A Generalisation of Jacobi's Fundamental Formulae

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1. Jacobi obtained his well-known formulae by a purely algebraic method[1], but it was not until H. J. S. Smith had obtained them similarly[2], but by the use of a more symmetrical notation, that they were put into the form by which they are known today.

The basic formula is

$$\begin{aligned}2\vartheta_3(w)\,\vartheta_3(x)\,\vartheta_3(y)\,\vartheta_3(z)\\ =\vartheta_3(w')\,\vartheta_3(x')\,\vartheta_3(y')\,\vartheta_3(z')+\vartheta_4(w')\,\vartheta_4(x')\,\vartheta_4(y')\,\vartheta_4(z')\\ +\vartheta_2(w')\,\vartheta_2(x')\,\vartheta_2(y')\,\vartheta_2(z')-\vartheta_1(w')\,\vartheta_1(x')\,\vartheta_1(y')\,\vartheta_1(z'),\end{aligned}$$

where

$$\begin{aligned}2w' &= -w+x+y+z,\\ 2x' &= w-x+y+z,\\ 2y' &= w+x-y+z,\\ 2z' &= w+x+y-z.\end{aligned}$$

The other formulae may be obtained from this by the addition of suitable half periods to the variables.

The formulae in this paper are obtained by a generalisation of Smith's method and, so far as I am aware, they have not been given previously. They relate a product of $2N$ theta-functions in one set of variables to a sum of N^2 terms in the other set, each term containing a product of $2N$ theta-functions. For the general case, with $N > 2$, they do not have such a wide range of application as Jacobi's formulae, but nevertheless they possess an elegance which makes them worth recording.

2. To simplify the notation we will adopt the following conventions throughout. We suppose that N is a positive integer, and we write the sum

$$x_1+x_2+\ldots+x_{2N}=\sum_{j=1}^{2N} x_j$$

and omit the limits of summation when no risk of confusion occurs. We also write similar expressions in m, m^2, n, n^2, mx, nX as $\Sigma\, m_j$, $\Sigma\, m_j{}^2$, $\Sigma\, n_j$,

[1] C. G. J. Jacobi, *Ges. Werke*, I, pp. 502–6.

[2] H. J. S. Smith, *Proc. London Math. Soc.*, I (1866), 1–12.

$\Sigma\, n_j^2$, $\Sigma\, m_j\, x_j$, $\Sigma\, n_j\, X_j$. A product of $2N$ terms of this kind is indicated in the same manner, for instance $\prod_{j=1}^{2N} \vartheta_3(x_j)$ is written as $\Pi\, \vartheta_3(x_j)$.

3. Consider the set of equations

$$N(X_i + x_i) = \Sigma\, x_j, \tag{1}$$

$$N(n_i + m_i) = \Sigma\, m_j, \tag{2}$$

where i runs from 1 to $2N$.

It will be seen immediately that

$$\Sigma\, n_j = \Sigma\, m_j, \quad \Sigma\, n_j^2 = \Sigma\, m_j^2, \quad \Sigma\, n_j\, X_j = \Sigma\, m_j\, x_j.$$

We now take the product of $2N$ ϑ_3s, expand them in series and use the relations we have just obtained. We get

$$\begin{aligned}\Pi\, \vartheta_3(x_j) &= \sum_{n_j=-\infty}^{\infty} \cdots \sum_{(j=1,2,\ldots,2N)} q^{\Sigma n_j^2} e^{2i\, \Sigma n_j X_j} \\ &= \sum_{m_j} \cdots \sum_{(j=1,2,\ldots,2N)} q^{\Sigma m_j^2} e^{\Sigma m_j x_j}.\end{aligned}$$

The summation in this last term will be over all the values of m_j corresponding to the values of n_j and, from (2), these will be of the form: an integer plus $(\Sigma\, m_j)/N$. We separate this sum into N sets of terms S_r, where $r = 0, 1, 2, \ldots, N-1$, in which $\Sigma\, m_j \equiv r \pmod N$.

Then

$$\Pi\, \vartheta_3(X_j) = \sum_{r=0}^{N-1} S_r$$

and

$$S_r = \sum_{m_j} \cdots \sum_{(j=1,2,\ldots,2N)} q^{\Sigma m_j^2} e^{2i\, \Sigma m_j x_j},$$

which is summed over all m_j such that $m_j =$ an integer plus r/N and $\Sigma\, m_j \equiv r \pmod N$.

We can introduce a factor

$$\frac{1}{N} \sum_{s=0}^{N-1} e^{2\pi i(\Sigma m_j - r)s/N}$$

into S_r, and sum over all m_j equal to an integer plus r/N, since this factor is unity if $\Sigma\, m_j \equiv r \pmod N$, and is the sum of the N-th roots of unity if $\Sigma\, m_j \not\equiv r \pmod N$.

We write then

$$S_r = \frac{1}{N} \sum_{s=0}^{N-1} \sum_{l_j=-\infty}^{\infty} \cdots \sum_{(m_j = l_j + r/N)} e^{2\pi i(\Sigma m_j - r)s/N} q^{\Sigma m_j^2} e^{2i\, \Sigma m_j x_j}.$$

In this formula, and in what follows, the l_j are integers corresponding to the integer parts of the m_j, and they are summed from $-\infty$ to $+\infty$.

Putting $m_j = l_j + r/N$ we get

$$S_r = \frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi i r s/N} \sum_{l_j=-\infty}^{\infty} \ldots \sum_{(j=1,2,\ldots,2N)} e^{(2\pi i s/N)\,.\,\Sigma(l_j+r/N)}\, q^{\Sigma(l_j+r/N)^2} \times \exp\{2i\Sigma(l_j+r/N)x_j\}.$$

Remembering that $q = e^{\pi i\tau}$ we can write this as

$$\frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi i r s/N}\prod e^{2irx_j/N}\, e^{2\pi i r s/N^2}\, q^{r^2/N^2} \sum_{l_j=-\infty}^{\infty} q^{l_j^2}\exp\left\{2il_j\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right)\right\}$$

$$= \frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi i r s/N}\prod e^{2irx_j/N}\, e^{2\pi i r s/N^2}\, q^{r^2/N^2}\,\vartheta_3\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right)$$

$$= \frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi i r s/N}\prod \exp\left\{\frac{2ir}{N}\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right)\right\} q^{-r^2/N^2}\,\vartheta_3\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right).$$

Our relation becomes finally

$$N\prod_{j=1}^{2N}\vartheta_3(X_j)$$
$$= \sum_{r,s,=0}^{N-1}\sum e^{-2\pi i r s/N}\prod_{j=1}^{2N}\exp\left\{\frac{2ir}{N}\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right)\right\} q^{-r^2/N^2}\,\vartheta_3\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right).$$

Three other formulae are immediately derived from this by increasing all the X_j by $\frac{1}{2}\pi$, $\frac{1}{2}\pi\tau$, $\frac{1}{2}\pi+\frac{1}{2}\pi\tau$, so that the x_j are also increased by $\frac{1}{2}\pi$, $\frac{1}{2}\pi\tau$, $\frac{1}{2}\pi+\frac{1}{2}\pi\tau$.

These formulae are

$$N\prod_{j=1}^{2N}\vartheta_4(X_j)$$
$$= \sum_{r,s=0}^{N-1}\sum e^{-2\pi i r s/N}\prod_{j=1}^{2N}\exp\left\{\frac{2ir}{N}\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right)\right\} q^{-r^2/N^2}\,\vartheta_4\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right),$$

$$N\prod_{j=1}^{2N}\vartheta_2(X_j)$$
$$= \sum_{r,s=0}^{N-1}\sum e^{-2\pi i r s/N}\prod_{j=1}^{2N}\exp\left\{\frac{2ir}{N}\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right)\right\} q^{r(N-r)/N^2}\,\vartheta_2\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right),$$

$$N\prod_{j=1}^{2N}\vartheta_1(X_j)$$
$$= \sum_{r,s=0}^{N-1}\sum e^{-2\pi i r s/N}\prod_{j=1}^{2N}\exp\left\{\frac{2ir}{N}\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right)\right\} q^{r(N-r)/N^2}\,\vartheta_1\left(x_j+\frac{\pi s}{N}+\frac{\pi r\tau}{N}\right).$$

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