(u_i) = x_0$, where π_i is the projection of P_i onto M. Let $c(u_i)$ be the point of P_i obtained by parallel displacement of u_i along c with respect to the connection defined by ω_i. Since (u_1, u_2) is a point of $P_1 \circ P_2$ over x_0, we can define similarly $c(u_1, u_2)$, i.e., the point of $P_1 \circ P_2$ obtained by parallel displacement of (u_1, u_2) along c with respect to the connection defined by the product of ω_1 and ω_2. Then we obtain easily

$$c(u_1, u_2) = (c(u_1), c(u_2)).$$

Suppose now that c is a closed curve starting from x_0. Then there exists a unique element $s_i \in G_i$ such that

$$c(u_i) = u_i s_i ;$$

it is by definition an element of the holonomy group with reference point u_i associated with the curve c. It follows evidently that

$$c(u_1, u_2) = (u_1, u_2)(s_1, s_2).$$

Hence

PROPOSITION 4. *Let* h *be the holonomy group with reference point* u_i *of the connection defined by* ω_i $(i = 1, 2)$ *and* h *the holonomy group with reference point* (u_1, u_2) *of the product of connections* ω_1 *and* ω_2. *Then*

(1) $h \subseteq h_1 \times h_2$;

(2) *The natural projection* $\varphi_i : h_1 \times h_2 \to h_i$ *maps* h *onto* h_i $(i = 1, 2)$.

Remark. h is not necessarily equal to $h_1 \times h_2$.

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Let $P_{n,k}$ (resp. $P^*_{n,k}$) denote $O(n+k)/\{1\}\times O(k)$ (resp. $O(n+k)/O(n)\times\{1\}$) which is a principal fibre bundle over $M_{n,k}$ with group $O(n)$ (resp. $O(k)$). We apply the above argument to the case where $P_1 = P_{n,k}$ and $P_2 = P^*_{n,k}$. We shall show that the principal fibre bundle $O(n+k)$ over $M_{n,k}$ is isomorphic to $P_{n,k}\circ P^*_{n,k}$. Let ν_1 (resp. ν_2) be the natural projection of $O(n+k)$ onto $P_{n,k}$ (resp. $P^*_{n,k}$). Then the map $\nu_1\times\nu_2$ of $O(n+k)$ into $P_{n,k}\times P^*_{n,k}$ defined by

$$(\nu_1\times\nu_2)(s) = (\nu_1(s),\ \nu_2(s)) \qquad s\in O(n+k)$$

will give an isomorphism of $O(n+k)$ onto $P_{n,k}\circ P^*_{n,k}$. Let π, π_1, π_2 be the projections of the bundles $O(n+k)$, $P_{n,k}$, $P^*_{n,k}$ onto $M_{n,k}$ respectively. Then

$$\pi_1\circ\nu_1(s) = \pi_2\circ\nu_2(s) = \pi(s) \qquad \text{for all } s\in O(n+k),$$

which shows that $\nu_1\times\nu_2$ maps $O(n+k)$ *into* $P_{n,k}\circ P^*_{n,k}$.

Suppose that $(\nu_1\times\nu_2)(s) = (\nu_1\times\nu_2)(s')$ for some s, $s'\in O(n+k)$. From $\nu_i(s) = \nu_i(s')$ $(i=1, 2)$, it follows that there exist $s_1\in O(n)\times\{1\}$ and $s_2\in\{1\}\times O(k)$ such that $s' = ss_1$ and $s' = ss_2$. Since $O(n)\times\{1\}$ and $\{1\}\times O(k)$ intersect only at the unit, we get that $s_1 = s_2 =$ the unit. Hence $s = s'$, which proves that $\nu_1\times\nu_2$ is a *one-to-one* map. Let u_1 and u_2 be arbitrary points of $P_{n,k}$ and $P^*_{n,k}$ such that $\pi_1(u_1) = \pi_2(u_2)$. Let s_1 and s_2 be elements of $O(n+k)$ such that $\nu_i(s_i) = u_i$ for $i=1, 2$. Then

$$\pi(s_1) = \pi_1\circ\nu_1(s_1) = \pi_2\circ\nu_2(s_2) = \pi(s_2).$$

Hence there exist elements $s_3\in O(n)\times\{1\}$ and $s_4\in\{1\}\times O(k)$ such that $s_1 = s_2s_3s_4$. Put $s = s_1s_4^{-1} = s_2s_3$. Then

$$\nu_1(s) = \nu_1(s_1s_4^{-1}) = \nu_1(s_1) = u_1,$$
$$\nu_2(s) = \nu_2(s_2s_3) = \nu_2(s_2) = u_2.$$

This completes the proof of the following

PROPOSITION 5. *The bundle $O(n+k)$ over $M_{n,k}$ is naturally isomorphic to the bundle $P_{n,k}\circ P^*_{n,k}$.*

We have shown in the previous paper [7] that the $\mathfrak{o}(n)$-component ω of the left invariant linear differential form $\theta = \omega+\eta+\zeta$ on $O(n+k)$ induces an $\mathfrak{o}(n)$-valued differential form on $P_{n,k}$, which we denote also by ω and which defines a connection (called the *canonical connection*) in the bundle $P_{n,k}$ over $M_{n,k}$. Similarly we can prove that the $\mathfrak{o}(k)$-component η of θ induces an $\mathfrak{o}(k)$-valued differential form on $P^*_{n,k}$, which we shall denote by the same letter η and which defines a connection in $P^*_{n,k}$ (by definition, the *canonical connection in* $P^*_{n,k}$). Then

PROPOSITION 6. *Under the natural isomorphism between the bundles* $O(n+k)$ *over* $M_{n,k}$ *and* $P_{n,k} \circ P^*_{n,k}$, *the canonical connection in* $O(n+k)$ *corresponds to the product of the canonical connections in* $P_{n,k}$ *and* $P^*_{n,k}$.

Now we shall study the holonomy group of the canonical connections in $O(n+k)$, $P_{n,k}$ and $P^*_{n,k}$. Because of Theorem 1, the study of the canonical connection in $O(n+k)$ can be reduced to that of the invariant Riemannian connection on $M_{n,k}$. Since $M_{n,k}$ is an irreducible symmetric space in the sense of E. Cartan, the *restricted* holonomy group of the invariant Riemannian connection on $M_{n,k}$ coincides with the connected component of the unit of the linear isotropic subgroup of the group of isometries [2], [10], in this case, $SO(n) \otimes SO(k)$.

In order to obtain the (non-restricted) holonomy group, we have to investigate the canonical connection more carefully. By the above argument and Theorem 1, the restricted holonomy group of the canonical connection in $O(n+k)$ is $SO(n) \times SO(k)$. Taking the unit of $O(n+k)$ as a reference point, we consider the set P_0 of all points in $O(n+k)$ which can be joined to the unit by horizontal curves [1], [8] with respect to the canonical connection in $O(n+k)$. The set P_0 is a principal fibre bundle over $M_{n,k}$ whose structure group is the holonomy group of the canonical connection [1], [8]. Since P_0 is arcwise connected, it is a submanifold of $SO(n+k)$. On the other hand, it is of the same dimension as $SO(n+k)$, because the connected component of the structure group of P_0 has the same dimension as $SO(n+k) \cap (O(n) \times O(k))$. As easily seen, P_0 cannot be a proper open submanifold of $SO(n+k)$. Hence P_0 coincides with $SO(n+k)$. The holonomy group of the canonical connection in $O(n+k)$ is, therefore, $SO(n+k) \cap (O(n) \times O(k))$. From Prop. 4 we obtain immediately the holonomy group of the canonical connection in $P_{n,k}$ (resp. $P^*_{n,k}$). Thus we have obtained the

THEOREM 2. *The holonomy groups of the canonical connections in* $O(n+k)$, $P_{n,k}$ *and* $P^*_{n,k}$ *are respectively* $SO(n+k) \cap (O(n) \times O(k))$, $O(n)$ *and* $O(k)$.

From the above theorem it follows that the holonomy group of the invariant

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Riemannian connection on $M_{n,k}$ is the image of $SO(n+k) \cap (O(n) \times O(k))$ under the natural isomorphism of $O(n) \times O(k)$ onto $O(n) \otimes O(k)$ and is decomposed into two connected components: one contains the unit and the other contains the following element:

$$\underset{((n,n)-\text{type})}{\begin{pmatrix} 1 & & 0 & \\ & \cdots & & \\ 0 & & 1 & \\ & & & -1 \end{pmatrix}} \otimes \underset{((k,k)-\text{type})}{\begin{pmatrix} 1 & & 0 & \\ & \cdots & & \\ 0 & & 1 & \\ & & & -1 \end{pmatrix}},$$

whose determinant is obviously $(-1)^{n+k}$. As it is well known, $M_{n,k}$ is orientable if and only if $n+k$ is even.

§ 4. Curvature forms of canonical connections

From the Maurer-Cartan equation of $\theta = \omega + \eta + \zeta$, we obtain

$$d\omega = -\frac{1}{2}[\omega, \omega] - \frac{1}{2}[\zeta, \zeta]_1$$

$$d\eta = -\frac{1}{2}[\eta, \eta] - \frac{1}{2}[\zeta, \zeta]_2,$$

where $[\zeta, \zeta]_1$ and $[\zeta, \zeta]_2$ are respectively the $\mathfrak{o}(n)$- and $\mathfrak{o}(k)$-components of $[\zeta, \zeta]$. Therefore the curvature forms Ω_1 and Ω_2 of the canonical connections in $P_{n,k}$ and $P^*_{n,k}$ are given by[1)]

$$\Omega_1 = -\frac{1}{2}[\zeta, \zeta]_1$$

$$\Omega_2 = -\frac{1}{2}[\zeta, \zeta]_2.$$

Now we shall calculate the curvature form of the invariant Riemannian connection on $M_{n,k}$ defined by

$$\gamma = \omega^* \otimes I_k + I_n \otimes \eta^*.$$

We observe that, for any matrices A, B of degree n and any matrix C of degree k,

$$[A \otimes I_k,\ B \otimes I_k] = [A, B] \otimes I_k$$

$$[A \otimes I_k,\ I_n \otimes C] = A \otimes C - A \otimes C = 0.$$

Hence

$$[\gamma, \gamma] = [\omega^*, \omega^*] \otimes I_k + I_n \otimes [\eta^*, \eta^*]$$

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1) More precisely, Ω_1 (resp. Ω_2) is the form on $P_{n,k}$ (resp. $P^*_{n,k}$) induced from the form $-\frac{1}{2}[\zeta, \zeta]_1$ (resp. $-\frac{1}{2}[\zeta, \zeta]_2$) on $O(n+k)$.

and

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = d\omega^* \otimes I_k + I_n \otimes d\eta^* + \frac{1}{2}[\omega^*, \omega^*] \otimes I_k + \frac{1}{2} I_n \otimes [\eta^*, \eta^*]$$
$$= \Omega_1^* \otimes I_k + I_n \otimes \Omega_2^*,$$

where Ω_i^* is the form on p^* corresponding to Ω_i.

§ 5. Characteristic classes

Let P be an arbitrary principal fibre bundle over a manifold M with Lie group G and with projection π. Let $\mathfrak{g}$ be the Lie algebra of G. A polynomial function f defined on $\mathfrak{g}$ is *invariant by G* if

$$f(ad(s)\cdot g) = f(g) \qquad \text{for all } g \in \mathfrak{g} \text{ and } s \in G.$$

Suppose there is given a connection in P and let Ω be its curvature form. Then Ω is a $\mathfrak{g}$-valued 2-form on P. The composite $f(\Omega)$ is a real valued differential form of degree $2r$ if f is a homogeneous polynomial of degree r. From the property

$$\Omega(\bar{u}_1\bar{s}_1, \ \bar{u}_2\bar{s}_2) = s^{-1}\Omega(\bar{u}_1, \ \bar{u}_2)s \qquad \text{for all } \bar{u}_1, \ \bar{u}_2 \in T_u(P),$$
$$\bar{s}_1, \ \bar{s}_2 \in T_s(G),$$

we conclude that, if f is invariant by G, then there exists a unique differential form $f(\Omega)^*$ on M such that

$$\pi^*(f(\Omega)^*) = f(\Omega).$$

It is known [3], [4] that $f(\Omega)^*$ is a closed form and the cohomology class to which it belongs is independent of the choice of connections. The cohomology class of $f(\Omega)^*$ is a *characteristic class* of the bundle P.

Remark 1. The form $f(\Omega)$ is always the coboundary of a certain form on P; however $f(\Omega)^*$ is not, in general, cohomologous to zero in M.

Remark 2. If M is a Riemannian (resp. Hermitian) manifold and P is the bundle of orthogonal frames (resp. unitary frams) over M, then the Pontrjagin classes (resp. Chern classes) are characteristic classes of P in the above defined sense.

§ 6. Symmetric functions

First we recall some known results in the theory of symmetric functions. We shall consider three types of symmetric functions of n quantities $a_1, \ldots, a_n$.

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The *symmetric functions* $p_1, \ldots, p_n$ *of the first type* are associated with the equation whose roots are the reciprocals of the quantities a_i;

$$f(x) \equiv \Pi(1-a_ix) = 1 - p_1x + p_2x^2 - \ldots + (-1)^n p_n x^n.$$

The function p_r is the sum of the $\binom{n}{r}$ products of r *different* quantities a_i. Define $p_r = 0$ for $r > n$.

The *symmetric functions* $q_1, \ldots, q_n, \ldots$ *of the second type* can be obtained by formal expansion of $1/f(x)$;

$$1/f(x) = 1/\Pi(1-a_rx) = \Pi(1 + a_rx + a_r^2x^2 + \ldots)$$
$$= 1 + q_1x + q_2x^2 + \ldots + q_nx^n + \ldots .$$

Then q_r is the sum of the homogeneous products of degree r of the quantities $a_1, \ldots, a_n$.

The *symmetric functions* $t_1, \ldots, t_n, \ldots$ *of the third type* are defined by

$$t_r = \sum_{i=1}^n a_i^r.$$

The following formulae are fundamental [9].

(I) $$p_r - p_{r-1}q_1 + p_{r-2}q_2 - \ldots + (-1)^r q_r = 0$$

(II) $$r! \cdot p_r = \begin{vmatrix} t_1 & 1 & & & \\ t_2 & t_1 & 2 & & \\ t_3 & t_2 & t_1 & 3 & \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ t_r & t_{r-1} & \cdots & \cdots & t_1 \end{vmatrix}$$

(III) $$r! \cdot q_r = \begin{vmatrix} t_1 & -1 & & & \\ t_2 & t_1 & -2 & & \\ t_3 & t_2 & t_1 & -3 & \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ t_r & t_{r-1} & \cdots & \cdots & t_1 \end{vmatrix}$$

Let $B = (b_j^i)$ be a matrix of degree n over the field of real numbers and let C be a non-singular matrix of degree n over the field of complex numbers such that $(a_j^i) \equiv A = C^{-1}BC$ is a triangular matrix:

$$a_j^i = 0 \quad \text{if} \quad i > j.$$

Put

$$a_1 = a_1^1, \ldots, a_n = a_n^n.$$

We shall express the symmetric functions p_r, t_r of the n quantities $a_1, \ldots,$

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a_n in terms of elements b^i_j of the matrix B. Put

$$p'_r(B) = \frac{1}{r!}\sum_{i,j}\delta(i_1, \ldots, i_r ; j_1, \ldots, j_r) b^{i_1}_{j_1} \ldots b^{i_r}_{j_r},$$

where $\delta(i_1, \ldots, i_r ; j_1, \ldots, j_r)$ is the generalized Kronecker δ. From $A = C^{-1}BC$, it follows that $p'_r(A) = p'_r(B)$. This fact can be proved as follows. A linear transformation B of the n-dimensional vector space V (over the field of complex numbers) induces a linear transformation B_r of $\wedge^r V$, where $\wedge^r V$ is the space of homogeneous elements of degree r of the exterior algebra $\wedge V$ over V. Then $p'_r(B)$ is nothing but the trace of B_r. This proves our assertion. Now we shall prove that $p'_r(A) = p_r$. Since A is triangular, we have

$$p'_r(A) = \frac{1}{r!}\sum_{i_1 \leq j_1, \ldots, i_r \leq j_r}\delta(i_1, \ldots, i_r ; j_1, \ldots, j_r) a^{i_1}_{j_1} \ldots a^{i_r}_{j_r}.$$

It is easy to see that, if $i_1 \leq j_1, \ldots, i_r \leq j_r$, then

$$\delta(i_1, \ldots, i_r ; j_1, \ldots, j_r) = 0 \quad \text{unless} \quad i_1 = j_1, \ldots, i_r = j_r.$$

Hence we have that $p'_r(A) = p_r$. Finally we have obtained

$$\text{(IV)} \qquad p_r = \frac{1}{r!}\sum\delta(i_1, \ldots, i_r ; j_1, \ldots, j_r) b^{i_1}_{j_1} \ldots b^{i_r}_{j_r}.$$

Next we shall calculate the symmetric functions t_r of the quantitites $a_1, \ldots, a_n$. Since $A^r \equiv AA \ldots A = C^{-1}B^rC$, we have

$$\mathrm{trace}(A^r) = \mathrm{trace}(B^r).$$

It is clear that $t_r = \mathrm{trace}(A^r)$. On the other hand,

$$\mathrm{trace}(B^r)\sum b^{i_1}_{i_2} b^{i_2}_{i_3} \ldots b^{i_r}_{i_1}.$$

Hence

$$\text{(V)} \qquad t_r = \sum b^{i_1}_{i_2} b^{i_2}_{i_3} \ldots b^{i_r}_{i_1}.$$

Notice that, if B is skew-symmetric and r is odd, then both p_r and t_r vanish. The definition of $q'_r(B)$ is clear from (III) and (V).

§ 7. Characteristic classes of $P_{n,k}$ and $P^*_{n,k}$

Let X be an arbitrary element of the Lie algebra $\mathfrak{o}(n)$. We define $p'_r(X)$ as in the preceding section. Then p'_r is a polynomial invariant by $O(n)$. Replacing X by the curvature form Ω_1 of the canonical connection in $P_{n,k}$, we obtain

$$p'_r(\Omega_1) = \frac{1}{r!}\sum\delta(i_1, \ldots, i_r ; j_1, \ldots, j_r)\Omega_1{}^{i_1}{}_{j_1} \wedge \ldots \wedge \Omega_1{}^{i_r}{}_{j_r}.$$

Let $\tilde{p}_r$ be the form on $M_{n,k}$ such that

$$\pi_1^*(\tilde{p}_r) = p'_r(\Omega_1).$$

Similarly we obtain

$$t'_r(\Omega_1) = \sum \Omega_{1\,i_2}^{i_1} \wedge \Omega_{1\,i_3}^{i_2} \wedge \ldots \wedge \Omega_{1\,i_1}^{i_r}$$

and define the form $\tilde{t}_r$ on $M_{n,k}$ such that

$$\pi_1^*(\tilde{t}_r) = t'_r(\Omega_1).$$

We define also the form $\tilde{q}_r$ on $M_{n,k}$ such that

$$\pi_1^*(\tilde{q}_r) = q'_r(\Omega_1).$$

The differential forms $\tilde{p}_r$, $\tilde{t}_r$ and $\tilde{q}_r$ on $M_{n,k}$ are all closed and their cohomology classes are independent of choice of connections in $P_{n,k}$.

Let Y be an arbitrary element of the Lie algebra $\mathfrak{o}(k)$. We define $p_r'^*(Y)$ in the same way (just replacing n by k). Replacing Y by the curvature form Ω_2 of the canonical connection in $P^*_{n,k}$, we obtain

$$P_r'^*(\Omega_2) = \frac{1}{r!}\sum \delta(i_1, \ldots, i_r;\ j_1, \ldots, j_r)\Omega_{2\,j_1}^{i_1} \wedge \ldots \wedge \Omega_{2\,j_r}^{i_r}.$$

Let $\tilde{p}_r^*$ be the form on $M_{n,k}$ such that

$$\pi_2^*(\tilde{p}_r^*) = p_r'^*(\Omega_2).$$

We define similarly the forms $\tilde{t}_r^*$ and $\tilde{q}_r^*$ on $M_{n,k}$.

Firstly we shall prove that

$$\tilde{t}_r + \tilde{t}_r^* = 0.$$

For this purpose, it suffices to show that

$$\pi^*(\tilde{t}_r) + \pi^*(\tilde{t}_r^*) = 0,$$

where π is the natural projection of $O(n+k)$ onto $M_{n,k}$. The left invariant $O(n+k)$-valued differential form θ on $O(n+k)$ is a skew-symmetric matrix differential form $(\theta^i_j)_{i,j=1,\ldots,n+k}$. We have

$$d\theta^\alpha_\beta = -\sum_{\gamma=1}^{n}\theta^\alpha_\gamma \wedge \theta^\gamma_\beta - \sum_{\lambda=n+1}^{n+k}\theta^\alpha_\lambda \wedge \theta^\lambda_\beta, \qquad \alpha, \beta = 1, \ldots, n$$
$$d\theta^\lambda_\mu = -\sum_{\nu=n+1}^{n+k}\theta^\lambda_\nu \wedge \theta^\nu_\mu - \sum_{\alpha=1}^{n}\theta^\lambda_\alpha \wedge \theta^\alpha_\mu, \qquad \lambda, \mu = n+1, \ldots, n+k.$$

Hence

$$\pi^*(\tilde{t}_r) = (-1)^r \sum (\theta^{\alpha_1}_{\lambda_1} \wedge \theta^{\lambda_1}_{\alpha_2}) \wedge (\theta^{\alpha_2}_{\lambda_2} \wedge \theta^{\lambda_2}_{\alpha_3}) \wedge \ldots \wedge (\theta^{\alpha_r}_{\lambda_r} \wedge \theta^{\lambda_r}_{\alpha_1})$$
$$\pi^*(\tilde{t}_r^*) = (-1)^r \sum (\theta^{\lambda_1}_{\alpha_2} \wedge \theta^{\alpha_2}_{\lambda_2}) \wedge (\theta^{\lambda_2}_{\alpha_3} \wedge \theta^{\alpha_3}_{\lambda_3}) \wedge \ldots \wedge (\theta^{\lambda_r}_{\alpha_1} \wedge \theta^{\alpha_1}_{\lambda_1}),$$

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where the summations are taken over the indices $\alpha_1, \ldots, \alpha_r$ which run from 1 to n and the indices $\lambda_1, \ldots, \lambda_r$ which run from $n+1$ to $n+k$. It is now evident that $\pi^*(\tilde{t}_r) = -\pi^*(\tilde{t}_r^*)$. We have proved the following duality theorem.

THEOREM 3. $\tilde{t}_r + \tilde{t}_r^* = 0$.

Now we shall find the corresponding relationship between $\tilde{p}_r$ and $\tilde{p}_r^*$. From the formula (III) in § 6 and Theorem 3, it follows that

$$r!\cdot\tilde{q}_r = \begin{vmatrix} -\tilde{t}_1^* & -1 & & \\ -\tilde{t}_2^* & -\tilde{t}_1^* & -2 & \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\tilde{t}_r^* & -\tilde{t}_{r-1}^* & \cdots & -\tilde{t}_1^* \end{vmatrix}$$

Hence

$$\tilde{q}_r = (-1)^r \tilde{p}_r^*.$$

Now, from (I) in § 6, we obtain the

THEOREM 3′. $\tilde{p}_r + \tilde{p}_{r-1}\tilde{p}_1^* + \tilde{p}_{r-2}\tilde{p}_2^* + \ldots + \tilde{p}_r^* = 0$.

Remark. As we have seen in the preceding section, the forms $\tilde{p}_r$, $\tilde{p}_r^*$, $\tilde{t}_r$ and $\tilde{t}_r^*$ vanish identically if r is odd.

§ 8. Characteristic classes of a manifold

Let M be an n-dimensional Riemannian manifold imbedded isometrically in the $(n+k)$-dimensional Euclidean space. Let P be the bundle of (tangent) orthogonal frames over M and let P^* be the bundle of normal orthogonal frames over M. Then P and P^* are principal fibre bundles over M with group $O(n)$ and $O(k)$ respectively. Let h_1 be the natural bundle map of P into $P_{n,k}$. We have shown in the previous paper [7] that the connection in P induced by h_1 from the canonical connection in $P_{n,k}$ is nothing but the Riemannian connection on M. Let h_2 be the natural bundle map of P^* into $P_{n,k}^*$. Then h_2 and the canonical connection in $P_{n,k}^*$ induce a connection in P^*, which we shall call the *normal connection*. It seems that the normal connection has been considered implicitly in classical differential geometry.

Both h_1 and h_2 induce the same mapping h of the base space M into the base space $M_{n,k}$. Put

$$p_r(M) = h^*(\tilde{p}_r), \qquad q_r(M) = h^*(\tilde{q}_r), \qquad t_r(M) = h^*(\tilde{t}_r),$$

$$p_r^*(M) = h^*(\tilde{p}_r^*), \qquad q_r^*(M) = h^*(\tilde{q}_r^*), \qquad t_r^*(M) = h^*(\tilde{t}_r^*).$$

The differential forms $p_r(M)$ and $p_r^*(M)$ on M are called the $2r$-th *Pontrjagin cocycle*[2] and the $2r$-th *normal Pontrjagin cocycle* of the imbedded Riemannian manifold M. The cohomology classes of $p_r(M)$ and $p_r^*(M)$ are called the $2r$-th *Pontrjagin class* and the $2r$-th *normal Pontrjagin class* of M respectively. From Theorems 3 and 3′, we obtain the

THEOREM 4. $t_r(M) + t_r^*(M) = 0$,

$$p_r(M) + p_{r-1}(M)\cdot p_1^*(M) + \ldots + p_r^*(M) = 0.$$

Since the forms $t_r(M)$ are completely determined by the curvature form of the Riemannian connection on M, so are the forms $t_r^*(M)$. Since the forms $p_r^*(M)$ and $q_r^*(M)$ are polynomials of $t^*(M)$ (see (II) and (III) in § 6), they depend only on the curvature form of the Riemannian connection on M. Hence

THEOREM 5. *The differential forms $p_r^*(M)$, $q_r^*(M)$ and $t_r^*(M)$ are all invariants of the Riemannian connection on M.*

We shall explain the meaning of the above theorem. Suppose $j : M \to R^{n+k}$ and $j' : M \to R^{n+k'}$ and $j' : M \to R^{n+k'}$ be isometrical imbeddings of an n-dimensional Riemannian space M. If $k' \leq k$, then we may consider j' as an isometrical imbedding of M into R^{n+k}. In general, there may not exist any motion ψ of R^{n+k} such that $j' = \psi \cdot j$. The above theorem states that the forms $p_r^*(M)$, $q_r^*(M)$, $t_r^*(M)$ do not depend on j but depend only on the Riemannian connection on M.

As we have remarked in § 5, the cohomology classes of $p_r(M)$, $q_r(M)$, $t_r(M)$ are differentiable invariants of P, hence they are differentiable invariants of M. From Theorem 4, it follows that the cohomology class of $t_r^*(M)$ is also a differentiable invariant of M. Since the forms $p_r^*(M)$, $q_r^*(M)$ are polynomials of $t_r^*(M)$ (see (II) and (III) in § 6), their cohomology classes are also differentiable invariants of M.

THEOREM 6. *The cohomology classes of $p_r(M)$, $q_r(M)$, $t_r(M)$, $p_r^*(M)$, $q_r^*(M)$ and $t_r^*(M)$ depend only on the differentiable structure of the manifold M.*

Remark. We understand by "differentiable structure" "C^∞-differentiable

[2] Our definition of Pontrjagin cocycles (and normal Pontrjagin cocycles) is different from that of [4, 5] by constant factors. In the following theorem 4, these constant factors are cancelled out.

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structure". However the above theorem is true for the C^3-differentiable structure.

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