Adv. Appl. Prob. **48,** 1211–1234 (2016)
*doi:*10.1017/apr.2016.71

BRANCHING PROCESSES IN GENERALIZED AUTOREGRESSIVE CONDITIONAL ENVIRONMENTS

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Abstract

Branching processes in random environments have been widely studied and applied to population growth systems to model the spread of epidemics, infectious diseases, cancerous tumor growth, and social network traffic. However, Ebola virus, tuberculosis infections, and avian flu grow or change at rates that vary with time—at peak rates during pandemic time periods, while at low rates when near extinction. The branching processes in generalized autoregressive conditional environments we propose provide a novel approach to branching processes that allows for such time-varying random environments and instances of peak growth and near extinction-type rates. Offspring distributions we consider to illustrate the model include the generalized Poisson, binomial, and negative binomial integer-valued GARCH models. We establish conditions on the environmental process that guarantee stationarity and ergodicity of the mean offspring number and environmental processes and provide equations from which their variances, autocorrelation, and cross-correlation functions can be deduced. Furthermore, we present results on fundamental questions of importance to these processes—the survival-extinction dichotomy, growth behavior, necessary and sufficient conditions for noncertain extinction, characterization of the phase transition between the subcritical and supercritical regimes, and survival behavior in each phase and at criticality.

Keywords: Branching processes in random environment; GARCH; Galton–Watson process; extinction; phase transition; limit theorems

2010 Mathematics Subject Classification: Primary 60J80
Secondary 62M10; 60G10; 60F05

1. Introduction

Branching processes in random environments have been extensively studied and applied to population growth systems to model the spread of epidemics, infectious diseases, cancerous cell or tumor growth, and social network traffic. However, Ebola viruses, tuberculosis infections, avian flu, and diseases that can turn resistant to treatment and become chronic illnesses notoriously spread, grow, or change at rates that vary with time—at peak rates during pandemic time periods, while at low rates when near extinction or in remission—behaviors that the known branching processes in random environments cannot accommodate. The branching process (BP) with dynamic random environments that is proposed here (defined in Section 2), called a *branching process in generalized autoregressive conditional environments* (GARCE BP), provides a novel and simple approach to branching processes that allows for such time-varying random environments and instances of peak growth and near extinction-type rates, as observed in filovirus outbreaks and a number of biomedical and other applications where branching processes have been applied directly or indirectly as an embedded process. The latter

Received 2 June 2015; revision received 7 February 2016.
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scenario is exemplified by branching random walks, contact processes, and voter models within the interacting particle systems family [14]. It should be evident that in situations such as propagation of data and information, paper citations, and social media and network traffic, where these processes have been or can be applied, data or network transmission rates may spike at times and dwindle at others. Since the proposed branching processes reside at the interface to time series, inferential and predictive time series techniques are readily available and implementable, which is not the case for the known branching processes in random environments. The GARCE BP can also accommodate recently studied epigenetic and starvation health inheritance models [17], [20], where strikingly the inheritance process not only depends on the current random environment but also on the environment that existed one, two, or three generations in the past, without DNA involvement.

At the time of this writing, the largest Ebola outbreak ever takes its dramatic course in Guinea, Liberia, and Sierra Leone. Ebola is one of the world's most deadly diseases that can kill the majority of those infected within days. The outbreak has been traced back to its first suspected case, a two-year old child who died on December 6, 2013, after being sick for four days, in Guéckédou, a Meliandou village. So did the child's sister, mother, grandmother, and the village midwife after hospitalization shortly after. A couple of months later, it became evident that the number of infected cases and deaths were increasing exponentially after the disease span out of control earlier. In Figure 1 we display cumulative and approximately weekly numbers of infections (suspected, probable, and confirmed cases) and deaths attributed to the virus during March 24, 2014 through October 14, 2014 overall and by region. The plots use approximately weekly data drawn from the website www.cdc.gov of the Centers for Disease Control and Prevention (accessed at multiple times during 2014). After drastic interventions, in November of 2014 the number of Ebola infections and death cases began to slow and the death rate, calculated as the ratio of (total number of deaths) divided by (total number of infections) at a given time, had decreased to approximately 0.36 from an initial rate close to 0.7 during March through July of the same year. Later in October after the cutoff date of October 14, 2014, the data reported exhibited apparent inaccuracies as the cumulative total of infection cases decreased between weeks for a couple of weeks. It is obvious from the graphs of the Ebola cases over time that the infection and death rates at a given time depend on preceding values of the transmission rate and death rate, respectively. Furthermore, it is evident that the Ebola disease had been spreading at a rate in the supercritical regime but would have to settle down in the subcritical or critical phase for the contagion to be contained, as we will see in the sequel from the study of the model at hand. The infection rates are also impacted by external factors such as the proportion of people who are immune to the virus, virus mutations, treatments such as blood transfusions from survivors, and control measures like border closing, management of the outbreak, risk aversion, incidences of noncontrolled existing Ebola cases, and precautions taken by those who care for the infected and health personnel.

In Figures 2 and 3 we display two simulations of a Poisson GARCE BP, which may serve as a model for the weekly number of Ebola infected cases and deaths during an outbreak, along with its average offspring number, Poisson rate, logarithm of the Poisson rate over time, and sample autocorrelation functions of the population size and average offspring number. Thus, two possible trajectories are shown for the weekly number of Ebola infected cases. The model parameters were fixed so that the model is near the critical value between certain extinction and noncertain extinction. In this branching process model that evolves in a random environment, the intensity parameter of the distribution of the environmental sequence depends on past values of the parameter and offspring numbers or observed cases. In Theorem 10

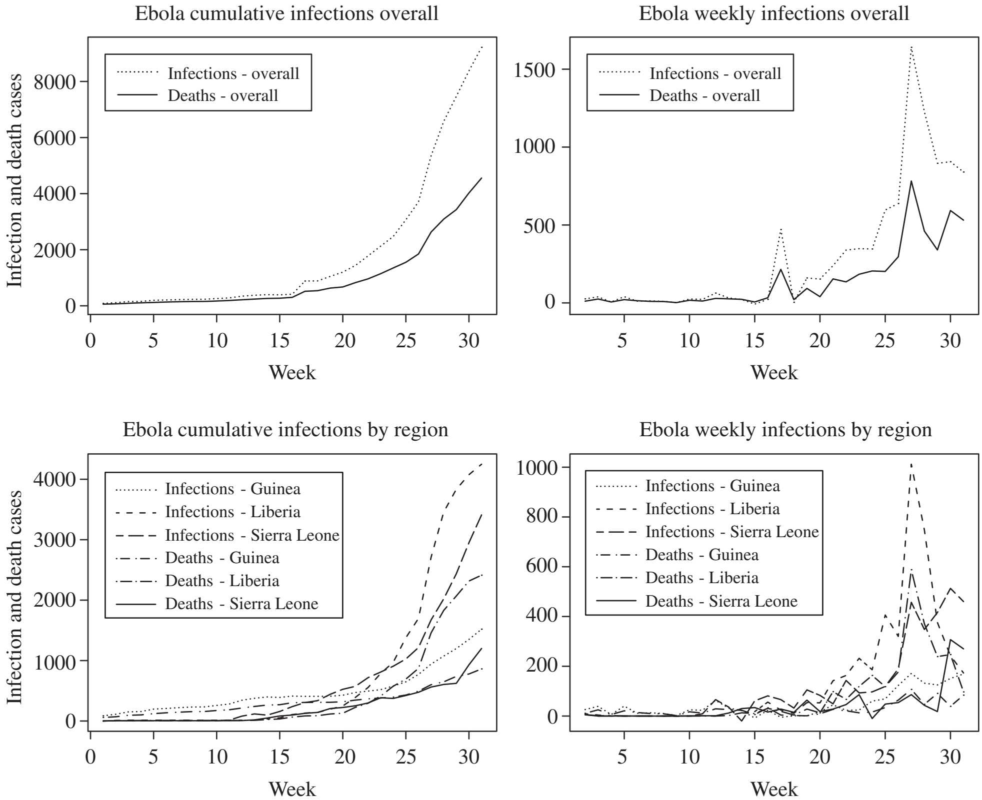

FIGURE 1: Cumulative and weekly numbers of Ebola infections and deaths by week from March 24, 2014 through October 14, 2014 overall and by region (Guinea, Liberia, and Sierra Leone).

we will establish that, when in the supercritical regime, conditional on survival, the spread of the disease grows exponentially indefinitely, whereas in the subcritical and critical regimes, the spread is halted with probability 1 for almost all random environments and the infection becomes extinct rather rapidly. As the example trajectories illustrate, the branching process can evolve in the supercritical phase for some time, switch between the supercritical and subcritical phases, and eventually transition to the subcritical phase to become extinct.

While here the focus is on introducing and motivating the GARCE BP model and studying its properties, Ebola outbreak data analyses prior to and after the substantial intervention are presented and discussed in [11]. The latter research explores the aspects of detecting and estimating an intervention effect in the GARCE branching processes. During an outbreak, a timely assessment as to whether an intervention has a sufficient impact to stabilize and eventually end it is equally important as early detection and accurate prediction of the magnitude of the outbreak several months before chaos and disarray take over. The GARCE branching processes proposed by the author in early 2014, when applied to the weekly Ebola virus data, were able to adequately address both of these problems in a timely fashion during the 2014 outbreak in West Africa.

The GARCE BP finds applications, as well, in the context of epigenetic inheritance and starvation health inheritance. The former example might play a role in health problems such

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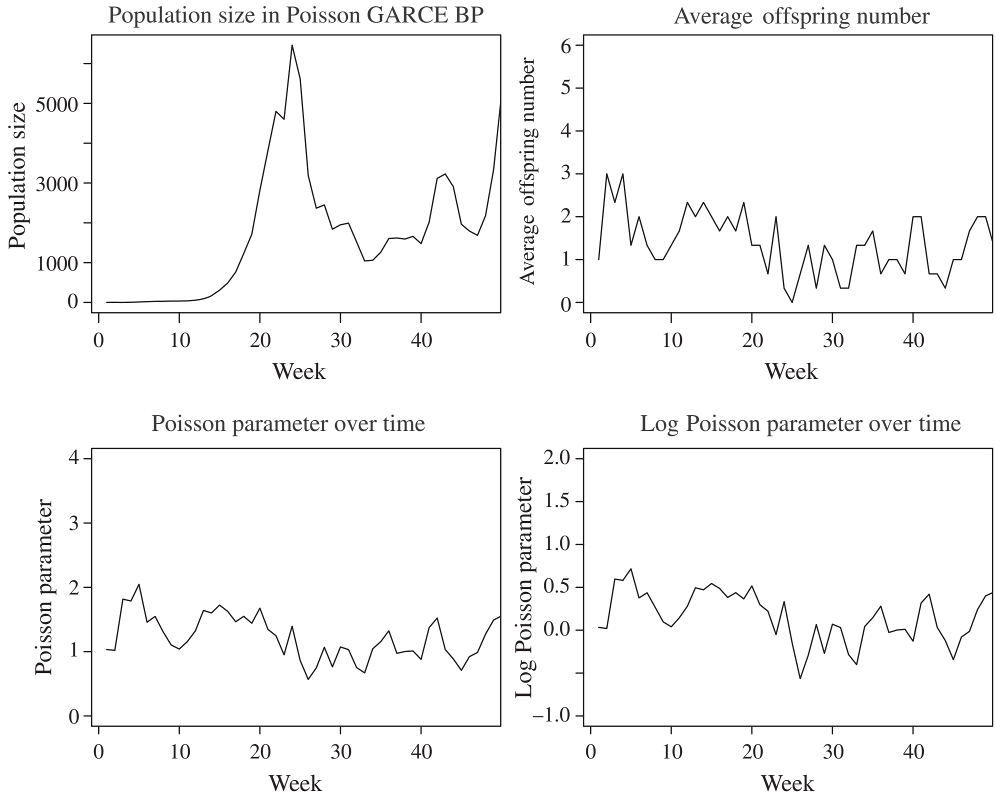

FIGURE 2: Simulated realization of a Poisson GARCE(1, 1) BP with $\alpha_0 = 0.31$, $\alpha_1 = 0.4$, $\beta_1 = 0.3$, $K = 3$, and $\alpha_0/(1 - \alpha_1 - \beta_1) = 1.033$ along with its average offspring number, Poisson rate, and logarithm of the Poisson rate over time.

as obesity and diabetes and more generally, the evolution of species. Conceivably, the health of someone's children may thus be affected by what his/her great-grandmother was exposed to during pregnancy. Recent findings [20] in animals indicate that pollutants, stress, diet, and other environmental factors can cause persistent changes in the mix of epigenetic marks in chromosomes. The induced epigenetic modifications that cause disease and reproductive problems are acquired, without any change in the DNA sequence, and passed on to later generations along with any resulting health risks. The second inheritance example where present changes in the environmental sequence are thought to be inherited through at least three consecutive generations is the starvation model described in [17]. According to human famines and animal studies, starvation can affect the health of descendants and offer an explanation on how such acquired traits might be transmitted from one generation to the next. If modeled by a GARCE BP, such inheritance would play out differently depending on whether the process is supercritical or (sub)critical.

Another large area of research where the integer-valued generalized autoregressive conditional heteroscedastic (INGARCH) models upon which the GARCE branching processes are built are of vital importance consists of clinical trials in drug development with categorical endpoints measured over time. Traditionally, in randomized, controlled trials (RCTs), patients are treated homogeneously and patients' responses are analyzed under the assumption of

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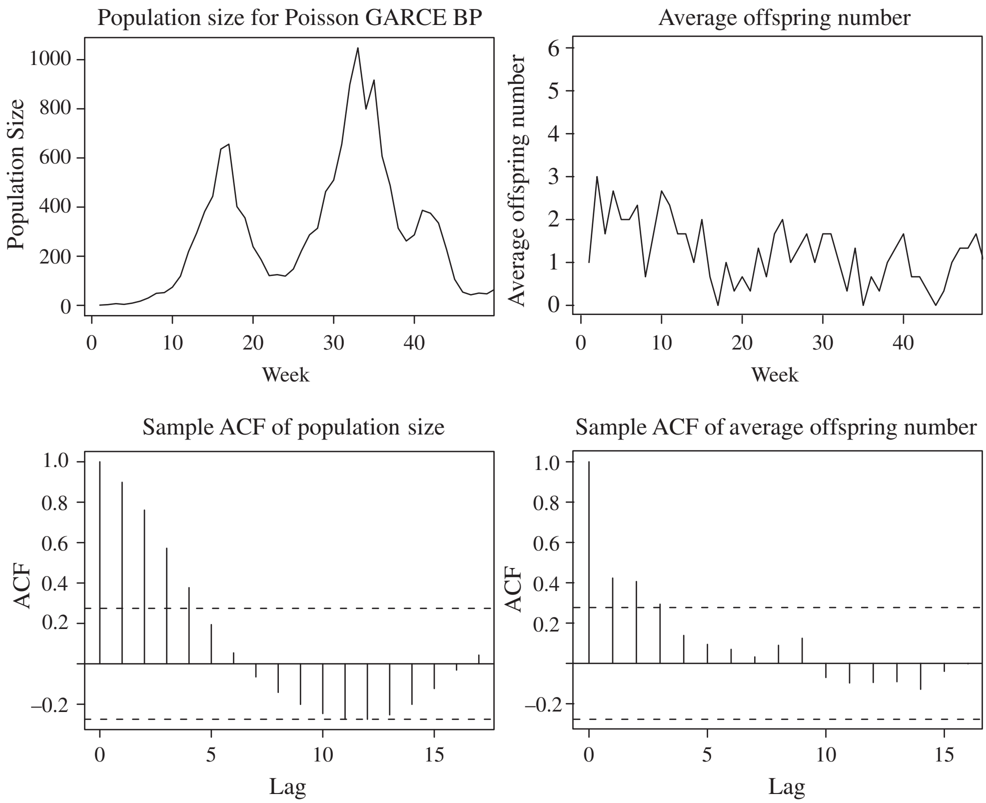

FIGURE 3: Simulated realization of a Poisson GARCE(1, 1) BP with $\alpha_0 = 0.31$, $\alpha_1 = 0.4$, $\beta_1 = 0.3$, $K = 3$, and $\alpha_0/(1 - \alpha_1 - \beta_1) = 1.033$ along with its average offspring number over time and the sample autocorrelation functions (ACFs) of the population size and average offspring number.

homogeneity within each treatment group. In personalized medicine but also in double-blind RCTs, more flexible statistical models are required where each patient has her or his own response trajectory over time, the response at a given time point depends on her or his response at preceding time points and intrinsic response rate that evolves over time. Examples of disease areas in drug development that exhibit categorical clinical endpoints are low-back or other pain, osteoarthritis, inflammatory skin disease, and diabetic foot ulcers that can lead to severe complications and occurs in 15% of all patients with diabetes. The endpoints range from improvement or change on a physician's or patient's global assessment of pain, pain intensity, or quality of life (e.g. 11-point numerical pain rating scale, 5-point scale, etc.) to the achievement of a clinical response such as reaching a prespecified least level of change or percent change from baseline of wound area at a given week. Of interest might be whether a patient responds to treatment, experiences a relapse, and returns to pretreatment levels after discontinuing therapy. The simulation of a binomial INGARCH model in Figure 4 illustrates a possible scenario of a RCT with categorical endpoint, measured weekly over 30 weeks, with 20 patients randomized in a 1:1 ratio to an active treatment arm and a placebo arm. The top two and bottom left graphs show the individual response curves over time for the active, control, and combined groups, whereas the bottom right plot separately displays the mean response

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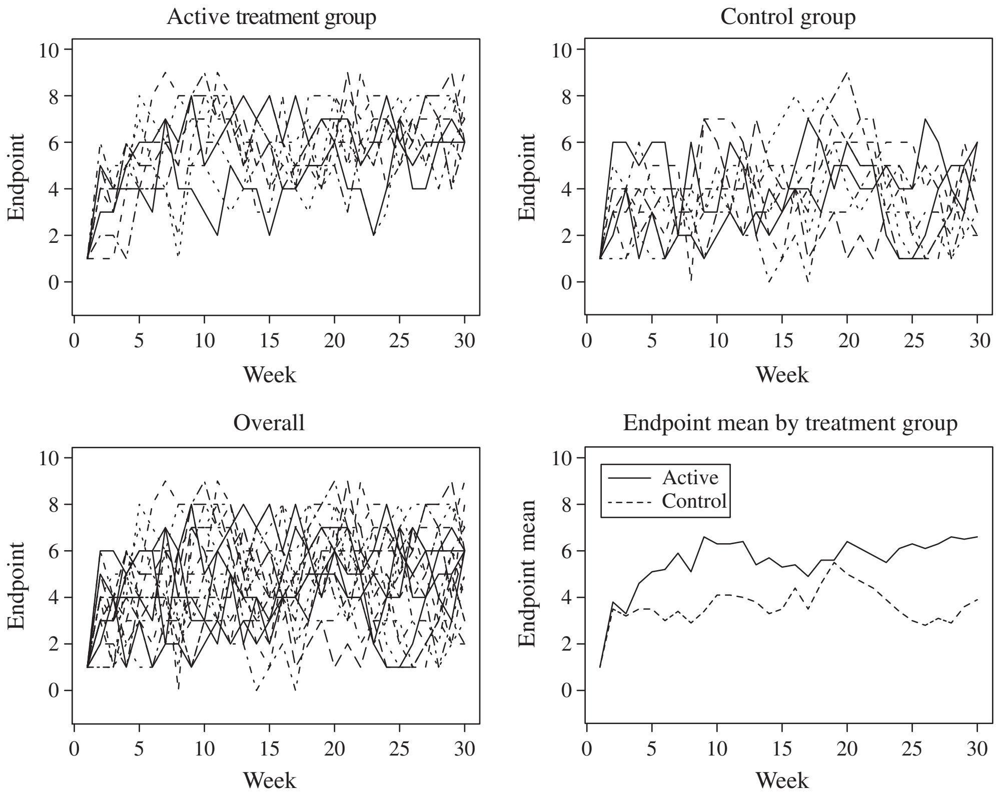

FIGURE 4: Simulated randomized clinical trial with categorical endpoint on a 10-point scale, measured weekly over 30 weeks, with 20 patients randomized in a 1:1 ratio to an active treatment arm and placebo arm. Ten realizations per group were sampled from a binomial INGARCH(1, 1) model with $m = 9$ and $\alpha_0 = 1.91, \alpha_1 = 0.4, \beta_1 = 0.3$ (active group) and $\alpha_0 = 1.61, \alpha_1 = 0.4, \beta_1 = 0.3$ (control group). The means of the stationary models are 6.37 and 5.37.

over time for the active and control groups. The models were chosen so that the resulting group means of the stationary models are equal to 6.37 and 5.37, respectively, which may model an effective treatment. While the individual curves can have large variability and whether a patient improves sensitively depends on the chosen assessment time point, the mean response of the active group tends to be superior to the one of the control group at most times. Yet there is some probability that the mean control response exceeds the mean active response, as seen in Figure 4 towards the end of the treatment period. When this happens, there is an increased chance for this scenario to occur for a few consecutive time points in view of the autoregressive structure of the model.

We mention that another research direction, which is pursued by the author in a separate paper, is to model the spatial spread of the infection over time in a related model, called GARCE branching random walk.

The rest of the paper is structured as follows. In Section 2 we introduce the GARCE branching processes and discuss examples with Poisson, generalized Poisson, negative-binomial, and binomial INGARCH offspring distributions. For these, in Sections 3, 4, and 5, we present conditions on the environmental process that guarantee strict or weak stationarity of the mean offspring number and environmental processes. Furthermore, we provide recurrence relations from which the variances, autocorrelation, and cross-correlation functions of these processes can be derived. Additionally in Section 3, we establish ergodicity of the joint mean offspring number and environmental process in the case of a generalized Poisson GARCE BP, discuss asymptotic properties of the maximum likelihood estimator for the model parameter vector, and briefly mention an approach to forecast future values of the branching process. Section 6 is devoted to fundamental questions of importance to the GARCE branching processes. It contains the main results on the survival–extinction dichotomy (Proposition 1), necessary and sufficient conditions for noncertain extinction, the extinction–explosion dichotomy (Theorem 11), characterizing conditions for subcriticality and supercriticality (Lemma 1), and the survival behavior in these two phases and at the phase transition (classification in Theorem 10). Finally, in Section 7 we examine limit theorems for the normalized GARCE BP including necessary and sufficient conditions for a nondegenerate limit and its properties in the supercritical case, paralleling the celebrated results of Athreya and Karlin [2] in 1971 and Kesten and Stigum [12] in 1966, and the extended Kesten–Stigum condition for the GARCE BP (Theorems 12–15).

2. GARCE branching processes

Consider a branching process $\{Z_t\}_{t\geq 0}$ for the population size Z_t at time t to be a sequence of nonnegative integer-valued random variables with initial population size $Z_0 > 0$ and with each of the Z_t members reproducing offspring according to a common offspring distribution with parameter λ_t and associated probability generating function (PGF) $\varphi_{\lambda_t}(s)$. In the model that has been referred to as the Smith–Wilkinson model in the literature (e.g. described in [3, p. 249]), at each time $t = 0, 1, 2, \ldots$, the function $\varphi_{\lambda_t}(s)$ is assumed to be chosen independently at random from a collection of PGFs with a specified time-homogeneous distribution. Our interest will be in a branching process with *dynamic random environments,* that is, a branching process with random environments (BPRE) where the distribution from which the random $\varphi_{\lambda_t}(s)$ is sampled evolves dynamically at any time and the dynamics of one of its distributional parameters is governed by a recurrence relation. As related in Section 1 above, we stress that the branching processes with dynamic random environments examined here are distinctively different from the BPRE that have been vigorously investigated in the literature starting with the work in [21]–[23] and [28] in 1968–1971 and [1] and [2] in 1971.

To set the stage, some more notation is needed. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{M}$ designate the collection of probability distributions on the nonnegative integers

$$\left\{\{p_i\}_{i=0}^{\infty}, \sum_{i\geq 0} ip_i < \infty, 0 \leq p_0 + p_1 < 1\right\}.$$

Let $\{\lambda_t(\omega)\}_{t\geq 0}$ be a sequence of mappings from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(\mathcal{M}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra in $\mathcal{M}$ generated by the usual topology. For any such mapping $\lambda = \lambda(\omega) = \{\lambda_t(\omega)\}_{t\geq 0}$ from Ω to $\mathcal{M}$, define the PGF

$$\varphi_\lambda(s) = \sum_{i=0}^{\infty} p_i(\lambda)s^i, \qquad |s| \leq 1, \tag{1}$$

where $\{p_i(\lambda)\}_{i\geq 0}$ is the probability distribution associated with λ. Moreover, let $\sigma(D)$ be the sub-σ-algebra of $\mathcal{F}$ generated by a given collection D of random variables (RVs) on $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, let $\{Y_t(\omega)\}_{t\geq 0}$ be a sequence of nonnegative real-valued RVs defined on $(\Omega, \mathcal{F}, \mathbb{P})$

and define the σ-algebras

$$\mathcal{F}_t(\lambda) = \sigma(\lambda_0, \lambda_1, \ldots, \lambda_t), \qquad \mathcal{F}(\lambda) = \sigma(\lambda),$$
$$\mathcal{F}_{t,z,y}(\lambda) = \sigma(\lambda_0, \lambda_1, \ldots, \lambda_t, Z_0, Z_1, \ldots, Z_t, Y_0, Y_1, \ldots, Y_t). \tag{2}$$

Definition 1. A *branching process* $\{Z_t\}_{t\geq 0}$ *in generalized autoregressive conditional environments* $\{\lambda_t\}_{t\geq 0}$ of order p and q or GARCE(p,q) BP is a process $\{Z_t\}_{t\geq 0}$ with environmental process $\{\lambda_t\}_{t\geq 0}$ that satisfies the recurrence relations

$$\mathbb{E}(s^{Z_{t+1}} \mid \mathcal{F}_{t,z,y}) = [\varphi_{\lambda_{t+1}}(s)]^{Z_t} \quad \text{almost surely (a.s.),} \tag{3a}$$

$$\lambda_{t+1} = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t+1-i} + \sum_{j=1}^{q} \beta_j \lambda_{t+1-j} \tag{3b}$$

with $Y_{t+1-i} = (1/K)\sum_{k=1}^{K} X_{t+1-i,k}$ (for the role of $\{Y_t\}$, see Remark 1(iii) below), where the random $(t+1)$th generation offspring numbers $X_{t+1,k}$ in the sum $Z_{t+1} = \sum_{k=1}^{Z_t} X_{t+1,k}$ are conditionally independent for $t \geq 1$, given $\mathcal{F}_{t,z,y} = \mathcal{F}_{t,z,y}(\lambda)$, identically distributed, and specified by their common PGF $\varphi_{\lambda_{t+1}}(s)$ ($\varphi_\lambda(s)$ depends on $(\lambda_0, \lambda_1, \ldots, \lambda_{t+1})$ only through λ_{t+1}). Here, K is a fixed, suitably chosen, small positive integer, $\alpha_0 > 0$, $\alpha_p, \beta_q > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$ for $i = 1, 2, \ldots, p-1$, $j = 1, 2, \ldots, q-1$, $p \geq 1$ and $q \geq 0$, and when $Z_t < K$, use similarly defined imaginary offspring numbers for $X_{t+1,k}$ for each $Z_t < k \leq K$ to define Y_{t+1}. Additionally, $\{Z_t\}_{t\geq 0}$ is required to satisfy the property that for any set of integers $1 \leq t_1 < t_2 < \cdots < t_k$ and $|s_i| \leq 1$ for $i = 1, 2, \ldots, k$,

$$\mathbb{E}(s_1^{Z_{t_1}} \cdots s_k^{Z_{t_k}} \mid \mathcal{F}(\lambda), Z_0 = m) = [\mathbb{E}(s_1^{Z_{t_1}} \cdots s_k^{Z_{t_k}} \mid \mathcal{F}(\lambda), Z_0 = 1)]^m \quad \text{a.s.} \tag{4}$$

Thus, conditional on the environments, the distribution of $\{Z_t\}_{t\geq 1}$ with $Z_0 = m$ is almost surely identical to the law of the sum of m independent processes that are distributed as $\{Z_t\}_{t\geq 1}$ with $Z_0 = 1$. Assume the starting values $\lambda_t = \lambda_0$ for some real constant $\lambda_0 > 0$, $Y_t = 0$ for all $t \leq 0$, and $Y_1 = Z_1$. Other suitable starting values for $\{\lambda_t\}$ are also possible. Observe that Z_{t+1} is the sum of Z_t independent RVs $X_{t+1,1}, X_{t+1,2}, \ldots, X_{t+1,Z_t}$, conditional on the past $\mathcal{F}_{t,z,y}$, and the nonnegative RVs $X_{t+1,k}$ are the $(t+1)$th generation offspring numbers of the Z_t particles of generation t. Furthermore, note that the *mean offspring number* Y_{t+1} per parent among the first K parents in generation t satisfies the conditional environmental equation or GARCE equation $\lambda_{t+1} = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t+1-i} + \sum_{j=1}^{q} \beta_j \lambda_{t+1-j}$ in (3b). The existence proof of a process $\{Z_t\}_{t\geq 0}$ that obeys the postulated defining relations is routine and rests on the Harris construction [8]. A key observation that substantially simplifies the structure of this branching process with dynamic random environments is that, conditionally on the environmental process $\{\lambda_t\}_{t\geq 0}$, the process $\{Z_t\}_{t\geq 0}$ is Markovian and has independent lines of descent. A few other comments are in order.

Remark 1. (i) The GARCE BP is a BPRE whose environmental and mean offspring number processes exhibit autoregressive serial dependence structure. Both latter processes display nonlinear behavior and allow for clustering of extreme values, even while being stationary.

(ii) In this paper the principal interest lies in the long-run properties of the GARCE BP $\{Z_t\}$ and environmental process $\{\lambda_t\}$. These do not depend on the starting values of the processes $\{\lambda_t\}$ and $\{Y_t\}$. Various choices for the latter are reasonable and sensible. For example, set λ_0 equal to 1, 2, or μ, say, where $\mu = \mathbb{E}(\lambda_t)$ for a stationary process $\{\lambda_t\}$, and $Y_1 = Z_1$, $Y_2 = Z_2/Z_1$, $Y_3 = Z_3/Z_2$, say, and for $t \geq 4$, use Y_t as defined in (3b).

(iii) The sequence of variables $\{Y_t\}_{t\geq 0}$, which captures some of the past values of the offspring numbers $\{X_{t,k}\}_{t\geq 0}$ for $k \geq 1$ (K from each of the p previous generations), are incorporated in the GARCE equation in order to inject randomness to the environmental process $\{\lambda_{t+1}\}$. Thus, the environment, which provides the intrinsic parameter of the branching process, is updated with information about the most recent offspring numbers. We will see that for the Ebola virus example, the intrinsic parameter is the infection rate. An alternative definition to $\{Y_t\}_{t\geq 0}$ that comes to mind is $\{Z_{t+1}/Z_t\}_{t\geq 0}$. Yet various issues would arise when incorporating it while defining the GARCE BP. For instance, as $Z_t \to \infty$ as $t \to \infty$, the strong law of large numbers implies that $Z_{t+1}/Z_t \to \lambda_{t+1}$ a.s. and, thus, the environmental process becomes nonrandom, whereas when $Z_t \to 0$ as $t \to \infty$, the environmental process $\{\lambda_{t+1}\}$ does not enjoy certain desirable properties such as second-order stationarity.

From the definitions we derive the conditional mean and variance of Y_t and Z_t as

$$\begin{aligned}
\mathbb{E}(Y_t \mid \mathcal{F}_{t-1,z,y}) &= \mathbb{E}(X_{t,1} \mid \mathcal{F}_{t-1,z,y}),\\
\operatorname{var}(Y_t \mid \mathcal{F}_{t-1,z,y}) &= \frac{1}{K}\operatorname{var}(X_{t,1} \mid \mathcal{F}_{t-1,z,y}),\\
\mathbb{E}(Z_t \mid \mathcal{F}_{t-1,z,y}) &= Z_{t-1}\mathbb{E}(X_{t,1} \mid \mathcal{F}_{t-1,z,y}),\\
\operatorname{var}(Z_t \mid \mathcal{F}_{t-1,z,y}) &= Z_{t-1}\operatorname{var}(X_{t,1} \mid \mathcal{F}_{t-1,z,y}).
\end{aligned} \tag{5}$$

When examining the features and behavior of the GARCE BP $\{Z_t\}$, the focus will be on the Poisson, generalized Poisson (GP), negative binomial (NB), and binomial INGARCH models as its offspring distribution associated with (1). In each case, the environment λ_t is defined so that it represents the expected offspring number at time t.

Example 1. (*Poisson INGARCH offspring distribution.*) For fixed $t \geq 1$ and each $1 \leq k \leq Z_t$, $X_{t+1,k} \mid \mathcal{F}_{t,z,y} \sim \mathcal{P}(\lambda_{t+1})$ is conditionally independent from $X_{t+1,j}$ for $1 \leq j \neq k \leq Z_t$, where $\mathcal{P}(\lambda_{t+1})$ denotes the Poisson distribution with parameter λ_{t+1}, which satisfies the GARCE equation in (3b). Heinen [9] in 2003 and Ferland *et al.* [7] in 2006 proposed the Poisson INGARCH(p, q) model for time series of counts to reflect nonlinear behavior and clustering of outliers in count data and a continuous and stationary accompanying intensity process. This and related models met with considerable interest (see [29]–[31] and the extensive references therein). These models applied to interacting particles and growth models such as the branching processes with random environments may have vast potential for future research developments if the rapid evolution of the GARCH-type models offer any indication. After Engle [6] and Bollerslev [4] initiated these models, they have become overwhelmingly popular in econometrics and finance.

Write $X_t := X_{t,1}$ for ease of notation. Thanks to the Poisson moments and (5), the unconditional means and variances of the mean offspring number, number of offspring, and GARCE branching processes Y_t, X_{t-1}, and Z_t have the expressions

$$\begin{aligned}
\mathbb{E}(Y_t) = \mathbb{E}(X_t) &= \mathbb{E}(\lambda_t), \qquad \mathbb{E}(Z_t) = \mathbb{E}(Z_{t-1}\lambda_t),\\
\operatorname{var}(Y_t) &= \frac{1}{K}\operatorname{var}(X_t)\\
&= \mathbb{E}[\operatorname{var}(Y_t \mid \mathcal{F}_{t-1,z,y})] + \operatorname{var}[\mathbb{E}(Y_t \mid \mathcal{F}_{t-1,z,y})]\\
&= \frac{1}{K}\mathbb{E}(\lambda_t) + \operatorname{var}(\lambda_t),\\
\operatorname{var}(Z_t) &= \mathbb{E}(Z_{t-1}\lambda_t) + \operatorname{var}(Z_{t-1}\lambda_t).
\end{aligned}$$

Example 2. (*GP INGARCH offspring distribution.*) For fixed $t \geq 1$ and each $1 \leq k \leq Z_t$, $X_{t+1,k} \mid \mathcal{F}_{t,z,y} \sim \mathcal{GP}(\lambda^*_{t+1}, \kappa)$ is conditionally independent from $X_{t+1,j}$ for $1 \leq j \neq k \leq Z_t$, where $\mathcal{GP}(\lambda^*_{t+1}, \kappa)$ denotes the generalized Poisson distribution with parameters κ with $\max(-1, -\lambda^*_{t+1}/4) < \kappa < 1$ and λ^*_{t+1} given by $\lambda^*_{t+1}/(1-\kappa) = \lambda_{t+1}$ for λ_{t+1} being governed by (3b). This model referred to as GP INGARCH(p, q) that was studied in [30] in 2012 allows for overdispersion or underdispersion of the conditional process $\{X_t\}$, depending on whether $\kappa > 0$ or $\kappa < 0$ (over-dispersion (under-dispersion) when the variability is larger (smaller) than the mean). Borrowing the expressions for the conditional mean and variance of a GP INGARCH RV from [30], we collect, in view of (5),

$$\mathbb{E}(Y_t \mid \mathcal{F}_{t-1,z,y}) = \frac{\lambda^*_t}{1-\kappa} = \lambda_t, \qquad \operatorname{var}(Y_t \mid \mathcal{F}_{t-1,z,y}) = \frac{\lambda^*_t}{K(1-\kappa)^3} = \frac{\phi^2}{K}\lambda_t,$$

where $\phi = 1/(1-\kappa)$. Consequently, again thanks to (5), the expressions for the unconditional means $\mathbb{E}(Y_t)$, $\mathbb{E}(X_t)$, and $\mathbb{E}(Z_t)$ are identical to those stated in Example 1 for Poisson INGARCH offspring, whereas the variances of Y_t, X_t, and Z_t are

$$\begin{aligned} \operatorname{var}(Y_t) &= \frac{1}{K}\operatorname{var}(X_t) = \frac{\phi^2}{K}\mathbb{E}(\lambda_t) + \operatorname{var}(\lambda_t), \\ \operatorname{var}(Z_t) &= \phi^2\mathbb{E}(Z_{t-1}\lambda_t) + \operatorname{var}(Z_{t-1}\lambda_t). \end{aligned} \tag{6}$$

Example 3. (*NB INGARCH offspring distribution.*) For fixed $t \geq 1$ and each $1 \leq k \leq Z_t$, $X_{t+1,k} \mid \mathcal{F}_{t,z,y} \sim \mathcal{NB}(r, p_{t+1})$ is conditionally independent from $X_{t+1,j}$ for $1 \leq j \neq k \leq Z_t$, where $\mathcal{NB}(r, p_{t+1})$ denotes the negative binomial distribution with parameters r for some positive integer r and $p_{t+1} \in (0, 1)$ given by $r(1-p_{t+1})/p_{t+1} = \lambda_{t+1}$ for λ_{t+1} obeying the GARCE equation in (3b). This model is called NB INGARCH(p, q), and referred to as geometric INGARCH(p, q) if $r = 1$, was investigated in [29] in 2011 for the purpose of allowing for over-dispersion (i.e. variability is larger than mean) in integer-valued time series and potential extreme observations. We assume that the conditional probability distribution of $X_{t+1,k}$, given $\mathcal{F}_{t,z,y}$ has the form

$$\mathbb{P}(X_{t+1,k} = y \mid \mathcal{F}_{t,z,y}) = \binom{y+r-1}{r-1} p^r_{t+1}(1-p_{t+1})^y, \qquad y = 0, 1, \dots,$$
$$p_{t+1} = \frac{1}{1+\lambda_{t+1}}, \qquad q_{t+1} = 1 - p_{t+1} = \frac{\lambda_{t+1}}{1+\lambda_{t+1}}.$$

Employing the expressions for the conditional moments of a NB INGARCH RV from [29] in conjunction with (5), we obtain the conditional mean and variance of Y_t,

$$\mathbb{E}(Y_t \mid \mathcal{F}_{t-1,z,y}) = \mathbb{E}(X_t \mid \mathcal{F}_{t-1,z,y}) = \lambda_t, \qquad \operatorname{var}(Y_t \mid \mathcal{F}_{t-1,z,y}) = \frac{\lambda_t}{K}\left(1 + \frac{\lambda_t}{r}\right),$$

where $\lambda_t = r(1-p_t)/p_t$. The expressions for the unconditional means and variances of Y_t, X_t, and Z_t are

$$\begin{aligned} \mathbb{E}(Y_t) = \mathbb{E}(X_t) &= \mathbb{E}(\lambda_t), \qquad \mathbb{E}(Z_t) = \mathbb{E}(Z_{t-1}\lambda_t), \\ \operatorname{var}(Y_t) = \frac{1}{K}\operatorname{var}(X_t) &= \mathbb{E}\left[\frac{\lambda_t}{K}\left(1 + \frac{\lambda_t}{r}\right)\right] + \operatorname{var}(\lambda_t), \\ \operatorname{var}(Z_t) &= \mathbb{E}\left[Z_{t-1}\frac{\lambda_t}{K}\left(1 + \frac{\lambda_t}{r}\right)\right] + \operatorname{var}(Z_{t-1}\lambda_t). \end{aligned} \tag{7}$$

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Example 4. (*Binomial INGARCH offspring distribution.*) For fixed $t \geq 1$ and each $1 \leq k \leq Z_t$, $X_{t+1,k} \mid \mathcal{F}_{t,z,y} \sim \mathcal{B}(m, p_{t+1})$ is conditionally independent from $X_{t+1,j}$ for $1 \leq j \neq k \leq Z_t$, where $\mathcal{B}(m, p_{t+1})$ denotes the binomial distribution with parameters m for some positive integer m and $p_{t+1} \in (0, 1)$ given by $mp_{t+1} = \lambda_{t+1}$ for λ_{t+1} being governed by (3b). In view of the requirement that $mp_{t+1} \in (0, m)$, it is necessary that $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$. When studying the binomial GARCE (BIN GARCE) BP, the only interesting case is $m \geq 2$, since the $m = 1$ case leads to almost sure extinction. Evidently, the conditional mean and variance of Y_t are

$$\mathbb{E}(Y_t \mid \mathcal{F}_{t-1,z,y}) = \lambda_t = mp_t,$$
$$\operatorname{var}(Y_t \mid \mathcal{F}_{t-1,z,y}) = \frac{1}{K}\operatorname{var}(X_t \mid \mathcal{F}_{t-1,z,y}) = \frac{1}{K}mp_t(1-p_t) = \frac{1}{K}\lambda_t\left(1 - \frac{\lambda_t}{m}\right).$$

With these expressions in mind, we derive the following expressions for the unconditional means and variances of Y_t, X_t, and Z_t:

$$\mathbb{E}(Y_t) = \mathbb{E}(X_t) = \mathbb{E}(\lambda_t), \qquad \mathbb{E}(Z_t) = \mathbb{E}(Z_{t-1}\lambda_t),$$
$$\operatorname{var}(Y_t) = \frac{1}{K}\operatorname{var}(X_t) = \frac{1}{K}\mathbb{E}(\lambda_t)\left(1 - \frac{\mathbb{E}(\lambda_t)}{m}\right) + \operatorname{var}(\lambda_t)\left(1 - \frac{1}{mK}\right),$$
$$\operatorname{var}(Z_t) = \mathbb{E}\left[Z_{t-1}\lambda_t\left(1 - \frac{\lambda_t}{m}\right)\right] + \operatorname{var}(Z_{t-1}\lambda_t)$$

(also see Theorem 8). In the next three sections we will investigate some of the key properties of these models.

3. GP GARCE BP

In this section we provide conditions for the existence and strict stationarity of the mean offspring number process $\{Y_t\}_{t\in\mathbb{Z}}$ and environmental process $\{\lambda_t\}_{t\in\mathbb{Z}}$ and ergodicity of the process $\{(Y_t, \lambda_t)\}_{t\in\mathbb{N}}$ for the GP GARCE(p, q) BP $\{Z_t\}_{t\in\mathbb{Z}}$. We also examine the autocorrelation and cross-correlation structure of the processes $\{Y_t\}_{t\in\mathbb{Z}}$ and $\{\lambda_t\}_{t\in\mathbb{Z}}$ and present recurrence relations they satisfy and expressions for their first- and second-order moments. It suffices to examine the GP GARCE BP, since the Poisson GARCE BP is the special case $\kappa = 0$. The first result features a sufficient condition that assures the existence of a unique stationary process $\{Y_t\}_{t\in\mathbb{Z}}$.

Theorem 1. *If $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$ for the GP* GARCE(p, q) *BP $\{Z_t\}_{t\geq 0}$, then there exists a unique strictly stationary mean offspring number process $\{Y_t\}_{t\in\mathbb{Z}}$ that satisfies the GARCE equation (3b) prior to the random extinction event $Z_T = 0$ (if any). Furthermore, the first two moments of $\{Y_t\}_{t\in\mathbb{Z}}$ are finite and their expressions are given by*

$$\mathbb{E}(Y_t) = \mathbb{E}(\lambda_t) = \mu := \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j}, \qquad \operatorname{var}(Y_t) = \operatorname{var}(\lambda_t) + \frac{\mu\phi^2}{K}.$$

Proof. Adopting the techniques relied on in [7] and the proof of [30, Theorem 1], we will prove the statements of the theorem. Define the polynomials $D(B) = 1 - \beta_1 B - \beta_2 B^2 - \cdots - \beta_q B^q$ and $G(B) = \alpha_1 B + \alpha_2 B^2 + \cdots + \alpha_p B^p$, where B denotes the backshift operator. Denote

$$\lambda_t = D^{-1}(B)(\alpha_0 + G(B)Y_t) = \alpha_0 D^{-1}(1) + H(B)Y_t,$$

https://doi.org/10.1017/apr.2016.71 Published online by Cambridge University Press

where $H(B) = D^{-1}(B)G(B) = \sum_{j=1}^{\infty} \psi_j B^j$. Define K independent copies $X_{t,k}^{(n)}$ for $t, n \in \mathbb{Z}$, and $1 \le k \le K$ of the random variable $X_t^{(n)}$ as constructed and defined in [30, Equation (3.2)]. Thus, for fixed t and n, the $X_{t,k}^{(n)}$ for $1 \le k \le K$ are independent and identically distributed RVs. It was argued in [30] that the expectation and variance of each $X_{t,k}^{(n)}$ are well defined and $\mathbb{E}(X_{t,1}^{(n)})$ does not depend on t. Write $\mu_n = \mathbb{E}(X_{t,1}^{(n)})$. As in [30], note that $\mu_k = 0$ if $k < 0$. In addition,

$$\mu_n = \psi_0 + \sum_{j=1}^{\infty} \psi_j \mu_{n-j} = \alpha_0 D^{-1}(1) + H(B)\mu_n,$$

$$\lim_{n\to\infty} \mu_n = \frac{\alpha_0 D^{-1}(1)}{1 - \sum_{j=1}^{\infty} \psi_j} = \frac{\alpha_0}{D(1) - G(1)} = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j} \tag{8}$$

if this limit exists. The reasoning in the proof of [30, Theorem 1] establishes that, for each $1 \le k \le K$, the sequence $\{X_{t,k}^{(n)}\}_{n\in\mathbb{Z}}$ has an almost sure limit $X_{t,k}$ for each t, that, for each n, the process $\{X_{t,k}^{(n)}\}_{t\in\mathbb{Z}}$ is a strictly stationary process, and, thus, $\{X_{t,k}\}_{t\in\mathbb{Z}}$ is strictly stationary. The same reasoning applies to $Y_t^{(n)} = (1/K) \sum_{k=1}^{K} X_{t-1,k}^{(n)}$, since $Y_t^{(n)}$ is the average of a fixed number of RVs $X_{t-1,k}^{(n)}$. Hence, we conclude that the sequence $\{Y_t^{(n)}\}_{n\in\mathbb{Z}}$ has an almost sure limit Y_t for each t, the process $\{Y_t^{(n)}\}_{t\in\mathbb{Z}}$ is a strictly stationary process for each n, and, thus, $\{Y_t\}_{t\in\mathbb{Z}}$ is a strictly stationary process.

Furthermore, it was shown in [30] that $\mathbb{E}(X_t^2) \le C$ for some positive finite constant C, which implies that the first two moments of X_t are finite. Consequently, $\mathbb{E}(Y_t^2) \le C/K < \infty$. It follows that the first two moments of Y_t are finite. Invoking arguments in parallel to those in [7, Proposition 5 and Section 2.6] (see also [30]) leads to the conclusion that, for each $k \ge 1$, the $X_{t,k} \mid \mathcal{F}_{t-1,z,y} \sim \mathcal{GP}(\lambda_t^*, \kappa)$, and the processes $\{Y_t\}$ and $\{\lambda_t\}$ follow the GARCE equation in (3b) with $\lambda_t^*/(1-\kappa) = \lambda_t$.

Finally, if $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$ then it is evident from (6) and (8) that the unconditional mean of Y_t is given by

$$\mathbb{E}(Y_t) = \mathbb{E}(X_t) = \mathbb{E}(\lambda_t) = \mu = \lim_{n\to\infty} \mu_n,$$

where an expression for $\lim_{n\to\infty} \mu_n$ is stated on the right-hand side of (8), and the unconditional variance of Y_t is as claimed, as well. This completes the proof. □

We point out that GARCE(1, 1) branching processes with $\alpha_1 + \beta_1 = 1$, so called IGARCE branching processes, are nonstationary. For stationary processes $\{Y_t\}_{t\in\mathbb{Z}}$ and $\{\lambda_t\}_{t\in\mathbb{Z}}$, denote their respective autocovariance function (ACVF) $\{\gamma_Y(k) = \operatorname{cov}(Y_{t+k}, Y_t)\}_{k\ge0}$ and $\{\gamma_\lambda(k) = \operatorname{cov}(\lambda_{t+k}, \lambda_t)\}_{k\ge0}$ and their cross-covariance function (CVF) $\{\gamma_{Y\lambda}(k) = \operatorname{cov}(Y_{t+k}, \lambda_t)\}_{k\ge0}$. The following theorem provides a set of equations from which the variances, autocorrelation functions, and cross-correlation function of $\{Y_t\}$ and $\{\lambda_t\}$ can be deduced.

Theorem 2. *Assume that the GP* GARCE(p,q) *BP* $\{Z_t\}_{t\ge0}$ *satisfies (3b) with* $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j < 1$, *and, thus, the process* $\{Y_t\}_{t\in\mathbb{Z}}$ *is stationary. Then the ACVFs* $\{\gamma_Y(k)\}_{k\ge0}$ *and* $\{\gamma_\lambda(k)\}_{k\ge0}$ *obey the linear equations*

$$\gamma_Y(k) = \sum_{i=1}^{p} \alpha_i \gamma_Y(|k-i|) + \sum_{j=1}^{\min(k-1,q)} \beta_j \gamma_Y(k-j) + \sum_{j=k}^{q} \beta_j \gamma_\lambda(j-k) \quad \text{for } k \ge 1,$$

$$\gamma_\lambda(k) = \sum_{i=1}^{\min(k,p)} \alpha_i \gamma_\lambda(k-i) + \sum_{i=k+1}^{p} \alpha_i \gamma_Y(i-k) + \sum_{j=1}^{q} \beta_j \gamma_\lambda(|k-j|) \quad k \ge 0,$$

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and the cross-CVF $\{\gamma_{Y\lambda}(k)\}_{k\ge 0}$ *is given by*

$$\gamma_{Y\lambda}(k) = \begin{cases} \gamma_\lambda(k) & \text{for } k \ge 0, \\ \gamma_Y(k) & \text{for } k < 0. \end{cases}$$

Proof. These assertions were proved in [27, Theorem 1], where here the processes $\{Y_t\}$ and $\{\lambda_t\}$ replace the processes $\{X_t\}$ and $\{M_t\}$, respectively, in [27]. □

Example 5. Consider the GP GARCE(1, 1) BP with $\alpha_1 + \beta_1 < 1$ with stationary mean offspring number process $\{Y_t\}$. Arguments that parallel those in [27] lead to the following expressions for the variances and ACFs of Y_t and λ_t:

$$\operatorname{var}(Y_t) = \frac{\mu\phi^2}{K}\frac{1-\beta_1(2\alpha_1+\beta_1)}{1-(\alpha_1+\beta_1)^2}, \qquad \operatorname{var}(\lambda_t) = \frac{\mu\phi^2}{K}\frac{\alpha_1^2}{1-(\alpha_1+\beta_1)^2} \quad \text{for } t \ge 0,$$
$$\rho_Y(k) = (\alpha_1+\beta_1)^{k-1}\alpha_1\frac{1-\beta_1(\alpha_1+\beta_1)}{1-\beta_1(2\alpha_1+\beta_1)} \quad \text{for } k \ge 1,$$
$$\rho_\lambda(k) = (\alpha_1+\beta_1)^k \quad \text{for } k \ge 0.$$

These may be verified in several straightforward algebra steps by invoking Theorem 2 for various choices of k. A convenient order to do so consists of deriving the expressions for $\rho_\lambda(k)$, $\gamma_\lambda(0)$, $\gamma_Y(0)$, and $\rho_Y(k)$. The arguments are omitted here in view of space considerations.

Remark 2. Observe that the mean offspring number process $\{Y_t\}$ of the GP GARCE(p,q) BP can be represented as an ARMA(m,q) process with $m = \max(p,q)$ and an innovation sequence $\{U_t = Y_t - \lambda_t\}_t$ that is a martingale sequence and is white noise (uncorrelated RVs) with $\operatorname{var}(U_t) = \operatorname{var}(Y_t)$: we obtain $Y_t = \alpha_0 + \sum_{i=1}^m(\alpha_i+\beta_i)Y_{t-i} + U_t - \sum_{j=1}^q \beta_j U_{t-j}$. However, there are some restrictions in treating the process $\{Y_t\}$ as an ARMA model and applying techniques that are employed for ARMA, since $\{Y_t\}$ does not take continuous values but is a discrete-valued process.

We continue to examine $\{Z_t\}$ in the special case when $q = 0$.

Corollary 1. *Assume that the process* $\{Y_t\}_{t\in\mathbb{Z}}$ *for the GP* GARCE(p) *BP be second-order stationary. Then the ACVF* $\{\gamma_Y(k)\}_{k\ge 0}$ *satisfies the equations given by*

$$\gamma_Y(k) = \sum_{i=1}^p \alpha_i\gamma_Y(|k-i|), \qquad k \ge 1.$$

Proof. These statements are an immediate consequence of [27, Theorem 1]. Note that the same equations were stated in [27, Corollary 1] and [30, Corollary 1]. □

Example 6. Consider the GP GARCE(1) BP with $\alpha_1 < 1$, a special case of Example 5 with $p = 1$ and $q = 0$. The first two cumulants of Y_t are $\kappa_1 = \mu = \alpha_0/(1-\alpha_1)$ and $\kappa_2 = \operatorname{var}(Y_t) = \phi^2\alpha_0/(K(1-\alpha_1)(1-\alpha_1^2))$.

In the sequel, we will be concerned with the GP GARCE(1, 1) BP that is governed by

$$\frac{\lambda_{t+1}^*}{1-\kappa} = \lambda_{t+1} = \alpha_0 + \alpha_1 Y_t + \beta_1\lambda_t. \tag{9}$$

In Sections 6 and 7 we will study the survival behavior of the GARCE BP and properties of a normalized process under the assumption of stationarity and ergodicity of the bivariate

https://doi.org/10.1017/apr.2016.71 Published online by Cambridge University Press

process $\{(Y_t, \lambda_t)\}_t$. The ergodicity feature also is crucial to the asymptotic theory of the conditional maximum likelihood estimators in the GARCE BP model (see [29] and [30]), as we will see in Theorem 4 below. In the next result we show that the processes $\{Y_t\}_t$ and $\{(Y_t, \lambda_t)\}_t$ for the GP GARCE(1, 1) BP are (geometrically) ergodic if $\alpha_1 + \beta_1 < 1$. For this purpose, it is convenient to use a two-sided stationary version of each process with time domain $\mathbb{Z}$ in place of $\mathbb{N}$. It exists thanks to Kolmogorov's extension theorem.

Theorem 3. *Suppose that the GP* GARCE(1, 1) *BP satisfy (9) with* $\alpha_1 + \beta_1 < 1$. *Then the process* $\{(Y_t, \lambda_t)\}_{t\in\mathbb{Z}}$ *has a unique stationary distribution and the processes* $\{Y_t\}_{t\in\mathbb{Z}}$ *and* $\{(Y_t, \lambda_t)\}_{t\in\mathbb{Z}}$ *are ergodic.*

Proof. The arguments of proof run in parallel to those carried out to prove [16, Theorems 2.1 and 3.1] and are omitted here. Some main ideas are as follows. A key ingredient is the additivity property of the GP distribution, that is, if $X_j \sim \mathcal{GP}(\lambda^*, \cdot)$ for each $1 \le j \le K$ and $Y = K^{-1}\sum_{j=1}^{K} X_j$, it follows that $K \cdot Y \sim \mathcal{GP}(K\lambda^*, \cdot)$. Coupling between two versions of processes $\{(Y'_t, \lambda'_t)\}_t$ and $\{(Y''_t, \lambda''_t)\}_t$ is performed on a suitable common probability space, with two starting values λ'_1 and λ''_1 being independently sampled from the stationary distribution. We can show that $\{Y_t\}_{t\in\mathbb{Z}}$ is absolutely regular with coefficient

$$\beta(t) = \mathbb{E}\Big[\sup_{A\in\sigma(Y_t, Y_{t+1},\dots)} |\mathbb{P}(A \mid \sigma(\dots, Y_{-1}, Y_0)) - \mathbb{P}(A)|\Big] \le \frac{2\mu(\alpha_1 + \beta_1)^{t-1}}{1-\beta_1} \quad \text{for } t \ge 1.$$

This implies that $\{Y_t\}_{t\in\mathbb{Z}}$ is strongly mixing, and, in turn, ergodic. Finally, we can establish that the random intensities $\{\lambda_t\}_{t\in\mathbb{Z}}$ can be represented as measurable functionals of past variables of $\{Y_t\}_{t\in\mathbb{Z}}$. This suffices to conclude that $\{(Y_t, \lambda_t)\}_{t\in\mathbb{Z}}$ is ergodic, as well. □

It is noted in [16] that, while $\{Y_t\}_{t\in\mathbb{Z}}$ is strongly mixing and ergodic, $\{(Y_t, \lambda_t)\}_{t\in\mathbb{Z}}$ and $\{\lambda_t\}_{t\in\mathbb{Z}}$ may not be strongly mixing. Furthermore, we remark that for the GP INGARCH(1, 1), an ergodicity result is stated in [30, Theorem 3].

The next result sheds some light on the estimation of the GARCE BP model parameters, which is part of the model building that can be used to generate forecasts for future values of the GARCE BP. We first produce predicted values of the environmental process $\{\lambda_t\}$ that are obtained from the GARCE recurrence relation in (3b) and then apply bootstrapping techniques that are commonly relied on in forecasts in GARCH models in order to predict future values for the processes $\{X_{t,i}\}$ and $\{Z_t\}$. Assume that we have n observations $Y_1, Y_2, \dots, Y_n$ that were generated from the GP GARCE(1, 1) BP. Recall that $\phi = 1/(1-\kappa)$. The conditional likelihood function is given by

$$\prod_{t=2}^{n} \frac{K\lambda_t[K\lambda_t + (\phi-1)KY_t]^{KY_t-1}\phi^{-KY_t}}{(KY_t)!} \exp\bigg\{-\frac{K\lambda_t + (\phi-1)KY_t}{\phi}\bigg\}.$$

We present various asymptotic properties of the maximum likelihood estimator (MLE) for the model parameter vector. For the description of the algorithm and a proof, see [30].

Theorem 4. *Suppose that the GP* GARCE(1, 1) *BP satisfy (9) with* $\alpha_1 + \beta_1 < 1$. *Denote* $\theta = (1/(1-\kappa), \alpha_0, \alpha_1, \beta_1)'$ *and let* θ° *denote the true value of* θ. *Under the assumptions stated in [30, Theorem 5] adapted to the process* $\{Y_t\}$, *the MLE* $\hat\theta$ *is unique, consistent, and asymptotically normal, thus,*

$$\sqrt{n}\,(\hat\theta - \theta^\circ) \xrightarrow{\text{D}} \mathcal{N}(0, G^{-1}),$$

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where G is given in [30], modulo slight modification required for the process $\{Y_t\}$ *and* '$\xrightarrow{\mathrm{D}}$' *denotes convergence in distribution.*

We conclude this section with the finding that all moments of $\{Y_t\}$ are finite.

Theorem 5. *Suppose that the GP* GARCE$(1, 1)$ *BP* $\{Z_t\}_{t\geq 0}$ *satisfy (9). Then all of the moments of* Y_t *are finite if and only if* $\alpha_1 + \beta_1 < 1$.

Proof. This can be inferred from [30, Theorem 4] since Y_t is the average of a fixed number of independent and identically distributed RVs that follow the same distribution as X_t. □

4. NB GARCE BP

We continue to illuminate the NB GARCE(p, q) BP and are concerned with conditions for the weak stationarity of $\{Y_t\}_{t\in\mathbb{Z}}$ and $\{\lambda_t\}_{t\in\mathbb{Z}}$. Furthermore, we provide recurrence relations that the ACFs and cross-CVF obey and derive expressions for their first- and second-order moments.

Theorem 6. *A necessary condition for the NB* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *to exhibit a weakly stationary mean offspring number process* $\{Y_t\}_{t\in\mathbb{Z}}$ *and environmental process* $\{\lambda_t\}_{t\in\mathbb{Z}}$ *prior to the random extinction event* $Z_T = 0$ *(if any) is that* $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j < 1$. *If the process* $\{Y_t\}_{t\in\mathbb{Z}}$ *is first-order stationary then its mean is given by*

$$\mathbb{E}(Y_t) = \mathbb{E}(\lambda_t) = \mu := \frac{\alpha_0}{1 - \sum_{i=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_j}.$$

Furthermore, the variances of $\{Y_t\}_{t\in\mathbb{Z}}$ *and* $\{\lambda_t\}_{t\in\mathbb{Z}}$ *are related by*

$$\mathrm{var}(Y_t) = \bigg(\frac{\mu}{K}\bigg)\bigg(1 + \frac{\mu}{r}\bigg) + \mathrm{var}(\lambda_t)\bigg(1 + \frac{1}{rK}\bigg).$$

Proof. The proof of the statements relating to the weak stationarity proceeds in parallel to those for the GP GARCE(p, q) BP in [30, Theorem 1 and Proposition 1], since $\{Y_t\}_{t\in\mathbb{Z}}$ satisfies the same GARCE equation (3b) (also compare to [29, Theorem 1]). Moreover, by virtue of (7), an elementary exercise leads to the relation

$$\gamma_Y(0) = \frac{1}{K}\mathbb{E}\bigg[\lambda_t\bigg(1 + \frac{\lambda_t}{r}\bigg)\bigg] + \mathrm{var}(\lambda_t) = \frac{\mu}{rK}(r + \mu) + \bigg(1 + \frac{1}{rK}\bigg)\gamma_\lambda(0),$$

which achieves the proof. □

The ACFs and cross-CVF of $\{Y_t\}$ and $\{\lambda_t\}$ satisfy the same relations as those of the GP GARCE(p, q) BP (Theorem 2) from which the variances and ACFs of $\{Y_t\}$ and $\{\lambda_t\}$ can be inferred.

Theorem 7. *Assume that the NB* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *satisfies* $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j < 1$ *and has a mean-stationary process* $\{Y_t\}_{t\in\mathbb{Z}}$. *The ACVFs* $\{\gamma_Y(k)\}_{k\geq 0}$ *and* $\{\gamma_\lambda(k)\}_{k\geq 0}$ *obey the linear equations in Theorem 2 and the cross-CVF* $\{\gamma_{Y\lambda}(k)\}_k$ *is as stated in Theorem 2.*

Example 7. Consider the special case of a NB GARCE$(1, 1)$ BP with a weakly stationary mean offspring number process $\{Y_t\}_{t\in\mathbb{Z}}$. Write $\mu = \alpha_0/(1 - \alpha_1 - \beta_1)$. We obtain

$$\mathrm{var}(Y_t) = \frac{\mu}{K}\bigg(1 + \frac{\mu}{r}\bigg)\frac{1 - \beta_1(2\alpha_1 + \beta_1)}{1 - (\alpha_1 + \beta_1)^2 - \alpha_1^2/rK},$$

$$\mathrm{var}(\lambda_t) = \frac{\alpha_1^2\mu}{K}\frac{1 + \mu/r}{1 - (\alpha_1 + \beta_1)^2 - \alpha_1^2/rK}$$

for $t \geq 0$ and the same expressions for the autocorrelations $\rho_Y(k)$ and $\rho_\lambda(k)$ for $k \geq 1$ and $k \geq 0$, respectively, as displayed in Example 5 for the GP GARCE(1, 1) BP. The derivations follow from a series of algebra steps in parallel to those needed in Example 5, in conjunction with the relation between $\operatorname{var}(Y_t)$ and $\operatorname{var}(\lambda_t)$ stated in Theorem 6. The details are omitted.

A corollary to Theorem 7 for the NB GARCE(p) BP is as follows.

Corollary 2. *Assume that the NB* GARCE(p) *BP* $\{Z_t\}_{t\geq 0}$ *has a weakly stationary* $\{Y_t\}_{t\in\mathbb{Z}}$*. Then the ACVF* $\{\gamma_Y(k)\}_{k\geq 0}$ *satisfies the recurrence equations*

$$\gamma_Y(k) = \sum_{i=1}^{p} \alpha_i \gamma_Y(|k-i|), \qquad k \geq 1.$$

Remark 3. To the best of the author's knowledge, the existence, strict stationarity, and ergodicity of the NBGARCH process are open problems (see [29, Section 6]). Therefore, the same aspects are unresolved questions for the mean offspring number process $\{Y_t\}_t$ for the NB GARCE(p, q) BP.

5. BIN GARCE BP

In this section we examine the stationarity and ACFs and cross-CVF of the processes $\{Y_t\}_{t\in\mathbb{Z}}$ and $\{\lambda_t\}_{t\in\mathbb{Z}}$ for the BIN GARCE(p, q) BP $\{Z_t\}_{t\geq 0}$.

Theorem 8. *A necessary condition for the BIN* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *to exhibit a weakly stationary mean offspring number process* $\{Y_t\}_{t\in\mathbb{Z}}$ *prior to the random extinction event* $Z_T = 0$ *(if any) is that* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$*. Furthermore, the first two moments of* $\{Y_t\}_{t\in\mathbb{Z}}$ *and* $\{\lambda_t\}_{t\in\mathbb{Z}}$ *can be expressed as*

$$\mathbb{E}(Y_t) = \mathbb{E}(\lambda_t) = \mu := \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j},$$

$$\operatorname{var}(Y_t) = \frac{\mu}{K}\left(1 - \frac{\mu}{m}\right) + \operatorname{var}(\lambda_t)\left(1 - \frac{1}{mK}\right).$$

Proof. The assertions about the weak stationarity of $\{Y_t\}_{t\in\mathbb{Z}}$ are demonstrated by similar reasoning as for the GP GARCE(p, q) BP, as outlined in the proof of Theorem 1, since $\{Y_t\}_{t\in\mathbb{Z}}$ satisfies the same GARCE equation (3b). A few computational steps in conjunction with the observation that $\mathbb{E}(\lambda_t) = m\mathbb{E}(p_t)$ and with the conditional variance formula derive the relationship between $\operatorname{var}(Y_t)$ and $\operatorname{var}(\lambda_t)$ as follows:

$$\begin{aligned}
\operatorname{var}(Y_t) &= K^{-1}\mathbb{E}[\operatorname{var}(X_{t-1} \mid \mathcal{F}_{t-1,z,y})] + \operatorname{var}(\lambda_t) \\
&= \frac{m}{K}[\mathbb{E}(p_t) - \mathbb{E}(p_t^2)] + \operatorname{var}(\lambda_t) \\
&= \frac{m}{K}\mathbb{E}(p_t)(1 - \mathbb{E}(p_t)) + \operatorname{var}(\lambda_t)\left(1 - \frac{1}{mK}\right) \\
&= \frac{\mu}{K}\left(1 - \frac{\mu}{m}\right) + \operatorname{var}(\lambda_t)\left(1 - \frac{1}{mK}\right).
\end{aligned}$$

This completes the proof. □

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The ACFs and cross-CVF of $\{Y_t\}$ and $\{\lambda_t\}$ satisfy the same recurrence relations as those of the GP GARCE(p, q) BP, stated in Theorem 2, which allows for derivation of their variances, ACFs, and cross-correlation function. The next corollary has a proof that is identical to the one of Theorem 2.

Corollary 3. *Assume that the BIN* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *satisfies* $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j < 1$ *and has a mean-stationary process* $\{Y_t\}_{t\in\mathbb{Z}}$*. Then the ACVFs* $\{\gamma_Y(k)\}_{k\geq 0}$ *and* $\{\gamma_\lambda(k)\}_{k\geq 0}$ *obey the linear equations given in Theorem 2 and the cross-CVF* $\{\gamma_{Y\lambda}(k)\}_k$ *is as stated in Theorem 2.*

Example 8. Consider the BIN GARCE(1, 1) BP with $\alpha_1 + \beta_1 < 1$ with its mean offspring number process $\{Y_t\}$ in the stationary regime. We obtain the following expressions for the variances of Y_t and λ_t:

$$\operatorname{var}(Y_t) = \frac{\mu}{K}\left(1 - \frac{\mu}{m}\right)\frac{1 - \beta_1(2\alpha_1 + \beta_1)}{1 - (\alpha_1 + \beta_1)^2 + \alpha_1^2/mK} \quad \text{for } k \geq 1,$$

$$\operatorname{var}(\lambda_t) = \frac{\mu}{K}\left(1 - \frac{\mu}{m}\right)\frac{\alpha_1^2}{1 - (\alpha_1 + \beta_1)^2 + \alpha_1^2/mK} \quad \text{for } k \geq 0.$$

These expressions are obtained in the same fashion as those we obtained for the GP GARCE(1,1) BP in Example 5. While the variance formulae are slightly different because the variances are related differently and an adapted formula for $\gamma_Y(1) = (\alpha_1\mu/K)(1 - \mu/m)([1 - \beta_1(\alpha_1 + \beta_1)]/[1 - (\alpha_1 + \beta_1)^2 + \alpha_1^2/mK])$ emerges, we observe that the formulae for the ACFs ρ_Y and ρ_λ are identical to those for the GP GARCE(1, 1) BP in Example 5.

We remark that in the special case when $q = 0$, the ACVF $\{\gamma_Y(k)\}_k$ satisfies the recurrence relation stated in Corollary 1. In addition, note that the process $\{Y_t\}$ for the BIN GARCE(p, q) BP has the same ARMA representation as the GP GARCE(p, q), which can be viewed in the remark following Example 5.

6. Certain and noncertain extinction

In this section we turn to investigate the survival–extinction dichotomy, provide necessary and sufficient conditions for noncertain extinction of the GARCE BP $\{Z_t\}$, show that it becomes extinct or explodes with probability 1, identify conditions for subcriticality and supercriticality, and study the survival behavior in these two phases and at the phase transition. For the purpose of developing and applying ideas, we follow the exposition in [1] and overview in [3] on BPREs. For the remainder of this article, we shall impose the following standing hypothesis, unless otherwise stated.

Assumption 1. *The process* $\{(Y_t, \lambda_t)\}_{t\in\mathbb{Z}}$ *for the* GARCE(p, q) *BP is strictly stationary and ergodic.*

We observe that if the process $\{\lambda_t\}$ is started with its stationary law, the resulting observed process $\{Y_t\}$ will be stationary. Recollect the notation $\lambda = (\lambda_0, \lambda_1, \dots) = \{\lambda_t\}_{t\geq 0}$ for the environmental process. Denote the extinction set $\mathcal{E}$ and conditional and unconditional extinction probabilities $q_k(\lambda)$ and q_k, when we initially start with k individuals, by

$$\mathcal{E} = \{\omega\colon Z_t(\omega) = 0 \text{ for some } t\}, \tag{10}$$
$$q_k(\lambda) = \mathbb{P}(\mathcal{E} \mid \mathcal{F}(\lambda),\ Z_0 = k), \qquad q_k = \mathbb{P}(\mathcal{E} \mid Z_0 = k),$$

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where we recall $\mathcal{F}(\lambda)$ from (2). A consequence of the relations in (3a) and (4) is

$$\mathbb{E}(s^{Z_{t+1}} \mid \mathcal{F}(\lambda),\ Z_0 = k) = [\varphi_{\lambda_1}(\varphi_{\lambda_2}(\cdots \varphi_{\lambda_{t+1}}(s)\cdots))]^k.$$

Obviously,

$$q_k(\lambda) = q_1(\lambda)^k \quad \text{a.s.}, \qquad q_k = \mathbb{E}[q_1(\lambda)^k],$$

which establishes that $\{q_k\}_{k\geq 1}$ is a moment sequence. For the sequel, let us write $q(\lambda) = q_1(\lambda)$. Since the sequence of events $\mathcal{E}_t = \{\omega\colon Z_t(\omega) = 0\}$ increases to $\mathcal{E}$, we conclude that

$$q(\lambda) = \lim_{t\to\infty} \varphi_{\lambda_1}(\varphi_{\lambda_2}(\cdots \varphi_{\lambda_t}(0)\cdots)).$$

An immediate consequence of this equation is the following crucial functional equation:

$$q(\lambda) = \varphi_{\lambda_1}(q(T\lambda)) \tag{11}$$

(see [1] and [22]), where T denotes the back-shift transformation

$$T\lambda = T(\lambda_0, \lambda_1, \ldots) = (\lambda_1, \lambda_2, \ldots).$$

An observation that was shown in [1, Theorem 6] is that $q(\lambda)$ is the *minimal* solution to (11), and when $\mathbb{P}(q(\lambda) < 1) = 1$, it is the *unique* solution. Specifically, if $\tilde{q}(\lambda)$ is any RV that satisfies $\tilde{q}(\lambda) = \varphi_{\lambda_1}(\tilde{q}(T\lambda))$, then we have $\mathbb{P}(q(\lambda) \leq \tilde{q}(\lambda)) = 1$, and when $\mathbb{P}(\tilde{q}(\lambda) < 1) = 1$, additionally, $\mathbb{P}(q(\lambda) = \tilde{q}(\lambda)) = 1$ is implied.

Equation (11) together with the basic assumption $\mathbb{P}(\lambda_1 \in \mathcal{M}) = 1$ imply at once that the event $\{q(\lambda) = 1\}$ is shift invariant, that is, the two events $\{\omega\colon\ q(\lambda) = 1\}$ and $\{\omega\colon\ q(T\lambda) = 1\}$ coincide with probability 1 (see [1, Proposition 1]). Since the event $\{q(\lambda) = 1\}$ is preserved under T, under the assumption that the shift transformation T is *stationary* and *ergodic*, the zero–one law applies to $\{q(\lambda) = 1\}$, that is, $\mathbb{P}(q(\lambda) = 1) = 0$ or $\mathbb{P}(q(\lambda) = 1) = 1$. In other words, when extinction of the BPRE $\{Z_t\}_{t\geq 0}$ happens, it occurs almost surely with respect to the probability measure of the environmental process.

Proposition 1. *Suppose that the* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *satisfies Assumption 1. Then, we have* $\mathbb{P}(q(\lambda) = 1) = 0$ or $\mathbb{P}(q(\lambda) = 1) = 1$.

We continue to summarize necessary and sufficient conditions for $\mathbb{P}(q(\lambda) < 1) = 1$, a positive survival probability $\{\lambda_t\}$-a.s. This scenario is referred to as ‘noncertain extinction’ in the BPRE literature. Define the random quantities

$$V_\lambda = \log \varphi'_{\lambda_1}(1), \qquad U_\lambda = -\log(1 - \varphi_{\lambda_1}(0)) = -\log(1 - p_0(\lambda_1)),$$
$$Q_\lambda = \log\biggl(\frac{1 - q(\lambda)}{1 - q(T\lambda)}\biggr), \tag{12}$$

where $p_0(\cdot)$ denotes the probability of 0 offspring. We will refer to the GARCE BP $\{Z_t\}$ as *supercritical, critical*, or *subcritical* depending on whether $\mathbb{E}(V_\lambda) > 0$, $= 0$, or < 0, respectively, where V_λ is defined in (12). In the next lemma we present explicit expressions for V_λ in the case of the GP, NB, and BIN GARCE BP, and, thus, characterize the phase transition between the subcritical and supercritical phases of the process.

Lemma 1. *Suppose the processes* $\{Y_t\}_{t\in\mathbb{Z}}$ *and* $\{\lambda_t\}_{t\in\mathbb{Z}}$ *for the GP, NB or BIN* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *be strictly stationary. Then*

$$\mathbb{E}(V_\lambda) = \mathbb{E}[\log(\lambda_1)],$$

and if $p_1 = p(\lambda_1)$ *denotes the parameter of the negative binomial and binomial distributions,*

$$\mathbb{E}(U_\lambda) = \begin{cases} -\mathbb{E}[\log(1-\exp(-\lambda_1))], & \text{(13a)} \\ -\mathbb{E}[\log(1-p_1^r)] = -\mathbb{E}\Big[\log\Big(1-\Big(\dfrac{r}{r+\lambda_1}\Big)^r\Big)\Big], & \text{(13b)} \\ -\mathbb{E}[\log(1-(1-p_1)^m)] = -\mathbb{E}\Big[\log\Big(1-\Big(1-\dfrac{\lambda_1}{m}\Big)^m\Big)\Big]. & \text{(13c)} \end{cases}$$

Proof. (i) We begin with examining the GP GARCE(p, q) BP (13a) in its stationary regime. The PGF of a generalized Poisson RV with parameters λ and κ is given by $\varphi_{\lambda^*}(s) = \exp(\lambda(z-1))$ for $|s| \leq 1$, where $z = s\exp(\kappa(z-1))$. A straightforward calculation yields $\varphi'_\lambda(1) = \lambda^*/(1-\kappa) = \lambda$. Thus, in view of the definition in (12), we have $V_\lambda = \log(\lambda_1^*/(1-\kappa)) = \log(\lambda_1)$. Moreover, observe that $\varphi_{\lambda_1}(0) = \exp(-\lambda_1)$ and recall the definition of U_λ from (12). This verifies the first claims about V_λ and U_λ.

(ii) Next, consider the NB GARCE(p, q) BP (13b) in the stationary regime. The PGF of a NB RV with parameters r and p is given by $\varphi_\lambda(s) = [p/(1-(1-p)s)]^r$ for $|s| \leq 1$. A simple calculation provides $\varphi'_\lambda(1) = r(1-p)/p = \lambda$. Therefore, $V_\lambda = \log(\lambda_1)$. Additionally, $\varphi_{\lambda_1}(0) = p_1^r$ with $p_1 = r/(r+\lambda_1)$. This immediately establishes the claims for V_λ and U_λ.

(iii) At last, we turn to the BIN GARCE(p, q) BP (13c) in the stationary regime. The PGF of a binomial RV with parameters m and p is given by $\varphi_\lambda(s) = [ps + (1-p)]^m$ for $|s| \leq 1$. Differentiation yields $\varphi'_\lambda(1) = mp = \lambda$. Therefore, we obtain $V_\lambda = \log(\lambda_1)$. Furthermore, note that $\varphi_{\lambda_1}(0) = (1-p_1)^m = (1-\lambda_1/m)^m$. Hence, this verifies the third assertions stated about V_λ and U_λ and completes the proof. □

Write $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$. We present sufficient conditions for almost sure extinction of the GARCE BP.

Theorem 9. *Suppose that the* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *satisfies Assumption 1.*

(i) *If* $\mathbb{E}(V_\lambda^+) < \infty$ *then*

- $\mathbb{E}(V_\lambda^+) \leq \mathbb{E}(V_\lambda^-) \leq \infty$ *implies* $\mathbb{P}(q(\lambda) = 1) = 1$, *while*
- $\mathbb{E}(V_\lambda^+) > \mathbb{E}(V_\lambda^-)$ *and* $\mathbb{E}(U_\lambda) < \infty$ *imply* $\mathbb{P}(q(\lambda) = 1) = 0$.

(ii) *If* $\mathbb{E}(V_\lambda^+) = \infty$ *then* $\mathbb{E}(U_\lambda) < \infty$ *implies* $\mathbb{P}(q(\lambda) = 1) = 0$.

Proof. In light of Assumption 1, the claims follow at once from [1, Corollary 1, Section 2, p. 1507] and [1, Theorem 3]. □

Remark 4. (i) (*Necessary condition for noncertain extinction.*) We have $\mathbb{P}(q(\lambda) < 1) = 1$ and $\mathbb{E}(V_\lambda^+) > 0$ imply $\mathbb{E}|V_\lambda| < \infty$, $\mathbb{E}(V_\lambda) > 0$, $\mathbb{E}|Q_\lambda| < \infty$, and $\mathbb{E}(Q_\lambda) = 0$ [1, Theorem 1]. For the necessity of $\mathbb{E}(V_\lambda) > 0$, see also Theorems 9 and 10(iii) below. Tanny [24] in 1977 provided the weaker condition $\lim_{t\to\infty} t^{-1}\log(1-p_0(\lambda_t)) = 0$ than $\mathbb{E}|U_\lambda| < \infty$, along with $\mathbb{E}(V_\lambda) > 0$, that is necessary but not sufficient, where $p_0(\lambda_1)$ denotes the probability of 0 offspring. Note that the condition $\mathbb{E}|U_\lambda| < \infty$, along with $\mathbb{E}(V_\lambda) > 0$, is sufficient (presented in [1, Theorem 1], also a consequence of a later result in [24]).

(ii) (*Sufficient condition for noncertain extinction.*) Suppose that $\mathbb{E}(U_\lambda) < \infty$ and $\mathbb{E}(V_\lambda^-) < \mathbb{E}(V_\lambda^+) \leq \infty$ hold. Then $\mathbb{P}(q(\lambda) < 1) = 1$ [1, Theorem 3]. However, Tanny [24] pointed out that $\mathbb{E}|U_\lambda| < \infty$ is not a necessary condition for $\mathbb{P}(q(\lambda) < 1) = 1$, while being sufficient, and gave an example of noncertain extinction where $\mathbb{E}(V_\lambda) > 0$ and $\mathbb{E}|U_\lambda| = \infty$.

(iii) (*Necessary and sufficient condition for noncertain extinction.*) Coffey and Tanny [5, Theorem 1] in 1983 furnished a necessary and sufficient condition for $\mathbb{P}(q(\lambda) < 1) = 1$, together with $\mathbb{E}(V_\lambda) < \infty$, as follows. There is a function $\nu(\lambda)$ taking values in the positive integers such that

(i) $\mathbb{E}[\log(\sum_{k=0}^{\nu(\lambda)-1} kp_k(\lambda_1) + \nu(\lambda)\sum_{k=\nu(\lambda)}^{\infty} p_k(\lambda_1))] > 0$,

(ii) $\lim_{t\to\infty} t^{-1}\log\nu(T^t\lambda) = 0$ with probability 1, where $p_k(\lambda_1)$ denotes the probability of k offspring under the stationary offspring distribution. According to the first statement, the truncated GARCE BP is supercritical, while the second assertion states that the truncation points grow more slowly than any exponential sequence.

(iv) (*Open cases.*) It was noted in [3] that the following cases appear to be open:

(i) $\mathbb{E}(V_\lambda^+) = \infty$ and $\mathbb{E}(V_\lambda^-) = \infty$, and

(ii) $\infty \geq \mathbb{E}(V_\lambda^+) > \mathbb{E}(V_\lambda^-)$ and $\mathbb{E}(U_\lambda) = \infty$.

The following classification result sheds light on the survival behaviors of the subcritical, critical, and supercritical processes. For the subcritical and critical GARCE BP, extinction is certain almost surely relative to the environment, unless we are in a special degenerate case for the critical process that is not interesting. Noncertain extinction is only possible in the case of a supercritical GARCE BP. If the process survives indefinitely, a law of large numbers holds.

Theorem 10. (Classification.) *Suppose that the* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *satisfies Assumption 1 and* $\mathbb{E}(V_\lambda)$ *exist.*

(i) *If* $\mathbb{E}(V_\lambda) < 0$ *then* $\mathbb{P}(q(\lambda) = 1) = 1$.

(ii) *If* $\mathbb{E}(V_\lambda) = 0$ *then either* $\mathbb{P}(q(\lambda) = 1) = 1$ *or* $\mathbb{P}(p_1(\lambda_1) = 1) = 1$, *where* $p_1(\lambda_1)$ *denotes the probability of one offspring under the stationary offspring distribution. The event* $\mathbb{P}(p_1(\lambda_1) = 1) = 1$ *implies that* $\mathbb{P}(Z_t \equiv 1 \text{ for all } t \mid Z_0 = 1, \mathcal{F}(\lambda)) = 1$ *with probability* 1.

(iii) (Law of large numbers.) *If* $\mathbb{E}(V_\lambda) > 0$ *then* $\lim_{t\to\infty} t^{-1}\log Z_t = \mathbb{E}(V_\lambda)$ *almost everywhere on the event* $\{\omega\colon Z_t(\omega) \to \infty \text{ as } t \to \infty\}$.

Proof. These statements are consequences of [24, Theorem 5.5]. □

Hence, noncertain extinction is precluded, unless the GARCE BP $\{Z_t\}$ is supercritical. In addition, the boundary $\mathbb{E}(V_\lambda) = 0$ delineates the phase transition between the two phases of certain extinction and noncertain extinction, where extinction occurs with certainty at the phase transition. Easy upper and lower bounds on the phase transition between certain extinction and noncertain extinction can be derived in terms of the first two moments of the environment.

Remark 5. (i) *(Upper bound on phase transition.)* Observe that, by virtue of the Jensen inequality, $\mathbb{E}[\log(\lambda_1)] \leq \log[\mathbb{E}(\lambda_1)] = \log\mu$ for the GARCE(p, q) BP in its stationary regime, where μ denotes the mean offspring number. Hence, $\mu < 1$ implies that $\{Z_t\}$ is subcritical, while $\mu \leq 1$ implies that $\{Z_t\}$ is not supercritical.

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(ii) *(Lower bound on phase transition.)* On the other hand, if we rely on the Taylor expansion of the function $\log(x)$ at $x_0 = 1$, straightforward calculations yield $\log(\lambda) \geq (\lambda-1) - \frac{1}{2}(\lambda-1)^2 = -\frac{3}{2} + 2\lambda - \frac{1}{2}\lambda^2$, and, thus,

$$\mathbb{E}[\log(\lambda_1)] \geq -\tfrac{3}{2} + 2\mathbb{E}(\lambda_1) - \tfrac{1}{2}\mathbb{E}(\lambda_1^2) = C_\mu - \tfrac{1}{2}\operatorname{var}(\lambda_1)$$

for some real constant C_μ given as $C_\mu = -\frac{3}{2} + 2\mu - \mu^2/2$ for the GARCE(p, q) BP. Thus, $C_\mu - \operatorname{var}(\lambda_1)/2 > 0$ implies that $\{Z_t\}$ is supercritical, whereas $C_\mu - \operatorname{var}(\lambda_1)/2 \geq 0$ implies that $\{Z_t\}$ is not subcritical. Expressions for μ and $\operatorname{var}(\lambda_1)$ can be found in Sections 3–5 for the models studied.

The well-known result for the Bienaymé–Galton–Watson (BGW) process that it either explodes or becomes extinct with probability 1 continues to be valid for the GARCE(p, q) BP. A look at the extinction–explosion dichotomy will conclude this section.

Theorem 11. *Suppose that the* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *satisfies Assumption 1. Then either*

$$\mathbb{P}(Z_t \to 0 \text{ or } Z_t \to \infty \mid \mathcal{F}(\lambda)) = 1,$$

independently of Z_0*, or* $\mathbb{P}(Z_t \equiv 1 \text{ for all } t \mid Z_0 = 1,\ \mathcal{F}(\lambda)) = 1$ *with probability* 1.

Proof. For a proof of this result, see [24, Theorem 5.3]. The same result under the hypothesis that $\mathbb{E}(V_\lambda^+) < \infty$ is contained in [1, Corollary 1, Section 3, p. 1515] (see also [1, Theorems 5–7]). □

7. Limit theorems

At last, we are interested in limit theorems for the normalized GARCE BP and necessary and sufficient conditions for a nondegenerate limit. We also examine the extended Kesten–Stigum condition $\mathbb{E}(Z_1 \log^+ Z_1/\varphi'_{\lambda_1}(1)) < \infty$ for the GARCE BP $\{Z_t\}_{t\geq 0}$ as it relates to a nondegenerate limit. We begin with providing hypotheses that assure the existence of a normalizing sequence for $\{Z_t\}_{t\geq 0}$ so that the normalized process converges to a limit with probability 1 that is finite with probability 1 and nonzero away from the extinction set of $\{Z_t\}_{t\geq 0}$. For a supercritical BGW process, Seneta [18] in 1968 demonstrated the existence of a normalizing sequence $\{c_t\}_{t\in\mathbb{N}}$ such that $c_t^{-1} Z_t$ converges in distribution to a nonnegative RV W that is nonzero except on the extinction set of the BGW process. In 1971 Heyde [10] relied on an exponential martingale to extend this result to almost sure convergence. In 1978 Tanny [25] proved an analogue of Seneta's result [18] for BPRE and strengthened an earlier version that was presented in [19] in 1975, which did not preclude infinite values of W happening with positive probability and zero values of W away from the extinction set. The following theorem is an adaptation of Tanny's results [25] in 1978 to the GARCE branching processes, whose proof is in [25, Theorem 1].

Theorem 12. *Consider a* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *under Assumption 1 that satisfies* $\mathbb{E}|V_\lambda| < \infty$*. Then there exists a sequence of RVs* $\{c_t(\lambda)\}_{t\in\mathbb{Z}}$ *that only depend on* λ *with the following properties:*

(i) $\lim_{t\to\infty} Z_t/c_t(\lambda) = W$ *with probability* 1*,*

(ii) $\mathbb{P}(W = 0 \mid \mathcal{F}(\lambda)) = q(\lambda)$ *a.s.,*

(iii) $\mathbb{P}(W < \infty \mid \mathcal{F}(\lambda)) = 1$ *with probability* 1*,*

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(iv) $\lim_{t\to\infty} t^{-1}\log c_t(\lambda) = \mathbb{E}(V_\lambda)$ *with probability* 1, *and*

(v) *as* $t \to \infty$, *the sequence* $c_{t+1}(\lambda)/c_t(\lambda)$ *converges in distribution to* V_λ.

Note that for all subsequent results, we will focus on the supercritical GARCE BP. We continue with another limit result. A proof is presented in [25, Theorem 2]. Let $\{W_t^\circ\}_{t\in\mathbb{N}}$ denote the reduced branching process that we obtain by restricting ourselves to the branching subprocess that consists of the members of the GARCE BP that have infinite descent.

Theorem 13. *Suppose that the* GARCE(p, q) *BP* $\{Z_t\}_{t\ge 0}$ *is supercritical and Assumption 1 hold. Then for almost all* λ, *we have*

$$\mathbb{P}\bigg(\mathcal{E}^c,\ \lim_{t\to\infty}\frac{W_t^\circ}{1-q(T^t\lambda)Z_t} = 1 \ \bigg|\ \mathcal{F}(\lambda)\bigg) = \mathbb{P}(\mathcal{E}^c \mid \mathcal{F}(\lambda)) = 1 - q(\lambda),$$

where $\mathcal{E}^c = \{\omega\colon\ Z_t(\omega) \to \infty \text{ as } t \to \infty\}$ *denotes the complement of the event* $\mathcal{E}$ *stated in (10).*

We now specify a normalizing sequence that transforms the normalized GARCE BP into a martingale sequence. Denote

$$W_t = \frac{Z_t}{\pi_t}, \qquad \pi_t = \prod_{j=1}^{t}\varphi'_{\lambda_j}(1) \quad \text{for } t \ge 1,\ \pi_0 = Z_0 = 1. \tag{14}$$

The subsequent martingale theorem rests on the one demonstrated in [2, Theorem 1] in 1971, and follows along the ideas in [12] in 1966.

Theorem 14. *Suppose that the* GARCE(p, q) *BP* $\{Z_t\}_{t\ge 0}$ *with* $Z_0 = 1$ *is supercritical and satisfies Assumption 1. Let* $\{\mathcal{F}_{t,z}\}$ *denote the filtration stated in (2). Then the process* $\{W_t\}_{t\ge 0}$, *defined in (14), is a nonnegative* $\{\mathcal{F}_{t,z}\}$*-martingale and the limit* $\lim_{t\to\infty} W_t = W$ *exists almost surely. In addition, the condition*

$$\mathbb{E}\bigg(\frac{Z_1\log(Z_1)}{\varphi'_{\lambda_1}(1)}\bigg) < \infty$$

implies that

(i) $\lim_{t\to\infty}\mathbb{E}(\exp(-uW_t) \mid \mathcal{F}(\lambda)) \equiv \psi(u,\lambda)$ *is the unique solution of the functional equation*

$$\psi(u,\lambda) = \varphi_{\lambda_1}\bigg(\psi\bigg(\frac{u}{\varphi'_{\lambda_1}(1)},\ T\lambda\bigg)\bigg) \quad a.s.$$

among those solutions obeying $\lim_{u\downarrow 0} u^{-1}[1-\psi(u,\lambda)] = 1$,

(ii) $\mathbb{E}(W \mid \mathcal{F}(\lambda)) = 1$,

(iii) $\mathbb{P}(W = 0 \mid \mathcal{F}(\lambda)) = q(\lambda)$ *a.s.*

The last result surrounds the extended Kesten–Stigum condition [12] for the supercritical GARCE(p, q) branching processes, which is sufficient but not necessary for W to be nondegenerate, and provides a necessary and sufficient condition for W to be nondegenerate. The statements are analogous to those demonstrated in [26] the proofs in [26, Theorems 1, 3, and 4].

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Theorem 15. *Suppose that the* GARCE(p, q) *BP* $\{Z_t\}_{t\geq 0}$ *is supercritical with* $\mathbb{E}|V_\lambda| < \infty$ *and Assumption 1 hold. Let* $F_t(\cdot \mid \mathcal{F}_t(\lambda))$ *denote the cumulative conditional offspring distribution in the* t*th generation of the GARCE BP, given* $\mathcal{F}_t(\lambda)$*, where* $\mathcal{F}_t(\lambda)$ *is as stated in (2). Let* $\{W_t\}$ *be as stated in (14).*

(i) (Necessary and sufficient condition for nondegenerate W.) *If, for each* $Q > 0$*, the environmental process* $\lambda = \{\lambda_t\}_{t\in\mathbb{N}}$ *satisfies*

$$\sum_{t=1}^{\infty} \frac{1}{\varphi'_{\lambda_{2t}}(1)} \int_{\mathrm{e}^t Q}^{\infty} x \, \mathrm{d}F_{2t}(x \mid \mathcal{F}_t(\lambda)) < \infty \tag{15}$$

almost everywhere on

$$\mathcal{B} = \bigg\{\lambda \colon \sum_{t=1}^{\infty} (\varphi'_{\lambda_t}(1))^{-1} \int_{\mathrm{e}^t Q}^{\infty} x \, \mathrm{d}F_t(x \mid \mathcal{F}_t(\lambda)) < \infty\bigg\},$$

then $\lim_{t\to\infty} W_t = W$ *exists with probability* 1 *and* W *is nondegenerate if and only if*

$$\sum_{t=1}^{\infty} \frac{1}{\varphi'_{\lambda_t}(1)} \int_{\pi_{t+1}}^{\infty} x \, \mathrm{d}F_t(x \mid \mathcal{F}_t(\lambda)) < \infty \tag{16}$$

with probability 1*, where* π_t *is defined in (14). Furthermore, when* W *is nondegenerate, then* $\mathbb{P}(W = 0 \mid \mathcal{F}(\lambda)) = q(\lambda)$ *with probability* 1 *and* $\mathbb{E}(W \mid \mathcal{F}(\lambda)) = 1$ *with probability* 1*.*

(ii) (Sufficient condition for nondegenerate W.) *If (15) holds then* $\lim_{t\to\infty} W_t = W$ *exists with probability* 1 *and the condition*

$$\mathbb{E}\bigg(\frac{Z_1 \log^+ Z_1}{\varphi'_{\lambda_1}(1)}\bigg) < \infty \tag{17}$$

is a sufficient condition for W *to be nondegenerate.*

(iii) (Alternate form of a necessary and sufficient condition for nondegenerate W.) *Condition (16) is valid if and only if there exists some real constant* $\delta > 0$ *such that*

$$\sum_{t=1}^{\infty} \frac{1}{\varphi'_{\lambda_t}(1)} \int_{\mathrm{e}^{t\delta}}^{\infty} x \, \mathrm{d}F_t(x \mid \mathcal{F}_t(\lambda)) < \infty \tag{18}$$

with probability 1*. Additionally, if the condition in (18) holds for some* $\delta > 0$*, it holds for all* $\delta > 0$*.*

Remark 6. (i) For BGW processes, the condition in (16) simplifies to the well-known Kesten–Stigum result [12], which says that $\mathbb{E}(Z_1 \log^+ Z_1) < \infty$ is a necessary and sufficient condition for a nondegenerate W. It was earlier studied in [13] in 1959 and has since been intensively investigated in the literature (for a later proof among others, see [15]).

(ii) We point out that the extended Kesten–Stigum condition in (17) is unnecessary for W to be nondegenerate for a stationary and ergodic BPRE. We can see this in [26, Example 3.1, p. 137] with nondegenerate W, $\mathbb{E}|V_\lambda| < \infty$, and $\mathbb{E}(Z_1 \log^+ Z_1/\varphi'_{\lambda_1}(1)) = \infty$.

Acknowledgement

I am grateful to the anonymous referee for helpful comments and suggestions.

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