Combinatorics, Probability and Computing (2014) 23, 686–716.
doi:10.1017/S0963548314000315

Part Sizes of Smooth Supercritical Compositional Structures

EDWARD A. BENDER[1] and ZHICHENG GAO[2†]

[1]Department of Mathematics, UCSD, La Jolla, CA 92093-0112, USA
(e-mail: ebender@ucsd.edu)

[2]School of Mathematics and Statistics, Carleton University, Ottawa, Ontario K1S5B6, Canada
(e-mail: zgao@math.carleton.ca)

Received 2 February 2013; revised 23 September 2013; first published online 9 July 2014

We define the notion of smooth supercritical compositional structures. Two well-known examples are compositions and graphs of given genus. The 'parts' of a graph are the subgraphs that are maximal trees. We show that large part sizes have asymptotically geometric distributions. This leads to asymptotically independent Poisson variables for numbers of various large parts. In many cases this leads to asymptotic formulas for the probability of being gap-free and for the expected values of the largest part sizes, number of distinct parts and number of parts of multiplicity k.

2010 *Mathematics subject classification*: Primary 05A16
Secondary 05A15, 05A05, 05C05, 05C10, 05C30

1. Introduction

Many combinatorial structures are composed of *supports* and *parts* (or *components*). For example, graphs are composed of a set of connected components, permutations are composed of a set of cycles, compositions are composed of a sequence of integers, functional digraphs are composed of a sequence of rooted trees. We study part sizes in structures produced by the sequence construction $S(P(x))$ when the composition is supercritical and satisfies some smoothness conditions (see Definitions 1.3 and 1.4 below for a precise meaning).

Convention 1.1 (sets of integers). The natural numbers are denoted by $\mathbb{N}$. Subsets of $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$ are denoted by letters such as I, J and K.

† Research supported by NSERC.

Convention 1.2 (generating functions and power series). Power series are denoted by capital letters and the coefficients by the corresponding lower-case letters appropriately subscripted. If a power series is thought of as a generating function, a bar is placed over the letter indicating the coefficients. Thus

$$F(x) = \sum_{n\geqslant 0} f_n x^n = \sum_{n\geqslant 0} \bar{f}_n x^n \quad \text{for an ordinary generating function, and}$$

$$F(x) = \sum_{n\geqslant 0} f_n x^n = \sum_{n\geqslant 0} \bar{f}_n x^n/n! \quad \text{for an exponential generating function.}$$

In either case, $\rho(F)$ is its radius of convergence and $[x^n]\,F(x) = f_n$, the coefficient of x^n.

The set of combinatorial structures associated with a generating function is denoted by the corresponding script letter, *e.g.*, $\mathcal{A}$ with $A(x)$ and $\mathcal{P}$ with $P(x)$.

Definition 1.3 (compositional families). A *compositional family* consists of a *support* or *core* generating function $S(x)$ and a *part* or *component* generating function $P(x)$ with $p_0 = 0$. The generating function for the family is $A(x) = S(P(x))$. All of these generating functions are either ordinary or exponential.

We note that the relation $A(x) = S(P(x))$ indicates that each structure in the family $\mathcal{A}$ is constructed using a support from $\mathcal{S}$ and a *sequence* of parts from $\mathcal{P}$. Such a construction is usually referred to as the 'sequence construction'. If we think of the power of x in the generating functions as keeping track of 'size', then the size of a structure is the sum of the sizes of its parts.

Definition 1.4 (smooth supercritical). Let $\mathcal{A}$ be a family of compositional structures with part generating function $P(x)$ and support generating function $S(x)$. Let $A(x) = S(P(x))$ and $g_{n,k} = [x^n]S^{(k)}(P(x))$. We call $\mathcal{A}$ *smooth supercritical* if the following conditions hold.

(a) It is supercritical, that is, there is a $0 < r < \rho(P)$ such that $\rho(S) = P(r)$.
(b) There is a constant $\delta > 0$ such that $a_n/a_{n+t} \to r^t$ uniformly for $|t| \leqslant n^\delta$.
(c) For each fixed positive integer k, $g_{n,k}/g_{n+1,k} \sim r$.

Note that $r = \rho(S^{(k)}(P(x)))$ for any fixed non-negative integer k. Also, since $g_{n,0} = a_n$, (c) follows from (b) when $k = 0$.

Verification of (b) and (c) may be difficult. Theorem 1.12 establishes them using rather weak information about the coefficients of $S(x)$.

Example 1.5 (some smooth supercritical families). For compositions where the parts must lie in some set $\mathcal{P}$,

$$S(x) = \frac{1}{1-x} \quad \text{and} \quad P(x) = \sum_{p\in\mathcal{P}} x^p.$$

For all sets $\mathcal{P}$, this is supercritical. Smoothness holds if $\gcd(i-j \mid p_i p_j \neq 0) = 1$. In Section 2 we consider more general supports. For example, if the parts form a square array, $S(x) = \sum x^{k^2}$.

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

Runs in words on an alphabet can be studied via $S(P(x))$, where $S(x)$ is the generating function for words without runs and $P(x) = x/(1-x)$ is the generating function for a run. Placing a run in a word is done by replicating the letter at which it is placed. We study runs in Example 3.2.

A *smooth graph* is a graph where every vertex has degree at least two. We can then replace each vertex by a rooted tree to obtain all graphs In this case, $\rho(S) = 0$ and so is outside our considerations. However, we are able to discuss graphs of a given genus. One might expect that 'functional digraphs' and rooted maps could also be discussed; however, there are difficulties. Graphs are discussed in Section 3.

The 'exponential formula' for labelled structures in terms of components, states that $A(x) = e^{P(x)}$, where $P(x)$ counts parts and $S(x) = e^x$ is the exponential generating function for sets. This is usually referred to as the 'set construction', and is never supercritical since $\rho(e^x) = \infty$.

Gourdon [17] studied the distribution of the largest component size in a general compositional structure when the generating functions are of 'algebraic–logarithmic' type. Unlike the present paper, he considers the subcritical and critical cases as well as the supercritical case. We obtain additional results about the components for a much broader class of supercritical structures; however, our results for the largest part are less precise than Gourdon's for algebraic–logarithmic generating functions. Theorems 1.6, 1.10 and 1.11 below contain results for smooth supercritical families of compositional structures under increasingly restrictive conditions on $\mathcal{P}$. These are of a probabilistic nature and so only involve relative numbers. All except Theorem 1.6(a) implicitly assume $\rho(P) < \infty$.

In contrast, Theorem 1.13 provides asymptotics for the number of structures based on assumptions about s_k, the number of supports. As mentioned earlier, Theorem 1.12 proves smoothness under relatively weak assumptions on the s_k.

Theorem 1.6 (geometric and Poisson behaviour). *Let $S(x)$, $P(x)$ and $A(x) = S(P(x))$ be the generating functions for a smooth supercritical compositional family.*

(a) *Let $\mathcal{Q}$ be a non-empty subset of $\mathcal{P}$. Select a structure of size n uniformly at random and let $X_{\mathcal{Q}}(n)$ be the number of parts in the structure that are in $\mathcal{Q}$. Then*

$$\mathsf{E}(X_{\mathcal{Q}}(n)) \sim nQ(r)/(rP'(r)) \quad \text{and} \quad \operatorname{Var}(X_{\mathcal{Q}}(n)) = o(n^2).$$

Furthermore, the distribution of $X_{\mathcal{Q}}(n)$ depends only on $Q(x)$, that is, the number of parts of various sizes in $\mathcal{Q}$. In particular, if X_k keeps track of a single part of size k, then $\mathsf{E}(X_k(n)) \sim nr^{k-1}/P'(r)$. Furthermore, $X_{\mathcal{P}}$ keeps track of the total number of parts in the structure and

$$\lim_{n\to\infty} \frac{\mathsf{E}(X_k(n))}{\mathsf{E}(X_{\mathcal{P}}(n))} = \frac{r^k}{P(r)},$$

a geometric-like behaviour in k.

(b) *Let $f(x)$, β and the infinite set $J \subseteq \mathbb{N}$ be such that $f'(x) = o(1)$ as $x \to \infty$ and $p_j \sim e^{f(j)}\beta^{-j}$ as $j \to \infty$ through J. Let ζ_j be the number of parts of size j in a random*

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

structure of size n. Let $\alpha = \beta/r$ and define $\sigma(n)$ implicitly by

$$\sigma(n) = \frac{\ln(n/rP'(r)) + f(\sigma(n))}{\ln\alpha} \quad \text{or equivalently by} \quad \alpha^{\sigma(n)} e^{-f(\sigma(n))} = \frac{n}{rP'(r)}. \tag{1.1}$$

Then there is a function $\omega(n) \to \infty$ such that the random variables

$$\{\zeta_j : \sigma(n) - \omega(n) \leqslant j \leqslant n;\ j \in J\}$$

are asymptotically independent Poisson random variables with means $\mu_j = \alpha^{\sigma(n)-j}$, where convergence is in total variation distance.

Remark 1.7. Here is an interpretation of the somewhat mysterious $\sigma(n)$. We illegally treat asymptotics as exact and $\sigma(n)$ as an integer in J. Since there are p_j possible parts of size j, the expected number of parts of size j is $p_j \mathsf{E}(X_j(n))$, which equals μ_j for a Poisson distribution. Replace $\mathsf{E}(X_j(n))$ with the asymptotic estimate in (a) and replace p_j with its asymptotic estimate in (b). We find that $\mu_j = 1$ when $j = \sigma(n)$.

Remark 1.8. If J is very sparse, the behaviour in (b) may be trivial for most n. However, in combinatorial situations one often has $p_j \sim C(\ln j)^a j^b \rho(P)^{-j}$ for some C, a and b and $j \in \mathbb{N}$. In that case,

$$\sigma(n) = \frac{\ln(n/rP'(r)) + a\ln(\ln(\ln n)) + b\ln(\ln n)}{\ln\alpha} + o(1),$$

where $\alpha = \rho(P)/r$ and we may ignore the $o(1)$ in $\sigma(n)$.

Definition 1.9 (gap-free and $I(\mathcal{P})$). Let $I(\mathcal{P}) = \{k \mid p_k \neq 0\}$. A structure is said to be *gap-free* if, whenever it has a part of size k, it has a part of size j for all $j \in I(\mathcal{P})$ less than k.

One may choose to ignore whether or not parts of sizes less than some fixed k_0 are present in the definition of gap-free. By Lemma 5.5 below, this does not affect our asymptotic results.

Theorem 1.10 (rough estimates). *Let $S(x)$, $P(x)$ and $A(x) = S(P(x))$ be the generating functions for a smooth supercritical compositional family with $\rho(P) < \infty$. Suppose there is an $f(x)$ such that $f'(x) = o(1)$ as $x \to \infty$ and $p_n \sim e^{f(n)}\rho(P)^{-n}$ as $n \to \infty$ through $I(\mathcal{P})$. Define $\alpha = \rho(P)/r$ and define $\sigma(n)$ by* (1.1). *Let the random variable M_n be the maximum part size in a random structure of size n.*

(a) *If $\omega_i(n)$ are any functions such that $\omega_i(n) \to \infty$ and $\sigma(n) - \omega_1(n) \in I(\mathcal{P})$, then*

$$\Pr\big(\sigma(n) - \omega_1(n) \leqslant M_n \leqslant \sigma(n) + \omega_2(n)\big) \to 1.$$

(b) *The probability that a random structure of size n is gap-free is bounded away from zero as $n \to \infty$.*

When $\mathbb{N} \setminus I(\mathcal{P})$ is finite, we can obtain more accurate results. We state these next along with some additional results.

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

Theorem 1.11 (largest part, distinct parts, gaps, *etc.*). *Let $S(x)$, $P(x)$ and $A(x) = S(P(x))$ be the generating functions for a smooth supercritical compositional family with $\rho(P) < \infty$. Suppose $v = |\mathbb{N} \setminus I(\mathcal{P})|$ is finite, and $p_n \sim e^{f(n)}\rho(P)^{-n}$, where $f(x)$ satisfies $f'(x) = o(1)$ as $x \to \infty$.*

Let $\alpha = \rho(P)/r$, let log *denote logarithm to the base α, and let $\sigma(n)$ be given by* (1.1). *Let $\gamma \doteq 0.577216$ be Euler's constant, and let*

$$P_k(x) = \log e \sum_{\ell \neq 0} \Gamma(k + 2i\pi\ell \log e) \exp(-2i\ell\pi x), \tag{1.2}$$

a periodic function of x with period 1.

(a) *Let the random variable M_n be the size of the maximum part in a random structure of size n. For any function $\omega_a(n)$ such that $\omega_a(n) \to \infty$ as $n \to \infty$, $|M_n - \sigma(n)| < \omega_a(n)$ a.a.s. Furthermore,*

$$\mathsf{E}(M_n) = \sigma(n) + \gamma \log e - \log(\alpha - 1) + \frac{1}{2} + P_0\big(\sigma(n) + 1 - \log(\alpha - 1)\big) + o(1).$$

(b) *Let the random variable D_n be the number of distinct part sizes in a random structure of size n. For any function $\omega_b(n)$ such that $\omega_b(n) \to \infty$ as $n \to \infty$, $|D_n - \sigma(n)| < \omega_b(n)$ a.a.s. Furthermore,*

$$\mathsf{E}(D_n) + v = \sigma(n) + \gamma \log e - \frac{1}{2} + P_0(\sigma(n)) + o(1).$$

(c) *Let $q_n(k)$ be the probability that a random structure of size n is gap-free and has largest part k. There is a function $\omega_c(n) \to \infty$ as $n \to \infty$ such that*

$$q_n(k) \sim \exp\left(\frac{-\alpha^{\sigma(n)-k}}{\alpha - 1}\right) \prod_{j \leqslant k} \big(1 - \exp\big(-\alpha^{\sigma(n)-j}\big)\big) \tag{1.3}$$

uniformly for $|k - \sigma(n)| < \omega_c(n)$. Furthermore, for any constant B, the minimum of $q_n(k)$ over $|k - \sigma(n)| < B$ is bounded away from zero.

(d) *Let q_n be the probability that a random structure of size n is gap-free. Then q_n is asymptotic to the sum of the right-hand side of* (1.3), *where the sum may be restricted to $|k - \sigma(n)| < \omega_d(n)$ for any $\omega_d(n) \to \infty$ as $n \to \infty$. Furthermore, $q_n \sim b_m$, where*

$$m = \left\lfloor \frac{\alpha^{\sigma(n)+1}}{\alpha - 1} \right\rfloor$$

and

$$b_m = \begin{cases} 1 & \text{if } m = 0, \\ \displaystyle\sum_{k=0}^{m-1} b_k \binom{m}{k} (1/\alpha)^k (1 - 1/\alpha)^{m-k} & \text{if } m > 0. \end{cases} \tag{1.4}$$

(e) *Let $g_n(k)$ be the probability that a random structure of size n has exactly k parts of maximum size. Then, for each fixed $k > 0$,*

$$g_n(k) \sim \frac{(\alpha - 1)^k}{k!\alpha^k} P_k\big(\sigma(n) + 1 - \log(\alpha - 1)\big) + \frac{(\alpha - 1)^k \log e}{k\alpha^k} \quad \text{as } n \to \infty.$$

(f) *Let $D_n(k)$ be the number of distinct part sizes that appear exactly k times in a random structure of size n. Then, for fixed $k > 0$,*

$$\mathsf{E}(D_n(k)) = \frac{P_k(\sigma(n))}{k!} + \frac{\log e}{k} + o(1) \quad \text{as } n \to \infty.$$

Let $m_n(k)$ be the probability that a randomly chosen part size in a random structure of size n has multiplicity k. For fixed k, $m_n(k) \sim \mathsf{E}(D_n(k))/\log n$.

We recall that $\Gamma(a+iy)$ goes to zero exponentially fast as $y \to \pm\infty$. Thus the sum (1.2) is dominated by the terms with small ℓ. Furthermore, for $1 < \alpha < 2$ the amplitude of the oscillation of $P_0(x)$ is less than 10^{-6} [18].

Theorem 1.12 (smooth supercriticality). *Let $P(x)$ and $S(x)$ be the part generating function and support generating function, respectively, of a family $\mathcal{A}$ of compositional structures. Let $s = \rho(S)$. If the following conditions hold, then $\mathcal{A}$ is smooth supercritical.*

(a) *There is a $0 < r < \rho(P) \leqslant \infty$ such that $\rho(S) = P(r)$.*
(b) $\gcd\{i - j \mid p_i p_j \neq 0\} = 1$.
(c) *There is an $\epsilon > 0$ and an infinite set $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ such that*
   (i) $k_{i+1} - k_i = O(k_i^{1-\epsilon})$,
   (ii) $s_k \leqslant \exp(O(k^{1-\epsilon}))s^{-k}$ *for all k,*
   (iii) $s_k \geqslant \exp(-O(k^{1-\epsilon}))s^{-k}$ *for $k \in K$.*

Theorem 1.13 (number of structures). *We make the same assumptions as Theorem 1.12 and also assume that there is a function $G(x)$ such that the following hold.*

(a) $G'(x) = o(G(x)/x)$ *as $x \to \infty$.*
(b) $\sum_{k \leqslant x} s_k \rho(S)^k = (x + o(\sqrt{x}))G(x)$ *as $x \to \infty$.*

Then $a_n \sim \mu r^{-n} G(\mu n)$, where $\mu = P(r)/rP'(r)$.

The rest of the paper is organized as follows. Section 2 discusses multi-dimensional compositions. Section 3 gives applications to some other families of compositional structures. Section 4 briefly discusses some open questions. Section 5 states some lemmas which are used to prove the theorems. The remaining sections contain the various proofs.

2. Multi-dimensional compositions

While higher-dimensional partitions have been extensively studied, higher-dimensional compositions have not been. One reason is that the choice of support for basic higher-dimensional partitions is clear: a Ferrers diagram of a partition one dimension lower. Furthermore, monotonicity requirements, which are restrictions *between* nearby parts, lead to interesting problems.

Ordinary compositions with fairly general local inter-part restrictions have been studied [5, 6], but we are aware of only the following two extensions to higher-dimensional supports.

Matrix compositions. Various authors have studied 'matrix compositions' (see, *e.g.*, [21, 22]). These have $d \times m$ rectangular supports, where d is fixed. Zero parts are allowed but all zero columns are forbidden. Because of this restriction on zeros, they do not fit the present paper. As pointed out in Example 7 of [5], rectangular compositions with fixed d and local restrictions can be converted to ordinary compositions with local restrictions by reading an array B top to bottom, left to right:

$$b_{1,1}, b_{2,1}, \dots, b_{d,1}, b_{1,2}, \dots, b_{d,m}.$$

The methods of [5] and [6] can be extended to allow zero parts as in matrix compositions.

Convex polyominoes. We plan to consider compositions with convex polyomino support and rather general local restrictions in a future paper [9].

If compositions have no inter-part restrictions and we want to study the parts, then $A(x) = S(P(x))$. We claim this is supercritical whenever $\mathcal{S}$ is infinite. Note first that $\mathcal{P}$ is simply the allowed part sizes and so $I(\mathcal{P}) = \mathcal{P}$ and $P(x) = \sum_{p\in\mathcal{P}} x^p$. Second, since $\rho(P) = \infty$ when $\mathcal{P}$ is finite and $\rho(P) = 1$ otherwise, $P(x)$ is unbounded as $x \to \rho(P)$ and so the family of structures is supercritical whenever $S(x)$ is not a polynomial. As the next example shows, rather general supports lead to smooth supercritical compositions.

Example 2.1 (a variety of supports). Since we usually have no direct knowledge of the behaviour of $S(P(x))$ but know a lot about $S(x)$, we apply Theorem 1.12. As a result, we make the following assumptions:

(a) $S(x)$ is not polynomial, *i.e.*, $\mathcal{S}$ is infinite,
(b) $\gcd(i - j \mid p_i p_j \neq 0) = 1$.

Since (a) implies Theorem 1.12(a) and (b) is a restatement of Theorem 1.12(b), it suffices to find a set K for Theorem 1.12(c). Since $p_k = 1$ for all $k \in \mathcal{P}$, Theorems 1.6, 1.10 and 1.11 will apply provided that $\mathcal{P}$ behaves as required in those theorems.

While the motivation for choosing a particular set of supports may be geometric (such as hypercubes), only the values of s_k matter. Here are a few examples of supports.

Partitions. Suppose s_k is the number of partitions of k in one or more dimensions and, possibly, with restrictions on parts. Often $\log s_k = \Theta(k^b)$ for some $0 < b < 1$, and so we can take $K = \mathbb{N}$, $s = 1$ and $1 - \epsilon = b$ in Theorem 1.12(c). (See [10] for d-dimensional partitions with $d > 2$.)

Compositions. It was shown in [5] that for many locally restricted compositions $s_k \sim Ar^{-k}$ for some $A > 0$ and $0 < r < 1$. Thus we let $K = \mathbb{N}$, $s = r$ and $\epsilon = 1$.

Hypercubes. For d-dimensional hypercubes, $s_k = 1$ if k is a dth power and $s_k = 0$ otherwise. In this case, we let K be the set of dth powers, $s = 1$ and $\epsilon = 1/d$.

Hyper-rectangles. For d-dimensional hyper-rectangles, s_k is the number of ways to write k as an ordered product of d factors. Thus $1 \leqslant s_k \leqslant k^{d-1}$, and so we can take $K = \mathbb{N}$, $s = 1$ and $\epsilon > 0$ any value less than 1. For $(d+f)$-dimensional hyper-rectangles with f

dimensions fixed, let φ be the product of the fixed dimensions. Then $1 \leqslant s_{\varphi k} \leqslant k^{d-1}$, and we proceed as before but with $K = \varphi\mathbb{N}$.

Polyominoes. The rather complicated generating function for convex polyominoes was found by Klarner and Rivest [19]. It was shown in [3] that its only singularity on its circle of convergence is a simple pole. Hence Theorem 1.12 applies. Convex polyomino supports for locally restricted compositions will be studied in [9]. Information about some other types of polyominoes can be found in [11].

Smooth supercritical structures. Lemma 5.6 below shows that such structures of size k can be used as supports for compositions. If we apply this to structures built from supports $\mathcal{S}_1$ and parts $\mathcal{P}_1$, the new support generating function is $S_2(x) = S_1(P_1(x))$. If $\mathcal{P}_2$ is the new set of parts, $A(x) = S_2(P_2(x)) = S_1(P_1(P_2(x)))$. When $\mathcal{P}_i$ are sets of integers, the result corresponds to placing a list of integers at each point in $\mathcal{S}_1$ and counting structures by the sum over all lists. The lengths of the lists must lie in $\mathcal{P}_1$ and the elements of the lists must lie in $\mathcal{P}_2$.

In many cases, Theorem 1.13 provides asymptotics for a_n. The cases of rectangular support and hypercube support require some work.

For rectangular support, s_k is the number of divisors of k. It is known [1, Theorem 3.3] that

$$H(x) = \sum_{k \leqslant x} s_k = x \ln x + (2\gamma - 1)x + O(\sqrt{x}),$$

which satisfies the assumption in Theorem 1.13 with $G(x) = \ln x + 2\gamma - 1$. Therefore $a_n \sim \mu r^{-n} \ln n$, where $P(r) = 1$ determines r and $\mu = 1/rP'(r)$. When all part sizes are allowed, $P(x) = x/(1-x)$ and so $r = 1/2$ and $\mu = 2$.

For hypercube support, non-zero s_k are too sparse to apply Theorem 1.13. However, we have the following result.

Theorem 2.2 (number of compositions with hypercube support). *Let the supports be the d-dimensional hypercubes.*

(a) *If $d = 2$, $\mathcal{P}$ is infinite and* $\gcd(i - j \mid p_i p_j \neq 0) = 1$, *then*

$$a_n \sim \frac{\mu r^{-n}}{\sqrt{2\pi\sigma^2 n}} \sum_{j=-\infty}^{\infty} \exp\left(\frac{-2\mu(j - \{\sqrt{\mu n}\})^2}{\sigma^2}\right),$$

where $r = P^{-1}(1)$, $\mu = 1/rP'(r)$, $\sigma^2 = r^2 P''(r)\mu^3 + \mu^2 - \mu$ and $\{x\}$ denotes the fractional part of x.

(b) *If $d = 2$ and $\mathcal{P} = \mathbb{N}$, then*

$$a_n \sim \frac{2^n}{\sqrt{2\pi n}} \sum_{j=-\infty}^{\infty} \exp\left(-4(j - \{\sqrt{n/2}\})^2\right).$$

(c) *If* $d > 2$ *and* $\mathcal{P} = \mathbb{N}$, *then*

$$a_n \sim \binom{n-1}{a^d-1} + \binom{n-1}{(a+1)^d-1},$$

where $a = \max\{i \mid 2i^d - 1 < n\}$. *One of the two binomial coefficients will usually be negligible.*

3. Other applications

We begin by discussing two of Gourdon's examples [17]. Then we look at a graph application where the parts are maximal trees. Since the generating function for the structures must have non-zero radius of convergence, we must limit the number of graphs in some manner. We discuss labelled graphs of fixed genus. Another possibility is functional digraphs, but this situation is critical rather than supercritical. Still another possibility is rooted maps. Two problems arise. First, $P(x) = T^2(x)$, the square of the tree generating function, and so no results are obtained about tree sizes directly. Second, the generating function is

$$A(x) = S(T^2(x))\left(1 + \frac{4xT'(x)}{T(x)}\right). \tag{3.1}$$

Although $S(T^2(x))$ is smooth supercritical, we are faced with an additional factor.

Example 3.1 (threads in rooted plane trees). This is based on Example 7 of [17]. Trees are unlabelled, rooted and plane. Direct edges toward the root. A *thread* is a path each of whose vertices except possibly the starting vertex, has indegree 1. We want to study the number of vertices in maximal length threads. Thus $P(x) = x/(1-x)$, the generating function for non-empty paths. The supports are those trees with no vertices of indegree 1. By the standard iterative construction of rooted plane trees

$$S(x) = x \sum_{d\in\mathcal{D}} S(x)^d,$$

where $\mathcal{D}$ consists of the allowed indegrees. Thus $0 \in \mathcal{D}$ and $1 \notin \mathcal{D}$. It follows that the dominant singularities of $S(x)$ are branch points at $\rho(S)\omega$ where ω is a $\gcd(\mathcal{D})$th root of unity and $w = \rho(S)$ is given implicitly by

$$\sum_{d\in\mathcal{D}} (d-1)w^d = 0 \quad \text{and then } 1/\rho(S) = \sum_{d\in\mathcal{D}} w^{d-1}.$$

Furthermore, if $a_n \neq 0$, then $n-1$ is a multiple of $\gcd(\mathcal{D})$. Theorem 1.12 applies with $k_i = 1 + \gcd(\mathcal{D})i$. Since $r/(1-r) = \rho(S)$, we have

$$\begin{aligned} r &= \frac{\rho(S)}{1+\rho(S)}, \qquad & P'(r) &= (1+\rho(S))^2, \\ \alpha &= \frac{1+\rho(S)}{\rho(S)}, \qquad & \sigma(n) &= \log_\alpha(n) - 1 + 2\log_\alpha(\alpha-1). \end{aligned} \tag{3.2}$$

As Gourdon notes, in the case of unary–binary trees, $\rho(S) = 1/2$ and $r = 1/3$.

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

We note that the equations in (3.2) depend only on the fact that $P(x) = x/(1-x)$.

Example 3.2 (runs in words). This is based on Example 9 of [17]. A *run* is a repeat of a single letter in a word on a finite alphabet A. We want to study lengths of runs in words of length n. The supports are Smirnov words (words without runs) and the parts are runs on a single fixed letter. A run of length k replaces the letter $i \in A$ to which it attached with the k-long subword $i \dots i$. If $N = |A|$, then $P(x) = x/(1-x)$, $A(x) = 1/(1-Nx) - 1$ and $S(x) = Nx/(1-(N-1)x)$. Since $A(x)$ has a simple pole, we have a smooth supercritical family and our theorems apply. Since (3.2) depends only on $\rho(S)$ and $P(x)$, it applies in this case as well.

We can restrict the set of words by limiting the letters that can follow a letter $i \in A$ to a subset A_i of A provided that two conditions hold. First, $i \in A_i$ for all $i \in A$. Second, for all $i, j \in A$ there is a word of the form $\cdots i \cdots j \cdots i \cdots$. In this case, the 0–1 transition matrix given by $m_{i,j} = 1$ if and only if $j \in A_i$ is such that M^k has all non-zero entries for sufficiently large k. Hence M has a unique maximum eigenvalue and so $A(x)$ has one dominant singularity and that is a simple pole. Thus $a_n \sim Cr^{-n}$ and $[x^n]A^{(k)}(x) \sim C(nr)^k r^{-n}$ for some C. We have

$$A^{(k)}(x) = \sum_{i \leqslant k} S^{(i)}(P(x)) F_{i,k}(x),$$

where $F_{i,k}(x)$ is a polynomial in the various derivatives $P^{(j)}(x) = j!(1-x)^{-j-1}$ and $F_{k,k}(x) = (P'(x))^k$. We will show by induction on k that

$$[x^n]\, S^{(k)}(P(x)) \sim C(nr(1-r)^2)^k r^{-n}, \tag{3.3}$$

and so the family is smooth supercritical. The case $k = 0$ is trivial. By the induction hypothesis and the fact that $\rho(P) > r$, it follows that

$$[x^n]\, A^{(k)}(x) = [x^n]\big(S^{(k)}(P(x))(1-x)^{-2k}\big) + O(n^{k-1} r^{-n}).$$

Hence

$$[x^n]\, S^{(k)}(P(x)) = [x^n]\big(A^{(k)}(x)(1-x)^{2k}\big) + O(n^{k-1} r^{-n}).$$

Equation (3.3) follows from Theorem 2 in [4].

Example 3.3 (runs of a particular letter). As in the previous example, we consider runs in words on an N letter alphabet A; however, now we are interested only in runs of a particular letter, say ℓ. There are no restrictions on the letters in the words.

Let $\varphi \notin A$ be a new letter. Let $\mathcal{S}$ be all words of the form $\varphi\vec{w}$, where $\vec{w}$ is a possibly empty word on the alphabet $A \setminus \{\ell\}$. Thus $S(x) = x/(1-(N-1)x)$. Let $\mathcal{P}$ be the set of words φz^k, where k is a non-negative integer. Thus $P(x) = x/(1-x)$. We attach φz^k to a letter i in a word by replacing i with iz^k. The structures $\mathcal{A}$ are then all words of the form $\varphi\vec{w}$, where $\vec{w} \in A^*$ is a possibly empty word in A. In other words $[x^{n+1}]\, S(P(x))$ is the number of words of length n in A^*. Our theorems apply.

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

Table 1.

$\|A\|$	n	R	$\mathsf{E}(m_n)$	$\bar{m}_n$	$\bar{M}_n$
2	2^{16}	400	15.33	15.35	16.35
3	3^{10}	100	9.66	9.69	10.72
3	3^{9}	100	8.66	8.67	9.75
4	4^{8}	100	7.71	7.70	8.71

Here and in the previous example, $P(x) = x/(1-x)$, and r is given by $1-(N-1)P(r) = 0$. Thus the results here are the same as those when we consider runs on all letters; however, the interpretation of word length n and run length k is different.

- In the previous example n was the length of a word $\vec{w}$ and k was the length of a run i^k for some $i \in A$.
- In this example, because of φ, $n-1$ is the length of a word $\vec{w}$ and $k-1$ is the length of a run ℓ^{k-1}.

To illustrate the difference we look at Theorem 1.11(b). For the previous example, it tells us that the expected number of distinct run lengths in a word of length n is

$$\sigma(n) + \gamma \log e - \frac{1}{2} + P_0(\sigma(n)) + o(1).$$

On the other hand, the expected number of distinct run lengths for a particular letter in a word of length n is

$$\sigma(n+1) + \gamma \log e - \frac{3}{2} + P_0(\sigma(n+1)) + o(1),$$

where $1/2$ was replaced by $3/2$ because the formula in Theorem 1.11(b) was counting $\varphi \in \mathcal{P}$, which corresponds to no run in ℓ and so should not be counted. Since $\sigma(n+1) = \sigma(n) + o(1)$, we conclude that the expected number of distinct run lengths in all letters and in a single letter differs by $1 + o(1)$. A similar result holds for $\mathsf{E}(M_n)$. Since we found these results surprising, we did Monte Carlo runs. The results in Table 1 agree with the theory. The $\bar{m}_n$ and $\bar{M}_n$ are the observed average maxima for runs of a single letter and all runs, respectively. The former was calculated by averaging maximum run lengths over all $|A|$ letters. The number of Monte Carlo runs is R, and $\mathsf{E}(m_n)$ is the value in Theorem 1.11(a) for runs of a single letter.

Example 3.4 (labelled graphs of given genus). Let Γ be a connected graph with loops and/or multiple edges allowed or not. The graph is *embedded* in a surface Σ (a closed 2-manifold) if all the components of $\Sigma - \Gamma$ are simply connected. An alternative definition removes the simply connected restriction on components, leading to a larger set of graphs for a given surface. To obtain a functional equation of the form we have analysed, we insist that the graph not be a tree. In all cases, forbidding trees does not change asymptotic results since trees are a vanishingly small fraction of all graphs of a given genus. (If connectedness of the graph is not required then this is no longer true, and we have not studied that situation.)

The asymptotic enumeration of the number of various types of connected labelled graphs embeddable in a surface (or oriented surface) of genus g has been determined. See [8] and [12]. In all cases the number with n vertices satisfies

$$\bar{a}_n = a_n n! \sim A n^t r^{-n} n!,$$

where A depends on the surface Σ and type of graph, t depends on the surface, and $r < 1/e$ depends on the type of graph. We have the exponential generating function relation $A(x) = S(T(x))$, where $T(x) = P(x)$ counts rooted labelled trees and $S(x)$ counts graphs with all vertices of degree at least two that are embeddable in Σ. (A loop contributes two to the degree of its vertex.)

Since $\bar{t}_n = n^{n-1}$, once we show that Definition 1.4 is satisfied with $P = T$, it will follow that Theorems 1.6, 1.10 and 1.11 apply.

It is well known that $T(x) = xe^{T(x)}$, where $\rho(T) = 1/e$ and $T(1/e) = 1$. Definitions 1.4(a) and 1.4(b) follow from the known asymptotics for $\bar{a}_n$. It follows from $T(x) = xe^{T(x)}$ that

$$T'(x) = \frac{e^{T(x)}}{1 - T(x)},$$

which is analytic and non-zero for $|x| < 1/e$. Condition (c) follows as in the previous example.

4. Some open problems

The critical and subcritical cases behave quite differently since, as Gourdon [17] showed for algebraic–logarithmic functions, the expected value of the largest part is proportional to n. We do not know to what extent our results can be adapted to these cases. Someone considering the critical case might want to start with [2].

Even in the supercritical case there are many questions, some of which we now discuss.

If one considers less general $S(x)$ and $P(x)$, it should be possible to obtain more precise results. For example, the variables $X_Q(n)$ in Theorem 1.6(a) should satisfy a central limit theorem, perhaps through the application of the Quasi-Powers Theorem [15, Theorem IX.8]. Instead, we opted to look for the most general conditions under which we could obtain interesting results.

If we restricted $S(x)$ and $P(x)$ further, we might have been able to use analytic tools and obtain distribution results as in [17, 24].

Almost all our results deal with the situation where $\rho(P)$ is finite. What can be said when $P(x)$ is entire? The use of $\rho(P)$ in formulas appears to cause a problem; however, (1.1) suggests the following possibility. The right-hand equation in (1.1) can be rewritten as

$$\frac{1}{r^{\sigma(n)} p_{\sigma(n)}} = \frac{n}{rP'(r)}, \tag{4.1}$$

which contains nothing infinite. In his Example 8, Gourdon [17] has exponential generating functions with $S(x) = 1/(1-x)$ and $P(x) = e^x - 1$ and, in our notation, observes that $\sigma(n) \sim (\ln n)/(\ln \ln n)$. We show that this agrees with our (4.1). Since $r = \ln 2$, $P'(r) = 2$ and $p_n = 1/n! \approx (e/n)^n$, (4.1) becomes approximately $(\sigma(n)/C)^{\sigma(n)} = n$ and so

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

$\ln n \approx \sigma(n) \ln \sigma(n)$, whence $\sigma(n) \approx \ln n / \ln \ln n$. (Some sort of smoothness is required for p_k so that talking about $p_{\sigma(n)}$ for non-integral $\sigma(n)$ is reasonable and so that theorems can be proved.)

Suppose we are only interested in considering the sizes of a subset of the parts. One possibility is that we are studying a subset $\mathcal{Q}$ of $\mathcal{P}$ for which q_k is well-behaved. For example, we might consider graphs by genus but, instead of all trees, be interested only in those maximal trees which are unary–binary trees. On the other hand, consider runs in words where we are only interested in runs on a proper subset B of the alphabet. Using the approach in Example 3.2, $p_k = 1$ for all k and the letter in the run is determined by the letter it replaces in the support. Hence there is no $\mathcal{Q}$. A way around this was used in Example 3.3 for keeping track of runs in a particular letter. We are unable to deal with the case $|B| > 1$ or with words having restrictions as in Example 3.2.

Of course one can ask about more general forms of the generating function equation. The maps in (3.1) are a simple case. Perhaps more complicated is the problem of keeping track of runs in words where we are only interested in those runs whose letters lie in $B \subset A$.

5. Five lemmas

Although the results and lemmas in this paper are similar to those in [6], the approach to proving the lemmas is rather different. In the earlier paper, where locally restricted compositions were studied, infinite matrices were used. This was possible because the core contained one object of each size, and it was necessary because of local restrictions. Neither the possibility nor the necessity applies here, so our approach to proving the lemmas is different.

Lemma 5.1 (concentration and tails). *Under the same assumptions as Theorem 1.6(a), the following are true.*

(a) *Theorem 1.6(a) is true.*
(b) *There are constants $C_j > 0$ depending on what $\mathcal{Q}$ contains such that*

$$\Pr(X_{\mathcal{Q}}(n) < C_1 n) < C_2(1 + C_3)^{-n} \quad \textit{for all } n.$$

(c) *Let ζ_k be the number of parts of size k in a random structure of size n and let δ be given by Definition 1.4(b). There is a constant B and, for any fixed $\epsilon > 0$, a constant $B(\epsilon)$ such that, as $n \to \infty$,*

$$\Pr(\zeta_k > 0) \leqslant \begin{cases} Bnr^k p_k & \textit{when } k \leqslant n^\delta, \\ B(\epsilon)n(r+\epsilon)^k p_k & \textit{for all } n \textit{ and } k. \end{cases}$$

Recall that each structure in $\mathcal{A}$ is composed of a support in $\mathcal{S}$ and a sequence of parts in $\mathcal{P}$. Within a sequence we can consider certain parts as *marked*. They then form a marked subsequence and the resulting structure is a marked structure with those parts marked.

Lemma 5.2 (marked structures). *Make the same assumptions as in Theorem 1.6(a). Let C be any positive constant. Let $\ell_i(n)$, $1 \leqslant i \leqslant k$, be k-tuples of positive integers such that* $\max_i(\ell_i(n)) < (\log n)^C$.

(a) *Let $L_i(n)$ be a part of size $\ell_i(n)$ and let $H(x)$ be the generating function for marked structures where the associated marked subsequence is $\mathbf{L}(n) = (L_1(n), \ldots, L_k(n))$. Then*

$$\bar{h}_n \sim \begin{cases} \bar{a}_n \dfrac{(n/rP'(r))^k r^s}{k!} & \text{in the unlabelled case,} \\ \bar{a}_n \dfrac{(n/rP'(r))^k r^s}{k!} \displaystyle\prod_{i=1}^{k} \dfrac{1}{\ell_i(n)!} & \text{in the labelled case,} \end{cases}$$

as $n \to \infty$, where $s = \ell_1(n) + \cdots + \ell_k(n)$. Furthermore, the rate of convergence does not depend on the specific $L_i(n)$ but only on the $\ell_i(n)$.

(b) *Let $H(x)$ be the generating function for all marked structures containing k marked parts such that the ith distinguished part has size $\ell_i(n)$. Then*

$$\bar{h}_n \sim \bar{a}_n \left(\prod_{i=1}^{k} p_{\ell_i} \right) \frac{(n/rP'(r))^k r^s}{k!} \quad \text{as } n \to \infty, \text{ where } s = \ell_1(n) + \cdots + \ell_k(n). \tag{5.1}$$

The following is Lemma 12 of [16].

Lemma 5.3 (characterization of Poisson). *Let $(m)_k = m(m-1)\cdots(m-k+1)$ denote the falling factorial. Suppose that $\zeta_1, \ldots, \zeta_n = \zeta_1(n), \ldots, \zeta_n(n)$ is a set of non-negative integer variables on a probability space Λ_n, $n = 1, 2, \ldots$, and there is a sequence of positive reals $\sigma(n)$ and constants $\alpha > 1$ and $0 < c < 1$ such that*

(i) $\sigma(n) \to \infty$ *and* $n - \sigma(n) \to \infty$,

(ii) *for any fixed positive integers ℓ, $m_1, \ldots, m_\ell$, and sequences $k_1(n) < k_2(n) < \cdots < k_\ell(n)$ with $|k_i(n) - \sigma(n)| = O(1)$, $1 \leqslant i \leqslant \ell$, we have*

$$\mathsf{E}\big((\zeta_{k_1(n)})_{m_1}(\zeta_{k_2(n)})_{m_2}\cdots(\zeta_{k_\ell(n)})_{m_\ell}\big) \sim \prod_{j=1}^{\ell} \alpha^{(\sigma(n)-k_j(n))m_j}, \tag{5.2}$$

(iii) $\Pr(\zeta_{k(n)} > 0) = O\big(c^{k(n)-\sigma(n)}\big)$ *uniformly for all $k(n) > \sigma(n)$.*

Then there exists a function $\omega(n) \to \infty$ so that for $k = \lfloor \sigma(n) - \omega(n) \rfloor$, the total variation distance between the distribution of $(\zeta_k, \zeta_{k+1}, \ldots, \zeta_n)$, and that of $(Z_k, Z_{k+1}, \ldots, Z_n)$ tends to 0, where the $Z_j = Z_j(n)$ are independent Poisson random variables with $\mathsf{E}Z_j = \alpha^{\sigma(n)-j}$.

Remark 5.4. We will apply Lemma 5.3 to obtain the Poisson result in Theorem 1.6(b). This is used together with Mellin transforms and a result of Hitczenko and Knopfmacher [18] on sequences of geometric i.i.d. random variables, to prove various parts of Theorem 1.11.

Lemma 5.5 (plentitude of small parts). *We assume the hypotheses and notation of Theorem 1.10. Let $j = O(\ln n)$, ζ_j be the number of occurrences of a part of size j in a random*

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

structure of size n. Let k be any fixed positive integer. Then

$$\Pr(\zeta_j < k) \leqslant B\big(D^{-n} + \big(nr^j p_j\big)^{-k}\big), \tag{5.3}$$

for some constants $B > 0$ *and* $D > 1$*. Furthermore, if* $\omega(n) \to \infty$*, then*

$$\Pr\Bigg(\bigvee_{\substack{j<\sigma(n)-\omega(n)\\ j\in I(\mathcal{P})}} (\zeta_j < k)\Bigg) \leqslant \sum_{\substack{j<\sigma(n)-\omega(n)\\ j\in I(\mathcal{P})}} \Pr(\zeta_j < k) = o(1). \tag{5.4}$$

Lemma 5.6 (inherited smoothness). *If* a_n *satisfies Definition 1.4(b), then it satisfies Theorem 1.12(c) with* $s_k = a_k$, $K = \mathbb{N}$ *and any* $\epsilon \leqslant \delta$.

6. Proof of the lemmas

We start with asymptotics for coefficients of various derivatives. Assume that $\emptyset \neq \mathcal{Q} \subseteq \mathcal{P}$. Let $R(x) = P(x) - Q(x)$ and $A(x,t) = S(Q(x)t + R(x))$. For fixed $k \geqslant 0$ and ℓ we have $(y)_k \sim (y+\ell)^k$ as $y \to \infty$. Our goal is to show that

$$[x^n]\, S^{(k)}(P(x)) \sim \left(\frac{n}{rP'(r)}\right)^k a_n \quad\text{and}\quad [x^n]\frac{\partial^k}{\partial t^k}A(x,t)\bigg|_{t=1} \sim \left(\frac{nQ(r)}{rP'(r)}\right)^k a_n. \tag{6.1}$$

We have

$$\frac{\partial^k}{\partial t^k}A(x,t)\bigg|_{t=1} = S^{(k)}(P(x))Q(x)^k \tag{6.2}$$

and

$$A^{(k)}(x) = \sum_{i=1}^{k} S^{(i)}(P(x))G_{i,k}(x),$$

where $G_{i,k}$ is a polynomial in the derivatives of P and $G_{k,k}(x) = (P'(x))^k$. Note that

$$[x^n]\, A^{(k)}(x) \sim n^k a_{n+k} \sim (n/r)^k a_n.$$

Applying Theorem 2 of [4] with $B(x)$ equal to $S^{(k)}(P(x))$ and $A(x)$ equal to the various polynomials in derivatives of P, we have

$$(n/r)^k a_n \sim \sum_{i=1}^{k} G_{i,k}(r)\; [x^n]\, S^{(i)}(P(x)).$$

An easy induction on k shows that $[x^n]\, S^{(k)}(P(x)) \sim (n/r)^k a_n / G_{k,k}(r)$ and so the left-hand side of (6.1) follows. The right-hand equation comes from applying Theorem 2 of [4] to (6.2) and then using the left-hand equation.

Proof of Lemma 5.1. We begin with (a). Include in $A(x)$ the variable t to keep track of whatever $X_{\mathcal{Q}}(n)$ is counting, and write $A(x,t)$. The expected value of $X_{\mathcal{Q}}(n)$ is

$$\mu_n = [x^n]\, A_t(x,1)/a_n,$$

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

and its second moment about zero is

$$v_n = [x^n]\,(A_{tt}(x,1) + A_t(x,1))/a_n.$$

It follows from (6.1) that

$$\mu_n \sim nQ(r)/rP'(r) \quad \text{and} \quad v_n \sim \big(nQ(r)/rP'(r)\big)^2 \sim \mu_n^2.$$

Since $\sigma_n^2 = v_n - \mu_n^2$, this proves strong concentration, completing the proof of (a).

We now prove (b). Recall that $R(x) = P(x) - Q(x)$. For each $0 < t \leqslant 1$, let $r(t)$ be the radius of convergence of $S(Q(x)t + R(x))$. Note that $r(1) = r < \rho(P)$ Since $Q(x)$ and $R(x)$ have positive derivatives for $0 < x < \rho(P)$, there is a $t_0 < 1$ such that $Q(x)t + R(x) = \rho(S)$ has a strictly decreasing solution $x = r(t) < \rho(P)$ for all $t > t_0$. Fix some $t \in (t_0, 1)$ and choose $\epsilon > 0$ such that

$$\frac{r(t) - \epsilon}{r + \epsilon} > 1.$$

Then $[x^n]\,S(Q(x)t + R(x)) \leqslant C_2(r(t) - \epsilon)^{-n}$ for some C_2 and all n, and the total number of structures of size n (divided by $n!$ in the labelled case) is bounded below by $C_2(r+\epsilon)^{-n}$ for n sufficiently large to guarantee $a_n > 0$. Thus the fraction of structures with $X_Q(n) < C_1 n$ is bounded above by

$$\frac{t^{-C_1 n} C_2(r(t) - \epsilon)^{-n}}{C_2(r+\epsilon)^{-n}}.$$

It follows that we can choose $C_1 > 0$ sufficiently small that

$$\frac{r + \epsilon}{t^{C_1}(r(t) - \epsilon)} < 1.$$

This proves (b).

We now prove (c). We can overcount structures with a part of size k by inserting such a part at random in a sequence of parts before putting it into a support set. This leads to the generating function

$$\sum_{m=1}^{\infty} s_m P(x)^{m-1}(m p_k x^k) = p_k x^k S'(P(x)).$$

By (6.1), we have $[x^n]\,(x^k S'(P(x))) \sim nB_1 a_{n+1-k}$ for some B_1 and all $n \geqslant k$. Hence the probability that a random structure of size n contains a part of size k is bounded above by

$$\frac{n p_k B_1 a_{n-k+1}}{a_n}.$$

The first upper bound in (c) follows from $a_{n-k+1}/a_n \sim r^{k-1}$ when $k = O(n^\delta)$. However, since $a_n/a_{n+1} \sim r$ as $n \to \infty$, we have

$$\frac{a_{n-k+1}}{a_n} < B_2(\epsilon)(r+\epsilon)^k \quad \text{for all } n \geqslant k \geqslant 1,$$

which gives the general upper bound. □

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

Proof of Lemma 5.2. We begin with (a) in the unlabelled case. When the k marked parts are removed from an m-long sequence of parts, it is split into $k+1$ subsequences, some of which may be empty. If the number of parts in these subsequences is $m_0, \ldots, m_k$, then $m_0 + \cdots + m_k = m - k$, a $k+1$ part composition of $m-k$ with empty parts allowed. There are $\binom{m}{k}$ of these. Thus

$$H(x) = \sum_m s_m \binom{m}{k} P(x)^{m-k} \prod_{i=1}^{k} x^{\ell_i(n)} = \sum_m s_m \binom{m}{k} P(x)^{m-k} x^s = \frac{x^s S^{(k)}(P(x))}{k!},$$

where $s = \ell_1(n) + \cdots + \ell_k(n)$. By this and (6.1),

$$[x^n]\, H(x) = \frac{1}{k!}\, [x^{n-s}]\, S^{(k)}(P(x)) \sim \frac{1}{k!} \left(\frac{n-s}{rP'(r)}\right)^k a_{n-s}.$$

Part (a) follows in the unlabelled case from $s < k(\log n)^C$ and Definition 1.4(b).

The proof of (a) for the labelled case is the same, except that factorials appear. Thus we obtain

$$\sum_m s_m \binom{m}{k} P(x)^{m-k} \prod_{i=1}^{k} \frac{x^{\ell_i(n)}}{\ell_i(n)!} = \left(\prod_{i=1}^{k} \frac{1}{\ell_i(n)!}\right) \frac{x^s S^{(k)}(P(x))}{k!},$$

and the remainder of the proof proceeds as in the unlabelled case since we are dealing with a_n and h_n, not $\bar{a}_n$ and $\bar{h}_n$.

For (b) we note that since a marked structure includes the marking of the parts, different parts lead to different marked structures and hence we can simply sum the result in (a) over the $\prod \bar{p}_{\ell_i(n)}$ different sequences. □

Proof of Lemma 5.5. In the following proof, any reference to a part size k, whether directly or in an index of summation, implicitly assumes that $k \in I(\mathcal{P})$, that is, $p_k > 0$.

Let λ be the smallest k such that $p_k > 0$. Let $\bar{a}_n[\![\text{statement}]\!]$ be the number of structures of size n for which the statement is true. We want to study $\bar{a}_n[\![\zeta_j < k]\!]/\bar{a}_n$ for $\lambda < j < \sigma(n) - \omega(n)$. By Lemma 5.1(b) there is a $\delta > 0$ such that $\bar{a}_n[\![\zeta_\lambda < \delta n]\!]/\bar{a}_n \leqslant C_1 D^{-n}$ for some C_1 and $D > 1$. Given a structure with $\zeta_\lambda \geqslant \delta n$ and $\zeta_j < k$, we can convert it into a structure with $\zeta_j \geqslant k$ by replacing k parts of size λ with parts of size j. This is a many-to-any conversion because there are $\bar{p}_\ell$ parts of size ℓ. There are at least $\binom{\delta n}{k}$ ways to select the position in the support. Each position can have any of $\bar{p}_\lambda$ parts, each of which can be replaced by any of $\bar{p}_j$ parts. The result has at most $2k-1$ parts of size j and we cannot tell which were produced by replacement. Hence each new structure could have arisen in at most $\binom{2k-1}{k}$ ways. For unlabelled structures, we have

$$\binom{\delta n}{k} \bar{a}_n[\![(\zeta_\lambda \geqslant \delta n) \wedge (\zeta_j < k)]\!]\, \frac{p_j^k}{p_\lambda^k} \leqslant \binom{2k-1}{k} \bar{a}_{n+k(j-\lambda)},$$

and so

$$\bar{a}_n[\![(\zeta_\lambda \geqslant \delta n) \wedge (\zeta_j < k)]\!]/\bar{a}_n \leqslant \frac{\binom{2k-1}{k}}{\binom{\delta n}{k}}\, \frac{p_\lambda^k}{p_j^k}\, \frac{a_{n+k(j-\lambda)}}{a_n}. \tag{6.3}$$

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

For labelled structures, using multinomial coefficients to take care of relabelling, we obtain

$$\binom{\delta n}{k}\bar{a}_n[\![(\zeta_\lambda \geqslant \delta n)\wedge(\zeta_j<k)]\!]\ \frac{\bar{p}_j^k\binom{n+k(j-\lambda)}{j,\cdots,j,n-k\lambda}}{\bar{p}_\lambda^k\binom{n}{\lambda,\cdots\lambda,n-k\lambda}} \leqslant \binom{2k-1}{k}\bar{a}_{n+k(j-\lambda)},$$

which also gives (6.3).

Since k is fixed and $j=O(\ln n)$, $a_{n+k(j-\lambda)}/a_n \sim r^{-k(j-\lambda)}$. Thus

$$\begin{aligned}\Pr(\zeta_j<k) &\leqslant \Pr(\zeta_\lambda<\delta n)+\frac{a_{n+k(j-\lambda)}}{a_n}\,\frac{\binom{2k-1}{k}}{\binom{\delta n}{k}}\left(\frac{p_\lambda}{p_j}\right)^k\\ &< C_1D^{-n}+r^{-k(j-\lambda)}(1+o(1))\left(\frac{4p_\lambda}{(\delta n-k)p_j}\right)^k\\ &\leqslant C_1D^{-n}+C_2(nr^jp_j)^{-k}.\end{aligned}$$

The inequality in (5.4) is straightforward. The sum of the first term in (5.3) is $o(1)$ since there are $o(n)$ terms in the sum. We now bound the sum of the second term. Since $p_j > C_3e^{f(j)}\rho(P)^{-j}$,

$$\frac{1}{nr^jp_j}<\frac{e^{-f(j)}\alpha^j}{C_3n}.$$

Since $\alpha=\rho(P)/r>1$ and $f'(x)=o(1)$, it follows that, for sufficiently large j, $e^{-f(j)}\alpha^j$ is an increasing function with the ratio of consecutive terms bounded away from 1. At the upper range of the summation on j,

$$\begin{aligned}e^{-f(j)}\alpha^j|_{j=\sigma(n)-\omega(n)} &= e^{-f(\sigma(n)-\omega(n))}\alpha^{\sigma(n)-\omega(n)}\\ &=\alpha^{-(1+o(1))\omega(n)}e^{-f(\sigma(n))}\alpha^{\sigma(n)}=\frac{n}{rP'(r)}\alpha^{-(1+o(1))\omega(n)},\end{aligned}$$

by the definition of σ. It follows that the sum is $o(1)$. □

Proof of Lemma 5.6. Fix k sufficiently large that $a_n>0$ for $n\geqslant k$. We have

$$\frac{a_n}{a_k}=\prod_{i=1}^{\ell}\frac{a_{n_{i+1}}}{a_{n_i}} \quad\text{where}\quad \begin{cases} n_1=k,\\ n_{i+1}-n_i=\lfloor n_i^\delta\rfloor \quad \text{for } 1\leqslant i<\ell,\\ n_{\ell+1}=n.\end{cases}$$

Using Definition 1.4(b), we may replace $a_{n_{i+1}}/a_{n_i}$ with $r^{n_i-n_{i+1}}(1+o(1))$ where the $o(1)$ is as $i\to\infty$. Since $(1+o(1))=\exp(o(1))$ and the product contains $O(n^{1-\delta})$ factors, we obtain

$$\frac{a_n}{a_k}=r^{k-n}\exp(o(n^{1-\delta}))\quad\text{as}\quad n\to\infty,$$

which completes the proof. □

7. Proof of Theorem 1.6

Since (a) has been proved in Lemma 5.1, we turn to (b).

Since J is infinite, $\beta\geqslant\rho(P)>r$ and thus $\alpha>1$. We will show that the three hypotheses (i)–(iii) of Lemma 5.3 are satisfied with $\sigma(n)$ and α as in the theorem and $1/\alpha<c<1$.

Hypothesis (i) is obvious.

For (ii), let $\ell, m_1, \ldots, m_\ell$ be fixed and let $k_1(n) < k_2(n) < \cdots < k_\ell(n)$ be sequences satisfying $k_i(n) = \sigma(n) + O(1)$ and $k_i(n) \in I(\mathcal{P})$. The expectation $\mathsf{E}\prod(\zeta_{k_i})_{m_i}$, when multiplied by $\bar{a}_n$, equals the number of structures in which m_i parts of size k_i have been marked and linearly ordered, $1 \leqslant i \leqslant \ell$. Let $m = \sum_i m_i$ and let $(L_1, \ldots, L_m)$ be one of the $\binom{m}{m_1,\ldots,m_\ell}$ possible linear orders of m_1, k_1, *etc.* Given a marked structure counted by Lemma 5.2, the linear orders may be imposed on the m marked parts in $m!$ ways. Hence,

$$\bar{a}_n \,\mathsf{E}\prod(\zeta_{k_i})_{m_i} \sim m!\left(\prod_{i=1}^{\ell} p_{k_i}^{m_i}\right)\frac{\bar{a}_n(n/rP'(r))^m r^s}{m!},$$

where $s = \sum_i m_i k_i$. Dividing both sides by $\bar{a}_n$ and using $p_j \sim e^{f(j)}\beta^{-j}$, we have

$$\mathsf{E}\prod(\zeta_{k_i})_{m_i} \sim \exp\left(\sum_{i=1}^{\ell} m_i f(k_i)\right)(n/rP'(r))^m(r/\beta)^s \sim \left(ne^{f(\sigma(n))}/rP'(r)\right)^m \alpha^{-s},$$

where the last result follows from $f'(x) = o(1)$ and $k_i - \sigma(n) = O(1)$. Since $s = \sum k_j(n)m_j$ and $\sigma(n)$ is given by (1.1), this confirms hypothesis (ii).

To prove that (iii) holds, we consider $k(n) < (\log n)^2$ and $k(n) \geqslant (\log n)^2$ separately.

For $k(n) < (\ln n)^2$, apply the first bound in Lemma 5.1(c) to obtain

$$\Pr(\zeta_{k(n)} > 0) = O\left(ne^{f(k(n))}\alpha^{-k(n)}\right) = O\left(\alpha^{\sigma(n)-k(n)}e^{o(k(n)-\sigma(n))}\right) = O\left(c^{k(n)-\sigma(n)}\right),$$

for some constant $1/\alpha < c < 1$ and the bound implied by the big-oh is uniform. This establishes (iii) for $k(n) < (\ln n)^2$.

For $k(n) \geqslant (\ln n)^2$ we apply the second bound in Lemma 5.1(c) to obtain

$$\ln(\Pr(\zeta_k(n) > 0)) = -(k(n) - \sigma(n))\ln\alpha + k(n)\ln(1 + \epsilon/r)) + o(k(n) - \sigma(n)) + O(1).$$

Noting $k(n) \geqslant (\ln n)^2$, $\sigma(n) = O(\ln n)$, and $\alpha > 1$, we have, by choosing ϵ close to 0,

$$\ln(\Pr(\zeta_{k(n)} > 0)) < -\delta(k(n) - \sigma(n))$$

for some positive constant δ. This completes the proof. □

8. Proof of Theorem 1.10

By Theorem 1.6(b), we can estimate the probability of the various events in what follows using products of probabilities based on independent Poisson variables. It is understood in what follows that the ranges of various products, sums and event conjunctions are restricted to elements of $I(\mathcal{P})$. Recall that $\alpha > 1$.

We begin with (a). Let j be the maximum number in $I(\mathcal{P})$ such that $j < \sigma(n) - \omega_1(n)$. Then

$$\Pr(M_n < \sigma(n) - \omega_1(n)) = \Pr(M_n < j) \leqslant \Pr(\zeta_j < 1) = o(1),$$

the last by (5.3).

Since a smaller $\omega_2(n)$ leads to a stronger conclusion, we can assume that $\omega_2(n)$ here is no larger than the $\omega(n)$ in Theorem 1.6(b). With $t = \sigma(n) + \omega_2(n)$, Theorem 1.6(b) gives

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

us

$$\Pr(M_n < t)) = \Pr\left(\bigwedge_{j=t}^{n}\{\zeta_j = 0\}\right) \sim \prod_{j=t}^{n} \exp(-\alpha^{\sigma(n)-j}) = \exp\left(-\sum_{j=t}^{n} \alpha^{\sigma(n)-j}\right)$$
$$\geqslant \exp\left(\frac{-\alpha^{\sigma(n)-t}}{1-\alpha^{-1}}\right) \tag{8.1}$$
$$= \exp\left(\frac{-\alpha^{-\omega_2(n)}}{1-\alpha^{-1}}\right) = \exp(o(1)) \sim 1.$$

This proves the other inequality of (a).

Let $k(n)$ be the largest element of $I(\mathcal{P})$ that is less than $\sigma(n)$. We prove (b) by showing that the probability of a gap-free structure of size n with largest part size $k(n)$ is bounded away from zero. Let $\omega(n) \to \infty$. By Lemma 5.5 the fraction of structures not containing all parts of size less than $\sigma(n) - \omega(n)$ is $o(1)$. Hence it suffices to estimate the probability of the event

$$Z_n = \left(\bigwedge_{j=\sigma(n)-\omega(n)}^{k(n)} \{\zeta_j > 0\}\right) \bigwedge \left(\bigwedge_{j=\sigma(n)}^{n} \{\zeta_j = 0\}\right).$$

By Theorem 1.6(b), we obtain

$$\Pr(Z_n) \sim \left(\prod_{j=\sigma(n)-\omega(n)}^{k(n)} (1-\exp(-\alpha^{\sigma(n)-j}))\right)\left(\prod_{j=\sigma(n)}^{n} \exp(-\alpha^{\sigma(n)-j})\right). \tag{8.2}$$

The second product is the same as the one estimated in (8.1) except that now $t = \sigma(n)$. Thus that product is bounded away from zero. For the first product in (8.2) we note that, for $x > 0$,

$$1 - e^{-x} = 1 - \frac{1}{e^x} > 1 - \frac{1}{1+x} = \frac{1}{1+1/x} > e^{-1/x}.$$

Hence the first product has a lower bound

$$\exp\left(-\sum_{j \leqslant \sigma(n)} \alpha^{j-\sigma(n)}\right) > \exp\left(\frac{-1}{1-\alpha^{-1}}\right),$$

which is again bounded away from zero. This completes the proof of (b). □

9. Proof of Theorem 1.11

Proof of Theorem 1.11(a,b). Whenever $\omega(n) \to \infty$, we have a.a.s.

$$\sigma(n) + \omega(n) > M_n \geqslant D_n > \sigma(n) - \omega(n),$$

where the first inequality follows from Theorem 1.10(a), the second is trivial and the last follows from (5.4) with $k = 1$. The a.a.s. claims about $|D_n - \sigma(n)|$ and $|M_n - \sigma(n)|$ in (a) and (b) follow.

We now prove the $\mathsf{E}(D_n)$ formula in (b). Let ζ_j be the number of occurrences of parts of size j. Let $\omega(n)$ be any function that goes to infinity. We assume n is so large that all

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

$j \geqslant \lfloor \sigma(n) - \omega(n) \rfloor$ are in $I(\mathcal{P})$. Note that

$$\mathsf{E}(D_n) = \sum_{j=1}^{n} \Pr(\zeta_j > 0).$$

We split the sum into a central sum F_c over $\lfloor \sigma(n) - \omega(n) \rfloor \leqslant j \leqslant \lfloor \sigma(n) + \omega(n) \rfloor$, a sum F_s over smaller j and a sum F_ℓ over larger j. By Theorem 1.6(b), the terms in F_c are

$$1 - \exp(-\alpha^{\sigma(n)-j}) + o(1)$$

and so, if $\omega(n) \to \infty$ sufficiently slowly,

$$F_c = o(1) + \sum_{j=\lfloor \sigma(n)-\omega(n) \rfloor}^{\lfloor \sigma(n)+\omega(n) \rfloor} t_j(n), \quad \text{where} \quad t_j(n) = 1 - \exp(-\alpha^{\sigma(n)-j}).$$

By (5.4) with $k = 1$, $F_s + v = \lfloor \sigma(n) - \omega(n) \rfloor - 1 + o(1)$. Since the same is true for the sum of $t_j(n)$ over $j < \lfloor \sigma(n) - \omega(n) \rfloor$,

$$F_s + v = o(1) + \sum_{j < \lfloor \sigma(n)-\omega(n) \rfloor} t_j(n).$$

We now use Lemma 5.1(c) to show that $F_\ell = o(1)$. Since $p_k \sim e^{f(k)} \rho(P)^{-k}$, there is a $C(\epsilon)$ such that

$$\Pr(\zeta_j > 0) \leqslant C(\epsilon) n e^{f(j)} (r/\rho(P))^j F(j) \quad \text{where} \quad F(j) = \begin{cases} 1 & \text{if } j < n^\delta, \\ (1+\epsilon/r)^j & \text{otherwise.} \end{cases}$$

Splitting the sum according to these two cases, one easily has that $F_\ell = o(1)$. Putting this all together,

$$\mathsf{E}(D_n) + v = \sum_{j \geqslant 0} \bigl(1 - \exp(-\alpha^{\sigma(n)-j})\bigr) - 1 + o(1).$$

Let

$$h(x) = \sum_{j \geqslant 0} \bigl(1 - \exp(-\alpha^{x-j})\bigr).$$

Then $\mathsf{E}(D_n) + v + 1 = h(\sigma(n)) + o(1)$. We use the standard Mellin transform. (See [15, p.765], and also their Example B.5, which treats $\alpha = 2$.) It follows that

$$h(x) = x + \gamma \log e + \frac{1}{2} + P_0(x) + o(1), \tag{9.1}$$

where $P_0(x)$ is given by (1.2). This proves Theorem 1.11(b).

For the maximum part size M_n, we proceed in a similar manner:

$$\begin{aligned} \mathsf{E}(M_n) &= \sum_{j=1}^{n} \Pr(M_n \geqslant j) \\ &= \sum_{j=1}^{\lfloor \sigma(n)-\omega(n) \rfloor - 1} 1 + \sum_{j=\lfloor \sigma(n)-\omega(n) \rfloor}^{n} \bigl(1 - \Pr(\wedge_{i \geqslant j}\{\zeta_j = 0\})\bigr) + o(1) \end{aligned}$$

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

$$= \sum_{j=1}^{\lfloor\sigma(n)-\omega(n)\rfloor-1} 1 + \sum_{j=\lfloor\sigma(n)-\omega(n)\rfloor}^{n} \left(1-\exp\left(-\frac{1}{\alpha-1}\alpha^{\sigma(n)+1-j}\right)\right) + o(1)$$
$$= \sum_{j\geqslant 0}^{n} \left(1-\exp\left(-\frac{1}{\alpha-1}\alpha^{\sigma(n)+1-j}\right)\right) - 1 + o(1)$$
$$= h\big(\sigma(n)+1-\log(\alpha-1)\big) - 1 + o(1).$$

With (9.1), this proves (a). □

Proof of Theorem 1.11(c). By Theorem 1.11(b) we can limit k to $|k-\sigma(n)|<\omega_b(n)$. By Theorem 1.6 there is some $\omega(n)\to\infty$ such that, uniformly for $|k-\sigma(n)|<\omega(n)$, we have

$$q_n(k) = p(n)\left(\prod_{j=h(n)}^{k}(1-\exp(-\alpha^{\sigma(n)-j}))\right)\left(\prod_{j=k+1}^{n}\exp(-\alpha^{\sigma(n)-j})\right) + o(1)$$
$$= p(n)\left(\prod_{j=h(n)}^{k}(1-\exp(-\alpha^{\sigma(n)-j}))\right)\exp\left(-\sum_{j=k+1}^{n}\alpha^{\sigma(n)-j}\right) + o(1),$$

where $h(n)=\lfloor\sigma(n)-\omega(n)\rfloor$ and $p(n)$ is the probability that a random structure of size n contains all part sizes in $I(\mathcal{P})$ less than $h(n)$. A little calculation shows that we may extend the final product to include $j<h(n)$. By (5.4) with $k=1$, $p(n)\sim 1$. This gives (1.3).

A little calculation produces a bound on (1.3) when $|k-\sigma(n)|<B$, completing the proof. □

Proof of Theorem 1.11(d). Let $\Gamma=(\Gamma_1,\Gamma_2,\ldots,\Gamma_m)$ be a sequence of i.i.d. geometric random variables with parameter $p=(\alpha-1)/\alpha$. Hitczenko and Knopfmacher [18] showed that the probability that the sequence Γ is gap-free is given by the b_m in our (1.4), and they established the oscillation of b_m when $p\neq 1/2$.

Let $\omega(m)$ go to infinity arbitrarily slowly with m. Let M'_m be the largest Γ_i.

As was shown in [18] (similar to Theorem 1.11(b,c)), all part sizes less than $\sigma(m)-\omega(m)$ are asymptotically almost surely present in Γ and $|M'_m-\sigma(m)|<\omega(m)$. Let

$$\zeta'_j = |\{i:\Gamma_i=j\}|, \qquad \lambda_j = m(\alpha-1)\alpha^{-j},$$
$$k^- = \lfloor\sigma(m)-\omega(m)\rfloor, \qquad k^+ = \lfloor\sigma(m)+\omega(m)\rfloor.$$

When $k^-\leqslant k\leqslant k^+$,

$$\Pr(\zeta'_j=k)\sim e^{-\lambda_j}\lambda_j^k/k!$$

by the standard Poisson approximation for i.i.d. rare random variables. Starting in the middle of page 26 of [6] it was shown that $\{\zeta'_j : k^-\leqslant j\leqslant k^+\}$ are asymptotically independent.

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

Thus, with $k = \max\{j : \zeta_j' > 0\}$,

$$b_m \sim \sum_{k=k^-}^{k^+} \left(\prod_{j=k+1}^{k^+} e^{-\lambda_j} \right) \left(\prod_{j=k^-}^{k} (1 - e^{-\lambda_j}) \right)$$
$$\sim \sum_{k=k^-}^{k^+} \exp(-m\alpha^{-k}) \prod_{j=k^-}^{k} \bigl(1 - \exp(-m(\alpha-1)\alpha^{-j})\bigr). \tag{9.2}$$

Equation (9.2) is the same as the sum of (1.3) if $m = \alpha^{\sigma(n)+1}/(\alpha-1)$. However, (9.2) was derived under the assumption that m is an integer. We now treat (9.2) as a function of real variable m, say $h(m)$, and show that $h'(m) = o(1)$ as $m \to \infty$. It then follows that $h(x) \sim h(\lfloor x \rfloor)$ as $x \to \infty$ and we will be done. Call the terms in the sum (9.2) $T_k(m)$. We have

$$|T_k'(m)| < \left| \frac{T_k'(m)}{T_k(m)} \right| = |(\ln(T_k(m))'| \leqslant \alpha^{-k} + \sum_{j=k^-}^{k} \frac{(\alpha-1)\alpha^{-j}}{\exp\bigl(m(\alpha-1)\alpha^{-j}\bigr) - 1}$$
$$< \alpha^{-k} + \sum_{j=k^-}^{k} \frac{(\alpha-1)\alpha^{-j}}{m(\alpha-1)\alpha^{-j}} \leqslant \alpha^{-k^-} + \frac{k-k^-+1}{m} < \frac{\omega_1(m)}{m},$$

for some $\omega_1(m) \to \infty$ much slower than m. Since there are only $2\omega(m)$ values for k, we have $h'(m) = o(1)$. □

Proof of Theorem 1.11(e). It follows from Theorem 1.6 that

$$g_n(k) \sim \sum_{j>\sigma(n)-\omega(n)} \Pr\Bigl(\{\zeta_j = k\} \bigwedge \Bigl(\bigwedge_{i>j} \{\zeta_i = 0\} \Bigr) \Bigr).$$

Setting $j = \ell + \lfloor \sigma(n) \rfloor$ and $\delta(n) = \sigma(n) - \lfloor \sigma(n) \rfloor$,

$$g_n(k) \sim \sum_{\ell=-\infty}^{\infty} \frac{\alpha^{-k(\ell-\delta(n))}}{k!} \prod_{i\geqslant \ell} \exp(-\alpha^{-(i-\delta(n))})$$
$$\sim \sum_{\ell=-\infty}^{\infty} \frac{\alpha^{-k(\ell-\delta(n))}}{k!} \exp\left(\frac{-\alpha^{-(\ell-1-\delta(n))}}{\alpha-1} \right).$$

It follows from Poisson's summation formula [23] that

$$g_n(k) \sim \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k!} \exp(-2\pi i \ell t) \alpha^{-k(t-\delta(n))} \exp\left(\frac{-\alpha^{-(t-1-\delta(n))}}{\alpha-1} \right) dt.$$

Setting

$$z = \frac{1}{\alpha-1} \alpha^{-(t-1-\delta(n))},$$

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

we have

$$\begin{aligned} g_n(k) &\sim \frac{\log e}{k!}\left(\frac{\alpha-1}{\alpha}\right)^k \sum_{\ell=-\infty}^{\infty} \exp\big(-2\pi i\ell(\delta(n)-\log(\alpha-1))\big) \int_0^\infty e^{-z} z^{k-1+2\pi i\ell \log e} dz \\ &\sim \frac{\log e}{k!}\left(\frac{\alpha-1}{\alpha}\right)^k \sum_{\ell=-\infty}^{\infty} \Gamma\big(k+2\pi i\ell \log e\big) \exp\big(-2\pi i\ell(\sigma(n)-\log(\alpha-1))\big) \\ &\sim \frac{1}{k!}\left(\frac{\alpha-1}{\alpha}\right)^k P_k\big(\sigma(n)-\log(\alpha-1)\big) + \frac{\log e}{k}\left(\frac{\alpha-1}{\alpha}\right)^k. \end{aligned}$$

This completes the proof of (e). □

Proof of Theorem 1.11(f). By (5.4) and Theorem 1.10(a) we may limit our attention to parts j, for which $|j-\sigma(n)| \leqslant \omega(n)$. By Theorem 1.6, the probability that part j appears with multiplicity k is asymptotically $e^{-\mu_j}\mu_j^k/k!$ where $\mu_j = \alpha^{\sigma(n)-j}$. Using the Poisson summation formula as in the proof of (c), the expected number of parts of multiplicity k is asymptotic to

$$\begin{aligned} \sum_j \exp(-\alpha^{\sigma(n)-j})\frac{\alpha^{k(\sigma(n)-j)}}{k!} &\sim \frac{1}{k!}\sum_{\ell=-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\big(-2i\pi\ell t-\alpha^{-(t-\delta(n))}\big)\alpha^{-k(t-\delta(n))}dt \\ &\sim \frac{\log e}{k!}\sum_{\ell=-\infty}^{\infty} \exp\big(-2i\pi\ell\sigma(n)\big)\Gamma\big(k+2i\pi\ell\log e\big) \\ &\sim \frac{P_k(\sigma(n))}{k!}+\frac{\log e}{k}. \end{aligned}$$

The claim about $m_n(k)$ follows from the fact that $m_n(k) = \mathsf{E}(D_n(k)/D_n)$ and the tight concentration of D_n in (c), an argument used by Louchard [20] for unrestricted compositions. □

10. Proof of Theorems 1.12 and 1.13

We begin with some asymptotic results common to both proofs.

Given m, k and v define

$$\mu(v) = \frac{vP'(v)}{P(v)}, \tag{10.1}$$

$$\sigma(v)^2 = \frac{v\,d\mu(v)}{dv} = \mu(v)-\mu(v)^2+\frac{v^2P''(v)}{P(v)} \quad \text{and} \quad t = m-\mu(v)k.$$

We make use of (2.9) in Drmota [13], setting $M=1$ and replacing the double sum by $O(r^{3/2}/k)$, under the assumption that is $o(1)$. Using our notation, (2.9) gives the following uniform estimate in m and k when v is restricted to a closed subinterval of $(0,\rho(P))$ and $t = o(k^{2/3})$:

$$[x^m]\,P(x)^k = \frac{P(v)^k}{v^m\sqrt{2\pi\sigma(v)^2k}}\left(\exp\left(\frac{-t^2}{2\sigma(v)^2k}\right)\big(1+O(t/k+t^2/k^2)\big)+O(1/k)\right). \tag{10.2}$$

(We note in passing that (2.1) of [13] should be $\gcd\{i-j \mid y_iy_j>0\}=1$.)

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

We note for later use that $\sigma(v)^2 > 0$ implies that $d\mu/dv > 0$. Since we had some difficulty with Drmota's proof that $\sigma^2 > 0$, we give another here. It is related to the $\bar{\kappa}_2$ in his (2.11), so it suffices to prove that the latter is positive. Since (2.11) can be thought of in terms of the probability generating function $(P(vx)/P(v))^n$, it suffices to note that $P(vx)/P(v)$ has non-zero variance because $P(x)$ has at least two non-zero coefficients.

We split the range of summation in $\sum_k s_k\,[x^n]P(x)^k$ into three regions as follows. With r as in Definition 1.4, let

$$\lambda(v) = 1/\mu(v) = P(v)/(vP'(v)) \quad \text{and} \quad k^* := k^*(n) = \lambda(r)n. \tag{10.3}$$

Fix $0 < \delta < \epsilon/2$ and let $C > 0$ be arbitrary.

- The *central region* consists of those k such that $|k - k^*| < Cn^{1-\delta}$.
- The *upper tail* consists of those k such that $k - k^* \geqslant Cn^{1-\delta}$.
- The *lower tail* consists of those k such that $k^* - k \geqslant Cn^{1-\delta}$.

Our goal is to show that a_n is asymptotic to the sum over the central region.

For u in a small neighbourhood of 1, let

$$v = v(u) = P^{-1}(s/u). \tag{10.4}$$

We note $v(1) = r$. Also (10.1) and (10.4) define λ as a function of u. Now

$$\frac{d\,\ln(u^\lambda v)}{du} = \frac{d\lambda}{du}\ln u + \frac{\lambda}{u} + \frac{1}{v}\frac{dv}{du}.$$

Since

$$\frac{dv}{du} = -\frac{P(v)}{uP'(v)} = -\frac{v\lambda}{u},$$

we have

$$\frac{d\,\ln(u^\lambda v)}{du} = \frac{d\lambda}{du}\ln u, \tag{10.5}$$

and

$$\frac{d\lambda}{du} = -\frac{v\lambda}{u}\frac{d\lambda}{dv} \geqslant B_1$$

for some positive constant B_1 and u in a neighbourhood of 1. In the following, the B_i denote some positive constants.

Bounding the sum in the upper region. Let $u_0 = 1 + Cn^{-\delta}$, $v_0 = v(u_0)$, and $\lambda_0 = \lambda(u_0)$. We have from (c(ii))

$$\begin{aligned}\sum_{k\geqslant\lambda_0 n} s_k\,[x^n]\,P(x)^k &\leqslant u_0^{-\lambda_0 n}\,[x^n]\sum_{k\geqslant\lambda_0 n} s_k u_0^k P(x)^k\\ &\leqslant \exp(O(n^{1-\epsilon}))u_0^{-\lambda_0 n}\,[x^n]\sum_{k\geqslant 0} s^{-k}u_0^k P(x)^k\\ &= \exp(O(n^{1-\epsilon}))u_0^{-\lambda_0 n}\,[x^n](1-u_0P(x)/s)^{-1}\\ &= \exp(O(n^{1-\epsilon}))(u_0^{\lambda_0}v_0)^{-n},\end{aligned}$$

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

where we used the fact that v_0 is the unique singularity of $(1-u_0P(x)/s)^{-1}$ on its circle of convergence and it is a simple pole. When $u=1$, we have $v(1)=r$ and $\lambda(1)=k^*/n$. Integrating (10.5) from $u=1$ to u_0, we obtain

$$\ln(u_0^{\lambda_0}v_0)-\ln r \geqslant B_1u(\ln u-1)\big|_{u=1}^{u=u_0} = B_1\big((1+Cn^{-\delta})\big(\ln(1+Cn^{-\delta})-1\big)+1\big).$$

Since $\ln(1+x)>x-x^2/2$, we have

$$\begin{aligned}\ln(u_0^{\lambda_0}v_0)-\ln r &\geqslant B_1\big((1+Cn^{-\delta})(Cn^{-\delta}-C^2n^{-2\delta}/2-1)+1\big)\\ &\geqslant B_2n^{-2\delta}-O(n^{-3\delta})\geqslant B_3n^{-2\delta}\quad \text{for sufficiently large } n.\end{aligned}$$

It follows that the upper tail is bounded by

$$\exp(O(n^{1-\epsilon}))r^{-n}\exp(-B_4n^{1-2\delta}) = r^{-n}\exp(-B_5n^{1-2\delta}). \tag{10.6}$$

To show that this is negligible, it suffices to find a sufficiently large term in the central region.

By (c(i,iii)) there is a k within $O(n^{1-\epsilon})$ of k^* such that the term is at least

$$\exp(O(k^{1-\epsilon}))\,s^{-k}\,[x^n](P(x)^k).$$

Apply (10.2) with $m=n$ and $t=0$ to obtain

$$[x^n](P(x)^k)\geqslant \frac{B_6P(v)^k}{v^nk^{1/2}}.$$

Since $k-k^*=O(n^{1-\epsilon})$, it follows from (10.3) that $v=v(1)+O(k/n-k^*/n)=r+O(n^{-\epsilon})$ and $P(v)=s(1+O(n^{-\epsilon}))$. Thus

$$s_k\,[x^n](P(x)^k)\geqslant \exp(O(k^{1-\epsilon}))\,s^{-k}\frac{B_7s^k}{r^nn^{1/2}}\exp(-B_8n^{1-\epsilon})\geqslant r^{-n}\exp(-B_9n^{1-\epsilon}).$$

Comparing this with (10.6) shows that the upper tail is negligible.

Bounding the sum in the lower region. The estimation for the lower region is similar. We now set $u_0=1-n^{-\delta}$ and note that

$$\begin{aligned}\sum_{k\leqslant\lambda_0n}s_k\,[x^n]\,P(x)^k &\leqslant u_0^{-\lambda_0n}\,[x^n]\sum_{k\leqslant\lambda_0n}s_ku_0^kP(x)^k\\ &\leqslant \exp(O(n^{1-\epsilon}))\,u_0^{-\lambda_0n}\,[x^n]\sum_{k\geqslant0}s^{-k}u_0^kP(x)^k\\ &= \exp(O(n^{1-\epsilon}))\,u_0^{-\lambda_0n}\,[x^n](1-u_0P(x)/s)^{-1}\\ &= \exp(O(n^{1-\epsilon}))\,(u_0^{\lambda_0}v_0)^{-n}.\end{aligned}$$

Integrating (10.5) from $u=u_0$ to $u=1$, we obtain

$$\ln(u_0^{\lambda_0}v_0)-\ln r\geqslant B_3n^{-2\delta}.$$

The rest follows from exactly the same argument as for the upper range.

Proof of Theorem 1.12. Definition 1.4(a) is just hypothesis (a). Definition 1.4(c) is an immediate consequence of Theorem 1 of [7]. To prove (b) we first show that the sum for

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

a_{n+t} can be limited to the same region for $t = O(n^\delta)$. Then we estimate the ratio of terms with the same k value in the two central region sums.

Adjusting the a_{n+t} sum's central region. Since n was arbitrary, the above argument shows that we can restrict the sum for a_{n+t} to its central region, which will differ slightly from the region for a_n. Since C was arbitrary and t in Definition 1.4(b) is small compared to n, we can choose a C for a_{n+t} so that its central region is included in the region for a_n. Since adding additional terms to the central region does not affect the asymptotics, we can use the same central region for a_n and a_{n+t}.

Ratio of the central sums. It suffices to show that the ratios of all pairs of terms with the same value of k and $s_k > 0$ approach r^t uniformly. Fix $0 < \beta < \min\{\delta, 1/2\}$ and assume $0 < t \leqslant n^\beta$. Since

$$\frac{s_k\ [x^n](P(x)^k)}{s_k\ [x^{n+t}](P(x)^k)} = \frac{[x^n](P(x)^k)}{[x^{n+t}](P(x)^k)},$$

we apply (10.2) with $m = n$ and $m = n + t$ where $\mu = n/k$. Thus, uniformly in the central region, we have

$$\begin{aligned}\frac{s_k\ [x^n](P(x)^k)}{s_k\ [x^{n+t}](P(x)^k)} &= \frac{v^t(1 + O(1/k))}{\exp(-t^2/2\sigma^2 k)(1 + O(t/k)) + O(1/k)}\\ &= v^t(1 + O(t^2/k)) = v^t(1 + o(1)).\end{aligned}$$

As $k - \mu n = O(n^{1-\delta})$, estimates like those following (10.4) lead to $v = r(1 + O(n^{-\delta}))$ uniformly and so $v^t = r^t\big(1 + O(\exp(B_{10} t n^{-\delta}))\big)$ uniformly, completing the proof. □

Proof of Theorem 1.13. Recall that $\delta < \epsilon/2$ and so $\delta < 1/2$. Define

$$w(n) = n^{1-\delta} \quad \text{and} \quad H(x) = \sum_{1 \leqslant k \leqslant x} s_k \rho^k(S).$$

Since $G'(t) = o(G(t)/t)$ as $t \to \infty$, we have, for $|x| \leqslant w(n)$,

$$G(\mu n + x) = G(\mu n) + o(xG(\mu n)/n) = G(\mu n) + o(n^{-\delta} G(\mu n)) \sim G(\mu n). \tag{10.7}$$

Thus, for $|x| \leqslant w(n)$,

$$H(\mu n + x) = (\mu n + x)G(\mu n + x) + o(n^{1/2} G(\mu n)). \tag{10.8}$$

Instead of (10.2), it is more convenient to use Drmota's equation (2.11) to estimate $[x^n]\,P(x)^k$. His ρ is our r, and his $\bar{\mu}$ is our μ. Setting $M = 1$, using the first part of his Remark 5 and converting to our notation, his (2.11) becomes

$$[x^n]\,P(x)^k = \frac{kP(r)^k}{nr^n\sqrt{2\pi\sigma^2 n}}\left(\exp\left(\frac{-t^2}{2\sigma^2 n}\right)\left(1 + \frac{b_{1,1}t}{n} + \frac{b_{2,3}t^3}{n^2}\right) + O(1/n)\right) \tag{10.9}$$

where the $b_{i,j}$ are some constants,

$$\mu = \frac{P(r)}{rP'(r)}, \quad t = k - \mu n \quad \text{and} \quad \sigma^2 = \frac{\mu^3 r^2 P''(r)}{P(r)} + \mu^2 - \mu.$$

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

As shown earlier in this section, it suffices to limit k to the central region when estimating

$$\sum_k s_k\,[x^n](P(x)^k).$$

Let $L_n = \mu n - w(n)$ and $U_n = \mu n + w(n)$. The central region is $\mathbb{N} \cap (L_n, U_n]$. We have

- $t/n = o(1)$ uniformly in the central region,
- $t^3/n^2 = o(1)$ uniformly for $|t| \leqslant n^{3/5}$,
- $\exp(-t^2/2\sigma^2 n)(t^3/n^2) = o(1/n)$ uniformly for $n^{3/5} < |t| \leqslant w(n)$.

Thus (10.9) implies

$$[x^n]\,P(x)^k = \frac{kP(r)^k}{nr^n\sqrt{2\pi\sigma^2 n}}\left(\exp\left(\frac{-t^2}{2\sigma^2 n}\right)(1+o(1)) + O(1/n)\right)$$

uniformly in the central region, and so

$$a_n \sim \frac{\mu r^{-n}}{\sqrt{2\pi\sigma^2 n}}\left(O\left(\frac{H(U_n)-H(L_n)}{n}\right) + \sum_{L_n<k\leqslant U_n} s_k \rho(S)^k \exp\left(\frac{-(k-\mu n)^2}{2\sigma^2 n}\right)(1+o(1))\right). \tag{10.10}$$

By (10.7) and (10.8),

$$H(U_n) - H(L_n) = 2w(n)G(\mu n) + O(n)o(n^{-\delta}G(\mu n)) + o(n^{1/2}G(\mu n)).$$

Thus $O((H(U_n)-H(L_n))/n) = o(G(\mu n))$, which we will see is small compared to the rest of (10.10).

Using Abel's summation formula [1, Theorem 4.2], we have

$$\sum_{L_n<k\leqslant U_n} s_k \rho(S)^k \exp\left(\frac{-(k-\mu n)^2}{2\sigma^2 n}\right) = \big(H(U_n)-H(L_n)\big)\exp\left(\frac{-w(n)^2}{2\sigma^2 n}\right) - \int_{L_n}^{U_n} H(x)\frac{d}{dx}\exp\left(\frac{-(x-\mu n)^2}{2\sigma^2 n}\right)dx.$$

By (10.7) and (10.8), the first term on the right-hand side is $\exp(-n^{1-2\delta}/2\sigma^2)O(nG(\mu n))$, which is $o(n^{-c}G(\mu n))$ for all c since $\delta < 1/2$. It follows from (10.8) that the error made in replacing $H(x)$ with $xG(x)$ in the integral is

$$o(n^{1/2}G(\mu n))\int_{-\infty}^{\infty}\left|\frac{d}{dx}\exp\left(\frac{-x^2}{2\sigma^2 n}\right)\right|dx = o(n^{1/2}G(\mu n)).$$

Thus

$$\sum_{L_n<k\leqslant U_n} s_k \rho(S)^k \exp\left(\frac{-(k-\mu n)^2}{2\sigma^2 n}\right) \sim -\int_{L_n}^{U_n} xG(x)\frac{d}{dx}s\exp\left(\frac{-(x-\mu n)^2}{2\sigma^2 n}\right)dx + o(n^{1/2}G(\mu n)).$$

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

Using integration by parts and (10.7), this becomes

$$
\begin{aligned}
&(U_nG(U_n) - L_nG(L_n))\exp\left(-\frac{w(n)^2}{2\sigma^2 n}\right)\\
&\qquad - \int_{L_n}^{U_n} (G(x) + xG'(x))\exp\left(-\frac{(x-\mu n)^2}{2\sigma^2 n}\right)dx + o(n^{1/2}G(\mu n))\\
&= \int_{L_n}^{U_n} G(x)(1+o(1))\exp\left(-\frac{(x-\mu n)^2}{2\sigma^2 n}\right)dx + o(n^{1/2}G(\mu n))\\
&= G(\mu n)(1+o(1))\int_{L_n}^{U_n} \exp\left(-\frac{(x-\mu n)^2}{2\sigma^2 n}\right)dx + o(n^{1/2}G(\mu n))\\
&\sim \sqrt{2\pi\sigma^2 n}\, G(\mu n).
\end{aligned}
$$

The theorem follows from this and (10.10). □

11. Proof of Theorem 2.2

We use (10.9) to prove (a). Here $P(r) = 1$. Since there are only about $n^{1/2}$ supports for compositions of n and since there is an i^2 within $O(n^{1/2})$ of any value μn, it follows that, for asymptotic purposes, we can sum (10.9) over those values of k for which $|t| < n^{3/5}$. This leads to

$$
a_n \sim \sum_{|i^2-\mu n|<n^{3/5}} \frac{\mu r^{-n}}{\sqrt{2\pi\sigma^2 n}} \exp\left(\frac{-(\mu n - i^2)^2}{2\sigma^2 n}\right).
$$

Set $x = \sqrt{\mu n}$, $\delta = \{x\}$, the fractional part of x, and $i = x + j - \delta$. Then

$$
i^2 - \mu n = (x + j - \delta)^2 - x^2 = 2x(j-\delta) + (j-\delta)^2.
$$

Hence $|i^2 - \mu n| < n^{3/5}$ implies $j = O(n^{1/10})$. Thus, uniformly in j over the range of summation,

$$
\frac{-(\mu n - i^2)^2}{2\sigma^2 n} = \frac{-(2x(j-\delta))^2}{2\sigma^2 n} + o(n^{1/5}/n) = \frac{-2\mu(j-\delta)^2}{\sigma^2} + O(n^{-4/5}).
$$

The sum can be extended over all i since the tail contribution is negligible.

Part (b) follows from (a) by noting that $P(x) = x/(1-x)$ and so, with a little calculation, $r = \mu = \sigma = 1/2$.

We now prove (c). We first note

$$
a_n = \sum_{i\geqslant 1} \binom{n-1}{i^d - 1}.
$$

Let $a = \max\{i \mid 2i^d - 1 < n\}$. We will show that

$$
\sum_{i\leqslant a} \binom{n-1}{i^d-1} \sim \binom{n-1}{a^d-1} \quad \text{and} \quad \sum_{i>a} \binom{n-1}{i^d-1} \sim \binom{n-1}{(a+1)^d-1}. \tag{11.1}
$$

https://doi.org/10.1017/S0963548314000315 Published online by Cambridge University Press

For $k \leqslant (n-1)/2$,

$$\frac{\binom{n-1}{k-j}}{\binom{n-1}{k}} = \prod_{i=0}^{j-1} \frac{k-j+i+1}{n-k+i} = \prod_{i=0}^{j-1}\left(1 - \frac{n-2k+j-1}{n-k+i}\right)$$
$$< \exp\left(-\sum_{i=0}^{j-1} \frac{n-2k+j-1}{n-k+i}\right) < \exp(-j^2/n).$$

Set $k = a^d - 1$ and $k - j \leqslant (a-1)^d - 1$. Since $a \approx (n/2)^{1/d}$, the smallest value of j is approximately da^{d-1} and so $j \geqslant Cn^{(d-1)/d}$ for some $C > 0$. Thus

$$\sum_{i<a} \binom{n-1}{i^d - 1} < \binom{n-1}{a^d - 1} n \exp(-Cn^{2(d-1)/d}/n) = \binom{n-1}{a^d - 1} n \exp(-Cn^{1-2/d}) = o(1).$$

A similar argument works for the second sum in (11.1). □

Acknowledgements

We thank the referees for their comments, which led to improvements in this paper.

References

[1] Apostol, T. (1976) *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer.

[2] Banderier, C., Flajolet, P., Schaeffer, G. and Soria, M. (2001) Random maps, coalescing saddles, singularity analysis and Airy phenomena. *Random Struct. Alg.* **19** 194–246.

[3] Bender, E. A. (1974) Convex n-ominoes. *Discrete Math.* **8** 219–226. (The value of f in (11) is incorrect. The correct value, $f = 2.67564$, is in the abstract.)

[4] Bender, E. A. (1974) Asymptotic methods in enumeration. *SIAM Review* **16** 485–515.

[5] Bender, E. A. and Canfield, E. R. (2009) Locally restricted compositions II: General restrictions and infinite matrices. *Electron. J. Combin.* **16** R108.

[6] Bender, E. A., Canfield, E. R. and Gao, Z. C. (2012) Locally restricted compositions IV: Nearly free large parts and gap-freeness. *Electron. J. Combin.* **19** #14.

[7] Bender, E. A., Canfield, E. R. and Richmond, L. B. (2008) Coefficients of functional composition often grow smoothly. *Electron. J. Combin.* **15** R21.

[8] Bender, E. A. and Gao, Z. C. (2011) Asymptotic enumeration of labelled graphs by genus. *Electron. J. Combin.* **18** #13.

[9] Bender, E. A. and Gao, Z. C. Locally restricted compositions V: Convex polyomino support. In preparation.

[10] Bhatia, D. P., Prasad, M. A. and Arora, D. (1997) Asymptotic results for the number of multidimensional partitions of an integer and directed compact lattice animals. *J. Phys. A: Math. Gen.* **30** 2281–2285.

[11] Bousquet-Mélou, M. and Rechnitzer, A. (2002) Lattice animals and heaps of dimers. *Discrete Math.* **258** 235–274.

[12] Chapuy, G., Fusy, É., Giménez, O., Mohar, B. and Noy, M. (2011) Asymptotic enumeration and limit laws for graphs of fixed genus. *J. Combin. Theory Ser. A* **118** 748–777.

[13] Drmota, M. (1994) A bivariate asymptotic expansion of coefficients of powers of generating functions. *Europ. J. Combin.* **15** 139-152.

[14] Flajolet, P. and Odlyzko, A. (1990) Singularity analysis of generating functions. *SIAM J. Discrete Math.* **3** 216–240.

[15] Flajolet, P. and Sedgewick, R. (2009) *Analytic Combinatorics*, Cambridge University Press.

[16] Gao, Z. C. and Wormald, N. C. (2000) The distribution of the maximum vertex degree in random planar maps. *J. Combin. Theory, Ser. A* **89** 201–230.

[17] Gourdon, X. (1998) Largest component in random combinatorial structures. *Discrete Math.* **180** 185-209.

[18] Hitczenko, P. and Knopfmacher, A. (2005) Gap-free compositions and gap-free samples of geometric random variables. *Discrete Math.* **294** 225–239.

[19] Klarner, D. A. and Rivest, R. L. (1974) Asymptotic bounds for the number of convex n-ominoes. *Discrete Math.* **8** 31–40.

[20] Louchard, G. (2003) The number of distinct part sizes of some multiplicity in compositions of an integer: A probabilistic analysis. In *Discrete Random Walks: Paris 2003*, pp. 155–170 (electronic).

[21] Louchard, G. (2008) Matrix compositions: A probabilistic analysis. *Pure Math. Appl.* **19** 127–146.

[22] Munarini, E., Poneti, M. and Rinaldi, S. (2009) Matrix compositions. *J. Integer Seq.* **12** #09.4.8.

[23] Olver, F. W. J. (1974) *Asymptotics and Special Functions*, Academic Press.

[24] Prodinger, H. and Wagner, S. Bootstrapping and double-exponential limit laws. http://mathsci.sun.ac.za/∼swagner/pub.html.