Math. Struct. in Comp. Science (2008), *vol.* 18, *pp.* 613–643.
doi:10.1017/S0960129508006695 First published online 26 February 2008 Printed in the United Kingdom

Monad compositions II: Kleisli strength

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Received 7 July 2006; revised 30 May 2007 and 24 October 2007

In this paper we introduce the concept of Kleisli strength for monads in an arbitrary symmetric monoidal category. This generalises the notion of commutative monad and gives us new examples, even in the cartesian-closed category of sets. We exploit the presence of Kleisli strength to derive methods for generating distributive laws. We also introduce linear equations to extend the results to certain quotient monads. Mechanisms are described for finding strengths that produce a large collection of new distributive laws, and consequently monad compositions, including the composition of monadic data types such as lists, trees, exceptions and state.

1. Introduction

A common construction in programming is to compose two data types, such as in constructing a list of records. Data types come equipped with certain operations and algebraic rules governing these operations. A simple but important question to ask is whether these operations continue to exist, and if the corresponding algebraic rules are preserved, when the data types are composed. In this paper we address this question by restricting our attention to a particular widely used group of data types, namely those that are monadic. In this case, the critical issue for the existence of such compositions is the presence of a distributive law.

Beck (1969) showed how monad compositions arise from distributive laws. While many papers about distributive laws have appeared since, including Adamek and Lawvere (2001), Koslowski (2005) and Marmolejo *et al.* (2002), less attention has been paid to general techniques for producing examples of these laws. In Manes and Mulry (2007), the current authors began a program of deriving such techniques in which critical use was made of the notion of a commutative monad. The methods described there could only be partially applied to some of the best-known data type constructor monads such as those for binary trees, lists, exceptions, and state, since all of these are non-commutative. There are few existing results showing that such well-known monads can compose in the sense of preserving their monadic structure, even though the corresponding data types are routinely composed in programming environments. For instance, it is currently not known if the list data constructor can compose with itself as a monad, a striking observation since it is common practice to work with lists of lists. The attempt to construct a composition of lists in King and Wadler (1993) failed: see Manes and Mulry (2007, Example 5.19). Weaker notions of composition can also be considered (see, for example, Luth and Ghani (2002), which is discussed in Manes and Mulry (2007, Remark 2.4.4)), but such notions are not addressed in this paper.

This paper attacks another aspect of the general problem of monad data type composition by introducing and utilising the notion of Kleisli strength for monads in an arbitrary symmetric monoidal category. We then derive a new set of schema for generating distributive laws that only require this weaker notion. In particular, some of the unsatisfactory issues described above are addressed. Now, all the aforementioned data type constructor monads, namely the list, binary tree, exception and state monad constructors, come equipped with Kleisli strengths, and in some cases more than one, leading in turn to multiple distributive laws. While the case of a distributive law for lists over itself is not resolved, a variety of examples of the composition of non-empty lists can be derived, as well as distributive laws on more complex data type constructors.

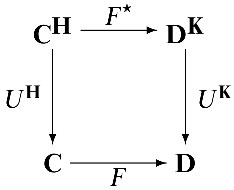
The remainder of this section reviews a few of the basic definitions and constructions related to monads, algebras and distributive laws, but also presents some new definitions. The paper assumes the reader is familiar with the general notions of category theory (Mac Lane 1971), and also depends heavily on Manes and Mulry (2007).

In Section 2 we define the notion of Kleisli strength and provide a number of examples, as well as methods, for generating Kleisli strength. As the examples illustrate, this notion is of independent interest and generalises the prior notion of a commutative monad. The notion of coherent Kleisli strength is also introduced, while its connection to monadic signatures with linear equations is addressed later in the paper. This section also provides a brief introduction to classical commutative monads and Kock strength, and contrasts these concepts with that of general Kleisli strength.

Section 3 discusses the free monad $\mathbf{\Sigma}^@$ generated by a signature Σ and establishes a result on the existence of distributive laws of the form $\mathbf{\Sigma}^@\mathbf{K} \to \mathbf{K}\mathbf{\Sigma}^@$ where $\mathbf{K}$ is a monad with Kleisli strength. Examples of new Kleisli strengths and consequent distributive laws for some specific signatures are also presented.

Section 4 uses coherent Kleisli strength in connection with monadic linear signatures to generate new distributive laws. The main theorem stresses universal algebra with *linear* equations (which have the same non-repeating set of variables on both sides); though only linear terms can be interpreted for algebras in a symmetric monoidal category, this restriction is required to produce our results even in the category of sets.

Let $\mathbf{C}$, $\mathbf{D}$ be categories with functor $F : \mathbf{C} \to \mathbf{D}$. let $\mathbf{H} = (H, \mu, \eta)$ be a monad in $\mathbf{C}$ and $\mathbf{K} = (K, \nu, \rho)$ be a monad in $\mathbf{D}$. We write $\mathbf{C}^\mathbf{H}$ for the category of $\mathbf{H}$-algebras with forgetful functor $U^\mathbf{H} : \mathbf{C}^\mathbf{H} \to \mathbf{C}$ and write $\mathbf{C}_\mathbf{H}$ for the Kleisli category of $\mathbf{H}$ with canonical functor $\imath_\mathbf{H} : \mathbf{C} \to \mathbf{C}_\mathbf{H}$, $\imath_\mathbf{H}(X \xrightarrow{f} Y) = X \xrightarrow{f} Y \xrightarrow{\eta_Y} HY$. An *algebra lift*



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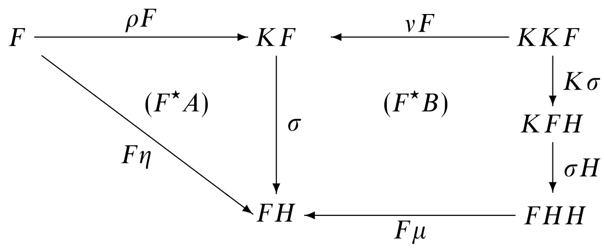
is classified by a natural transformation $\sigma : KF \to FH$ satisfying



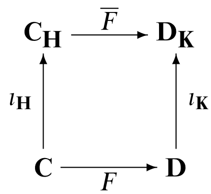
The correspondences between $F^\star$ and σ are given by

$$F^\star(A, \xi) \;=\; (FA,\; KFA \xrightarrow{\sigma_A} FHA \xrightarrow{F\xi} FA) \tag{1}$$

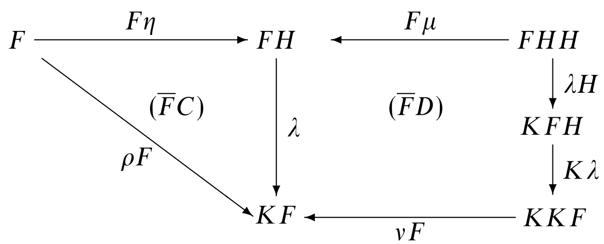
and, if $F^\star(HA, \mu_A) = (FHA,\; KFHA \xrightarrow{\gamma_A} FHA)$,

$$\sigma_A \;=\; KFA \xrightarrow{KF\eta_A} KFHA \xrightarrow{\gamma_A} FHA\,. \tag{2}$$

A *Kleisli lift*



is classified by a natural transformation $\lambda : FH \to KF$ satisfying



The correspondences between $\overline{F}$ and λ are given by

$$\overline{F}A = FA,\;\; \overline{F}(A \xrightarrow{\alpha} HB) \;=\; FA \xrightarrow{F\alpha} FHB \xrightarrow{\lambda_B} KFB \tag{3}$$

and

$$\lambda_A \;=\; \overline{F}(id_{HA})\,. \tag{4}$$

The transformations above are referred to as *lifting transformations*. When **C** and **D** coincide, a *monad map* $\sigma : \mathbf{K} \to \mathbf{H}$ is a natural transformation $\sigma : K \to H$ satisfying

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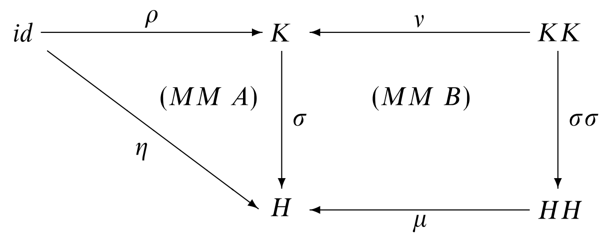


A monad map is exactly a lifting transformation for both an algebra lift and a Kleisli lift of the identity functor of **C**. The relationship between the algebra lift and the monad map σ is given by

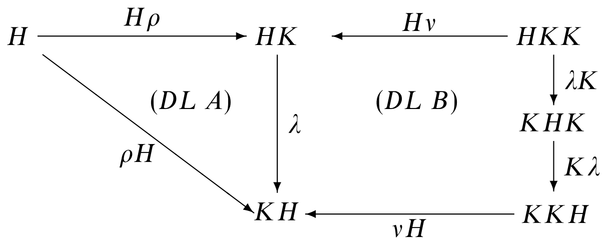
$$id^\star(A,\xi) \;=\; (A,\; KA \xrightarrow{\sigma_A} HA \xrightarrow{\xi} A) \tag{5}$$

and, if $id^\star(HA,\mu_A) = (HA,\; KHA \xrightarrow{\gamma_A} HA)$, we have

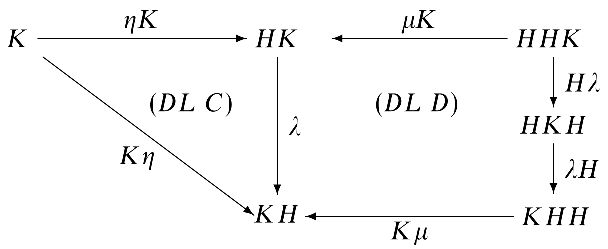
$$\sigma_A \;=\; KA \xrightarrow{K\eta_A} KHA \xrightarrow{\gamma_A} HA, \tag{6}$$

so σ_A is the unique **K**-homomorphic extension of η_A.

While algebra and Kleisli lifts are generally defined for different categories **C**, **D** as above, it is often useful to restrict attention to the case where both categories agree, as was the case for monad maps as well as in the following important definition. For monads **H,K** on category **C**, a *distributive law of* **H** *over* **K** is a natural transformation $\lambda : HK \to KH$ that classifies both an algebra lift and a Kleisli lift, that is, the following diagrams commute.



(Kleisli lift $\overline{H}$)



(Algebra lift $K^\star$)

The next few results are from Beck (1969).

Theorem 1.1. If $\lambda : HK \to KH$ is a distributive law of **H** over **K**, then

$$\mathbf{K} \circ_\lambda \mathbf{H} \;=\; (KH, (\nu\mu)(K\lambda H), \rho\eta) \tag{7}$$

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is a monad in $\mathbf{C}$ whose algebras are isomorphic to the category of all (A, ξ, θ) with (A, ξ) a $\mathbf{K}$-algebra and (A, θ) an $\mathbf{H}$-algebra such that the following *composite law* holds:

$$\begin{array}{ccc} HKA & \xrightarrow{\lambda_A} & KHA \\ & & \downarrow K\theta \\ H\xi \downarrow & (CL) & KA \\ & & \downarrow \xi \\ HA & \xrightarrow[\theta]{} & A \end{array}$$

Here, the morphisms $f : (A, \xi, \theta) \to (A', \xi', \theta')$ are simultaneous $\mathbf{H}$- and $\mathbf{K}$- homomorphisms. The $\mathbf{K} \circ_\lambda \mathbf{H}$-structure map corresponding to (A, ξ, θ) is $\xi(K\theta)$, but if (A, γ) is a $\mathbf{K} \circ_\lambda \mathbf{H}$-algebra, the corresponding composite structure (A, ξ, θ) is given by $\xi = \gamma(K\eta_A)$, $\theta = \gamma(\rho_{HA})$. The passage $\lambda \mapsto \mathbf{K} \circ_\lambda \mathbf{H}$ is a bijection from the class of distributive laws of $\mathbf{H}$ over $\mathbf{K}$ to the class of natural transformations $m : KHKH \to KH$, with $(KH, m, \rho\eta)$ a monad for which ρH, $K\eta$ are monad maps and $m(K\eta\rho H) = id_{KH}$. The inverse bijection is given by

$$\lambda = HK \xrightarrow{\rho HK\eta} KHKH \xrightarrow{m} KH. \tag{8}$$

Theorem 1.2. If $\lambda : HK \to KH$ is a distributive law, not only does K lift to $K^\star : \mathbf{C}^{\mathbf{H}} \to \mathbf{C}^{\mathbf{H}}$, but, additionally, for each $\mathbf{H}$-algebra (A, θ),

$$\begin{aligned} \rho_A &: (A, \theta) \longrightarrow K^\star(A, \theta) \\ \nu_A &: K^\star K^\star(A, \theta) \longrightarrow K^\star(A, \theta) \end{aligned}$$

are $\mathbf{H}$-homomorphisms, so the entire monad $\mathbf{K}$ lifts to a monad $\mathbf{K}^\star$ in $\mathbf{C}^{\mathbf{H}}$. The passage from distributive laws λ to lifted monads $(K^\star, \rho, \nu)$ in $\mathbf{C}^{\mathbf{H}}$ is bijective. The algebras over the lifted monad are exactly the composite algebras of $\mathbf{K} \circ_\lambda \mathbf{H}$, but now with forgetful functor $\mathbf{C}^{\mathbf{K} \circ_\lambda \mathbf{H}} \to \mathbf{C}^{\mathbf{K}}$.

Since $(K\theta)\lambda_A$ is the structure map of $K^\star(A, \theta)$, the composite law simply asserts that

$$K^\star(A, \theta) \xrightarrow{\xi} (A, \theta) \text{ is an } \mathbf{H}\text{-homomorphism.} \tag{9}$$

The category of sets and total functions will be denoted **Set**.

Example 1.3. The *list monad* $\mathbf{L} = (L, \mu, \eta)$ in **Set** is important in this paper. It is defined as follows:

$$\begin{aligned} LA &= A^* && \text{(the set of all lists of elements of A)} \\ \eta_A(x) &= [x] \\ \alpha^\#[x_1,, \dots x_n] &= \alpha(x_1) \mathbin{+\!\!+} \cdots \mathbin{+\!\!+} \alpha(x_n) && (\mathbin{+\!\!+} = \text{concatenation}). \end{aligned}$$

The algebras of the list monad are monoids. $\mathbf{L}^+ = (L^+, \mu, \eta)$ is the submonad of non-empty lists whose algebras are semigroups.

The motivating example for the term 'distributive law' is $\mathbf{K}$ the monad for abelian groups, $\mathbf{L}$ the list monad (whose algebras are monoids) and $\lambda : LK \to KL$ the distributive law that converts a product of sums to a sum of products. The composite algebras are rings. Because it is common terminology that products distribute over sums, the monad **L** for products should distribute over the monad **K** for sums, that is, λ should be a distributive law of **L** over **K**. Note that in Manes and Mulry (2007, Definition 2.20), we mistakenly said '**K** over **H**' rather than '**H** over **K**' – we thank a referee for calling this to our attention, and the terminology has been corrected in this paper.

Example 1.4. The *power set monad* $\mathbf{P} = (P, \mu, \eta)$ in **Set** is defined by

$$PX = \{A : A \subset X\}, \quad \eta_X(x) = \{x\}, \quad \mu(\mathscr{A}) = \bigcup \mathscr{A}\,.$$

The algebras are complete sup-semilattices (which are the same objects as complete lattices, but the morphisms are required only to preserve suprema). $\mathbf{P_0} = (P_0, \mu, \eta)$ is the submonad of **P** with $P_0X = \{A : A \subset X,\ A \text{ finite}\}$. The $\mathbf{P_0}$-algebras are sup-semilattices.

Example 1.5. Let M be a monoid with unit e. This induces a monad (T, μ, η) in **Set** with $TX = M \times X$, $\eta_X\, x = (e, x)$ and $\mu_X(m, n, x) = (mn, x)$. The algebras of this monad are the left M-sets with equivariant maps. For example, M might be a monoid of 'certificates', or a semilattice of reliability values whose unit 1 represents 'guaranteed reliability', or simple strings on an alphabet. We will refer to this as the *M-set monad*.

Example 1.6. Let Exc be a non-empty set of 'exceptions'. The resulting *exceptions monad* in **Set** is (M, μ, η) with $MX = X + Exc$, η the first coproduct injection and $\mu_X : X + Exc + Exc \to X + Exc$ mapping x to itself and e from either copy of Exc to e. The algebras are sets equipped with an Exc-indexed family of constants, and the morphisms preserve these constants, that is, $f(e) = e$.

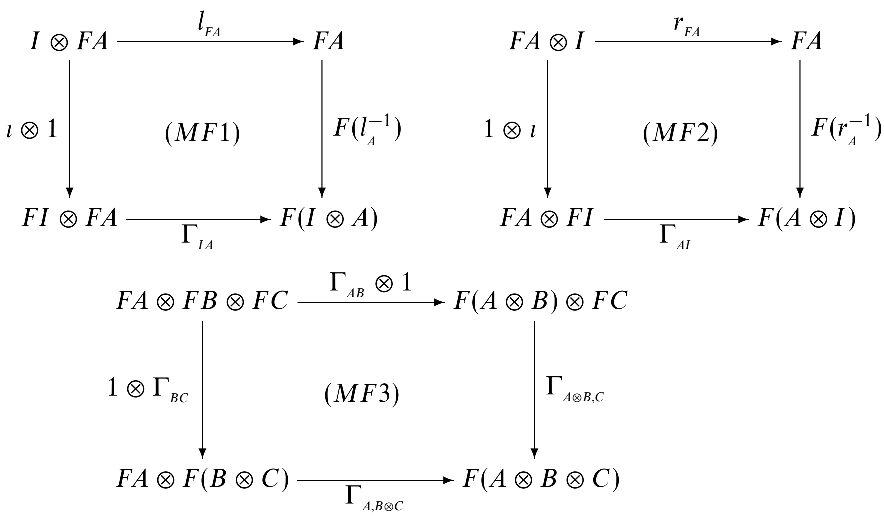
Example 1.7. The *binary tree monad* $\mathbf{V} = (V, \mu, \eta)$ in **Set** is defined as follows. VX consists of binary trees whose values (from X) are located in its leaves. We use E to denote an empty tree, Lx to denote a trivial tree (that is, a leaf) with value x and $N(v1, v2)$ to denote a tree consisting of left and right subtrees $v1$ and $v2$, respectively. We have $\eta(x) = Lx$, while $\mu E = E$, $\mu(Lt) = t$ and $\mu(N(t_1, t_2)) = N(\mu(t_1), \mu(t_2))$ for $t \in VX, t_1, t_2 \in VVX$. The algebras are sets equipped with an identity and a binary operation, with morphisms preserving both. $\mathbf{V}^+ = (V^+, \mu, \eta)$ denotes the submonad of non-empty binary trees.

2. Kleisli strength

2.1. *Definitions and examples*

For many of the general results that follow we will work in the setting of a symmetric monoidal category $\mathscr{V}$ with tensor $\otimes$ and tensor unit I. An important special case, which will occur often, is when $\otimes = \times$ is cartesian product and I is a terminal object. In this case we say that $\mathscr{V}$ is a *cartesian category*, noting that no further assumptions are made about finite limits.

Definition 2.1. Recall (Eilenberg and Kelly 1966, pages 473–474) that a *monoidal functor* $\mathscr{V} \to \mathscr{V}$ is $(F, \Gamma, \imath)$, with $F : \mathscr{V} \to \mathscr{V}$ a functor, $\imath : I \to FI$ a morphism and $\Gamma_{AB} : FA \otimes FB \to F(A \otimes B)$ a natural transformation, satisfying the axioms:



We will often suppress the associativity isomorphisms $\alpha_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, the unitary isomorphisms $l_A : I \otimes A \to A$ and $r_A : A \otimes I \to A$, and the symmetry isomorphisms $c_{AB} : A \otimes B \to B \otimes A$ unless they help make the exposition clearer. For example, $\cong: FA \to F(I \otimes A)$ with F a functor is understood to be $F\, l_A$.

We shall use the abbreviation $\otimes_n x$ for $x \otimes \cdots \otimes x$ (n times), where x can be either an object or a map. When x is an object and $n = 0$, we have $\otimes_n x = I$. In general, $\otimes_1 x = x$. We also use the same symbol for the n-fold tensor functor, so $\otimes_n(V_1, \ldots, V_n) = V_1 \otimes \cdots \otimes V_n$.

In this section, we fix a monad $\mathbf{K} = (K, \nu, \rho)$ on $\mathscr{V}$, noting that when $\mathscr{V}$ is a symmetric monoidal category, K is not required to be a monoidal functor.

For $n \geqslant 0$, consider the cartesian power category $\mathscr{V}^n$ with objects $(V_1, \ldots, V_n)$ and morphisms $(f_1, \ldots, f_n) : (V_1, \ldots, V_n) \to (W_1, \ldots, W_n)$. $\mathbf{K}$ induces a monad $\mathbf{K^{(n)}}$ on $\mathscr{V}^n$ through

$$
\begin{aligned}
K^{(n)}(V_1, \ldots, V_n) &= (K V_1, \ldots, K V_n) \\
\rho_{(V_1, \ldots, V_n)} &= (\rho_{V_1}, \ldots, \rho_{V_n}) \\
\nu_{(V_1, \ldots, V_n)} &= (\nu_{V_1}, \ldots, \nu_{V_n}).
\end{aligned}
$$

We denote the resulting Kleisli category of $\mathbf{K^{(n)}}$ simply as $\mathscr{V}^n_{\mathbf{K}}$.

The next definition generalises Mulry (1994, Example 2.3).

Definition 2.2. A *Kleisli strength on* $\mathbf{K}$ *of order* $n \geqslant 0$ is a natural transformation $\Gamma^n_{V_1, \cdots, V_n} : K V_1 \otimes \cdots \otimes K V_n \to K(V_1 \otimes \cdots \otimes V_n)$ that classifies a Kleisli lift $\overline{\otimes_n} : \mathscr{V}^n_{\mathbf{K}} \to \mathscr{V}_{\mathbf{K}}$ of the n-fold tensor functor $\otimes_n : \mathscr{V}^n \to \mathscr{V}$. Thus Γ^n is a natural transformation satisfying ($\Gamma^n A$) and ($\Gamma^n B$) as shown in the following diagram (note that V- subscripts are dropped to improve readability):

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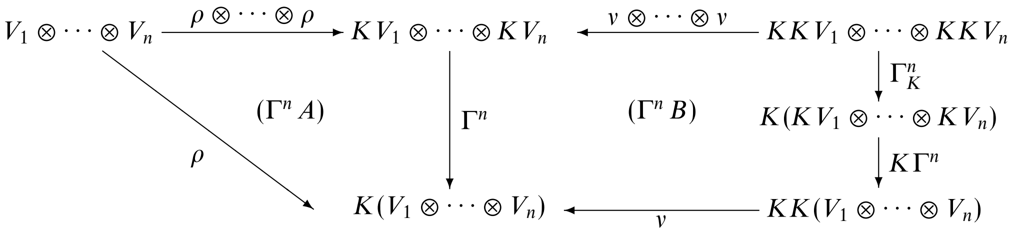
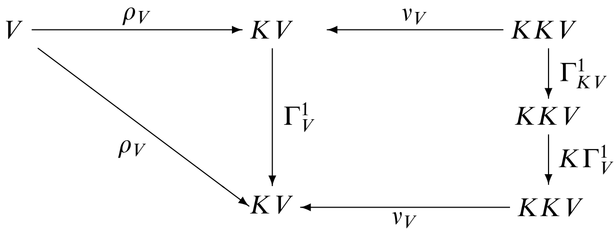


We will now clarify the definition in the special cases $n = 0, 1$. For $n = 0$, the empty tensor product is I, so $(\Gamma^0 A)$ stipulates that $\Gamma^0 : I \to KI$ coincides with ρ_I. In that case, $(\Gamma^0 B)$ holds. Hence, there is always a unique Kleisli strengh of order 0, namely ρ_I, whose Kleisli lift $\overline{\otimes_0} : 1 \to \mathscr{V}_{\mathbf{K}}$ is the object I. For $n = 1$, $\Gamma^1_V : KV \to KV$ must satisfy



So a Kleisli strength of order 1 is the same thing as a monad map $\mathbf{K} \to \mathbf{K}$ since the above diagrams are just (MMA) and (MMB). As such, there is always at least one Kleisli strength of order 1, namely $\Gamma^1_V = id_{KV}$. Notice that if $\Gamma^1_{KV} = id_{KKV}$, the diagrams state that Γ^1_V is the unique homomorphic extension of ρ_V, which forces $\Gamma^1_V = id_{KV}$ as well.

Example 2.3. Let $\mathbf{K} = \mathbf{L}$ be the list monad of Example 1.3. Then the reverse transformation $rev : L \to L$ is a Kleisli strength of order 1 on $\mathbf{L}$.

The reason that $(\Gamma^1 B)$ commutes is that to reverse a list such as $[a,b,c,d,e]$ one can divide it into blocks, for example, $[[a,b,c],[d,e]]$ and first reverse the blocks $[[d,e],[a,b,c]]$, then reverse the contents of each block $[[e,d],[c,b,a]]$ and, finally, flatten the result to get the reverse $[e,d,c,b,a]$ of the original list. Since id_L is also a Kleisli strength, as pointed out above, this example demonstrates that Kleisli strengths are not unique.

Example 2.4. Let $\mathbf{K} = \mathbf{P_0}$ be the finite power set monad of Example 1.4. Then $\Gamma^n(A_1,\ldots,A_n) = A_1 \times \cdots \times A_n$ is a Kleisli strength of order $n \geqslant 0$, where $A_i \in P_0 V_i$ for $1 \leqslant i \leqslant n$.

Example 2.5. Let $\mathbf{M}$ be the exceptions monad of Example 1.6, with Exc any non-empty set. If $a \in Exc$, then for each $n \geqslant 0$ the monad admits a Kleisli strength Γ^n, where $\Gamma^n(x_1,\ldots,x_n) = (x_1,\ldots,x_n)$, if all of the x_i are in X, and equals a otherwise. It is trivial to check that this is a Kleisli strength of order n and that if $a \neq b$ where $b \in Exc$, then the new Γ^n generated by b differs from the one generated by a. Thus for any cardinal κ, there exists a monad $\mathbf{K}$ with at least κ different Kleisli strengths.

Example 2.6. Much as in the previous example, let $\mathbf{K}$ be the '*add a bottom*'or lifting monad, that is, the monad defined on **Dom**, the category of Scott domains and (total)

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continuous maps (Scott 1970), $KA = A_\perp$. Then Γ^n is a Kleisli strength of order n ($n \geqslant 1$) where $\Gamma^n(x_1 \dots x_n) = (x_1 \dots x_n)$ if all of the x_i are in X, and equals $\perp$ otherwise.

The last strength provides a simple approach to showing that cartesian products in **Dom** induce them in **pDom**, the category of (possibly) bottomless domains and partial continuous maps. Partial maps $f : A \rightharpoonup B$ and $g : A \rightharpoonup C$ **pDom** induce total maps $f* : A \to B_\perp$ and $g* : A \to C_\perp$ in **Dom**. The partial map $A \rightharpoonup (B \times C)$ then corresponds to the total map $\Gamma^2(f*, g*)$ where $(f*, g*) : A \to B_\perp \times C_\perp$ is the unique total map in **Dom** (Mulry 1998).

Example 2.7. Let $\mathbf{V} = (V, \mu, \eta)$ be the binary trees monad of Example 1.7. Then V has a Kleisli strength of order 1, $ref : V \to V$, denoting reflection. Using the notation of Example 1.7, we define ref by

$$\begin{aligned} ref(E) &= E \\ ref(La) &= La \\ ref(N(t_1, t_2)) &= N(ref(t_2), ref(t_1)) . \end{aligned}$$

The reader is left to check that ref is a Kleisli strength. We further note that there is a monad map $list : V \to L$ that converts a binary tree into a list of values (read left to right) so that ref extends rev in the sense that $list(ref) = rev(list)$. This is a special case of more general constructions detailed in Examples 2.22 and 3.5.

Example 2.8. For a fixed set A, consider the exponential monad $\mathbf{M}$ where $MB = B^A$, $\eta(b) = \lambda a.b$, $\mu(F) = \lambda a.F(a)(a)$, that is, $\mathbf{M}$ is the A-indexed cartesian power monad of the identity monad (see Proposition 2.19 below). M has a Kleisli strength of order n where $\Gamma^n(f_1 \dots f_n) = \lambda a.(f_1(a) \dots f_n(a))$.

Condition ($\Gamma^n A$) is easy to check and left to the reader. Condition ($\Gamma^n B$) holds because, for example in the case of $n = 2$,

$$(\lambda a.\lambda b.(Fab, Gab))(x)(x) = ((\lambda a.Faa), (\lambda b.Gbb))(x) = (Fxx, Gxx) .$$

The case of general n is the same.

Example 2.9. Consider the M-set monad $KX = M \times X$ of Example 1.5. Defining Γ^2 by

$$\Gamma^2((m_1, x), (m_2, y)) = (m_1 m_2, x, y))$$

does not generally create a Kleisli strength as (Γ B) may fail. If M is a commutative monoid, however, it satisfies the rule

$$m_1 m_2 m_3 m_4 = m_1 m_3 m_2 m_4 ,$$

in which case Γ forms a Kleisli strength of order 2.

Proposition 2.10. Let Γ^2 be a Kleisli strength on $\mathbf{K}$ of order 2. Then

$$\Gamma^3_{XYZ} \;=\; KX \otimes KY \otimes KZ \xrightarrow{\Gamma^2 \otimes 1} K(X \otimes Y) \otimes KZ \xrightarrow{\Gamma^2} K(X \otimes Y \otimes Z)$$

is a Kleisli strength on $\mathbf{K}$ of order 3. The same process produces Γ^4 from Γ^3, Γ^5 from Γ^4, and so on, producing a Kleisli strength Γ^n of order n for all $n \geqslant 2$.

Proof. We show that Γ^3 satisfies the axioms. The same proof works for Γ^n to Γ^{n+1}. For axiom $(\Gamma^3 A)$,

$$\begin{aligned} \Gamma^3_{XYZ}(\rho_X \otimes \rho_Y \otimes \rho_Z) &= \Gamma^2_{X\otimes Y,Z}(\Gamma^2_{X,Y} \otimes id_{KZ})(\rho_X \otimes \rho_Y \otimes \rho_Z) \\ &= \Gamma^2_{X\otimes Y,Z}(\rho_{X\otimes Y} \otimes \rho_Z) && \text{(by } (\Gamma^2 A)) \\ &= \rho_{X\otimes Y\otimes Z} && \text{(by } (\Gamma^2 A)). \end{aligned}$$

For $(\Gamma^3 B)$, we have the following proof. The value

$$\begin{aligned} &\Gamma^3_{XYZ}(\nu_X \otimes \nu_Y \otimes \nu_Z) \\ &\quad = \Gamma^2_{X\otimes Y,Z}(\Gamma^2_{X,Y} \otimes id_{KZ})(\nu_X \otimes \nu_Y \otimes \nu_Z) \\ &\quad = \Gamma^2_{X\otimes Y,Z}(\nu_{X\otimes Y} \otimes \nu_Z)(K\Gamma^2_{XY} \otimes K(id_{KZ}))(\Gamma^2_{KX,KY} \otimes id) \\ &\qquad\qquad \text{(by } (\Gamma^2 B) \otimes (1\,\nu = \nu 1)) \\ &\quad = \nu_{X\otimes Y\otimes Z}(K\Gamma^2_{X\otimes Y,Z})\Gamma^2_{K(X\otimes Y),KZ}(K\Gamma^2_{XY} \otimes K(id_{KZ}))(\Gamma^2_{KX,KY} \otimes id) \\ &\qquad\qquad \text{(by } (\Gamma^2 B)) \\ &\quad = \nu_{X\otimes Y\otimes Z}(K\Gamma^2_{X\otimes Y,Z})K(\Gamma^2_{XY} \otimes id_{KZ})\Gamma^2_{KX\otimes KY,KX}(\Gamma^2_{KX,KY} \otimes id) \\ &\qquad\qquad \text{(by the naturality of } \Gamma^2). \\ &\quad = \nu_{X\otimes Y\otimes Z}(K\Gamma^3_{XYZ})(\Gamma^3_{KX,KY,KZ}) \end{aligned}$$

□

Note that in Example 2.4, Γ^n arises from Γ^2 by the process of the previous proposition for $n \geqslant 2$.

One can also define a Kleisli strength of order 3 in a similar way using $\Gamma^3 = \Gamma^2 \circ (1 \otimes \Gamma^2)$, and then $\Gamma^4 = \Gamma^2 \circ (1 \otimes \Gamma^3)$, and so on, to produce a Kleisli strength Γ^n of order n for all $n \geqslant 2$. This construction of Kleisli strength of order n generally differs from the one above (see Example 2.18), unless Kleisli strength Γ^2 is associative, which we define next.

Definition 2.11. A Kleisli strength of order 2 on **K** is *associative* if it satisfies axiom $(MF\,3)$ for a monoidal functor.

By the above discussion, we immediately have the following corollary.

Corollary 2.12. Let Γ^2 be a Kleisli strength on **K** of order 2. Then there are two canonical ways of generating a Kleisli strength of order n for $n \geqslant 3$. When Γ^2 is associative, the two approaches generate the same Kleisli strength of order $n \geqslant 3$.

The following notion will be useful later in the paper.

Definition 2.13. Let **K** be a monad in $\mathscr{V}$. A *coherent family of Kleisli strengths* on **K** is a family $(\Gamma^k : k = 1, 2, 3, \ldots)$ with each Γ^k a Kleisli strength of order k on **K**, all subject to $\Gamma^1 = id$ and $\Gamma^n(\Gamma^{k_1} \otimes \cdots \otimes \Gamma^{k_n}) = \Gamma^{k_1+\cdots+k_n}$.

Example 2.14. Let Γ^2 be an associative Kleisli strength of order 2 on **K**. Let $\Gamma^1 = id$ and, for $n > 1$, let Γ^n be derived from Γ^2 by either method of Corollary 2.12. Then (Γ^k) is a coherent family of Kleisli strengths.

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2.2. *Generating Kleisli strength*

Now we shall show how monad maps can be used to create Kleisli strength. Recall that a monad map $\mathbf{S} \to \mathbf{T}$ on category $\mathscr{V}$ classifies a Kleisli lift $\overline{id} : \mathscr{V}_{\mathbf{S}} \to \mathscr{V}_{\mathbf{T}}$ (as is also clear by regarding a monad map as a Kleisli strength of order 1). We use the following result from Mulry (1994).

Lemma 2.15. If $\lambda : FH \to KF$ and $\sigma : GK \to LG$ are two Kleisli lifting transformations of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ where monads H, K, L are associated with **C, D, E** respectively, then the composite functor GF also has a Kleisli lifting transformation, namely $\sigma_F \circ G\lambda$.

Proposition 2.16. Let $\alpha_1, \ldots, \alpha_n : \mathbf{K} \to \mathbf{H}$ ($n \geqslant 1$), $\beta : \mathbf{H} \to \mathbf{K}$ be monad maps. Let Γ be a Kleisli strength of order n on $\mathbf{H}$. Then

$$KV_1 \otimes \cdots \otimes KV_n \xrightarrow{\times_n \alpha_i} HV_1 \otimes \cdots \otimes HV_n \xrightarrow{\Gamma} H(V_1 \otimes \cdots \otimes HV_n) \xrightarrow{\beta} K(V_1 \otimes \cdots \otimes V_n)$$

is a Kleisli strength of order n on $\mathbf{K}$.

Proof. Let α_i classify the Kleisli lift $\overline{id_i} : \mathscr{V}_{\mathbf{K}} \to \mathscr{V}_{\mathbf{H}}$, let β classify the Kleisli lift $\overline{id} : \mathscr{V}_{\mathbf{H}} \to \mathscr{V}_{\mathbf{K}}$ and let Γ classify the Kleisli lift $\overline{\otimes_n} : \mathscr{V}^n_{\mathbf{H}} \to \mathscr{V}_{\mathbf{H}}$. Then the composite

$$\mathscr{V}^n_{\mathbf{K}} \xrightarrow{\otimes_n \overline{id_i}} \mathscr{V}^n_{\mathbf{H}} \xrightarrow{\overline{\otimes_n}} \mathscr{V}_{\mathbf{H}} \xrightarrow{\overline{id}} \mathscr{V}_{\mathbf{K}}$$

lifts the n-fold tensor functor $\mathscr{V}^n \to \mathscr{V}$, and thus classifies a Kleisli strength on $\mathbf{K}$. The proposition now follows directly from Lemma 2.15. □

Corollary 2.17. Let $\alpha_1, \ldots, \alpha_n : \mathbf{K} \to \mathbf{id}$ be monad maps ($n \geqslant 1$). Then

$$KV_1 \otimes \cdots \otimes KV_n \xrightarrow{\alpha_{1,V_1} \otimes \cdots \otimes \alpha_{n,V_n}} V_1 \otimes \cdots \otimes V_n \xrightarrow{\rho_{V_1 \otimes \cdots \otimes V_n}} K(V_1 \otimes \cdots \otimes V_n)$$

is a Kleisli strength on $\mathbf{K}$ of order n.

Proof. $id : V_1 \otimes \cdots \otimes V_n \to V_1 \otimes \cdots \otimes V_n$ is a Kleisli strength of order n on $\mathbf{id}$ and $\rho : \mathbf{id} \to \mathbf{K}$ is a monad map. □

Example 2.18. Let $\mathbf{L}^+$ be the non-empty list monad of Example 1.3. Then both *fst* and *lst* : $\mathbf{L}^+ \to id$, which choose the first and last, respectively, elements of a non-empty list, are monad maps. These generate four different Kleisli strengths of order 2 on the monad through Corollary 2.17, two of which, $\eta \circ (fst \times fst)$ and $\eta \circ (lst \times lst)$, are associative.

For instance, if we define $\Gamma^2 = \eta \circ (lst \times fst)$, then

$$\Gamma^2([a_1, a_2, a_3], [b_1, b_2, b_3]) = [(a_3, b_1)] .$$

This Γ^2 is not associative as

$$\Gamma^2(\Gamma^2 \times 1)([a_1, a_2], [b_1, b_2], [c_1, c_2]) = [(a_2, b_1, c_1)] ,$$

while

$$\Gamma^2(1 \times \Gamma^2)([a_1, a_2], [b_1, b_2], [c_1, c_2]) = [(a_2, b_2, c_1)] .$$

The next two results give further methods for constructing new Kleisli strengths from old ones. The proofs are routine diagram chases, which we leave to the reader.

Proposition 2.19. Let $\mathscr{V}$ have J-indexed products. Let $n \geqslant 1$, and consider $(\mathbf{K}_j, \Gamma_j)$ $(j \in J)$ with $\mathbf{K}_j = (K_j, v_j, \rho_j)$ a monad in $\mathscr{V}$ and Γ_j a Kleisli strength of order n on $\mathbf{K}_j$. Let $\mathbf{K} = (K, v, \rho)$ be the cartesian product monad $KA = \prod K_j A$, $pr_j\, \rho = \rho_j$, $pr_j\, v = KK \xrightarrow{pr_j\, pr_j} K_j K_j \xrightarrow{v_j} K_j$. Then $\Gamma_{A_1 \cdots A_n} : \times_n KA_j \to K(\times_n A_j)$ defined by $pr_j\, \Gamma = \Gamma_j\, (\times_n\, pr_j)$ is a Kleisli strength of order n on $\mathbf{K}$.

Example 2.20. Trivially, the identity monad **id** has a Kleisli strength of order n defined as $\Gamma^n = id : V_1 \otimes \cdots \otimes V_n \to V_1 \otimes \cdots \otimes V_n$. When **C** is **Set**, we may form the product monad $\mathbf{R} = \mathbf{id} \times \mathbf{id}$, which is exactly the *rectangular bands monad* of Manes and Mulry (2007, Example 2.1.7). Here $\eta(x) = (x, x)$ and $\mu(a, b, c, d) = (a, d)$. By Proposition 2.19, R has a Kleisli strength, which is defined for the case of $n = 2$ as $\Gamma((a, b), (c, d)) = ((a, c)(b, d))$.

Proposition 2.21. Let $\alpha : \mathbf{K} \to \mathbf{H}$ be a monad map and $n \geqslant 1$. Suppose also that we are given families Γ, Γ' and commutative squares

$$\begin{array}{ccc} \otimes_n KA_i & \xrightarrow{\ \Gamma'_{A_1 \cdots A_n}\ } & K(\otimes_n A_i) \\ \Big\downarrow {\scriptstyle \otimes_n \alpha_{A_i}} & & \Big\downarrow {\scriptstyle \alpha_{\otimes_n A_i}} \\ \otimes_n HA_i & \xrightarrow[\ \Gamma_{A_1 \cdots A_n}\ ]{} & H(\otimes_n A_i) \end{array}$$

(*cf.* the category $ps_n(\mathscr{V})$ in Definition 2.23 below). Then the following hold:

(1) If Γ is a Kleisli strength of order n on **H** and if α is pointwise monic, then Γ' (which is not *a priori* assumed natural) is a Kleisli strength of order n on **K**.
(2) If Γ' is a Kleisli strength of order n on **K** and if $\otimes_n \alpha\alpha : \otimes_n KK \to \otimes_n HH$ is pointwise epic, then Γ is a Kleisli strength of order n on **H**.

Example 2.22. Consider the monad map $list : V \to L$ and Kleisli strength ref of Example 2.7. Since $list$ is pointwise epic, the previous result provides an easy proof that rev is a Kleisli strength of order 1 also.

Assuming the monads $\mathbf{K}, \mathbf{H}$ have Kleisli strength Γ^K, Γ^H, respectively, of the same order n, one can construct the map $\Gamma^{KH} : (K\Gamma^H)\Gamma^K : \otimes_n KHA_n \to KH(\otimes_n A_n)$. If a distributive law $\lambda : HK \to KH$ exists, does this make Γ^{KH} a Kleisli strength of order n on the composite monad KH? In general the answer is no, but when λ preserves pre-strengths, as in Proposition 2.25 below, the answer is yes.

Definition 2.23. For fixed $n \geqslant 1$, we use $ps_n(\mathscr{V})$ to denote the category whose objects are pairs (F, Γ^F) with $F : \mathscr{V} \to \mathscr{V}$ a functor and $\Gamma^F_{V_1 \cdots V_n} : FV_1 \otimes \cdots \otimes FV_n \to F(V_1 \otimes \cdots \otimes V_n)$ a natural transformation. (Such a Γ^F is called a *pre-strength* on F, which explains the 'ps' in $ps_n(\mathscr{V})$.) A morphism $\alpha : (F, \Gamma^F) \to (G, \Gamma^G)$ is a natural transformation $\alpha : F \to G$ such

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that the following square commutes:

$$\begin{array}{ccc} FV_1 \otimes \cdots \otimes FV_n & \xrightarrow{\alpha_{V_1} \otimes \cdots \otimes \alpha_{V_n}} & GV_1 \otimes \cdots \otimes GV_n \\ \Big\downarrow{\scriptstyle \Gamma^F_{V_1 \cdots V_n}} & & \Big\downarrow{\scriptstyle \Gamma^G_{V_1 \cdots V_n}} \\ F(V_1 \otimes \cdots \otimes V_n) & \xrightarrow[\alpha_{V_1 \otimes \cdots \otimes V_n}]{} & G(V_1 \otimes \cdots \otimes V_n) \end{array}$$

When α is such a morphism we say α *preserves pre-strengths*. It is obvious that $ps_n(\mathscr{V})$ is a category under vertical composition of natural transformations, but it is also a 2-category under the horizontal composition of the endofunctor category of $\mathscr{V}$. Specifically, (F, Γ^F), (G, Γ^G) have the composition (GF, Γ^{GF}) where $\Gamma^{GF}_{V_1 \otimes \cdots \otimes V_n}$ equals

$$GFV_1 \otimes \cdots \otimes GFV_n \xrightarrow{\Gamma^G_{FV_1 \cdots FV_n}} G(FV_1 \otimes \cdots \otimes FV_n) \xrightarrow{G\Gamma^F_{V_1 \cdots V_n}} GF(V_1 \otimes \cdots \otimes V_n).$$

We leave the reader to show that if $\alpha : (F, \Gamma^F) \to (F', \Gamma^{F'})$ and $\beta : (G, \Gamma^G) \to (G', \Gamma^{G'})$ preserve pre-strengths, then so does $\beta\alpha : (G, \Gamma^G)(F, \Gamma^F) \to (G', \Gamma^{G'})(F', \Gamma^{F'})$. In the special case where F and G are monads with Γ^F and Γ^G Kleisli strengths of order n, we say α is a *map of Kleisli strengths.*

Example 2.24. The monad map *list* : $V \to L$ of Example 2.7 is a map of Kleisli strengths where *ref* and *rev* are Kleisli strengths of order 1 on V and L, respectively.

Proposition 2.25. Let $\mathbf{K} = (K, v, \rho)$, $\mathbf{H} = (H, \mu, \eta)$ be monads in $\mathscr{V}$, and let Γ^K be a Kleisli strength for $\mathbf{K}$ and Γ^H be a Kleisli strength for $\mathbf{H}$, both of order $n \geqslant 1$. Let $\lambda : HK \to KH$ be a distributive law of $\mathbf{H}$ over $\mathbf{K}$ that preserves these strengths. Then $(K\Gamma^H)\Gamma^K$ is a Kleisli strength of order n for the composite monad $\mathbf{K} \circ_\lambda \mathbf{H}$.

Proof. To keep the notation compact, we again write the proof for $n = 2$, noting, as before, that exactly the same proof works in all cases. We must show that (ΓA, ΓB) hold. Recall the monad structure of $\mathbf{K} \circ_\lambda \mathbf{H}$ from (7). For (ΓA),

$$\begin{aligned} &(K\Gamma^H_{VW})\,\Gamma^K_{HV,HW}\,(\rho_{HV}\,\eta_V \otimes \rho_{HW}\,\eta_W) \\ &\qquad = (K\Gamma^H_{VW})\,\Gamma^K_{HV,HW}\,(\rho_{HV} \otimes \rho_{HW})(\eta_V \otimes \eta_W) \\ &\qquad = (K\Gamma^H_{VW})\,\rho_{HV \otimes HW}\,(\eta_V \otimes \eta_W) && \text{(by } (\Gamma A)) \\ &\qquad = \rho_{H(V \otimes W)}\,\Gamma^H_{VW}\,(\eta_V \otimes \eta_W) && (\rho \text{ natural}) \\ &\qquad = \rho_{H(V \otimes W)}\,\eta_{V \otimes W} && \text{(by } (\Gamma^H A)) \end{aligned}$$

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The proof for (ΓB) is more tedious, and is given as follows:

$$
\begin{aligned}
&(K\Gamma^H_{VW})(\Gamma^K_{HV,HW})((v\mu)_V \otimes (v\mu)_W)(K\lambda_{HV} \otimes K\lambda_{HW})\\
&\quad = (K\Gamma^H_{VW})(\Gamma^K_{HV,HW})(v_{HV} \otimes v_{HW})(KK\mu_V \otimes KK\mu_W)(K\lambda_{HV} \otimes K\lambda_{HW})\\
&\qquad\qquad \text{(by horizontal composition)}\\
&\quad = (K\Gamma^H_{VW})\, v_{HV\otimes HW}\, (K\Gamma^K_{HV,HW})\,\Gamma^K_{KHV,KHW}\,(KK\mu_V \otimes KK\mu_W)\\
&\qquad (K\lambda_{HV} \otimes K\lambda_{HW})\\
&\qquad\qquad \text{(by } (\Gamma^K B))\\
&\quad = v_{H(V\otimes W)}\,(KK\Gamma^H_{VW})(K\Gamma^K_{HV,HW})\,\Gamma^K_{KHV,KHW}\,(KK\mu_V \otimes KK\mu_W)\\
&\qquad (K\lambda_{HV} \otimes K\lambda_{HW})\\
&\qquad\qquad \text{(since } v \text{ is natural)}\\
&\quad = v_{H(V\otimes W)}\,(KK\Gamma^H_{VW})(KK(\mu_V \otimes \mu_W))(K\Gamma^K_{HHV,HHW})(\Gamma^K_{KHHV,KHHW})\\
&\qquad (K\lambda_{HV} \otimes K\lambda_{HW})\\
&\qquad\qquad \text{(since } (K\Gamma^K)\Gamma^K \text{ is natural)}\\
&\quad = v_{H(V\otimes W)}\,(KK\mu_{V\otimes W})(KKH\Gamma^H_{VW})(KK\Gamma^H_{HV,HW})(K\Gamma^K_{HHV,HHW})\\
&\qquad (\Gamma^K_{KHHV,KHHW})(K\lambda_{HV} \otimes K\lambda_{HW})\\
&\qquad\qquad \text{(by } (\Gamma^H B))\\
&\quad = (v\mu)_{V\otimes W}\,(KKH\Gamma^H_{VW})(KK\Gamma^H_{HV,HW})(K\Gamma^K_{HHV,HHW})(\Gamma^K_{KHHV,KHHW})\\
&\qquad (K\lambda_{HV} \otimes K\lambda_{HW})\\
&\qquad\qquad \text{(by horizontal composition)}\\
&\quad = (v\mu)_{V\otimes W}\,(KKH\Gamma^H_{VW})(KK\Gamma^H_{HV,HW})(K\Gamma^K_{HHV,HHW})(K(\lambda_{HV} \otimes \lambda_{HW}))\\
&\qquad \Gamma^K_{HKHV,HKHW}\\
&\qquad\qquad \text{(since } \Gamma^k \text{ is natural)}\\
&\quad = (v\mu)_{V\otimes W}\,(KKH\Gamma^H_{VW})(K\lambda_{HV\otimes HW})(KH\Gamma^K_{HV,HW})(K\Gamma^H_{KHV,KHW})\\
&\qquad \Gamma^K_{HKHV,HKHW}\\
&\qquad\qquad \text{(since } \lambda \text{ preserves pre-strengths)}\\
&\quad = (v\mu)_{V\otimes W}\,(K\lambda_{H(V\otimes W)})(KHK\Gamma^H_{VW})(KH\Gamma^K_{HV,HW})(K\Gamma^H_{KHV,KHW})\\
&\qquad \Gamma^K_{HKHV,HKHW}\\
&\qquad\qquad \text{(since } \lambda \text{ is natural).}
\end{aligned}
$$

This completes the proof. □

Example 2.26. Let P_0, H be the finite power set and exceptions $(\_ + 1)$ monads both equipped with Kleisli strength of order n as in Examples 2.4 and 2.5. There is a distributive law $\lambda_A : PA + 1 \to P(A+1)$ where $\lambda(B) = B$ for B a subset of A and $\{*\}$ for exception $*$ (see Manes and Mulry (2007)). That λ preserves these strengths follows immediately from the equation $B \times \{*\} = B$, so the composite $P(\_ + 1)$ again has Kleisli strength of order n. When $n = 2$, for instance, $\Gamma(A_0, B_0) = \{(a,b)^* | a \in A_0, b \in B_0\}$ where $(a,b)^* = *$ if either a or b is $*$, and is (a,b) otherwise.

2.3. *Homomorphisms on Kleisli strength*

In this section we describe a notion of homomorphism relative to Kleisli strengths, which we will apply in Section 4.

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Definition 2.27. Let (A_i, ξ_i) be **K**-algebras, $i = 1, \ldots, n$, and (C, θ) be a **K**-algebra. Let Γ be a Kleisli strength on **K** of order n and let $f : A_1 \otimes \cdots \otimes A_n \to C$ be a morphism in $\mathscr{V}$. We say that f is a **Γ**-*homomorphism* if the following square commutes, where $f^\# = \theta(Kf)$ is the unique **K**-homomorphic extension of f:

$$\begin{array}{ccc} KA_1 \otimes \cdots \otimes KA_n & \xrightarrow{\Gamma_{A_1 \cdots A_n}} & K(A_1 \otimes \cdots \otimes A_n) \\ {\scriptstyle \xi_1 \otimes \cdots \otimes \xi_n} \downarrow & (\Gamma\text{-homo}) & \downarrow {\scriptstyle f^\#} \\ A_1 \otimes \cdots \otimes A_n & \xrightarrow[f]{} & C \end{array}$$

Lemma 2.28. If $f : (A_1, \xi_1) \otimes \cdots \otimes (A_n, \xi_n) \to (C, \theta)$ is a Γ-homomorphism and $g : (C, \theta) \to (D, \gamma)$ is a **K**-homomorphism, then gf is a Γ-homomorphism.

Proof. $g\,f\,(\xi_1 \otimes \cdots \otimes \xi_n) = g\,\theta\,(Kf)\,\Gamma = \gamma\,(Kg)\,(Kf)\,\Gamma = \gamma\,K(gf)\,\Gamma = (gf)^\#\,\Gamma.$ □

Theorem 2.29. If Γ is a Kleisli strength of order n on **K** and $f : A_1 \otimes \cdots \otimes A_n \to C$ is a morphism in $\mathscr{V}$ with (C, θ) a **K**-algebra, then there exists a unique Γ-homomorphism

$$(KA_1, \nu_{A_1}) \otimes \cdots \otimes (KA_n, \nu_{A_n}) \xrightarrow{\tilde{f}} (C, \theta)$$

with $\tilde{f}\,(\rho_{A_1} \otimes \cdots \otimes \rho_{A_n}) = f$.

Proof. If ψ is such a Γ-homomorphism then

$$\begin{aligned} \psi &= \psi\,(\nu_{A_1} \otimes \cdots \otimes \nu_{A_n})\,(K\rho_{A_1} \otimes \cdots \otimes K\rho_{A_n}) \\ &= \psi^\#\,\Gamma_{KA_1,\ldots,KA_n}\,(K\rho_{A_1} \otimes \cdots \otimes K\rho_{A_n}) && (\Gamma\text{-homo}) \\ &= \psi^\#\,K(\rho_{A_1} \otimes \cdots \otimes \rho_{A_n})\,\Gamma_{A_1 \cdots A_n} && (\Gamma \text{ is natural}) \\ &= \theta\,(K\psi)\,K(\rho_{A_1} \otimes \cdots \otimes \rho_{A_n})\,\Gamma_{A_1 \cdots A_n} \\ &= \theta\,(Kf)\,\Gamma_{A_1 \cdots A_n} \quad = \quad f^\#\,\Gamma_{A_1 \cdots A_n}\,, \end{aligned}$$

which proves uniqueness and forces us to define

$$\tilde{f} \;=\; KA_1 \otimes \cdots \otimes KA_n \xrightarrow{\Gamma_{A_1 \cdots A_n}} K(A_1 \otimes \cdots \otimes A_n) \xrightarrow{f^\#} C\,. \tag{10}$$

Then

$$\begin{aligned} \tilde{f}\,(\rho_{A_1} \otimes \cdots \otimes \rho_{A_n}) &= f^\#\,\rho_{A_1 \otimes \cdots \otimes A_n} && (\text{by } (\Gamma^n\,A)) \\ &= f\,. \end{aligned}$$

Furthermore, $\Gamma_{A_1 \cdots A_n} : (KA_1, \nu_{A_1}) \otimes \cdots \otimes (KA_n, \nu_{A_n}) \to (C, \theta)$ is a Γ-homomorphism since this is precisely $(\Gamma^n\,B)$. Thus $\tilde{f}$ is a Γ-homomorphism by Lemma 2.28. □

It follows that $\Gamma_{A_1 \cdots A_n}$ may be regarded as the unique Γ-homomorphism extending $\rho_{A_1 \otimes \cdots \otimes A_n}$.

Example 2.30. The reverse monad map on lists $rev : \mathbf{L} \to \mathbf{L}$ of Example 2.3 is a Kleisli strength of order 1. A *rev*-homomorphism between monoids is just an antihomomorphism: $f(xy) = (fy)(fx)$, $f(1) = 1$.

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Coherent families (Definition 2.13) enjoy the following composition property:

Lemma 2.31. Let **K** be a monad in $\mathscr{V}$ and let (Γ^k) be a coherent family of Kleisli strengths on **K**. Let (A_{ij}, ξ_{ij}), (B_i, θ_i), (C, γ) be **K**-algebras and suppose we are given

$$\begin{array}{rl} A_{i1} \otimes \cdots \otimes A_{ik_i} \xrightarrow{g_i} B_i & \quad \Gamma^{k_i}\text{-homomorphisms} \\ B_1 \otimes \cdots \otimes B_n \xrightarrow{f} C & \quad \Gamma^n\text{-homomorphism.} \end{array}$$

Then $f(g_1 \otimes \cdots \otimes g_n)$ is a $\Gamma^{k_1+\cdots+k_n}$-homomorphism.

Proof.

$$\begin{aligned} f(\otimes_n g_i)(\otimes_n \otimes_{k_i} \xi_{ij}) &= f(\otimes_n \theta_i)(\otimes_n K g_i)(\otimes_n \Gamma^{k_i}) && (g_i\ \Gamma\text{-homomorphism}) \\ &= \theta(Kf)\Gamma^n(\otimes_n K g_i)(\otimes_n \Gamma^{k_i}) && (f\ \Gamma\text{-homomorphism}) \\ &= \theta(Kf)K(\otimes_n g_i)\Gamma^n(\otimes_n \Gamma^{k_i}) && (\Gamma^n \text{ natural}) \\ &= \theta(Kf)K(\otimes_n g_i)\Gamma^{k_1+\cdots+k_n} && (\Gamma \text{ coherent}). \end{aligned}$$

□

2.4. *Classical commutative monads and Kock strength*

In this section we contrast our results to previous work on commutative monads. We begin with a result that will prove useful in what follows.

Lemma 2.32. Let Γ be a Kleisli strength of order 2 for **K**. Define natural transformations *lst*, *rst* by

$$\begin{aligned} lst_{AB} &= KA \otimes B \xrightarrow{1 \otimes \rho_B} KA \otimes KB \xrightarrow{\Gamma_{AB}} K(A \otimes B) \\ rst_{AB} &= A \otimes KB \xrightarrow{\rho_A \otimes 1} KA \otimes KB \xrightarrow{\Gamma_{AB}} K(A \otimes B). \end{aligned}$$

Then

$$\Gamma_{AB} = KA \otimes KB \xrightarrow{lst_{A,KB}} K(A \otimes KB) \xrightarrow{(rst_{AB})^\#} K(A \otimes B). \tag{11}$$

Similarly, $\Gamma_{AB} = (lst_{AB})^\# rst_{KA,B}$.

Proof.

$$\begin{aligned} (rst_{AB})^\#\, lst_{A,KB} &= v_{A\otimes B}(K\Gamma_{AB})K(\rho_A \otimes 1_{KB})\Gamma_{A,KB}(1_A \otimes \rho_{KB}) \\ &= v_{A\otimes B}(K\Gamma_{AB})\Gamma_{KA,KB}(K\rho_A \otimes 1_{KKB})(1_A \otimes \rho_{KB}) && (\Gamma \text{ natural}) \\ &= \Gamma_{AB}(v_A \otimes v_B)(K\rho_A \otimes 1_{KKB})(1_A \otimes \rho_{KB}) && (\Gamma\ B) \\ &= \Gamma_{AB}(v_A \otimes v_B)(\rho_A \otimes \rho_B) \\ &= \Gamma_{AB} && (\mathbf{K} \text{ monad}). \end{aligned}$$

□

Definition 2.33. A *Kock strength* on **K** is a Kleisli strength of order 2 on **K** such that (K, Γ, ρ_I) is a monoidal functor.

In Kock (1970), Anders Kock assumed $\mathscr{V}$ to be a symmetric monoidal *closed* category. Regarding such a $\mathscr{V}$ as a $\mathscr{V}$-category, Kock required K to be a $\mathscr{V}$-functor, and then required v and ρ to be $\mathscr{V}$-natural. The $\mathscr{V}$-functor structure transforms under adjointness

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to natural transformations of the form lst, rst as in the proof above. One can then define $l\Gamma = rst^{\#}\, lst$ and $r\Gamma = lst^{\#}\, rst$. Kock proved (Kock 1970, Theorem 2.13) that $(K, l\Gamma, \rho_I)$ and $(K, r\Gamma, \rho_I)$ are monoidal functors. He then defined **K** to be *commutative* if $l\Gamma = r\Gamma$. It follows that $l\Gamma = \Gamma = r\Gamma$ is a Kleisli strength of order 2 on **K**.

Our definition of Kock strength is then a natural generalisation of Kock's commutative monads to symmetric monoidal categories that are not necessarily closed. It is folklore that 'strength is unique in **Set**'. Here is a more precise statement.

Theorem 2.34. In the cartesian closed category **Set**, a monad **K** admits at most one Kock strength.

Proof. Let Γ be a Kock strength with induced lst, rst as in the proof of Lemma 2.32. For $a \in A$, $b \in B$, let $in_b : A \to A \times B$, $a \mapsto (a, b)$ and let $in_a : B \to A \times B$, $b \mapsto (a, b)$. For $\omega \in KA$, $b \in B$, we think of b as a function $b : 1 \to B$. We have

$$\begin{aligned}
lst_{AB}\,(\omega, b) &= \Gamma_{AB}(id \times \rho_B)\,(\omega, b) & \\
&= \Gamma_{AB}\,(id \times \rho_B)\,(id \times b)\,\omega & \\
&= \Gamma_{AB}\,(id \times Kb)\,(1 \times \rho_1)\,\omega & (\rho \text{ natural}) \\
&= (K\, in_b)\,\Gamma_{A,1}\,(1 \times \rho_1)\,\omega & (\Gamma \text{ natural}) \\
&= (K\, in_b)\,\omega & (MF\,2).
\end{aligned}$$

Similarly, $(MF\,1)$ implies $rst_{AB}(a, \omega) = (K\, in_a)\,\omega$. But Lemma 2.32 shows that Γ is determined by lst and rst. □

The monads with Kock strength in **Set** are precisely the commutative monads characterised by *Linton's Theorem* (Linton 1966), which may be paraphrased as follows.

Theorem 2.35. The following conditions on a monad **K** in **Set** are equivalent:

1. **K** has Kock strength.
2. Each function $f : X_1 \times \cdots \times X_n \to Y$ has a unique extension $KX_1 \times \cdots \times KX_n \to KY$, which is a homomorphism in each variable separately.
3. If (X, ξ), (Y, θ) are **K**-algebras, the set of **K**-homomorphisms $(X, \xi) \to (Y, \theta)$ is a **K**-subalgebra of the cartesian power $(Y, \theta)^X$.

Example 2.36. Let $MA = X + Exc$ be the exceptions monad of Example 1.6 where Exc has two elements, a and b. While monad M does admit two different Kleisi strengths of order 2 in Example 2.5, it does not admit a Kock strength. By the argument of Theorem 2.34, $l\Gamma(a, b) = a$ while $r\Gamma(a, b) = b$, so $l\Gamma \neq r\Gamma$. Thus there is no Kock strength, so M cannot be commutative in the sense of Kock.

We note that for Γ to be a Kock strength for a monad in **Set**, Γ-homomorphisms coincide with maps that are homomorphic in each variable separately, so condition (2) of Linton's theorem can be generalised – this is the content of Theorem 2.29. For Γ a Kleisli strength of order 2 on a monad **K** in a symmetric monoidal closed category, for each **K**-algebra (Y, θ) and object X the map $\theta^{\bullet} : K(Y^X) \to Y^X$, which corresponds under adjointness to

$$K(Y^X) \otimes X \xrightarrow{1 \otimes \rho_X} K(Y^X) \otimes KX \xrightarrow{\Gamma} K(Y^X \otimes X) \xrightarrow{K(ev)} KY\,,$$

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makes Y^X a $\mathbf{K}$-algebra, as we will show in Manes and Mulry (2008). When Γ is a Kock strength in **Set**, we have that $\theta^\bullet$ is the cartesian power. Even so, the third condition in Linton's theorem fails in general for Kleisli strength.

3. Monadic signatures

Definition 3.1. Let $\Sigma = (\Sigma_n : n = 0, 1, 2, \ldots)$ be a finitary signature, that is, a sequence of disjoint sets. A $\boldsymbol{\Sigma}$-*algebra* in $\mathscr{V}$ is a pair (X, δ) with X an object in $\mathscr{V}$ and $\delta = (\delta_\omega : \omega \in \Sigma_n)$ where, for $\omega \in \Sigma_n$, we have $\delta_\omega : \otimes_n X \to X$ is a $\mathscr{V}$-morphism. A Σ-*homomorphism* $f : (X, \delta) \to (X', \delta')$ is a $\mathscr{V}$-morphism $f : X \to X'$ such that the following square commutes for all $\omega \in \Sigma_n$:

$$\begin{array}{ccc} \otimes_n X & \xrightarrow{\otimes_n f} & \otimes_n X' \\ {\scriptstyle \delta_\omega}\downarrow & & \downarrow{\scriptstyle \delta'_\omega} \\ X & \xrightarrow[f]{} & X' \end{array}$$

The resulting category of Σ-algebras is written $\mathscr{V}^\Sigma$ with underlying functor $U^\Sigma : \mathscr{V}^\Sigma \to \mathscr{V}$. We say that Σ is *monadic* (in $\mathscr{V}$) if U^Σ is monadic, that is, there exists a monad $\boldsymbol{\Sigma}^@ = (\Sigma^@, \mu, \eta)$ and an isomorphism of categories Ψ over $\mathscr{V}$:

$$\begin{array}{ccc} \mathscr{V}^{\boldsymbol{\Sigma}^@} & \xrightarrow{\Psi} & \mathscr{V}^\Sigma \\ & \searrow \quad \swarrow {\scriptstyle U^\Sigma} & \\ & \mathscr{V} & \end{array}$$

It is well known that all finitary signatures are monadic in **Set**.

If $T : \mathscr{C} \to \mathscr{C}$ is an endofunctor, a T-*algebra* is a pair (X, ξ) with $\xi : TX \to X$. T-algebras form a category $\mathscr{C}^T$ over $\mathscr{C}$ with morphisms $f : (X, \xi) \to (Y, \theta)$ satisfying $\theta(Tf) = f\xi$. We say T generates a *free monad* $(\mathbf{T}^@, \gamma)$ if $\mathbf{T}^@ = (T^@, v, \rho)$ is a monad and $\gamma : T \to T^@$ is a natural transformation, subject to the universal property that for every monad $\mathbf{H} = (H, \mu, \eta)$ and natural transformation $u : T \to H$ there exists a unique monad map $\alpha : \mathbf{T}^@ \to \mathbf{H}$ with $\alpha\gamma = u$. It is known (Barr 1970, Proposition 5.2) that if $\mathscr{C}^T$ is monadic with $\mathscr{C}^T \cong \mathscr{C}^{\mathbf{T}^@}$ over $\mathscr{C}$, then there exists γ with $(\mathbf{T}^@, \gamma)$ the free monad generated by T.

Now consider a monadic finitary signature Σ with $\mathscr{V}^\Sigma \cong \mathscr{V}^{\boldsymbol{\Sigma}^@}$ with $\boldsymbol{\Sigma}^@ = (\Sigma^@, v, \rho)$. If the required coproducts exist, $\mathscr{V}^\Sigma \cong \mathscr{V}^T$ where $TA = \coprod_{\omega \in \Sigma_n} \otimes_n A$. In this case, $\boldsymbol{\Sigma}^@$ would be the free monad generated by T. This would be a useful tool for constructing strengths in view of results such as Proposition 2.16 since then monad maps out of $\boldsymbol{\Sigma}^@$ correspond to natural transformations out of T, which is a simpler idea. But natural transformations out of T are just families of natural transformations out of $\otimes_n id$ indexed by $\omega \in \Sigma_n$, and this construct is fully available without assuming coproducts. This leads to the following definition and proposition. While one could almost refer to Barr (1970) for the proof of the proposition, we will give a proof here to set down the specific constructions.

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Definition 3.2. If Σ is a finitary signature, a *free monad over* Σ is $(\mathbf{\Sigma}^{@}, \gamma)$ where $\mathbf{\Sigma}^{@} = (\Sigma^{@}, v, \rho)$ is a monad and $\gamma_\omega : \otimes_n id \to \Sigma^{@}$ is a family of natural transformations indexed by $\omega \in \Sigma_n$ with the universal property that if $\mathbf{H} = (H, \mu, \eta)$ is a monad and $u_\omega : \otimes_n id \to H$ is similarly-indexed family of natural transformations, then there exists a unique monad map $\alpha : \mathbf{\Sigma}^{@} \to \mathbf{H}$ with $\alpha\gamma_\omega = u_\omega$ for all $\omega \in \Sigma_n$.

Proposition 3.3. Let Σ be a monadic finitary signature with isomorphism $\Phi : \mathscr{V}^{\Sigma} \to \mathscr{V}^{\mathbf{\Sigma}^{@}}$ over $\mathscr{V}$, and $\mathbf{\Sigma}^{@} = (\Sigma^{@}, v, \rho)$. Let $\tau_\omega : \otimes_n \Sigma^{@} A \to \Sigma^{@} A$ describe $\Phi^{-1}(\Sigma^{@} A, v_A)$ and define $\gamma = (\gamma_\omega : \omega \in \Sigma_n)$ by

$$\gamma_{\omega,A} \;=\; \otimes_n A \xrightarrow{\otimes_n \rho_A} \otimes_n \Sigma^{@} A \xrightarrow{\tau_\omega} \Sigma^{@} A \,.$$

Then $(\mathbf{\Sigma}^{@}, \gamma)$ is the free monad generated by Σ.

Proof. Since $\Sigma^{@} f : (\Sigma^{@} A, v_A) \to (\Sigma^{@} B, v_B)$ is a Σ-homomorphism, τ_ω, and thus γ_ω, are natural transformations. Let $\mathbf{H} = (H, \mu, \eta)$ be a monad in $\mathscr{V}$ and $u_{\omega,A} : \otimes_n A \to HA$ be natural, $\omega \in \Sigma_n$. This induces an algebra lift $id^\star : \mathscr{V}^{\mathbf{H}} \to \mathscr{V}^{\Sigma}$ through

$$(A, \xi) \;\mapsto\; \otimes_n A \xrightarrow{u_{\omega,A}} HA \xrightarrow{\xi} A \,,$$

which maps $\mathbf{H}$-homomorphisms to Σ-homomorphisms since u is natural. The corresponding monad map $\alpha : \mathbf{\Sigma}^{@} \to \mathbf{H}$ is defined, as always, as the unique Σ-homomorphic extension of η. Here, this means that α_A is characterised by the equation $\alpha_A\,\rho_A = \eta_A$ and the following diagrams indexed by $\omega \in \Sigma_n$:

$$\begin{array}{ccc}
\otimes_n \Sigma^{@} A & \xrightarrow{\otimes_n \alpha_A} & \otimes_n HA \\
 & & \big\downarrow u_{\omega,HA} \\
\tau_\omega \big\downarrow & (A_\omega) & HHA \\
 & & \big\downarrow \mu_A \\
\Sigma^{@} A & \xrightarrow[\alpha_A]{} & HA
\end{array}$$

We have

$$\begin{aligned}
\alpha_A\,\gamma_{\omega,A} &= \alpha_A\,\tau_\omega\,(\otimes_n \rho_A) \\
&= \mu_A\,u_{\omega,HA}\,(\otimes_n \alpha_A)\,(\otimes_n \rho_A) \\
&= \mu_A\,u_{\omega,HA}\,(\otimes_n \eta_A) \\
&= \mu_A\,(H\eta_A)\,u_{\omega,A} \\
&= u_{\omega,A}\,.
\end{aligned}$$

For uniqueness, suppose $\beta : \mathbf{\Sigma}^{@} \to \mathbf{H}$ is a monad map with $\beta\,\gamma_\omega = u_\omega$ for all $\omega \in \Sigma_n$. Such a β classifies a functor $\mathscr{V}^{\mathbf{H}} \to \mathscr{V}^{\Sigma}$ over $\mathscr{V}$, so let $\delta_{\omega,A} : \otimes_n HA \to HA$ be the resulting Σ-algebra structure induced by (HA, μ_A). Then β_A is the unique map ψ with $\psi\rho_A = \eta_A$

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and such that the following family of squares commute:

$$\begin{CD} \otimes_n \Sigma^{@} A @>{\otimes_n \psi}>> \otimes_n HA \\ @V{\tau_{\omega,A}}VV (B_\omega) @VV{\delta_{\omega,A}}V \\ \Sigma^{@} A @>>{\psi}> HA \end{CD}$$

We have

$$\begin{aligned} u_{\omega,A} &= \beta_A \gamma_A \\ &= \beta_A \tau_{\omega,A} (\otimes \rho_A) \\ &= \delta_\omega (\otimes_n \alpha_A)(\otimes_n \alpha_A) \\ &= \delta_{\omega,A} (\otimes \eta_A), \end{aligned}$$

which leads to

$$\begin{aligned} \mu_A u_{\omega,HA} &= \mu_A \delta_{\omega,HA} (\otimes_n \eta_{HA}) \\ &= \mu_A \eta_{HA} \delta_{\omega,A} && (\delta \text{ is natural}) \\ &= \delta_{\omega,A}. \end{aligned}$$

Comparison of (A_ω) and (B_ω) then shows that $\beta_A = \psi = \alpha_A$. □

It is well known that a functor $\mathscr{C}^{\mathbf{T}} \to \mathscr{C}^{\mathbf{S}}$ over $\mathscr{C}$ is characterised by its values on the full subcategory of free algebras (TX, μ_X). This is because if (X, ξ) is a **T**-algebra, then $\xi = coeq(T\xi, \mu_X)$ is an absolute coequaliser in $\mathscr{C}$. Rather than elaborate this point, we give the following version of Proposition 3.3, which applies specifically to free monads.

Corollary 3.4. Let Σ be a monadic finitary signature and **H** be a monad in $\mathscr{V}$. Then there is a bijection between monad maps $\mathbf{\Sigma}^{@} \to \mathbf{H}$ and families of natural transformations $v_\omega : \otimes_n H \to H$ indexed by $\omega \in \Sigma_n$ subject to the commutativity of the squares

$$\begin{CD} \otimes_n HH @>{\otimes_n \mu}>> \otimes_n H \\ @V{v_\omega H}VV (C_\omega) @VV{v_\omega}V \\ HH @>>{\mu}> H \end{CD}$$

Proof. By Proposition 3.3, we need only show that such a v corresponds bijectively to families of natural transformations $u_\omega : \otimes_n id \to H$. Given u, we define

$$v_{\omega,A} \;=\; \otimes_n HA \xrightarrow{u_{\omega,HA}} HHA \xrightarrow{\mu_A} HA.$$

To see that such a v satisfies (C_ω), we have

$$\begin{aligned} v_\omega (\otimes_n \mu) &= \mu (u_\omega H)(\otimes_n \mu) \\ &= \mu (H\mu)(u_\omega HH) \\ &= \mu (\mu H)(u_\omega HH) \\ &= \mu (v_\omega H). \end{aligned}$$

Conversely, given v_ω, define $u_\omega = v_\omega(\otimes_n \eta)$. Then:

— If $u \mapsto v \mapsto \hat{u}$, then

$$\begin{aligned} \hat{u}_\omega &= v_\omega\,(\otimes_n \eta) \\ &= \mu\,(u_\omega H)\,(\otimes_n \eta) \\ &= \mu\,(H\eta)\,u_\omega \\ &= u_\omega\,. \end{aligned}$$

— If $v \mapsto u \mapsto \hat{v}$, then

$$\begin{aligned} \hat{v}_\omega &= \mu(uH) \\ &= \mu\,(v_\omega H)\,(\otimes_n \eta H) \\ &= v_\omega\,(\otimes_n \mu)\,(\otimes_n \eta H) && \text{(by } (C_\omega)) \\ &= v_\omega\,. \end{aligned}$$

This completes the proof. □

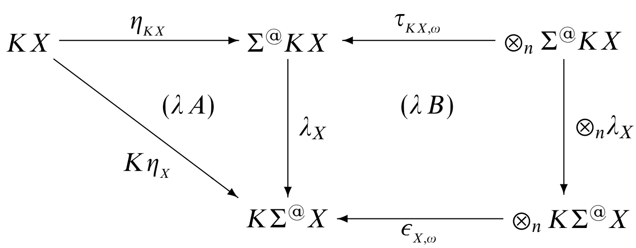
Example 3.5. We can now elaborate on the map *list* : $V \to L$ used in Example 2.7, which converts a tree into a list. Note that $\mathbf{V} = \mathbf{\Sigma}^@$ where $\Sigma_0 = \{E\}$ and $\Sigma_2 = \{N\}$. Consider the (obviously natural) concatenation map $+\!\!+ : L \times L \to L$, which satisfies the equation $\mu(l_1 +\!\!+ l_2) = \mu(l_1) +\!\!+ \mu(l_2)$. By Corollary 3.4, there is an induced monad map $V \to L$ that coincides with *list*.

The next theorem shows how the free monad generated by a signature interacts with Kleisli strengths on another monad to form a distributive law.

Theorem 3.6. Let Σ be a monadic finitary signature in $\mathscr{V}$. Let $\mathbf{K} = (K, \nu, \rho)$ be a monad in $\mathscr{V}$ equipped with a family of Kleisli strengths $(\Gamma^\omega : \omega \in \Sigma_n)$ with each Γ^ω of order n if $\omega \in \Sigma_n$. For X a $\mathscr{V}$-object, let $\Psi(\Sigma^@ X, \mu_X) = (\Sigma^@ X, \tau_X)$ (where τ and $\Psi = \Phi^{-1}$ are as in Proposition 3.3) so that $(K\Sigma^@ X, \epsilon_X)$ is a Σ-algebra where

$$\epsilon_{X,\omega} \;=\; \otimes_n K\Sigma^@ X \xrightarrow{\Gamma^\omega} K(\otimes_n \Sigma^@ X) \xrightarrow{K\tau_{X,\omega}} K\Sigma^@ X\,.$$

Let $\lambda : \Sigma^@ K \to K\Sigma^@$ be defined as the unique Σ-homomorphic extension of $K\eta$ such that λ is characterised by the diagrams



Then λ is a distributive law of $\mathbf{\Sigma}^@$ over $\mathbf{K}$.

Proof. We first define an algebra lift $K^\star : \mathscr{V}^\Sigma \to \mathscr{V}^\Sigma$ by $K^\star(X, \delta) = (KX, \delta^\star)$ where

$$\delta^\star \;=\; \otimes_n KX \xrightarrow{\Gamma^\omega} K(\otimes_n X) \xrightarrow{K\delta_\omega} KX\,.$$

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(Thus the lift of (1) is really $\Psi^{-1}K^\star\Psi$.) If $f : (X,\delta) \to (X',\delta')$ is a Σ-homomorphism, then $Kf : (KX,\delta^\star) \to (KX',\delta'^\star)$ is also a Σ-homomorphism because

$$\begin{aligned} \delta'^\star_\omega(\otimes_n Kf) &= (K\delta'_\omega)\,\Gamma^\omega_Y\,(\otimes_n Kf) \\ &= (K\delta'^\omega)(K(\otimes_n f))\Gamma^\omega_X && (\Gamma^\omega \text{ is natural}) \\ &= (Kf)(K\delta'^\omega)\,\Gamma^\omega_X && (f \text{ is a } \Sigma\text{-homomorphism}) \\ &= (Kf)\,\delta_\omega\,. \end{aligned}$$

Thus $K^\star$ is a well-defined algebra lift. As for any category of algebras over a monad, the unique $\mathbf{\Sigma}^@$-homomorphism $f^\# : (\Sigma^@ X,\mu_X) \to (Y,\theta)$ extending $f : X \to Y$ is $\Sigma^@ X \xrightarrow{\Sigma^@ f} \Sigma^@ Y \xrightarrow{\theta} Y$. Now, according to (2), $K^\star$ is classified by the natural transformation $\lambda = \Sigma^@ K \xrightarrow{\Sigma^@ K\eta} \Sigma^@ K\Sigma^@ \xrightarrow{\gamma} K\Sigma^@$ where γ_X is the $\mathbf{\Sigma}^@$-structure map of $\Psi^{-1}K^\star\Psi(\Sigma^@ X,\mu_X)$. But this is precisely the formula of the unique $\mathbf{\Sigma}^@$-homomorphic extension of $K\eta$. Thus λ is the same map as in $(\lambda A,\ \lambda B)$ above. By Theorem 1.2, it remains only to show that for (X,δ) a Σ-algebra, $\rho_X : (X,\delta) \to (KX,\delta^\star)$ and $v_X : (KKX,(\delta^\star)^\star) \to (KX,\delta^\star)$ are Σ-homomorphisms. Let $\omega \in \Sigma_n$. For ρ_X we have

$$\begin{aligned} \rho_X\,\delta_\omega &= (K\delta_\omega)\,\rho_{\otimes_n X} && (\rho \text{ is natural}) \\ &= (K\delta_\omega)\,\Gamma^\omega_{X\cdots X}\,(\otimes_n \rho_X) && (\text{by } (\Gamma^n A)) \\ &= \delta^\star_\omega\,(\otimes_n \rho_X)\,, \end{aligned}$$

while, for v_X, we have

$$\begin{aligned} v_X\,(\delta^\star)^\star_\omega &= v_X(KK\delta_\omega)(K\Gamma^\omega_{X\cdots X})(\Gamma^\omega_{KX\cdots KX}) \\ &= (K\delta_\omega)\,v_{\otimes_n X}(K\Gamma^\omega_{X\cdots X})(\Gamma^\omega_{KX\cdots KX}) && (v \text{ is natural}) \\ &= (K\delta_\omega)\,\Gamma^\omega_{X\cdots X}(\otimes_n v_X) && (\text{by } (\Gamma^n B)) \\ &= \delta^\star_\omega(\otimes v_X)\,. \end{aligned}$$

This completes the proof. □

Example 3.7. Let $\mathbf{L}$ be the list monad and α be a cardinal number. Then there exists a finitary signature Σ such that $\bigcup_n \Sigma_n$ has cardinal α and with the property that there exist at least 2^α different distributive laws of $\mathbf{\Sigma}^@$ over $\mathbf{L}$.

To see this, let Σ_1 be any set of cardinality α and define $\Sigma_n = \varnothing$ if $n \neq 1$. Then $\mathbf{\Sigma}^@$ is the monad of Example 1.5 with M the free monoid Σ_1^* generated by Σ_1. Fix a subset $A \subset \Sigma_1$. For $\omega \in A$, let $\Gamma^\omega_X : LX \to LX$ be the identity map and, for $\omega \notin A$, let Γ^ω_X be the reverse map as in Example 2.3. The reader may easily check that the distributive law λ of Theorem 3.6 satisfies

$$A \;=\; \{\omega \in \Sigma_1 : \lambda_A(\omega,[a,b]) = [(\omega,a),(\omega,b)]\}\,.$$

Lemma 3.8. Let Σ be a finitary signature and $id : \mathscr{V} \to \mathscr{V}$ be the identity functor. Suppose also that we are given a family $\gamma_\omega : \otimes_n id \to id$ of natural transformations indexed by $\omega \in \Sigma_n$. Let $\alpha_A : (\Sigma^@ A,\tau_A) \to (A,\gamma_A)$ be the unique Σ-homomorphic extension of id_A. Then $\alpha : \mathbf{\Sigma}^@ \to \mathbf{id}$ is a monad map.

Proof. We can apply Corollary 3.4 with $H = id$, since the diagram (C_ω) trivially commutes. □

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Example 3.9. Lemma 3.8 proves to be particularly useful as it provides a method for constructing distributive laws of the free Σ-monad over itself. If $\mathscr{V}$ is cartesian, the ith projection $\pi_i : \times_n id \to id$ is natural and so, for $\omega \in \Sigma_n$, we can choose γ_ω in Lemma 3.8 in n different ways. Thus we can apply Theorem 3.6 and Proposition 2.17 to produce Kleisli strengths and distributive laws of $\boldsymbol{\Sigma}^@$ over itself, for any finitary signature Σ.

For instance, let $\Sigma^@$ be the free monad in **Set** associated with signature $\Sigma_n = \{\cdot\}$ and all other $\Sigma_i = \varnothing$. An element t of $\Sigma^@ A$ can be viewed as an n-ary tree whose values (from A) are located in its leaves, thus generalising the non-empty binary tree monad of Example 1.7. We adopt the same notation in which Lx denotes a trivial tree and $N(v_1, \ldots v_n)$ denotes a tree with subtrees $v_1, \ldots v_n$. By Lemma 3.8, for each projection π_i we have a distinct monad map $\alpha_i : \boldsymbol{\Sigma}^@ \to \mathbf{id}$. By Proposition 2.17, we produce n^n different Kleisli strengths of order n, $\Gamma_{a_1 \ldots a_n} = \rho(\alpha_{a_1} \times \ldots \alpha_{a_n}) : (\Sigma^@ A)^n \to \Sigma^@(A^n)$, where each a_i represents a value from 1 to n. We now appeal to Theorem 3.6 to produce a distinct distributive law of $\boldsymbol{\Sigma}^@$ over itself, $\lambda_{a_1,\ldots a_n} : \Sigma^@\Sigma^@ \to \Sigma^@\Sigma^@$, for each of these strengths.

$\lambda_{a_1,\ldots a_n}(Lt) = t*$ where $t*$ has the same (n-shape) as t but every leaf value La is replaced by LLa:

$$\lambda_{a_1,\ldots a_n}(N(t_1, \ldots t_n)) = L(N(\alpha_{a_1}\lambda_{a_1,\ldots a_n}(t_1), \ldots, \alpha_{a_n}\lambda_{a_1,\ldots a_n}(t_n)) .$$

This produces distinct distributive laws, as we will show now. Let t denote the tree of the form

$$N(L(N(Lb_{1,1} \ldots Lb_{1,n})), \ldots L(N(Lb_{n,1} \ldots Lb_{n,n}))) .$$

Then

$$\lambda_{a_1,\ldots a_n}(t) = L(N(Lb_{1,a_1}, \ldots Lb_{n,a_n})) .$$

Clearly, if all the a_i and b_i differ, one derives n^n different distributive laws.

Example 3.10. In the previous example, if $n = 2$, then $\Sigma^@$ is the V^+ of Example 1.7. The monad maps *fst* and *lst* $: \mathbf{L}^+ \to id$ of Example 2.18 can be generalised to the monad V^+ where they are denoted *left* and *right* $: \mathbf{V}^+ \to id$, respectively. They are generated by the natural maps π_1 and $\pi_2 : id \times id \to id$, respectively. Using the notation of that example,

$$\begin{aligned} \mathit{left}\,(La) &= a \\ \mathit{left}\,(N(s,t)) &= \mathit{left}\,(s) . \end{aligned}$$

As in Example 3.9, these produce four different strengths of order 2, namely

$$\eta(\mathit{left} \times \mathit{left}) \qquad \eta(\mathit{left} \times \mathit{right}) \qquad \eta(\mathit{right} \times \mathit{left}) \qquad \eta(\mathit{right} \times \mathit{right}) .$$

These in turn generate 4 different recursive distributive laws $\lambda_{i,j} : V^+V^+ \to V^+V^+$ where $1 \leqslant i, j \leqslant 2$. Now the equations of the previous example become

$\lambda_{i,j}(Lt) = t*$ where $t*$ has the same shape as t but with every leaf value La replaced by LLa.

$$\lambda_{i,j}(N(s,t)) = L(N(\alpha_i\lambda_{i,j}(s), \alpha_j\lambda_{i,j}(t))) .$$

For instance, using the above formulas,

$$\lambda_{2,1}(L(N(N(La, Lb), Lc))) = N(N(LLa, LLb), LLc) ,$$

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while

$$\lambda_{2,1}(N(LN(La, Lb), N(LLc, LLd))) = LN(Lb, N(Lc, Ld))\,.$$

Example 3.11. In the case $n = 1$, the monad $\mathbf{\Sigma}^{@}$ of Example 3.9 will be denoted by $\mathbf{O}$ and satisfies the recursive equation $OA = A + OA$. Again using the notation L and N, we note that elements of OA are labelled 1-trees, so an element of OA has the form $N^n(La)$ where $n \geqslant 0$. We use this to describe the corresponding recursive monad map and distributive law.

First, note that the Kleisli 1-strength $m : O \to O$ is non-trivial (not the identity) and is defined by

$$\begin{aligned} m(La) &= La \\ m(N^n La) &= La \end{aligned}$$

Armed with the monad map m, we can now define a recursive distributive law $\lambda : OO \to OO$. Note that elements of OOA take the general form $N^n L(N^k La)$ where $n, k \geqslant 0$.

$$\begin{aligned} \lambda(L(N^k La)) &= N^k LLa \\ \lambda(N^n L(N^k La)) &= L(N^n La) \end{aligned}$$

For instance,

$$\lambda(LNNLa) = NNLLa\,,$$

while

$$\lambda L(NNL(N(La))) = L(NN(La))\,.$$

Example 3.12. Let **Top** be the category of topological spaces and continuous maps. For spaces X, Y, let $X \otimes Y$ be the set $X \times Y$ with topology of open sets those $U \subset X \times Y$ such that, for all $x_0 \in X$, $y_0 \in Y$, we have $\{x \in X : (x, y_0) \in U\}$ is open in X and $\{y \in Y : (x_0, y) \in U\}$ is open in Y. A function $f : X \otimes Y \to Z$ is continuous if and only if, with respect to the topologies on X, Y, f is separately continuous. This makes **Top** a symmetric monoidal category with unit the one-point space.

As jointly continuous maps are separately continuous, the cartesian product projections $\pi_i : \otimes_n id \to id$ are well defined (and clearly natural), so all of the remarks in Example 3.9 apply here as well, even though **Top** is not a cartesian category under $\otimes$.

Example 3.13. If Σ is a monadic finitary signature on cartesian $\mathscr{V}$, we have yet another way of generating a class of Kleisli strengths of order 1 on the free monad $\mathbf{\Sigma}^{@}$. Since any projection $\pi_i : \times_n \Sigma^{@} \to \Sigma^{@}$ is natural, so is the n-product of such projections. Composing with τ as in Proposition 3.3, we have for $\omega \in \Sigma_n$ that

$$v_\omega = \tau_\omega \circ (\times_n \pi_i) : (\times_n \Sigma^{@} A) \to (\times_n \Sigma^{@} A) \to (\Sigma^{@} A)$$

is again a natural family of maps v_ω. Furthermore, diagram (C_ω) of Corollary 3.4 commutes since

$$v_\omega(\times_n \mu) = \tau_\omega(\times_n \pi_i)(\times_n \mu) = \tau_\omega(\times_n \mu)(\times_n \pi_i) = (\mu)\tau_\omega(\times_n \pi_i) = (\mu)(v_\omega)\,.$$

Consequently, we have many different monad maps $\Sigma^{@} \to \Sigma^{@}$.

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For instance, the monad map *ref* of Example 1.7 corresponds to the map $\tau_2(\pi_2 \times \pi_1)$ for $\Sigma^@ = \mathbf{V}^+$. For the case of $\mathbf{V}^+$, four different monad maps $V^+ \to V^+$ are generated. The immediate question to ask is whether these new monad maps can be applied in the sense of Proposition 2.16 to the strengths of Example 3.9 to generate yet new strengths. The answer is no, composing produces exactly the same family of Kleisli strengths as found in Example 3.9. We leave the reader to check these details.

Example 3.14. The monad $\mathbf{K} = \mathbf{P_0}$ of Example 2.4 has a Kleisli strength of order n. By Theorem 3.6, there exists a distributive law $\lambda : \Sigma^@ P_0 \to P_0\Sigma^@$ for any monadic finitary signature Σ. For instance, if $\Sigma^@$ is V, then $\lambda : VP_0A \to P_0VA$ is defined by

$$\begin{aligned} \lambda E &= \{E\} \\ \lambda(LA_0) &= \{La : a \in A_0\} \\ \lambda(N(s,t)) &= \{N(s_i, t_j) : s_i \in \lambda(s), t_i \in \lambda(t)\}. \end{aligned}$$

Similarly, for the exponential monad M of Example 2.8, there exists a distributive law $\sigma : \Sigma^@ M \to M\Sigma^@$ for any monadic finitary signature Σ. Now when $\Sigma^@$ is V, the distributive law σ is defined to be

$$\begin{aligned} \sigma(E) &= \lambda a.E \\ \sigma(Lf) &= \lambda a.L(fa) \\ \sigma((N(t_1,t_2)) &= \lambda a.(N((\sigma t_1)a, (\sigma t_2)a)). \end{aligned}$$

Example 3.15. Let $MX = X + Exc$ be the exceptions monad of Example 1.6 and let $a \in Exc$, with induced Kleisli strength Γ_a as in Example 2.5. By Theorem 3.6, this strength generates a distributive law $V(X + Exc) \to VX + Exc$ defined by

$$\begin{aligned} \lambda_a(E) &= E \\ \lambda_a(Lx) &= x \quad \text{for all } x \in Exc \\ \lambda_a(t) &= t \quad \text{if no leaf of } t \text{ is of the form } Lx \text{ for } x \in Exc \\ \lambda_a(t) &= a \quad \text{otherwise.} \end{aligned}$$

It is clear that if a and b are distinct elements of Exc, then $\lambda_a \neq \lambda_b$, so for any cardinal α there are α different distributive laws.

4. Linear equations

Let Σ be a finitary signature. Let $v = \{v_1, v_2, v_2, \cdots\}$ be a fixed countably infinite set of variables.

Definition 4.1. The set $term_\Sigma$ of Σ-*terms* is defined recursively by:

— $v \subset term_\Sigma$

— if $t_1, \ldots, t_n \in term_\Sigma$ and $\omega \in \Sigma_n$, then $w[t_1, \ldots, t_n] \in term_\Sigma$.

For $t \in term_\Sigma$, the *list* $var(t) \in V^*$ *of variables of* t is defined recursively by

— $var(v_i) = [v_i]$

— $var(\omega[t_1, \ldots, t_n]) = var(t_1) + \cdots + var(t_n)$ where $+$ is list concatenation.

Note that if $\omega \in \Omega_0$, then $\omega \in term_\Sigma$ and $var(\omega) = 1$ is the empty list.

https://doi.org/10.1017/S0960129508006695 Published online by Cambridge University Press

Definition 4.2. We say that $t \in term_\Sigma$ is a *linear* term if $var(t)$ is repetition free. The subset of all linear terms in $term_\Sigma$ will be denoted lin_Σ. For $t \in lin_\Sigma$ of length n and A an object of $\mathscr{V}$, we write $|t| = n$, and let both A^t and A^n denote $\otimes_n A$, so $A^t = A^{|t|}$. A *linear* Σ-*equation* is an element (t, u) of $lin_\Sigma \times lin_\Sigma$ such that $var(t) = var(u)$. A *linear signature* is a pair (Σ, E) with E a set of linear Σ-equations.

Definition 4.3. Let Σ be a finitary signature and (A, δ) be a Σ-algebra. For each $t \in lin_\Sigma$, the δ-*interpretation* $\lt t\gt : A^t \to A$ of t is defined as follows:

— $\lt v_i\gt \; = \; A^{[v_i]} \xrightarrow{id} A$

— $\lt\omega[t_1, \ldots, t_n]\gt \; = A^{|t_1|+\cdots+|t_n|} \xrightarrow{\lt t_1\gt\otimes\cdots\otimes\lt t_n\gt} A^n \xrightarrow{\delta_\omega} A\,.$

(Note that if $\omega[t_1, \cdots, t_n]$ is linear, each t_i must be linear also, so the above is well defined.)

If (Σ, E) is a linear signature, a (Σ, E)-*algebra* is a Σ-algebra (A, δ) for which $\lt t\gt = \lt u\gt$ for all $(t, u) \in E$. This makes sense because $var(t) = var(u)$. The full subcategory of $\mathscr{V}^\Sigma$ of all (Σ, E)-algebras will be denoted $\mathscr{V}^{(\Sigma,E)}$. We say that the linear signature (Σ, E) is *monadic* if $\mathscr{V}^{(\Sigma,E)} \to \mathscr{V}$ is monadic.

If $\mathscr{V}$ is **Set**, it is well known that all linear signatures are monadic – use the Birkhoff variety theorem.

Example 4.4. The usual category of monoids in $\mathscr{V}$ is just $\mathscr{V}^{(\Sigma,E)}$ if $\Sigma_0 = \{1\}$, $\Sigma_2 = \{\star\}$ and E consists of the following linear equations (written in the usual way):

$$(x \star y) \star z \;=\; x \star (y \star z)$$
$$x \star 1 \;=\; x$$
$$1 \star x \;=\; x\,.$$

We now apply the theory of Γ-homomorphisms to linear signatures to obtain the following crucial lemma. While monads that admit Kleisli strength produce distributive laws for monads generated by monadic signatures, we need additional conditions to handle quotient monads. The notion of coherent strength (Definition 2.13) proves to be critical.

Lemma 4.5. Let (Σ, E) be a linear signature and (A, δ) be a (Σ, E)-algebra. Consider $(\mathbf{K}, \Gamma)$ where $\mathbf{K} = (K, v, \rho)$ is a monad in $\mathscr{V}$ and Γ is a coherent family of Kleisli strengths on $\mathbf{K}$. For $\omega \in \Sigma_n$, define

$$\epsilon_\omega \;=\; \otimes_n KA \xrightarrow{\Gamma^n} K(\otimes_n A) \xrightarrow{K\delta_\omega} KA\,. \tag{12}$$

Then (KA, ϵ) is a (Σ, E)-algebra.

Proof. By (10) in the proof of Theorem 2.29, ϵ_ω is a Γ-homomorphism, indeed the unique Γ-homomorphism extending $\otimes_n A \xrightarrow{\delta_\omega} A \xrightarrow{\rho_A} KA$. We shall show by induction on the definition of terms that for each linear term t, the ϵ-interpretation $\ll t\gg : (KA)^t \to KA$ is the unique Γ-homomorphism extending $\rho_A \;\lt t\gt : A^t \to KA$, where the latter $\lt t\gt$ is the δ-interpretation. Note that we use $\ll t\gg$ to distinguish the ϵ-interpretation from the δ-interpretation. $\ll v_i\gg = id : A \to A$ is a Γ-homomorphism since $\Gamma^1 = id$, and this is the basis step. For the inductive step, assume $\ll t_i \gg : KA^{t_i} \to KA$ is the Γ-homomorphic extension of ρ_A $<t_i> : A^{t_i} \to A$. Then for $\omega \in \Sigma_n$, $k = |t_1| + \cdots + |t_n|$, we have

$$\ll \omega[t_1, \ldots, t_n] \gg \;=\; \otimes_k KA^{t_i} \xrightarrow{\otimes_n \ll t_i \gg} \otimes_k KA \xrightarrow{\epsilon_\omega} KA$$

is a Γ-homomorphism by Lemma 2.31. To check the required behaviour on generators, we have

$$\begin{aligned} \ll \omega[t_1, \ldots, t_n] \gg \otimes_k \rho_A &= (K\delta_\omega)\Gamma(\otimes_n \ll t_i \gg)(\otimes_k \rho_A) \\ &= (K\delta_\omega)\Gamma(\otimes_n \rho_A)(\otimes_n <t_i>) && \text{(induction hypothesis)} \\ &= (K\delta_\omega)\rho_{\otimes_n A}(\otimes_n <t_i>) && (\Gamma A) \\ &= \rho_A\, \delta_\omega\, (\otimes_n <t_i>) && (\rho \text{ natural}) \\ &= \rho_A \;<\omega[t_1, \ldots, t_n]>\,. \end{aligned}$$

The desired result is now immediate from the uniqueness of Γ-homomorphic extension (Theorem 2.29). □

We can now show that a result similar to Theorem 3.6 applies to monadic finitary linear signatures.

Theorem 4.6. Let (Σ, E) be a monadic finitary linear signature, so there exists a monad **H** in $\mathscr{V}$ with $\mathscr{V}^{(\Sigma,E)} \cong \mathscr{V}^{\mathbf{H}}$ over $\mathscr{V}$. Let **K** be a monad in $\mathscr{V}$ with coherent family Γ of Kleisli strengths. Then $K^\star(A,\delta) = (KA, \epsilon)$, with ϵ defined by (12), is an algebra lift $K^\star : \mathscr{V}^{(\Sigma,E)} \to \mathscr{V}^{(\Sigma,E)}$ of K whose classifying transformation $\lambda : HK \to KH$ is a distributive law of **H** over **K**.

Proof. If (A,δ) is a (Σ,E)-algebra, (KA,ϵ) is a (Σ,E)-algebra by Lemma 4.5. To see that $K^\star$ is functorial, let $f : (A,\delta) \to (A',\delta')$ be a Σ-homomorphism. For $\omega \in \Sigma_n$, we must show that the following square commutes:

$$\begin{array}{ccc} \otimes_n KA & \xrightarrow{\otimes_n Kf} & \otimes_n KA' \\ {\scriptstyle \epsilon_\omega}\downarrow & & \downarrow{\scriptstyle \epsilon'_\omega} \\ KA & \xrightarrow[Kf]{} & KA' \end{array}$$

We have

$$\begin{aligned} (Kf)\epsilon_\omega &= (Kf)(K\delta_\omega)(\Gamma^n) \\ &= (K\delta'_\omega)K(\otimes_n f)(\Gamma^n) && \text{(since } f \text{ is a } \Sigma\text{-homomorphism)} \\ &= (K\delta'_\omega)(\Gamma^n)(\otimes_n Kf) && \text{(since } \Gamma \text{ is natural)} \\ &= \epsilon'_\omega(\otimes_n Kf)\,. \end{aligned}$$

To complete the proof, it suffices, by Theorem 1.2, to show that $\rho_A : (A,\delta) \to K^\star(A,\delta)$ and $\nu_A : K^\star K^\star(A,\delta) \to K^\star(A,\delta)$ are Σ-homomorphisms.

— For the first statement,

$$\begin{aligned} \epsilon_\omega\,(\otimes_n \rho_A) &= (K\delta_\omega)\,\Gamma^n\,(\otimes_n \rho_A) \\ &= (K\delta_\omega)\,\rho_{\otimes_n A} && (\Gamma A) \\ &= \rho_A\,\delta_\omega && (\rho \text{ natural}) \end{aligned}$$

— For the second statement,

$$\begin{aligned} \epsilon_\omega\,(\otimes_n \nu_A) &= (K\delta_\omega)\,\Gamma^n\,(\otimes_n \nu_A) \\ &= (K\delta_\omega)\,\nu_{K(\otimes_n A)}\,(K\Gamma^n)\,(\Gamma^n K) && (\Gamma B) \\ &= \nu_A\,(KK\delta_\omega)\,(K\Gamma^n)\,(\Gamma^n K) && (\nu \text{ natural}) \\ &= \nu_A\,(K\epsilon_\omega)(\Gamma^n K), \end{aligned}$$

as desired. □

We now ask whether we can relate the distributive laws of Theorems 3.6 and 4.6. The next result shows the answer is yes. First we recall a useful definition from Manes and Mulry (2007).

Definition 4.7. Let

$$\begin{aligned} \mathbf{H} &= (H,\mu,\eta) \\ \mathbf{H}' &= (H',\mu',\eta') \\ \mathbf{K} &= (K,\nu,\rho) \\ \mathbf{K}' &= (K',\nu',\rho') \end{aligned}$$

be monads in $\mathbf{C}$ and let $\lambda : HK \to KH$, $\lambda' : H'K' \to K'H'$ be distributive laws. A *morphism of distributive laws* $\lambda \to \lambda'$ is a pair (σ,τ) where $\sigma : \mathbf{H} \to \mathbf{H}'$, $\tau : \mathbf{K} \to \mathbf{K}'$ are monad maps such that the following square commutes:

$$\begin{array}{ccc} HK & \xrightarrow{\lambda} & KH \\ {\scriptstyle \sigma\tau}\downarrow & & \downarrow{\scriptstyle \tau\sigma} \\ H'K' & \xrightarrow[\lambda']{} & K'H' \end{array}$$

This gives a category of distributive laws with identities (id, id) and composition given by $(\sigma_1,\tau_1)(\sigma,\tau) = (\sigma_1\sigma,\tau_1\tau)$.

Corollary 4.8. Let (Σ, E), $\mathbf{H}' = (H',\mu',\eta')$, $\mathbf{K} = (K,\nu,\rho)$ and Γ be the same as in Theorem 4.6. Let $\lambda : \mathbf{\Sigma}^@\mathbf{K} \to \mathbf{K}\mathbf{\Sigma}^@$ be the distributive law of Theorem 3.6 and let $\lambda' : \mathbf{H}'\mathbf{K} \to \mathbf{K}\mathbf{H}'$ be the distributive law of Theorem 4.6. Then there exists a canonical monad map $\psi : \Sigma^@ \to H$ such that $(\psi, id) : \lambda \to \lambda'$ is a map of distributive laws.

Proof. Using $\tau'_\omega : (\otimes_n H') \to H'$ to denote the (Σ, E)-algebra associated to H'-algebra (H',μ'), we have that τ' satisfies (C_ω) by the definition of τ'(recall that τ denotes the corresponding Σ-algebra associated with monad $\Sigma^@$, see Theorem 3.6). By Corollary 3.4, there is a monad map $\psi : \Sigma^@ \to H'$, where $\mathbf{\Sigma}^@ = (\Sigma^@,\mu,\eta)$, so $\gamma'\psi = \alpha$ where $\alpha(\gamma')$ are the $(\Sigma^@H')$-algebras respectively associated with $K^\star\tau'$. Furthermore, by Theorem 4.6, the distributive law $\lambda' : H'K \to KH'$ is the one associated with $K^\star$ and defined by $K^\star(A,\delta) = (KA, (K\delta)\Gamma)$ acting on (Σ, E)-algebras, while the distributive law $\lambda : \Sigma^@K \to K\Sigma^@$ is generated by a $K^\star$ defined in exactly the same way, but now acting on Σ-algebras. Thus $K^\star\psi$ is a map of Σ-algebras, and hence of $\Sigma^@$-algebras, so $(K\psi)\gamma = \alpha(\Sigma^@ K\psi)$ where γ is the $\Sigma^@$-algebra associated to $K^\star\tau$. We have

$$\begin{aligned}
(K\psi)\lambda &= (K\psi)\gamma(\Sigma^@ K\eta) && \text{(by definition of } \lambda\text{)}\\
&= \alpha(\Sigma^@ K\psi)(\Sigma^@ K\eta) &&\\
&= \alpha(\Sigma^@ K\eta') && \text{(by MMA)}\\
&= (\gamma'\psi_{KH'})(\Sigma^@ K\eta') && \text{(by definition of } \alpha\text{)}\\
&= \gamma'(H'K\eta')\psi_K && \text{(by naturality of } \psi\text{)}\\
&= \lambda'\psi_K && \text{(by definition of } \lambda'\text{)}
\end{aligned}$$

so (ψ, id) is a map of distributive laws. □

Example 4.9. In Example 3.10, four different Kleisli strengths of order 2 are identified on the monad V^+, each generating a separate distributive law $\lambda_{i,j}$. Only two, however, $\eta(left \times left)$ and $\eta(right \times right)$, are coherent. Applying these two to Theorem 4.6 produces a family of two distributive laws $HV^+ \to V^+H$ where H is any quotient monad of V^+ generated by a set of linear equations E.

For example, if E consists of the single equation $(x \star y) \star z = x \star (y \star z)$, then H is the monad of non-empty lists L^+ where the canonical monad map ψ of Corollary 4.8 is exactly the monad map *list* of Example 3.5, which is now a map of distributive laws $list : \lambda_{i,i} \to \lambda'_{i,i}$ for $i = 1, 2$. The two distributive laws $\lambda'_{i,i} : L^+V^+ \to V^+L^+$ are defined by:

$$\lambda'_{i,i}[t] = t^{**} \qquad \text{(where } t^{**} \text{ has the same shape as } t \text{ but all the leaf values are changed from } La \text{ to } L[a]\text{)}$$

$$\lambda'_{i,i}(l_1 \mathbin{+\!\!\!+} l_2) = L(m_i\lambda'_{i,i}l_1 \mathbin{+\!\!\!+} m_i\lambda'_{i,i}l_2)\,.$$

For instance, $\lambda'_{i,i}[N(N(La, Lb), Lc), N(Ld, N(Le, Lf))] = L[a, d]$.

Example 4.10. In the previous example, replacing the monad V^+ with its quotient monad L^+ and the corresponding coherent Kleisli strengths with $\eta(fst \times fst)$ and $\eta(lst \times lst)$ (see Example 2.18), generates two distributive laws $\lambda'_{i,i} : L^+L^+ \to L^+L^+$, $i = 1, 2$. Now, $\lambda'_{i,i}[[a_1 \ldots a_n]] = [[a_1], \ldots [a_n]]$, while $\lambda'_{i,i}(l_1 \mathbin{+\!\!\!+} l_2) = [(m_i\lambda'_{i,i}l_1 \mathbin{+\!\!\!+} m_i\lambda'_{i,i}l_2)]$ where $m_1 = fst$ and $m_2 = lst$. For example, $\lambda'_{1,1}([[a], [b, c], [d]]) = [[a, b, d]]$. Note that both of these distributive laws differ from the one described in Koslowski (2005) and Manes and Mulry (2007), where, for instance, $\lambda'_{1,1}([[a], [b, c], [d]]) = [[a, b], [c, d]]$.

Example 4.11. In Example 3.14, distributive laws of the form $\lambda : \mathbf{\Sigma^@ P_0} \to \mathbf{P_0 \Sigma^@}$ were generated. The Kleisli strength of Example 2.4 is coherent as cartesian product is associative. Thus, we have distributive laws of the form $\lambda' : HP_0 \to P_0H$ for H any quotient monad of $\Sigma^@$ generated by a set of linear equations E. For instance, for V and L as in the previous example, we have a distributive law $\lambda' : LP_0 \to P_0L$ of $\mathbf{L}$ over $\mathbf{P_0}$ defined by $\lambda'_X([A_1, \ldots, A_n]) = \{[a_1, \ldots, a_n] : a_i \in A_i\}$. The canonical monad map $list : V \to L$ is again a map of distributive laws. For instance

$$\begin{aligned}
(P\,list)\lambda(N(A_0, B_0)) &= (P\,list)\{N(La, Lb) : a \in A_0, b \in B_0\}\\
&= \{[a, b] : a \in A_0, b \in B_0\}\,,
\end{aligned}$$

which agrees with

$$\lambda'(list_P)(N(A_0,B_0)) = \lambda'[A_0,B_0] = \{[a,b] : a \in A_0, b \in B_0\}.$$

Example 4.12. In Example 3.15, two distinct distributive laws, $\lambda_a, \lambda_b : V(X+Exc) \to VX + Exc$, associated with Kleisli strengths Γ_a and Γ_b on the exceptions monad, were generated. Both strengths are coherent and, consequently, we immediately have two distributive laws $\lambda'_a, \lambda'_b : L(X+Exc) \to (LX)+Exc$. For instance, λ'_a is defined by

$$\begin{aligned} \lambda'_a([\,]) &= [\,] & \\ \lambda'_a([x]) &= x & \text{for any element } x \text{ of } Exc \\ \lambda'_a(l) &= l & \text{if no element of } l \text{ is in } Exc \\ \lambda_a(l) &= a & \text{otherwise.} \end{aligned}$$

It is easy to generalise this to the case of Exc being a non-empty set of exceptions of any size.

Example 4.13. Manes and Mulry (2007) showed that the state monad (Moggi 1991), $\mathbf{M} = (\mathbf{M}, \nu, \rho)$ where $MA = (A \times S)^S$ was not commutative. For the case of Kleisli strength, the result is different. For instance, there are two Kleisli strengths of order two on monad M, Γ_1 and Γ_2. We define $\Gamma_i : (A \times S)^S \times (B \times S)^S \to (A \times B \times S)^S$ as follows:

$$\Gamma_1(t_1,t_2) = \lambda s.let(a,s_1) = t_1(s), let(b,s_2) = t_2(s), in(a,b,s_1)$$
$$\Gamma_2(t_1,t_2) = \lambda s.let(a,s_1) = t_1(s), let(b,s_2) = t_2(s), in(a,b,s_2)$$

It is a straightforward exercise to show that the Γ_i are Kleisli strengths. Furthermore, each is associative and so generates a coherent family by Example 2.14. For example, both $\Gamma_1(\Gamma_1 \times 1)(t_1,t_2,t_3)$ and $\Gamma_1(1 \times \Gamma_1)(t_1,t_2,t_3)$ are equal to

$$\lambda s.let(a,s_1) = t_1(s), let(b,s_2) = t_2(s), let(c,s_3) = t_3(s), in(a,b,c,s_1).$$

Consequently, Kleisi strength generates two distributive laws:

$$\lambda_i : V((A \times S)^S) \to (VA \times S)^S.$$

While coherence generates two more:

$$\lambda'_i : L((A \times S)^S) \to (LA \times S)^S.$$

Acknowlegements

We would like to thank the referees for their expert reading of this paper. In particular, we thank one referee for noting that the λ map in Manes and Mulry (2007, Example 2.4.7) does not generate a distributive law because the 'families monad' is, in fact, not a monad.

References

Ádamek, J. and Lawvere, F. W. (2001) How Algebraic is algebra? *Theory and Applications of Categories* **8** 253–283.

Barr, M. (1970) Coequalizers and free triples. *Mathematische Zeitschrift* **116** 307–322.

Beck, J. (1969) Distributive laws. *Springer-Verlag Lecture Notes in Mathematics* **80** 119–140.

Eilenberg, S. and Kelly, G. M. (1966) Closed categories. *Proceedings of the Conference on Categorical Algebra, La Jolla*, Springer-Verlag 421–562.

King, D. and Wadler, P. (1993) Combining monads. In: Launchbury, J. and Sansom, P. M. (eds.) *Proceedings of the 1992 Glasgow Workshop on Functional Programming*, Springer-Verlag 134–143.

Kock, A. (1970) Monads on symmetric monoidal closed categories. *Archiv der Mathematik* **21** 1–10.

Koslowski, J. (2005) A monadic approach to polycategories. *Theory and Applications of Categories* **14** 125–156.

Linton, F. E. J. (1966) Autonomous equational categories. *Journal of Mathematics and Mechanics* **15** 637–642.

Lüth, C. and Ghani, N. (2002) Composing Monads Using Coproducts. *International Conference on Functional Programming ICFP'02*, ACM Press 133–144.

Mac Lane, S. (1971) *Categories for the Working Mathematician*, Springer-Verlag.

Marmolejo, F., Rosebrugh, R. and Wood R. J. (2002) A basic distributive law. *Journal of Pure and Applied Algebra* **168** 209–226.

Manes, E. G. and Mulry, P. S. (2007) Monad Compositions I: General Constructions and Recursive Distributive laws. *Theory and Applications of Categories* **18** 172–208.

Manes, E. G. and Mulry, P. S. (2008) Monad Compositions III. (To appear.)

Moggi E. (1991) Notions of computations and monads. *Information and Computation* **93** 55–92.

Mulry P. S. (1994) Lifting theorems for Kleisli categories. *Springer-Verlag Lecture Notes in Computer Science* **802** 304–319.

Mulry P. S. (1998) Monads in semantics. *Electronic Notes in Theoretical Computer Science* **12** 275–286.

Scott D. (1970) Outline of a mathematical theory of computation. Technical Monograph PRG-2, Oxford University Computing Laboratory.